

# REASONING BY CONTRADICTION IN DYNAMIC GEOMETRY

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*This paper addresses contributions that dynamic geometry systems (DGSs) may give in reasoning by contradiction in geometry. We present analyses of three excerpts of students' work and use the notion of pseudo object, elaborated from previous research, to show some specificities of DGS in constructing proof by contradiction. In particular, we support the claim that a DGS can offer guidance in the solver's development of an indirect argument thanks to the potential it offers of both constructing certain properties robustly, and of helping the solver perceive pseudo objects.*

**Keywords:** Dynamic geometry; Indirect argument; Proof; Proof by contradiction; Pseudo object

Razonamiento por contradicción en geometría dinámica

*Este artículo aborda las contribuciones que los sistemas de geometría dinámica (DGSs) pueden dar al razonamiento por contradicción en geometría. Presentamos un análisis de tres extractos del trabajo de estudiantes y el uso de la noción de pseudo-objeto, elaborado a partir de investigaciones anteriores, para mostrar algunas especificidades del DGS en la construcción de pruebas por contradicción. En particular, afirmamos que un DGS puede orientar en el desarrollo de un argumento indirecto gracias a las posibilidades que ofrece tanto para construir sólidamente algunas propiedades como para ayudar a percibir los pseudo-objetos.*

**Términos clave:** Argumento indirecto; Geometría dinámica; Prueba; Prueba por contradicción; Pseudo-objeto

Literature shows that although much research has been conducted on the themes of proof and argumentation in mathematics education, rarely do the studies focus on particular proof structures, such as proof by contradiction. The research cen-

Baccaglini-Frank, A., Antonini, S., Leung, A., & Mariotti, M. A. (2013). Reasoning by contradiction in dynamic geometry. *PNA*, 7(2), 63-73.

tered on proof by contradiction has pointed to various difficulties it presents for students (see for example, Antonini & Mariotti, 2007, 2008; Leron, 1985; Mariotti & Antonini, 2006; Wu Yu, Lin, & Lee, 2003) especially the difficulties related to the formulation and interpretation of negation, to the managing of impossible mathematical objects, to the gap between contradiction and the proved statement.

Some literature takes into consideration contributions that DGSs may give to students' production of indirect arguments. Within the very little literature in this area, there is a study conducted by Leung and Lopez-Real that describes a proof by contradiction produced by two students working in a DGS. This case study triggered the development of a framework on theorem acquisition and justification in a DGS that the authors used to put together a scheme for "seeing" proof by contradiction in a DGS (Leung & Lopez-Real, 2002). We will illustrate aspects of this framework that we will make use of and develop further in the following section.

With the present paper we intend to contribute to better describe roles that a DGS can have in reasoning by contradiction. We will further elaborate and make use of notions from Leung and Lopez-Real's theoretical framework, in particular that of pseudo object, to analyze such roles. Moreover, we will provide analyses of three excerpts of students' work to show particular construction choices in a DGS can guide/promote significantly solvers' development of indirect arguments/reasoning by contradiction.

## METHODOLOGY

The data presented was collected during two different studies on the role of a DGS in processes of conjecture-generation and proof in the context of open problems in geometry. One study (Leung & Lopez-Real, 2002) was conducted with Form 4 (Grade 10) students in a band one secondary school in Hong Kong. Hong Kong's secondary schools are streamed according to students' ability. A band one school is for the most able students. The second study (Baccaglioni-Frank, 2010; Baccaglioni-Frank & Mariotti, 2010) was conducted with Italian high school students (16-18 years old) from three different *licei scientifici*. The participants of both studies had been working with dynamic geometry for at least a year prior to when the studies were carried out. Data was collected in the forms of audio and video tapes and transcriptions of the introductory lessons, Cabri-files worked on by the instructor and the students during the classroom activities, audio and video tapes, screenshots of the students' explorations, transcriptions of the task-based interviews, and the students' work on paper that was produced during the interviews.

## THE NOTION OF PSEUDO OBJECT

When working with paper and pencil and reasoning by contradiction, slight inaccuracies in the drawing allow the figure to represent properties, which a “proper” construction would not permit. For example, on paper, with no trouble one can assume to have drawn a triangle, of which two bisectors intersect at a right angle. In this case, one may easily be unaware of his/her assumption of contradictory properties, and it is completely up to him/her to become aware of a contradiction.

