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# Comparison between Optimal Control Allocation with Mixed Quadratic & Linear Programming Techniques

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**Abstract:** The paper provides a comparison between different control allocation techniques in over-actuated Autonomous Underwater Vehicles. The pseudoinverse, Linear Programming (LP), Quadratic Programming (QP), Mixed Integer Linear Programming (MILP) and Mixed Integer Quadratic Programming (MIQP) are evaluated in simulation on the *V-Fides* vehicle model. The MILP and MIQP techniques allow to include in their implementations a more detailed characterization of the non-linear static behaviour of the actuators. This customizability can be also exploited to improve the practical stability of the system. The metrics used for comparison include the maximum attainable forces and torques, the integral of the error allocation and the required thrusters effort. Our simulation results show that, in particular with respect to thrusters effort, MILP and MIQP are the preferred allocation methods. The computational complexity associated to both methods is not such to compromise their implementation in operating vehicles; in particular, the MILP version is currently implemented in the *V-Fides* vehicle.



*Keywords:* Optimal Control allocation, Quadratic Programming, Linear Programming, Mixed-Integer Linear Programming, Mixed-Integer Quadratic Programming

## 1. INTRODUCTION

In the last decade improvements to control allocation methods have been proposed to achieve more accuracy, efficiency and effectiveness in control systems. Terrestrial, aerospace and marine vehicles can be over-actuated systems to improve manoeuvrability and to add flexibility and robustness. In the review paper, Johansen and Fossen (2013) and reference therein, the multidisciplinarity of control allocation problem is highlighted: from aerospace to underwater vehicles, solutions have been proposed and the cross-disciplinary transfer of ideas has been important for mutual progress in each sector. One of the advantages of over-actuated vehicles is their redundancy, allowing to a full or partial recovery if a fault on an actuator or effector occurs (Sarkar et al., 2002; Caiti et al., 2015). A modular design separating the control system from the control allocation allows an easy and portable implementation (Johansen and Fossen, 2013). Despite the advantages, sometime a complex resolution method must be implemented to solve the control allocation problem. In order to obtain a solution among the multiple ones that can result from redundancy, several methods based on the formulation of an optimal problem have been proposed in past years: from the classic approach, where unconstrained $\ell_2$-norm minimum optimal problem is solved by *pseudoinverse* technique (Fossen and Sagatun, 1991), to more complex and customizable structures like Linear Programming (LP) and Quadratic Programming (QP) (Bodson, 2002; Enns, 1998; Bodson and Frost, 2011) where the force generated by actuators are considered linear regarding to commanded input. More computational demanding methods, based on Mixed-Integer formulations, can be considered to describe non-linear static characterization of the actuators. The main principle is to break in several points the non-linear function to obtain a piecewise linear form implementable into the optimal problem as linear constraints. In Bemporad and Morari (1999) is explained how the piecewise linear functions are implemented in Model Predictive Control framework, while in Bertsimas and Tsitsiklis (1997) a general formulation is exposed. A similar approach is implemented in Bolender and Doman (2004), where a Mixed Integer Linear Programming (MILP) optimal problem is described to solve the control allocation in a two-stage fashion, defining in each one a different cost functional. Moreover, a Mixed Integer Quadratic Programming (MIQP)-like formulation is proposed in Johansen et al. (2003) to solve the control allocation problem in marine vessels with rudder actuators, where the set of attainable thrust vectors is non-convex.

This paper presents a comparison between several control allocation techniques to evaluate the respective pros and cons as applied to an over-actuated Autonomous Underwater Vehicle (AUV). In particular pseudoinverse, LP, QP, MILP and MIQP are considered. But for the pseudoinverse, the other formulations exploit the customizability of the structure to improve the manoeuvrability and stability


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of the vehicle. The MILP and MIQP are implemented including the dead-zone behaviour of the actuators static response into the optimal problem. The following metrics are used for comparison: the maximum attainable forces and torques, the integral of the error allocation and the required thrusters effort. The test case vehicle is the one developed within *V-Fides Project* and illustrated in Caiti et al. (2014). The vehicle simulator handles the hydrodynamic forces, kinematic equations, system control with allocation module and a detailed characteristic of actuators response.

The paper is organized as follow: in the next section the formal statement of the various optimal allocation problems is given. In Section 3 the main feature of the vehicle are described, the metric used for comparison is formally stated, and simulation results on a typical vehicle survey mission are reported. In last section results are discussed and conclusions are given.

