Cascade representations for the Navier–Stokes equations

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1 Introduction

The basic equations governing the motion of a fluid are well understood. For simplicity, we shall refer to the case of an incompressible, constant density, viscous Newtonian fluid; the velocity vector field $u(t,x)$ and pressure scalar field $p(t,x)$ satisfy the classical Navier-Stokes equations (in dimension 3) with viscosity $\nu > 0$

$$\begin{aligned} \partial_t u + (u\cdot\nabla)u + \nabla p &= \nu\Delta u, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1}$$

with appropriate initial and boundary conditions depending on the problem. For relatively simple fluid motions, these equations give us a very good tool for simulations and physical understanding. But there are complex fluid motions, those usually called turbulent, where special features are experimentally or numerically observed which do not have a clear explanation yet from the Navier-Stokes equations. In a sense, there is something at the foundation of fluid dynamics that is still unclear. For later reference, let us mention that this happens when a certain parameter R, called Reynolds number, is very large.

Andrej Nikolaevič Kolmogorov, in his celebrated paper on turbulence [18], where he exposed very innovative ideas referred to as the K41 theory, used the following sentences to describe something which is a sort of idealization of experimental observations: "on the averaged flow are superposed the 'pulsations of the first order' consisting in disorderly displacements of separate fluid volumes [...] of

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diameters of the order of magnitude $l^{(1)} = l$ [...]. The pulsations of the first order are for very large R in their turn unsteady, and on them are superposed the pulsations of the second order with mixing path $l^{(2)} < l^{(1)}$. [...] the pulsations of the first order absorb the energy of the motion and pass it over successively to pulsations of higher orders."

This is an intuitive description of the so called *direct cascade*. Kinetic energy flows from larger scale structures (the "pulsations", often called eddies, which may be also thin vortex tubes or patches) to smaller ones, due to dynamical instabilities. Mathematics further idealizes these structures by means of the concept of Fourier component. So, opposite to regular fields, where Fourier components with small wave number k contain most of the energy and it decays very fast for large k's, in a turbulent fluid energy is distributed in a distinguished fashion between Fourier modes, with a sort of "long tail", also related to a poor regularity (at least in the limit of zero-viscosity).

At the same time, mostly in fluids with a 2D symmetry, it is experimentally observed that an *inverse cascade* takes place: energy contained in small scale structures cumulates to increase the energy of larger scale structures. Strictly speaking, in every non trivial fluid there are both kind of cascade, direct and inverse, but their intensity may be different. We suggest to read U. Frisch [13] for an extensive discussion of cascade models, Kolmogorov theory and turbulence.

The problem, thus, may be summarized as the question of understanding the interaction between modes, the exchange of energy between them, starting from the Navier-Stokes equations.

Without claiming that it solves this problem, we however like to review some "cascade representation" formulae for solutions to the Navier-Stokes equations, which are clearly based on the interaction between modes. Rabi Bhattacharya contributed to develop this interesting approach, that we shall review in next pages.

2 Fourier formulation of the Navier-Stokes equations

For simplicity, let us consider equations (1) on the torus $[0,2\pi]^3$. Write the Fourier series $u(t,x) = \sum_{k\in\mathbb{Z}^3} u_k(t)\, e^{ik\cdot x}$, $p(t,x) = \sum_{k\in\mathbb{Z}^3} p_k(t)\, e^{ik\cdot x}$ where $u_k(t)$ takes values in $\mathbb{C}^3$. The divergence–free condition $\operatorname{div} u = 0$ reads in Fourier variables as $k \cdot u_k(t) = 0$. By replacing the Fourier series into equations (1) and by projecting onto the plane orthogonal to k to get rid of the pressure, we get

$$\frac{d}{dt}u_k(t) + i \sum_{m+n=k} (u_m(t)\cdot k)\, \pi_k u_n(t) = -\nu |k|^2 u_k(t), \tag{2}$$

