CONJECTURING AND PROVING IN DYNAMIC GEOMETRY: THE ELABORATION OF SOME RESEARCH HYPOTHESES[1]

Anna Baccaglini-Frank∗ Maria alessandra Mariotti∗∗

∗Università degli Studi di Siena & University of New Hampshire

∗∗Università degli Studi di Siena

Research has shown that the tools provided by dynamic geometry systems impact students' approach to investigating open problems in Euclidean geometry. We particularly focus on types of processes that might be induced by certain uses of tools available in Cabri. Building on the work of Arzarello (Arzarello et al., 1998) and Olivero (1999, 2002), we have conceived a model describing some cognitive processes that may occur during the production of conjectures and proofs in a dynamic geometry environment and that might be related to the use of specific dragging schemes. Moreover, we hypothesize that such cognitive processes could be induced by introducing students to the use of dragging schemes.

Key words: conjecturing, dynamic geometry, dragging schemes, abductive processes, cognitive unity

INTRODUCTION

The contribution of a DGE to students' reasoning and proving is particularly evident during the investigation of open problems, since this process involves making conjectures (Mariotti, 2006). Instead of a static-conjecture built in a paper-and-pencil environment in a DGE a dynamic-conjecture [1] can be developed. Moreover, in a DGE, the invariant geometrical properties of a construction, which lead to conjectures, can easily be grasped. An interesting question is: what kind of support can a DGE provide first during the development of a conjecture and then during the production of a proof? The answer seems to depend on the nature of the problem. On one hand the ease to immediately grasp certain invariants seems to inhibit some argumentation processes that lead to finding useful elements for the construction of a proof. On the other hand, research has shown that a DGE can foster the learners' constructions and ways of thinking, and that it can help overcome some cognitive difficulties that students encounter with conjecturing and proving (e.g. Noss & Hoyles, 1996; Mariotti, 2002; De Villiers, 2004).

Building on the work of Olivero and Arzarello (Olivero, 1999; Arzarello et al., 1998), we have conceived a model of cognitive processes that can occur during the conjecturing stage of open problem investigations in a DGE. Through a qualitative study, our final goal is to give a detailed description of some cognitive processes related to conjecturing and proving, and of how a DGE might foster such processes,

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thus providing a base for further research and for the development of new curricular activities.

ORIGIN OF OUR HYPOTHESES

In the following paragraphs we will briefly outline the theoretical framework which the ideas are embedded in.

Semiotic Mediation and Semiotic Potential of an Artifact

A DGE like Cabri, which contains "objects" such as points, lines, circles, and ways to "manipulate" the objects, is a microworld (Papert, 1980; Balacheff & Kaput, 1996) built to resemble the mathematical world of Euclidean geometry. A key aspect of microworlds in mathematics education is that the "objects" included offer the opportunity for the user to experiment directly with the "mathematical objects" (Mariotti, 2005, 2006), because the logical reasoning behind the objects in the microworld is designed to be the same as that behind the real mathematical objects that they represent.

Recent research has developed the ideas of *tool of semiotic mediation* and of *semiotic potential of an artifact*:

> "...any artifact will be referred to as a tool of semiotic mediation as long as it is (or it is conceived to be) intentionally used by the teacher to mediate a mathematical content through a designed didactical intervention" (Bartolini Bussi & Mariotti, 2008).

Computers in general, and a DGE in particular, are considered to be tools of semiotic mediation (Mariotti, 2006; Bartolini Bussi & Mariotti, 2008). However, the mediation can occur successfully only if their semiotic potential is exploited. Therefore it becomes necessary to study ways that foster exploitation of such potential. This was a main goal we had in mind when we started developing our hypotheses.

A First Theoretical Model and the Dragging Schemes

The dragging tool can be activated by the user, through the mouse. It can determine the motion of different objects in fundamentally two ways: direct motion, and indirect motion. The *direct motion* of a base-element (for instance a point), that is an element from which the construction originates, represents the variation of this element in the plane. The *indirect motion* of an element occurs when a construction has been accomplished. In this case dragging the base-points will determine the motion of the new elements obtained through the construction. The use of dragging allows one to feel "motion dependency", which can be interpreted in terms of logical dependency within the geometrical context (Mariotti, 2002, p. 716). Starting from these phenomenological perspectives, a refined analysis of the dragging tool can highlight its semiotic potential that can be exploited by the teacher in school practice.