## 2. PROBLEM STATEMENT

The control allocation problem can be defined as finding a set of input commands to the actuators such that the forces and torques exerted on manoeuvrable Degrees Of Freedom (DOFs) of the system are equal to the desired ones computed by the control module. In over-actuated systems the control allocation problem may admit multiple solutions due to the actuation redundancy. Therefore, several approaches were proposed in the past years to reformulate the control allocation into an optimal problem based on a proper cost function.

### 2.1 Pseudoinverse

In the simple and classic approach (Fossen and Sagatun, 1991), the allocation problem for over-actuated vehicles is reformulated as a $\ell_2$-norm optimal problem, which can be formalized as:

$$\begin{aligned} \min_{f} \quad & f^T W f \\ \text{subject to } & \tau_d - Tf = 0 \end{aligned} \tag{1}$$

By defining as $n$ the number of actuators and $m$ the manoeuvrable DOFs, $f \in \mathbb{R}^n$ are the forces produced by the actuators and $W \in \mathbb{R}^{n\times n}$ is a weighting positive-definite matrix. The static transformation matrix $T \in \mathbb{R}^{m\times n}$ includes the information of the position and thrust axes of each actuator and it is employed to map the forces generated by each actuator in the total forces and torques exerted on the vehicle. Therefore, the objective of the optimal problem is to minimize the error between the desired generalized forces $\tau_d \in \mathbb{R}^m$ and the ones exerted on the vehicles. The solution to the minimization in (1), can be thus obtained via

$$\begin{aligned} T_w^\dagger &= W^{-1}T^T(TW^{-1}T^T)^{-1} \\ f &= T_w^\dagger \tau_d \end{aligned} \tag{2}$$

Note that the problem is unconstrained regarding the resulting forces, that means to have at disposal hypothetical unlimited forces from actuators. To achieve a feasible solution, the forces are chunked with the saturations imposed by the physical limitations of the actuators. A simple way to obtain the normalized input commands to the actuators, $u \in [-1, 1] \subset \mathbb{R}^n$, is to define the linear relation $f = Ku$, with $K \in \mathbb{R}^{n\times n}$ the diagonal matrix of the gains that characterize the static response of actuators.

$$u = K^{-1}T_w^\dagger \tau_d \tag{3}$$

A further practical approach can be adopted to introduce a more detailed static characteristic into the problem by introducing a look-up table downstream of the control allocation. Nevertheless, this formulation is somehow limiting since the only DOF available is the weighting matrix $W$.

### 2.2 Quadratic & Linear Programming

Control allocation can be cast as a minimization problem of the errors between the allocated and desired forces with respect to a chosen norm $\| \cdot \|_\ell$.

$$\min_{f} \| \tau_d - Tf \|_\ell \tag{4}$$

A first step to add more information about the actuators in the problem is to insert the saturation as constraints on the allocated forces. Choosing the $\ell_2$-norm, the resulting optimal problem is solved with quadratic programming methods and can be formalized as following.

$$\begin{aligned} \min_{\alpha_s} \quad & \alpha_s^T H_s \alpha_s \\ & \tau_d - Tf = \alpha_s \\ & f_{min} < f < f_{max} \end{aligned} \tag{5}$$

Where $\alpha_s \in \mathbb{R}^m$ is the vector of the residuals, $f_{min}$ and $f_{max}$ the lower and upper bounds respectively imposed by the saturations and $H_s \in \mathbb{R}^{m\times m}$ is the definite positive weighting matrix. The formulation (5) takes into account only the minimization of the residuals, thus multiple sub-optimal solutions may exist with respect to the forces, $f$. Therefore, in energy saving mindset the $\ell_2$-norm of the allocated forces is evaluated in the cost functional.

$$\begin{aligned} \min_{f,\alpha_s} \quad & f^T H_f f + \alpha_s^T H_s \alpha_s \\ & \tau_d - Tf = \alpha_s \\ & f_{min} < f < f_{max} \end{aligned} \tag{6}$$

The $H_f \in \mathbb{R}^{m\times m}$ is the definite positive weighting matrix associated to the forces generated by the actuators. Observing the formulations (1) and (6), besides the addition of the saturations into the problem there is one more DOF in tuning perspective, $H_s$. In the later section will be shown how the correct tuning of $H_s$ can improve the practical stability of the system by a judicious choice of the weighting matrix.