where the projection is defined as $\pi_k v = \left(I - \frac{k\otimes k}{|k|^2}\right)v$, for $v \in \mathbb{R}^3$, and we have used the identity $u_{k_1}(t)\cdot k_2 = u_{k_1}(t)\cdot k$, following from the divergence free condition. This equations are already full of information about the interaction between modes. The difficulty lies in the nature of the bi-linear map

$$(u_m,u_n)\longmapsto b_k(u_m,u_n):=i\big(u_m\cdot\tfrac{k}{|k|}\big)\pi_k u_n$$

which fulfills the easy bound $|b_k(u_m,u_n)|\leq|u_m|\,|u_n|$ but certainly has other more hidden algebraic properties of major importance, however not easy to exploit. From (2) one deduces the energy balance

$$\tfrac{1}{2}\tfrac{d}{dt}|u_k|^2+|k|\sum_{m+n=k}c^u_{k,m,n}|u_k|\,|u_m|\,|u_n|=-\nu|k|^2|u_k|^2$$

where the coefficients $c^u_{k,m,n}(t)$, depending on the solution, are given by

$$c^u_{k,m,n}(t)=b_k(\widetilde{u}_m(t),\widetilde{u}_n(t))\cdot\widetilde{u}_k(t)$$

where for any k we write $\widetilde{u}_k=\frac{u_k}{|u_k|}$. We have written the energy balance in this peculiar form for comparison with the identity (4) below. The difficulty here is that it is not clear when the energy flux is stronger in a direction more than another. As a comparison, let us mention the following simplified model (called dyadic model of turbulence, see [11], [3], [2]), made of scalar valued equation

$$\tfrac{d}{dt}X_n(t)=-\nu k_n^\alpha X_n(t)+k_{n-1}X_{n-1}^2(t)-k_nX_n(t)X_{n+1}(t)\tag{3}$$

where it is possible to understand very well the flux of energy between components, at least when all $X_n(t)$ are positive. Indeed here

$$\tfrac{1}{2}\tfrac{d}{dt}|X_n(t)|^2=-\nu k_n^\alpha|X_n(t)|+k_{n-1}X_{n-1}^2(t)X_n(t)-k_n|X_n(t)|^2X_{n+1}(t)\tag{4}$$

and thus, for solutions with all positive components, the energy $\frac{1}{2}|X_n(t)|^2$ of mode n increases due to mode $n-1$ and decreases due to mode $n+1$. The energy flux is from large scale to small scale structures.

For equations (2) this is still obscure. There are brilliant rigorous examples, however, in the literature, where something has been said. Let us mention A. Shnirelman [28], who uses the fact that Fourier pairs of modes (k_1,k_2), of the form $k_2\sim-k_1$, or more precisely $k_2=-k_1+k_0$, with small $k_0\in\mathbb{Z}^3$, produce an effect $b_{k_0}\left(u_{k_1},u_{k_2}\right)$ at mode k_0 close to the origin. For large $|k_1|$ (hence large $|k_2|$) we have a sort of inverse cascade, we have small scale structures which transfer energy to large scale structures. In [28] the inviscid equations are considered

$$\tfrac{d}{dt}u_k(t)+|k|\sum_{m+n=k}b_k(u_m,u_n)=f_k^N(t),\qquad u_k(0)=0$$

with $f^N=\left(f_k^N\right)_{k\in\mathbb{Z}^3}$ having high amplitude Fourier components at some k_1 and $k_2=-k_1+k_0$ with $|k_1|\sim N$ and $|k_0|\sim1$ (and $f_k^N=0$ for the other k's). As $N\to\infty$ the forcing term converges weakly to zero and the solution u^N maintains, due to the inverse cascade, a non-zero amplitude at k_0, producing in the limit a non-zero solution which started from the zero initial condition, without forcing term (in

particular, that solution is not energy preserving). The precise construction in [28] is obviously more elaborated than the short description given here.