The use of Cabri in the generation of conjectures is based on the interpretation of the dragging function in terms of logical control. In other words, the subject has to be capable of transforming perceptual data into a conditional relationship between hypothesis and thesis. The consciousness of the fact that the dragging process may reveal a relationship between geometric properties embedded in the Cabri figure directs the way of transforming and observing the screen image (Talmon & Yerushalmy, 2004). At the same time, that consciousness is needed to exploit some of the facilities offered by the software, like the 'locus of points' or 'point on object'. Such a consciousness is strictly related to the possibility of exploiting the heuristic potential of a DGE (Mariotti, 2006).

The theoretical model presented by Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 1998, 2002) addresses expert solvers' production of conjectures, and how abduction marks the transition from the conjecturing to the proving phase, when a passage from "ascending control" to "descending control" occurs. Abduction guides the transition, in that it seems to be key in allowing solvers to write conjectures in a logical 'if…then' form, a statement which is now ready to be proved. Arzarello et al.'s analysis of subjects' spontaneous development of dragging modalities led to the determination of a classification (Arzarello et al., 2002), which researchers have referred to as the "dragging schemes" (Olivero, 2002).

Abduction

In the previous section, the notion of abductive processes is mentioned. Peirce was the first to introduce the notion of abductive inference, and compare it with other inferences, such as deduction and induction. According to Peirce,

> "abduction looks at facts and looks for a theory to explain them, but it can only say a "might be", because it has a probabilistic nature. The general form of an abduction is: a fact A is observed; if C was true, then A would certainly be true; so, it is reasonable to assume C is true" (Peirce, 1960, p. 372).

Recently, researchers have renewed interest in abduction. In particular, Magnani defines abduction in a way that we find quite useful. According to him abduction is,

> "the process of inferring certain facts and/or laws and hypotheses that render some sentences plausible, that explain or discover some (eventually new) phenomenon or observation; it is the process of reasoning in which explanatory hypotheses are formed and evaluated" (Magnani, 2001, pp. 17-18).

Moreover, the following distinction of direct abduction versus creative abduction will be useful for our study. *Direct abduction* is when the "rule" used in the abductive process consists of a theorem that is already known to the student; while *creative abduction* is when the "rule" of the abduction consists of something new, that is not previously known by the student (see also Magnani, 2001; Thagard, 2006). Other researchers have studied various uses of abduction in mathematics education (Reid, 2003), and abductive processes in relation to transformational reasoning (Simon, 1996; Cifarelli, 1999; Ferrando, 2006). The basic idea is that an abductive inference may serve to organize, reorganize and transform problem solvers' actions (Cifarelli, 1999). Abductive processes have also been observed by Arzarello et al. (1998) during the development of conjectures when students were using the dragging schemes, as mentioned above. In the next section we describe how our work builds on that of Arzarello et al., trying to study in detail the processes that occur during the conjecturing stage in open problem investigations, how these processes may be fostered, and what they might lead to during the phase of proof production.

OUR HYPOTHESES

While Olivero, Arzarello, Paola, and Robutti (Olivero, 2000; Arzarello, et al., 2002) focused their attention on the subjects' use of the dragging schemes during the development of a conjecture, we concentrate on the abductive processes that may be induced by certain dragging schemes. Arzarello et al. observed that abduction occurs during solvers' use of the dragging schemes. Moreover, they claim that the production of conjectures is based on abductive processes. Thus, it seems that the use of certain dragging schemes may foster abductive processes, and, consequently, the production of conjectures. To some extent, the dragging schemes can be seen as cognitive artefacts (Norman,1991). We would like to investigate the relationship between the use of the dragging schemes and the development of abductive processes. In order to accomplish this investigation we need to induce solvers' use of dragging schemes, so we decided to introduce students to the specific dragging strategies. This way we seem to be able to induce the use of specific dragging schemes for the solution of open problems and, consequently, the appearance of abductive processes.