The minimization (4) can be cast as well as $\ell_1$-norm and obtain a comparable formulation of (6). Thus, the sum of the quadratic residuals in the cost functional (6) are substituted as sum of the absolute values, $\sum_{i=1}^m K_{s_i} \mid \alpha_{s_i} \mid + \sum_{i=1}^n K_{f_i} \mid f_i \mid$, where $K_s^T \in \mathbb{R}^m$ and $K_f^T \in \mathbb{R}^n$ are the positive weighting vector of the residuals $\alpha_s$ and $f$, respectively. In (Boyd and Vandenberghe, 2004)

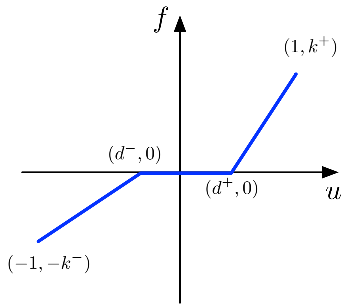


Fig. 1. An example of asymmetric characteristic defined by the positive, $k^+$, and negative, $k^-$, gains and a dead-zone in the range of input values $d^+$ and $d^-$.

is shown how the sum of absolute values included in the cost functional can be cast as LP with a *norm approximation* formulation. The resulting LP formulation of the optimal problem is as follows.

$$\begin{aligned} \min_{\alpha_s,\alpha_f} \quad & K_s\alpha_s + K_f\alpha_f \\ & -\alpha_s \leq \tau_d - Tf \leq \alpha_s \\ & -\alpha_f \leq f \leq \alpha_f \\ & f_{min} < f < f_{max} \\ & \alpha_s, \alpha_f \geq 0 \end{aligned} \tag{7}$$

In such procedure the variables $\alpha_s$ and $\alpha_f \in \mathbb{R}^n$ are the equivalent of the absolute value of the residuals and forces, respectively.
As in pseudinverse case, the QP and LP formulations are not suitable for implementing into the problem a more complex description of the actuators, i.e. in addition to the saturation, the relation is non-linear between input commands and forces, $f = K(u)$.

### *2.3 Mixed Integer Quadratic & Linear Programming*

The response of the actuators affected by non-linear behaviours can be approximated as piecewise linear function and introduced into the optimal control allocation problem as constraints. The mixed-integer formulations can describe multiple typologies of non-linear systems, as shown in Bemporad and Morari (1999). The logical statements can be converted from boolean algebraic to piecewise linear functions to solve an optimal non-linear problem with linear methods. In particular the authors are interested to describe the non-linearities in the static characteristic, such as dead-zones in neighbourhoods of the 0 input command as Fig. 1 shows. In Bertsimas and Tsitsiklis (1997) is proposed a set of equations to describe a piecewise linear function composed by $N$ segments. Each segment $i$ is described by the extremities $(X_i, Y_i)$ - $(X_{i+1}, Y_{i+1})$ on x-y plane and by one binary variable, $z_i \in \{0, 1\}$. In mutually exclusive way only one binary variable is selected, such that $\sum_{i=1}^{N} z_i = 1$. The piecewise linear function can be written as following:

$$\begin{aligned} & y = \sum_{i=1}^{N} w_i Y_i, \quad x = \sum_{i=1}^{N} w_i X_i \\ & \sum_{i=1}^{N} w_i = 1, \quad w_i \geq 0 \quad \forall i \in \{1, ..., N\} \\ & w_1 \leq z_1 \\ & w_i \leq z_{i-1} + z_i \quad \forall i \in \{2, ..., N-1\} \\ & w_N \leq z_{N-1} \\ & \sum_{i=1}^{N-1} z_i = 1, \quad z_i \in \{0, 1\} \end{aligned} \tag{8}$$

Where $w_i, i \in \{1, ..., N\}$ are the auxiliary variables and $z_i, i \in \{1, ..., N-1\}$ the binary variables. The implementation of the practical example shown in Fig. 1 can be done through the just presented formulation (8). The chosen segments are identified by the couples $(-1, -k^-)$, $(d^-, 0)$, $(d^+, 0)$, and $(1, k^+)$ in order to obtain the following representation.