3 Picard iteration and deterministic cascade representation

We may rewrite equation (2) as

$$u_k(t) = e^{-\nu|k|^2 t} u_k(0) + \int_0^t e^{-\nu|k|^2(t-s)} |k| \sum_{m+n=k} b_k(u_m(s), u_n(s))\, ds. \tag{5}$$

A natural scheme to prove for instance existence of solutions, or for the numerical approximation etc., is the iteration

$$u_k^{n+1}(t) = e^{-\nu|k|^2 t} u_k(0) + \int_0^t e^{-\nu|k|^2(t-s)} |k| \sum_{m+\ell=k} b_k(u_m^n(s), u_\ell^n(s))\, ds, \tag{6}$$

with $u_k^0(t) := e^{-\nu|k|^2 t} u_k(0)$. We follow here the presentation given by Gallavotti [14], Ch. 1, Section 12. Advanced results on this approach can be found in the paper by Bhattacharya et al. [4]. Since b_k is bi-linear, one can substitute $u_{k_1}^n(s)$ and $u_{k_2}^n(s)$ given in terms of $u_\cdot^{n-1}(\cdot)$ and $u_\cdot^0(\cdot)$ and so on, arriving at a series development based only on $u_\cdot^0(\cdot)$. For instance, just to have a rough idea, $u_k^2(t)$ is the sum of five terms, three of which are (the first one in place of the dots is just $u_k^0(t)$ but it is omitted for comparison with the picture below)

$$u_k^2(t) = \ldots + |k| \int_0^t e^{-\nu|k|^2(t-s)} \sum_{k_1+k_2=k} b_k(u_{k_1}^0(s), u_{k_2}^0(s))\, ds \tag{7}$$

$$+ |k| \int_0^t e^{-\nu|k|^2(t-s)} \sum_{k_1+k_2=k} |k_2| \int_0^s e^{-\nu|k_2|^2(s-r)} \cdot \cdot b_k\Big(\sum_{k_{21}+k_{22}=k_2} b_{k_2}(u_{k_{11}}^0(r), u_{k_{12}}^0(r)), u_{k_2}^0(s) \Big)\, dr\, ds + \ldots \tag{8}$$

$$+ |k| \int_0^t e^{-\nu|k|^2(t-s)} \sum_{k_1+k_2=k} |k_1|\,|k_2| \int_0^s \int_0^s e^{-\nu|k_1|^2(s-r)} e^{-\nu|k_2|^2(s-r')} \cdot \cdot b_k\Big(\sum_{k_{11}+k_{12}=k_1} b_{k_1}(u_{k_{11}}^0(r), u_{k_{12}}^0(r)), \sum_{k_{21}+k_{22}=k_2} b_{k_2}(u_{k_{21}}^0(r'), u_{k_{22}}^0(r')) \Big), \tag{9}$$

all given explicitly as suitable multi-linear combinations of $u_\cdot^0$. Figure 1 shows a (very rough) graphical representation of the three terms in equations (7), (8), (9).

In [14] the expansion is written symbolically as

$$u_k^n(t) = u_k^0(t) + \sum_{1 \le m \le 2^n} \sum_{\Theta \in m\text{-trees}} \Theta\left(k, t, u^0\right)$$

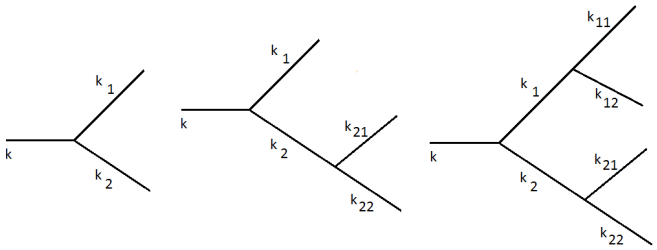

Fig. 1 Trees associated to the terms (7), (8), (9).

with a suitable definition of the set of all m-trees and of the operation $\Theta\left(k,t,u^0\right)$. Therefore, if a suitable limit takes place, we have the representation

$$u_k(t) = \sum_{m\geq 0} \sum_{\Theta\in m\text{-trees}} \Theta\left(k,t,u^0\right)$$

where, by convention, we write $\Theta\left(k,t,u^0\right) = u_k^0(t)$ if Θ is the 0-tree. For conceptual comparison with the probabilistic representations detailed below, we see here that, under proper conditions, one can express the solution as a series in terms of the initial condition.