Below is a hypothesis of what might occur as a solver, who has been introduced to the dragging schemes, approaches an open problem in a DGE.

- Step 1: conscious use of different dragging strategies to investigate the situation – after *wandering dragging*, in particular *dummy locus dragging* (or *lieu muet dragging*) to maintain a geometrical property of the figure (*intentionally induced invariance*, or III), and use of the *trace tool*.
- Step 2: consciousness of the locus (*lieu*) that appears through *lieu muet dragging* – this marks a shift in control from ascending to descending – and description of a second invariance (*invariance observed during dragging*, or IOD).
- Step 3: hypothesis of a conditional link between the III and the IOD, to explain the situation.
- Other forms of dragging may be performed: *line dragging*, *linked dragging*, and the *dragging test*.
- Step 4: formulation of a conjecture of the form 'if IOD then III' (product of the abduction).

- Step 5: production of a mathematical proof of the conjecture (or attempt of it). Potential re-formulation of the conjecture.

The Notion of *Path* and an Example

Another hypothesis that we advance is that there is a key element, the *path*, that plays a fundamental role in the abductive process. In this section, we will try to introduce the concept of *path* and its significance for the model.

One of the dragging schemes, *lieu muet dragging*, involves dragging a point with the intention of maintaining a given property of the figure (which becomes the III). Some regularity may appear during this dragging stage, leading to the discovery of particular constraints that the dragged point has to respect (expressed in the IOD). Because of their origin from dragging, such constraints may be interpreted as the property of the point to belong to a particular figure. In mathematical terminology, that of course may not be consistent with students' way of thinking, we can speak of a hidden locus (*lieu muet*). Such locus can be made explicit by the trace tool, through which it appears on the screen (*lieu parlante*). During lieu muet dragging the solver notices regularities of the point's movement and conceptualizes them as leading to an explicit object. We refer to this object as a *path* when the solver gains consciousness of it, as generated through dragging, and consciousness of its property that if the dragged point is on it, a geometrical property of the Cabri figure is maintained. In this sense a *path* is the reification (Sfard, 1991) of a *lieu* that can now be used in a "descending control" mode (Arzarello et al., 2002). Zooming into Step 2, above, we observe that this is the point of the process in which the notion of *path* arises, and we can add a Step 2bis to indicate the (potential) geometric interpretation of the *path*, in order to (potentially, after Step 3) perform *line dragging*, *linked dragging*, and the *dragging test along such path*.

We believe that the *path* plays an important role in relation to the abductive processes that can be used to develop conjectures in a DGE. In particular, recognition of a *path* can act as a bridge, fostering the formulation of a conjecture. In fact, the *path* can be used during the abductive processes, but then it may no longer appear (or it may appear in a different form) in the formulation of the conjecture. Below, we zoom into a way in which abductive processes may take place and lead to a derived conjecture, and then we provide an example of the model in use during an activity.

- Intentionally Induced Invariance (III): the solver tries to maintain a certain geometrical property.
- Invariance Observed during Dragging (IOD): the solver notices that when he/she drags a certain basic point X along the *path*, the III seems to be maintained.
- Product of abductive process: it becomes reasonable for the solver to assume that if point X lies on the *path* (description of the IOD), the III is true.

If the path is recognized as a particular geometrical figure *F*, the derived conjecture may be: if X lies on F, the III is true.

Activity: Draw three points A, M, K, then construct point B as the symmetric image of A with respect to M, and point C as the symmetric image of A with respect to K. Construct point D as the symmetric image of B with respect to K. Drag M and make conjectures about ABCD. Then try to prove your conjectures.