$$\begin{aligned} & f = \sum_{i=1}^{4} w_i F_i = -k^- w_1 + 0w_2 + 0w_3 + k^+ w_4 \\ & u = \sum_{i=1}^{4} w_i U_i = -1w_1 + d^- w_2 + d^+ w_3 + 1w_4 \\ & \sum_{i=1}^{4} w_i = 1 \\ & w_i \geq 0 \quad \forall i \in \{1, 2, 3, 4\} \\ & w_1 \leq z_1 \\ & w_i \leq z_{i-1} + z_i \quad \forall i \in \{2, 3\} \\ & w_4 \leq z_3 \\ & \sum_{i=1}^{3} z_i = 1, \quad z_i \quad \forall i \in \{0, 1\} \end{aligned} \tag{9}$$

The formulation (9) is introduced for each actuator in QP optimal problem to describe the static characteristic with dead-zone and asymmetric response. The (6) is modified in accordance to the MIQP problem:

$$\begin{aligned} \min_{u,\alpha_s} \quad & u^T H_u u + \alpha_s^T H_s \alpha_s \\ & \tau_d - Bf = \alpha_s \\ & u \in [-1, 1] \end{aligned} \tag{10}$$

Where $H_u \in \mathbb{R}^{m \times m}$ is the definite positive weighting matrix associated to the normalized input commands and the force vector $f$ is described by the equations (9) for each actuator.
Same approach is adopted in the LP formulation (7) to obtain the MILP version comprehensive of constraints (9).

$$\begin{aligned} \min_{\alpha_s,\alpha_u} \quad & K_s\alpha_s + K_u\alpha_u \\ & -\alpha_s \leq \tau_d - Tf \leq \alpha_s \\ & -\alpha_u \leq u \leq \alpha_u \\ & u \in [-1, 1] \\ & \alpha_s, \alpha_u \geq 0 \end{aligned} \tag{11}$$

The vector $\alpha_u \in \mathbb{R}^n$ is equivalent to the absolute values of the normalized input commands.

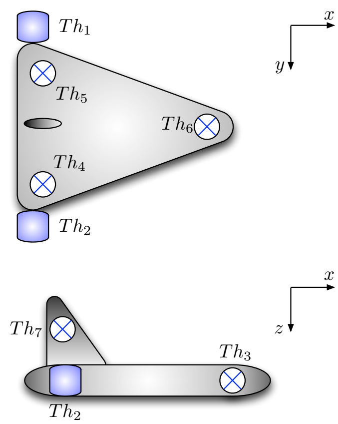


Fig. 2. Schematic representation of *V-Fides* vehicle. In the first figure is shown the view from above where $Th_1$, $Th_2$ are the propellers that thrust on surge axis; $Th_4$, $Th_5$ and $Th_6$ are the thrusters on heave axis. The lateral view, shown in the figure below, highlight the propellers that thrust on sway axis $Th_3$ and $Th_7$.

## 3. CASE STUDY

The following section presents the simulations of the vehicle developed within *V-Fides Project*, presented in Caiti et al. (2014), a 3000m deep rated and over-actuated AUV for exploration and environmental monitoring. The proposed control allocation methods: pseudo-inverse (2), QP (6), LP (7), MIQP (10) and MILP (11) have been tested, compared and evaluated on such simulations.

### *3.1 Simulator of the V-Fides Vehicle*

The vehicle has a triangle shape and is equipped with sensors for navigation aid like Doppler Velocity Log (DVL), tactical grade Inertial Measurement Unit (IMU) and an Ultra-Short Base Line (USBL). Also, in environmental monitoring missions, a mercury sensor provides the measures for evaluate the seawater pollution. Further, the vehicle is actuated by seven propellers disposed as shown in a schematic view in Fig. 2; this arrangement labels the vehicle as over-actuated and grants the manoeuvrability on the six DOFs. The thruster 1 and 2 are characterized by an asymmetric static response with respect to positive and negative values of input commands. Moreover during some simple experiments act to identify a more detailed static characterization of the thrusters, has been observed a non-linear behaviour in operative range and different values of dead-zone range also for same models of actuators (Fig. 3). The static characteristic of each thruster acquired during the experiments are implemented in model of the vehicle by means of a lookup table.