In a DGS a similar situation to that described in paper and pencil occurs when the solver constructs a figure with a robust property (Healy, 2000) while mentally imposing on it a contradictory property without a robust construction. By robust construction in a DGS, we mean a construction that can keep the desired properties of a figure invariant under dragging. What happens if, instead, the solver attempts to construct both properties robustly? However, the solver may be uncertain whether such a construction is possible or not, or s/he may realize the impossibility when interpreting the DGS’ feedback. Such feedback includes the making explicit, robustly, of all properties that are derived from the properties constructed robustly during the construction steps of the figure. This is the case we find particularly interesting. In this paper we report on ways of reasoning that seem to be induced by the feedback provided by the DGS.

As mentioned above, in a DGS no constructible figure can be realized by robust contradictory properties. So to represent a geometrical object with contradictory properties (at least) one property must not be constructed robustly, but only conceived (or projected onto the figure) by the solver. Therefore, the solver is completely in charge of conceiving any contradiction. In this paper, we will present three DGS cases of (attempts of) reasoning by contradiction by students that involve solvers projected non-constructible properties onto geometrical figures. To analyze these cases, we further elaborated Leung and Lopez-Real’s (2002) notion of pseudo object in a dynamic geometry environment as follows: A pseudo object is a geometrical figure associated to another geometrical figure either by construction or by projected-perception in such a way that it contains properties that are contradictory in the Euclidean theory.

We stress that the notion of pseudo object is solver-centered. Thus, the same dynamic figure can be a pseudo object for one solver, but not for another, depending on whether the solver has projected upon the geometrical objects contradictory properties. In this sense, any dynamic figure defined through a construction has the potential of becoming a pseudo object for any given solver. In DGS, this potentiality can be realized through a cognitive process of dragging in which conceiving a pseudo object is critical in reasoning by contradiction. To facilitate the analyses of this process, we introduce a notion of proto-pseudo object: A proto-pseudo object is a geometrical object that has the potential of becoming a pseudo object—such potential is exploited when the solver perceives a property of such object as being contradictory with respect to another of its properties.

Thus a proto-pseudo object can become an actual pseudo object once (and if) the solver consciously projects a property upon it that s/he is aware of as contradictory. We will use the notions of pseudo object and proto-pseudo object to show how DGSs seem to provide cognitive support in (a) offering the potential of constructing certain properties robustly, (b) generating feedback in the form of robustness of all properties that are consequences of the constructed ones, and (c) the possibility of dragging parts of the dynamic figure to explore compatibility between the robust properties and those the solver has projected upon it. As we shall see in the following cases, these features seem to guide solvers to conceiving pseudo objects in processes of reasoning by contradiction.

## THE ROLE OF DYNAMIC GEOMETRY IN THREE SOLUTION PROCESSES

Consider the following task from Leung and Lopez-Real's (2002) study.

*Given a quadrilateral in which the sum of the pairs of opposite angles is  $180^\circ$ , prove that it's cyclic.*

This task was given to the participants in Leung and Lopez-Real's study. We report on Hilda and Jane's solution.

### **Excerpt 1. The Case of Hilda and Jane**

Hilda and Jane constructed a quadrilateral ABCD of which the vertices A, B, D lie on the same circle with center E, while C, D, B lie on a distinct circle with center F. Then they marked the measures of the angle in A and in C as  $a$  and  $180^\circ - a$ , respectively, and proceeded to construct the quadrilateral BEDF (Figure 1).

If the labeling were Euclidean-correct this construction would not be possible since the circles would coincide, thus ABCD is biased and it leads to the existence of a non-degenerate quadrilateral BEDF, which Hilda and Jane conceive as a pseudo object. This can be seen both in Jane and Hilda's proof, and in an excerpt of the transcript of a follow-up interview the researchers had with the girls.

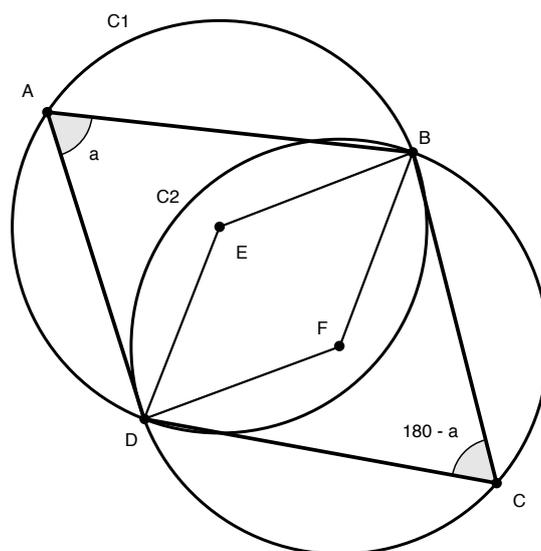


Figure 1. Diagram of the proof

Statement in the proof: “From the diagram we see that it has a contradiction as the sum of the opposite angles of the blue quadrilateral (EBFD) is  $360^\circ$  which is impossible.” We present an excerpt of the interview.