Beside [4], let us also mention the classical work of T. Kato [17] and the recent approach of Y. Sinai [30], see also [1, 29] and [15], which share something with the arguments above.

4 Stochastic cascade and majorizing kernels

In the expressions above we see (up to a factor $|k|$) the function $\nu|k|^2 e^{-\nu|k|^2 t}$ which is the exponential density with average $\frac{1}{\nu|k|^2}$. Along with the splitting structure of modes $k \longleftrightarrow k_1 + k_2$, this resembles a probabilistic approach to PDEs, with exponential waiting times and random branching. The idea of using branching processes as the underlying engine of probabilistic representations is not new, let us mention H. P. McKean pioneering work [21], as well as [31, 16]. In these papers branching is coupled with diffusion, and the stochastic representation is derived directly in the physical space, so that the linear operator is limited to generators of diffusions and the non–linearity is polynomial.

4.1 The stochastic cascade of Le Jan and Sznitman

Le Jan and Sznitman [19] have seen in these ingredients the opportunity to develop a probabilistic representation formula, that they have called stochastic cascade representation.

Let us give a brief outline of the method. Here we borrow the presentation of [6]. Consider PDEs on $\mathbb{R}^d$ with periodic boundary conditions, possibly vector valued with values in $\mathbb{R}^r$, of the type

$$\partial_t u = Au + F(u) + f, \tag{10}$$

where A is an operator with a complete set of eigenfunctions, F is a polynomial non–linearity (that for simplicity here we assume quadratic) in u and its derivatives, and f is a given driving function. The case of full space can be considered with similar ideas.

In short, the solution u is expanded into Fourier series, and the PDE is transformed into a system of countably many ODEs for the, possibly rescaled[1] Fourier coefficients $\chi(t) : \mathbb{Z}^d \to \mathbb{C}^r$ that solve an infinite dimensional system of $\mathbb{C}^r$-valued ODEs

$$\dot{\chi}_k = \lambda_k \Big[-\chi_k + C_b \sum_{m,n\in\mathbb{Z}^d} q_{k,m,n} B_{k,m,n}(\chi_m, \chi_n) + d_k \gamma_k(t) \Big]. \tag{11}$$

with $k \in \mathbb{Z}^d$. The constants $\lambda_k > 0$ (that will determine the rate of particle evolution), $q_{k,m,n}$, $d_k \in [0,1]$ (which will determine the probabilities of branching and dying), and $C_b \geq 0$ (the branching constant) are fixed, as are bi-linear operators $B_{k,m,n} : \mathbb{C}^r \times \mathbb{C}^r \to \mathbb{C}^r$ satisfying

$$|B_{k,m,n}(\chi, \chi')| \leq |\chi|\,|\chi'|$$

for all $\chi, \chi' \in \mathbb{C}^r$. In view of the probabilistic representation we assume

$$q_k + d_k = 1, \quad k \in \mathbb{Z}^d, \qquad \text{and} \qquad q_k \to 0, \quad \text{as } |k| \to \infty$$

where $q_k = \sum_{m,n\in Z^d} q_{k,m,n}$, and we consider the system above in its mild formulation. There is considerable flexibility when choosing the coefficients of the ODE system, and one can adjust the probabilities $q_{k,m,n}$, and d_k by adjusting the constant C_b and considering rescaled forcing data γ. In particular, in an equation where the probabilities q_k, d_k do not add up to 1, it is always possible to adjust d_k and the forcing data so that this constraint holds. Similarly, an equation with C_b replaced by bounded functions of k can be recast into the same form by forcing the k dependence into the probabilities q_k and d_k.