The Guidance Navigation and Control (GNC) system has been tested in simulation before the real implementation on the vehicle. The simulator represents the most accurate behaviour of the entire system that is possible to obtain with data in our possession. The Fig. 4 shows the block scheme of the system, partitioned in modules that reproduce the behaviour of the vehicle dynamic (including the hydrodynamic forces and added mass), the navigation filter, the guidance law, the control and the allocation. The guidance module of the Course Control System (CCS) is designed to follow a path described by the sequence of a given set of waypoints on x-y-z plane. The path-following task is accomplished with a based Line-Of-Sight (LOS) lookahead steering law and provides the desired surge ($u_d$), heave ($w_d$) and yaw ($r_d$) velocities to the control module. Further, the vehicle has to keep roll and pitch angles equal to zero during environmental monitoring surveys as well as the sway linear velocity. To achieve the desired velocities and angles, the control system is composed by a Proportional Integral Derivative (PID) on *roll*, *pitch* angles and a PI controller on *surge*, *sway*, *heave* and *yaw* velocities with anti-windup technique.

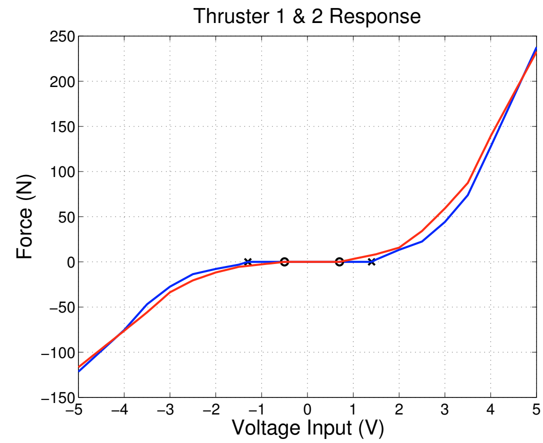


Fig. 3. The estimated static response of thrusters 1 (blue) and 2 (red) in forward and reverse mode with respect to voltage input range [-5:5] V. The marks 'x' and 'o' highlight the dead-zone range for thruster 1 and 2, respectively.

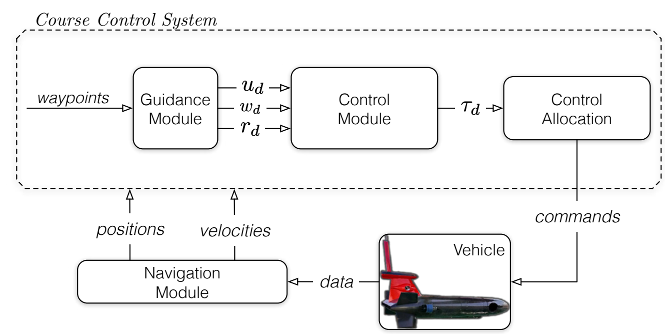


Fig. 4. Block scheme of the GNC system of the *V-Fides* vehicle. The navigation sensors provide data for the estimation of the velocity and position used in the *Course Control System* to compute the guidance law, the desired generalized forces and in last the input commands to the thrusters.

### *3.2 Scenario*

The comparison between the proposed control allocation methods have been evaluated on the path shown in Fig. 5, identified by the waypoints on x-y plane.

In order to simulate an uncertainty in the knowledge of the static response of the actuators and make a fair comparison between all the five techniques, in the pseudoinverse, LP and QP formulations have been added downstream a lookup table representing the inverse of the function shown in Fig. 1.

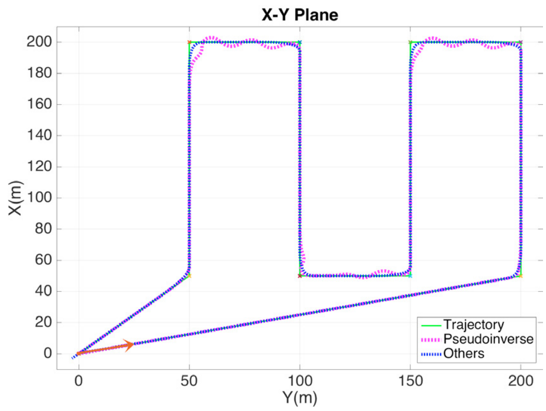


Fig. 5. (Green) The path to follow that reassemble a short typical survey. (Magenta) The path followed by the vehicle in the pseudoinverse case. (Blue) The path followed with the other allocation.