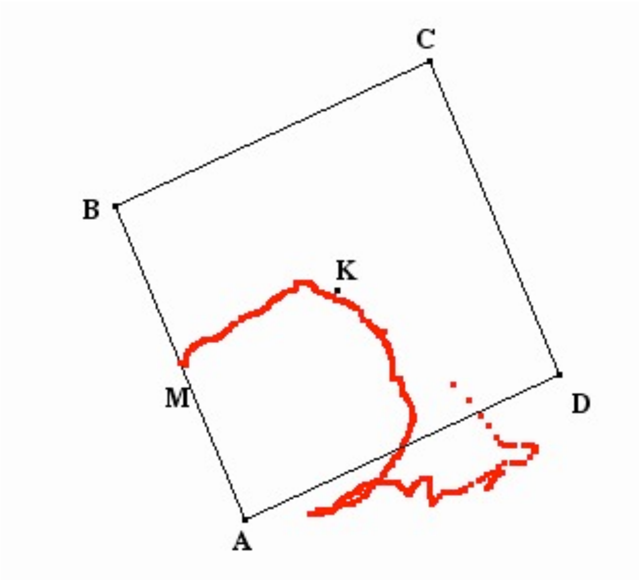

Fig 1: Dragging with the trace tool can help a student notice a *locus* (or *lieu*).

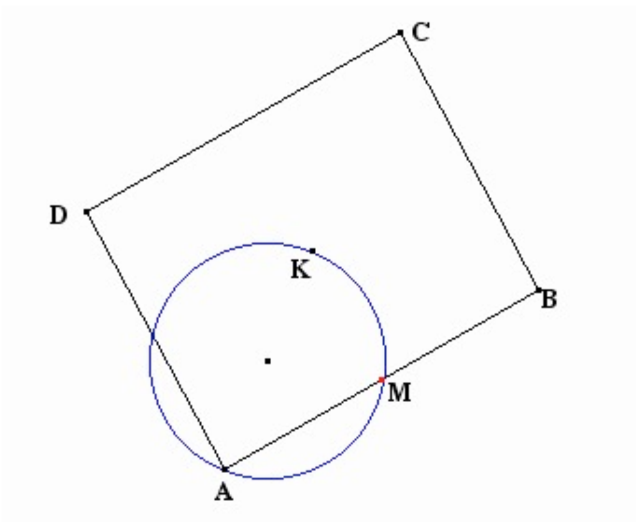

Fig 2: M is being dragged along the path (*line dragging*).

A Response [2]: Through *wandering dragging* solvers may notice that ABCD can become different types of parallelograms. In particular, they might notice that in some cases ABCD seems to be a rectangle (they can choose this as the III). With the intention of maintaining this property as an invariant, solvers might mark some configurations of M for which this seems to be true, and through the trace tool, try to drag maintaining the property, as shown in Fig 1. This can lead to noticing some regularity (IOD) in the movement of M, which might lead to awareness of an object along which to drag (the circle of diameter AK, potentially not yet recognized as "a circle"). At this point, when such awareness arises, we can speak of *path* with respect to the regularity of the movement of M.

If solvers recognize the path to be a familiar geometrical object, like in this case, they might be inclined to constructing it, as shown in Fig 2, and dragging along it (*line dragging*), or even linking the free point to it (*linked dragging*) and performing a *dragging test*. Through this abductive process, as an attempt at explaining the experienced situation, as Magnani describes (Magnani, 2001), solvers may hypothesize a conditional link between the III and IOD. At this point the abduction leads to a hypothesis of the form 'if IOD then III', leading to a conjecture like the following: "If M is on the circle of diameter AK, then ABCD is a rectangle," or (if they discover or derive a property of the base-points which is equivalent to M lying on the circle): "If AKM is a right triangle, ABCD is a rectangle."

In the case of the first conjecture, here is how we hypothesize the abduction (*creative abduction*) might go.

- III: ABCD is a rectangle.
- IOD: when M dragged along the *path*, fact A seems to be true. The path is a known geometric figure*:* the circle of diameter AK.
- Product of the abduction: If point M lies on the circle of diameter AK, ABCD is a rectangle.