We describe first the branching tree. Fix $k \in \mathbb{Z}^d$, a tree rooted at k is a system of particle positions, birth, branch and death times, defined inductively over the particles. At the root the birth time is zero, and the branching and death times are

[1] For instance, following Le Jan and Sznitman [19], for the three dimensional Navier–Stokes we set $\chi_k(t) = |k|^2 u_k(t)$.

exponential with rate λ_k. Given a tree, each particle, with position say k', either dies with probability $d_{k'}$, or disappear giving raise to two new independent particles, with positions m and n with probability $q_{k'mn}$. The new particles will have a lifespan distributed as independent exponential random variables with rates λ_m and λ_n. Notice that by construction, given a branching particle giving raise to two particles at positions m and n, and conditional to its genealogy, the two sub–trees generated are independent and with the same distribution of trees rooted at m and n. To ensure that the tree has only finitely many branches before a given time t, a sufficient condition is that $q_k \leq d_k$.

The solution of the system is represented by the expectation of a recursive functional R over a tree of branching particles. A branching event triggers the multiplication by $B_{k,m,n}$ of the two functionals corresponding to the two branches rooted at m, n, and a death event the evaluation of the external force. For instance if $r = 1$ and all the bi-linear forms $B_{k,m,n}$ coincide with the usual product in $\mathbb{C}$, then the evaluation over a tree $\mathscr{T}$ rooted at k is

$$R_t(\mathscr{T}) = C_b^{B_t} \prod_{\alpha \in D_t} \gamma_{k_\alpha}(t - t_\alpha) \prod_{\alpha : t \in [s_\alpha, t_\alpha)} \chi_{k_\alpha}(0),$$

where B_t and D_t are the number of particles that have branched and died, respectively, before time t. If $\chi(0) \in \ell^\infty(\mathbb{C}^r)$ and $\gamma \in L^\infty([0,T], \ell^\infty(\mathbb{C}^r))$, the representation formula for χ_k is given by the expectation of the functional on all trees rooted at k,

$$\chi_k(t) = \mathbb{E}_k[R_t],$$

whenever the expectation converges absolutely.

The Navier–Stokes equation in dimension three fit into the general scheme given above, with $d = r = 3$, and a suitable choice of the kernels q_{kmn} and of the products $B_{k,m,n}$ (here we take zero driving force). In particular, as we have already mentioned, Le Jan and Sznitman [19] take $\chi_k = |k|^2 u_k$.

4.2 Majorizing kernels

Rabi Bhattacharya and his group have considerably contributed to the analysis with a generalization of the stochastic cascade introduced above, see among others the papers [4, 5] and [24, 26, 33, 32, 10, 25], adding in particular a degree of freedom of conceptual importance, namely the *majorizing kernels*.

Let $h : \mathbb{Z}^3 \setminus \{0\} \to (0, \infty)$ be a function such that $(h * h)(k) \leq C|k|h(k)$ for some $C > 0$. Up to other details and some additional generality, a function h with the previous property is called a *majorizing kernel* (with exponent one). It generalizes the case $h(k) = 1/|k|^2$ treated by [19]. Following [4], given such h, setting $\chi_k(t) := u_k(t)/h(k)$, from (5) we get

$$\chi_k(t) = e^{-\nu|k|^2 t}\chi_k(0) + \int_0^t \nu|k|^2 e^{-\nu|k|^2 s} \sum_{m+n=k} \frac{h(m)h(n)}{\nu|k|h(k)} b_k(\chi_m(t-s),\chi_n(t-s))\,ds.$$

We introduce the Markov kernel $H_k(m,n) := \frac{h(m)h(n)}{(h*h)(k)}$ with support on the set of pairs (m,n) such that $m+n=k$. Setting $m(k) := \frac{(h*h)(k)}{\nu|k|h(k)}$ we have

$$\chi_k(t) = e^{-\nu|k|^2 t}\chi_k(0) + \nu|k|^2 \int_0^t e^{-\nu|k|^2 s} m(k) \sum_{k_1+k_2=k} H_k(k_1,k_2) b_k(\chi_{k_1}(t-s),\chi_{k_2}(t-s)).$$

The property of majorizing kernel guarantees that $m(k) \le 1$. Recall also that $|b_k(\chi_m,\chi_n)| \le |\chi_m|\,|\chi_n|$, already mentioned above. These two properties are of basic importance to control the convergence of the expected values described below.