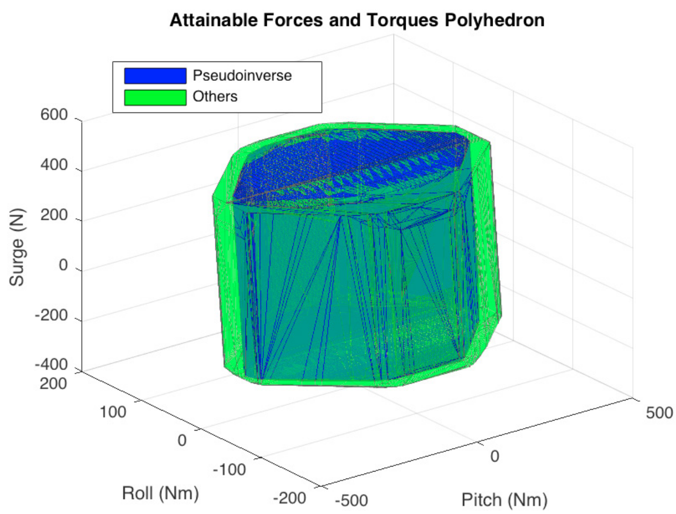


Fig. 6. An example of volume comparison between the pseudoinverse and the other techniques in the *Surge-Roll-Pitch* space.

In the proposed scenario, the vehicle is affected by a destabilizing Munk's moment and it is not straightforward to guarantee the stability w.r.t. the pitch when the thrusters that act on roll and pitch angles are saturated. In practical sense, prioritizing the torques allocation on the roll and pitch DOFs is instrumental in improving the stability of the vehicle. Thus, the gains $H_s$ and $K_s$ are tuned accordingly to the above considerations.

### 3.3 Evaluation Metrics and Simulation Results

The first comparison is based on the maximum achievable forces and torques. In particular, it is interesting to observe for each allocation technique which one guarantees the maximum volume defined on the space of the forces and torques along *Surge-Roll-Yaw*, *Surge-Pitch-Roll* and *Surge-Pitch-Yaw*. The volumes are calculated by choosing the space $\tau_d$ evenly distributed in a range defined on the base of the actuators saturations. Hence, the resulting allocated forces, $f$, are mapped on the vehicle by means of the transformation matrix, $T$. The Fig. 6 shows an example of volume comparison on the *Surge-Roll-Pitch* space, where the pseudoinverse achieves a worse result

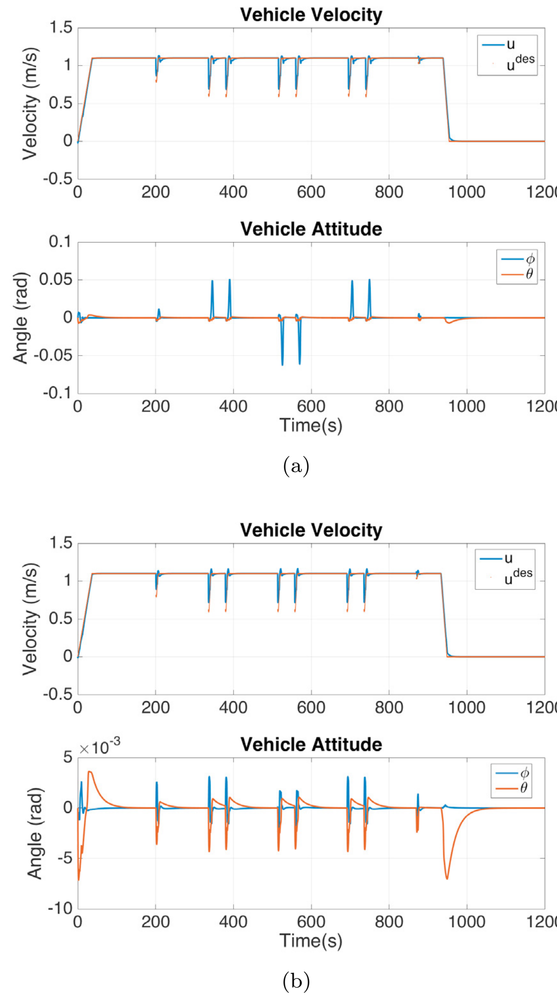


Fig. 7. The surge velocity and vehicle attitude (roll and pitch) during the simulations. a) Pseudoinverse. b) Others.

compared with the other techniques. Numerically, the Table 1 highlights how the quadratic formulations are slightly better than the linear ones. The loss of volume in the pseudoinverse case is mainly caused by the lack of saturation constraints into the optimal problem.