This product of the abduction coincides with a formulation of a conjecture. However, solvers might also perform a second abduction (this time a *direct abduction*) linking the property "M belongs to the circle" to a property of the base-points of the construction. In this case this may lead to a formulation of the conjecture like: "If the triangle AMK is a right triangle (with ∠AMK as the right angle), ABCD is a rectangle." In this case the further elaboration of the geometrical properties recognized in the path will have led to a key idea (Raman, 2003) of a possible proof. In particular, this idea together with that of triangles AMK and ABC being similar, should be enough for students to successfully provide a proof to their conjecture. In this sense, abductive processes involving the notion of *path* (as a reified concept the solver is aware of) might be a step towards the achievement of cognitive unity [3] (Boero, Garuti, & Mariotti, 1996; Pedemonte, 2003).

Some Research Questions

Given the hypotheses outlined above, we propose some general questions for a research study. First, it would be interesting to investigate what forms of reasoning (abductive, deductive, ...) are actually used (and how) during the conjecturing stage of an open problem in a DGE. In particular, if subjects use *lieu muet* dragging, what is the role of the *path*? Can our model be confirmed (even in a potentially modified version)? Second, how does a DGE contribute to the development of the proof of a conjecture? It would be interesting to compare the dragging schemes (if any) used during this stage to those used during the conjecturing stage. It might also be insightful to investigate the forms of reasoning used during the conjecturing stage in the cases in which subjects do produce a proof. Finally, it would be interesting to study whether it is possible to detect a relationship between the forms of reasoning analyzed, and, if possible, to describe such a relationship.

EXPERIMENTAL DESIGN AND POTENTIAL CONCLUSIONS

We propose to structure the study in the following general way: by a selection of the subjects, the introduction of the subjects to the dragging schemes, finally open-problem-activity-based interviews on pairs of students. We will use results from the pilot study to refine the model, the research questions, and the activities proposed during the interviews. In the results of this study we hope to be able to include: a description of some cognitive processes that occur during the conjecturing stage of the investigation of open problems in a DGE; and validation of the model (or of a revised version of it), or motivations for rejecting it as a useful descriptive model. Therefore, this study should help gain better comprehension of specific cognitive processes. In particular, we hope to gain some insight into how abductive processes may occur, whether they can be fostered by preliminary introduction of the dragging schemes, and how the notion of *path* may foster the formulation of conjectures.

A secondary objective is to gain insight into how a DGE contributes to the development of proofs. The activities proposed during the interviews will all be open problems in which students are asked to make conjectures and then try to prove them. The *path* might also play a role in the generation of a proof, in that it may be a part of the "reorganization and transformation" that occurs with abductive reasoning (Cifarelli, 1999). This might very well be new powerful tool for the solver to use in a potential proof (or solution of the problem) as an aid to gain cognitive unity, as mentioned above. In this case, it would be reasonable to hypothesize that if the dragging schemes were to foster abductive processes, and abductive processes were to foster cognitive unity, then introducing the tool of the dragging schemes to the students a priori might accelerate and facilitate the entire process of making a conjecture and reaching a proof for it.

If our hypotheses are confirmed, and the dragging schemes and the notion of *path* do contribute positively to the formulation of conjectures (and potentially of proofs), we will recognize them as tools of semiotic mediation, with a semiotic potential that could be exploited by teachers. In this case, teaching experiments, which introduce the dragging schemes at a class-level, should be carried out, in order to further investigate how the teacher can exploit the semiotic potential of the dragging schemes in the classroom practice. Later, large-scale quantitative research on the induction of cognitive processes through introduction of the dragging schemes could be conducted, with the didactic objective of implementing the teaching of the dragging schemes in school curricula.

NOTES

1. With "static" and "dynamic" referred to conjecture, here we intend to emphasize the nature of the conjecture's origin.

2. This is only one of the many possible responses leading to this specific conjecture. Of course different students might reach this conjecture in different ways. Moreover there are many different conjectures that students can formulate by focusing their attention on different geometric invariants (in this case, having ABCD be a kite, a rhombus, or a square).

3. Boero et al. introduce cognitive unity as follows: "During the production of the conjecture, the student progressively works out his/her statement through an intense argumentative activity functionally intermingled with the justification of the plausibility of his/her choices: during the subsequent proving stage, the student links up with his process in a coherent way, organizing some of the justifications ("arguments") produced during the construction of the statement according to a logical chain" (Boero, Garuti, & Mariotti, 1996, p.113).

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