When the stochastic cascade probabilistic scheme is applied to the last equation above, the solution is given by

$$u_k(t) = h(k)\mathbb{E}_k[R_t].$$

Up to details, the main result of [4] states that, when $\sup_{k\in\mathbb{Z}^3\setminus\{0\}} \frac{|u_k(0)|}{h(k)}$ is small enough (depending on ν), a unique solution exists of the Navier-Stokes equations, given by the probabilistic representation formula above. These theorems are competitive with those obtained by various authors using harmonic analysis tools, see for instance [9], and allow to capture several possible spaces of initial conditions, other than the pseudo-measures space of [19].

5 Pruning the trees

The major limitation of the previous two approaches (the deterministic and stochastic cascade representations) is that they apply only for relatively small (in suitable norms) initial conditions, since the series or the expected values have to converge similarly to a geometric series. There are cancellations, but it is very difficult to use them. Thus one has to majorize the complex multi-linear expressions by products of positive quantities; at the end this requires restrictive assumptions on data.

For a very simple model of those problems, namely a one-dimensional differential equation with quadratic non-linearity, it has been shown by F. Morandin [23] that Borel summability applies, a form of renormalization theory which takes advantage of cancellations. This approach allows one to treat arbitrary initial conditions. Unfortunately, at present, this technique did not find a proper extension even to two-dimensional ordinary differential equations.

The paper [6] provides both an explanation of the issue through a comparison equation, whose finiteness implies integrability of the recursive functional, and a way to avoid non–integrability by suitably pruning the tree.

5.1 The comparison equation

The comparison equation for the infinite dimensional system of ODEs is obtained essentially by neglecting any geometric information about the (vector) directions of the data in the system, namely, we consider a new infinite dimensional system by taking the norm of the data $|\chi_k(0)|$, γ_k and for the system (11), where each product $B_{k,n,m}$ is replaced by the standard product in $\mathbb{C}^r$, namely

$$\dot{\tilde{\chi}}_k = \lambda_k\big(-\tilde{\chi}_k + C_b \sum_{m,n\in\mathbb{Z}^d} q_{k,m,n}\tilde{\chi}_m\tilde{\chi}_n + d_k|\gamma_k|\big),$$

and let $\tilde{R}$ be the evaluation operator associated with the above equation. We now look for non-negative real solutions $\tilde{\chi}_k$, so that there is no issue in the definition of the expectation of $\tilde{R}$, as it takes values in the positive real numbers. Clearly, when the expectation of $\tilde{R}$ is finite, it provides a mild solution of the comparison equation. A sort of converse holds, as given in the following comparison theorem, see [6, Theorem 4.1].

Theorem 1. *If the expectations of $\tilde{R}$ are finite for all $t \in [0,T]$ and $k \in \mathbb{Z}^d$, then the expected values define a mild solution of the comparison equation.*

Conversely if there exists a finite mild solution of the comparison equation on $[0,T]$, then the expectations of the evaluation operator are finite for all $t \in [0,T]$ and $k \in \mathbb{Z}^d$.

Moreover, the probabilistic representation is the smallest positive solution of the comparison equation. Finally, the comparison $\mathbb{E}_k[|R_t|] \leq \mathbb{E}_k[\tilde{R}_t]$ holds, with equality whenever $|B_{k,m,n}(\chi,\chi')| = |\chi|\,|\chi'|$.

In other words the comparison equation essentially governs the convergence of the expectation of the original system (11), *but not the finiteness of the solutions of the original system* (10).

Let us consider a few examples. The first example is the most elementary, namely the one dimensional ODE $\dot{x} = -x + x^2$. The comparison equation is the same ODE, but with positive initial data only. Now we see immediately why the probabilistic representation blows up for initial data below -1, while the solution is global for the same initial data.