Table 1. Maximum achievable forces & torques

| Volume Percentage w.r.t. Pseudoinverse | LP % | QP % | MILP % | MIQP % |
|---|---|---|---|---|
| Surge-Roll-Yaw | 29.7 | 31.3 | 29.7 | 31.3 |
| Surge-Pitch-Roll | 31.5 | 32.7 | 31.5 | 32.7 |
| Surge-Pitch-Yaw | 8.7 | 9.1 | 8.7 | 9.1 |

In the following simulations, within the pseudoinverse case, have been observed how heavily the behaviour of the vehicle was affected by the tuning of the guidance law. The vehicle starts to lose quickly the stability on roll and pitch angles with the increasing demand of performance during the curves. Instead, with the other formulations, the mission is always concluded with success even in high demand of yaw velocities. The behaviour of the pseudoinverse case is owing to the impossibility to weight the roll and pitch DOFs and the shortage of maximum attainable torques. The Fig. 7 shows the vehicle behaviour during the simulations. In the Fig. 7(a) can be observed an initial phase of destabilizing effect on roll and pitch angle, which means the limitation of the pseudoinverse has been reached. Further, the trajectory of the vehicle have been also affected by the destabilizing effect as shown in Fig. 5. The Fig. 7(b) represents the behaviour in the other cases, remaining the same even in higher demands of yaw velocities.

The error allocation between the desired generalized forces, $\tau_d$, and the forces exerted on the vehicle is evaluated with Integral of Absolute Error (IAE) defined as $\int_0^{T_f} |\tau_d(t) - Tf|\, dt$. From Table 2 it can be observed the poor result of the pseudoinverse, instead the other techniques are similar, with a slightly better performance for the quadratic framework.

Table 2. Integral Absolute Error Allocation

| Error Allocation | Pseudo-inverse | LP | QP | MILP | MIQP |
|---|---|---|---|---|---|
| IAE | 365801 | 151848 | 145237 | 150325 | 144721 |
| Percentage w.r.t pseudoinverse | - | -58.5% | -60.3% | -58.9% | -60.4% |

In energy saving mindset, it is interesting to observe the usage of thrusters and whether the inclusion into the optimal problem of a better knowledge of the static response could improve the result. As before, an index Integral of Absolute Value (IAV) is defined as $\int_0^{T_f} |u(t)|\, dt$.

Table 3. Thrusters Usage

| Thruster Usage | Pseudo-inverse | LP | QP | MILP | MIQP |
|---|---|---|---|---|---|
| IAV | 22321 | 21729 | 22139 | 20239 | 20590 |
| Percentage w.r.t. pseudoinverse | - | -2.7% | -0.8% | -9.3% | -7.8% |

The Table 3 shows the MILP and MIQP obtain a better result, with an overall reduction of around $8 - 9\%$ w.r.t. the pseudoinverse technique.

### *3.4 Computational and Implementation Consideration*

Finally some considerations on the solver and the required computational effort: the various techniques have been implemented with the Gurobi Optimization toolbox, one of the fastest on the market. As expected, the Mixed-Integer approaches are slower and more computational demanding than the pseudoinverse, and the LP, QP methods. However, the problem at stake is not which is the fastest one, but if the slower ones can still be implemented in a real time control system without compromising the resources of the system itself. The MILP version is currently implemented on the *V-Fides Vehicle* on an embedded computer Intel Atom based, but with the GNU Linear Programming Kit (GLPK). Despite the modest computational power of the board, the control system runs at the required $10\,Hz$ without any issues.

## 4. DISCUSSION AND CONCLUSIONS

In the presented case study and with the metric used for the comparison, the pseudoinverse achieved always the worst performance in all the tests. The other four implemented control allocation methods are equivalent expect for the case of thrusters effort. The MIQP and MILP cases have achieved by far better performance w.r.t. LP and QP. The pseudo-inverse suffers from the lack of constraints describing the saturation of the actuators into the optimal problem, but also from the missing residuals in cost functional representing the 6-DOFs of the vehicle. Instead, the QP and LP formulations address such limitations, granting a better performance and enhance the practical stability, but are limited in representing a non-linear static response. The MILP and MIQP exploit the customizability, which allow to describe in more detailed way the actuators at the expense of the computational effort.

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