For a PDE example, consider the one dimensional Burgers equation

$$\partial_t u - \Delta u + (u\cdot\nabla)u = f,$$

with periodic boundary conditions and zero mean, where f is an external forcing. If we expand the solution in its Fourier coefficients u_k, define the weights $w_k = |k|^\alpha$, and set $\chi_k = w_k u_k$, $\lambda_k = |k|^2$ for $k \neq 0$, $q_{k,m,n} = C_b^{-1}\frac{|n|w_k}{\lambda_k w_m w_n}$, and $B_{k,m,n}(\chi,\chi') = -\mathtt{i}(\chi\cdot\frac{n}{|n|})\chi'$, whenever $m+n=k$ (and zero otherwise), then the Burgers system can be re-cast in the general form (11), if we choose $\alpha > 1$, C_b sufficiently large that $q_k < 1$ and define $d_k = 1 - q_k$ and $\gamma_k = (f_k w_k/\lambda_k d_k)$ for $k \in \mathbb{Z}^d$, $k \neq 0$.

Let us obtain now the comparison equation associated to the Burgers system. By defining $\tilde{u}_k = w_k^{-1}\tilde{\chi}_k$, we obtain a comparison equationthat in spatial coordinates reads

$$\partial_t \tilde{u} = \Delta\tilde{u} + \tilde{u}(-\Delta)^{\frac{1}{2}}\tilde{u} + \tilde{f}$$

where $\tilde{f}$ has Fourier coefficients $|f_k|$. Notice that this scalar comparison PDE is independent of the choice of weights, so in particular no majorizing kernel, as defined in the previous section, can fix the divergence. Similar conclusions can be given for the Navier–Stokes equations, but with a comparison equation that has a less simple and evocative comparison equation.

The scalar comparison equation associated to the Burgers system has quadratic growth and it is not difficult to show that, for instance with zero forcing and large enough initial data, solutions blow up in finite time, see for example [20] and the references therein for the case of branching with diffusion. Notice finally that, at least for $d = 1$, the equality in the last part of the Theorem above holds. This implies that the stochastic representation is well defined if and only if the corresponding comparison equation has a solution with finite Fourier coefficients.

5.2 Pruning the tree

To understand the idea of *pruning* introduced in [6], we consider the seemingly simple example $\dot{x} = -x + x^2$. The probabilistic representation of the solutions of this ODE is $x(t) = \mathbb{E}[u(0)^{N_t}]$, where N_t is the number of particles at time t of a simple rate one branching process starting from a single particle at time 0. The representation is well defined for all time $t \geq 0$ if and only if $|x(0)| \leq 1$, and is non well defined if $x(0) < -1$, where the ODE has global solutions. The underlying reason is that the absolute convergence of the expectation destroys the possible cancellations.

In order to try to take into account those cancellations, we try to keep the number of particles finite, so to approximate the expectation with a finite sum. To this aim we formulate a first modification of the branching process. Assign to each particle a label from the positive integers. Whenever a particle branches, the two offspring have label $n-1$. A particle with label 0 dies if tries to branch. Notice that this approximation is already given in [19] for their uniqueness proof (see also [4]). Let x_n be the expectation of the evaluation operator for these pruned trees. It turns out that the functions $(x_n)_{n\geq 0}$ satisfy the explicit iterative scheme

$$\dot{x}_n = -x_n + x_{n-1}^2,$$

that is the one dimensional counterpart of (6). Unfortunately, the limit as $n \to \infty$ of the above explicit iterative scheme fails to exists for large t if $x(0) < -1$.

To fully take into account the cancellations, we formulate a second and more effective modification. A branching particle with label n gives raise to two particles,

one with label $n-1$ and one *with the same label of its parent*[2]. The expectation of the evaluation operator this time leads to the semi–implicit iterative scheme

$$\dot{x}_n = -x_n + x_{n-1}x_n,$$

and it is straightforward to check that $x_n(t)$ is well defined for all n and t. Moreover, x_n converges to the solution $x(t)$ of the original ODE problem for each initial condition $u(0) \leq 1$. In other words we have the stochastic representation

$$u(t) = \lim_{n\to\infty} \mathbb{E}\big[x(0)^{N_t(n)}\big],$$

where $N_t(n)$ is the number of particles at time t originated by a root particle with label n.

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[2] In fact, when pruning a real tree, a good gardener always keeps the main growing direction of each main branch.

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