- 7 Interviewer: So before you did that presumably you first of all drew a circle through three of the points and then you did the same for these three points.
- 8 Hilda: Yes.
- 9 Interviewer: So then you marked these two centers. What did you say after that?
- 10 Hilda: Because the angle sum of a quadrilateral is 360 and these two [referring to  $\angle E$  and  $\angle F$ ] already add up to 360 so this is not possible.

Our analysis suggests that the quadrilateral ABCD is initially a proto-pseudo object, and it becomes a pseudo object for the solvers once EFBD is perceived as “not possible” (Line 10). EFBD possesses two contradictory properties which the solvers perceive simultaneously as (a) a quadrilateral with 4 angles whose sum is 360 degrees (as all convex quadrilaterals), and (b) a quadrilateral in which the sum of only two angles is 360 degrees —statement in proof and Line 10. This pseudo object EFBD contains the contradiction necessary for a proof by contradiction. The proof is completed by noticing that when EFBD is being dragged to degenerate (disappear) the two circles, C1 and C2 in Figure 1, coincide. In other words, the presence of the pseudo object implies the negation of the conclusion of the statement to prove. By arriving at the proof, the solvers are aware that their original quadrilateral ABCD also possess contradictory properties; that is, (a) its four vertices are on different circles and (b) the sum of two opposite angles is  $180^\circ$ . Hence the proto-pseudo object ABCD becomes a pseudo object.

The support offered by the DGS (in this case, Cabri) consisted in guiding the solvers' transition of ABCD from its status of proto-pseudo object to a genuine pseudo object. In this case the transition occurred through the perception of a pseudo object (EFBD) associated with ABCD. Finally, it seems quite remarkable that Hilda and Jane decided to construct two distinct circles (Line 7) through two sets of three points of the original quadrilateral, showing the case in which ABCD is cyclical in terms of coincidence of the two circles via dragging in DGS. In particular, they transform the problem, which does not contain any impossibility in its original statement, into a problem of constructing something impossible. This is the type of problem that Stefano and Giulio, and Tommaso and Simone were given in Baccaglioni-Frank and Mariotti (2010).

### **Excerpt 2. The Case of Stefano and Giulio**

Similarly to what we described above, in the following cases, awareness of the presence of a pseudo object determines the impossibility of a construction, thus validating a statement such as “this construction is impossible.” In the following two cases, we encounter similar solution processes to those described by Mariotti and Antonini (2009). The solvers conceive a (or various) “new” object(s) that is (are) used to “show” a contradiction. However, having the DGS at their disposal, the solvers make use of it in significant ways that we will describe. The task was as follows.

*Answer the following question: Is it possible to construct a triangle with two perpendicular angle bisectors? If so, provide steps for a construction. If not, explain why.*

Giulio and Stefano immediately advance the hypothesis that the construction is not possible, but quickly transit to constructing a figure in Cabri to try to explain their intuition.

- 1 *Stefano:* No, the only way is to have 90 degree angles... [unclear which these may be, as Ste was not constructing the figure nor looking at the screen.]
- 2 *Giulio:* That for a triangle is a bit difficult! [giggling]... So... they have to be.
- 3 *Stefano:* If triangles have 4 angles...
- 4 *Giulio:* No, I was about to say something silly...

Immediately Giulio starts constructing two perpendicular lines and refers to them as the bisectors of the triangle (Figure 2).

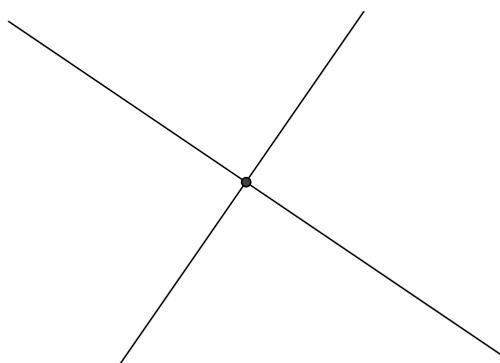


Figure 2. Bisectors of a triangle drawn by Giulio

- 5 *Stefano:* Yes, these are bisectors, right?
- 6 *Interviewer:* Yes.
- 7 *Giulio:* So, now we need to get... bisectors... how can we have an angle from the bisector?
- ...
- 8 *Giulio:* the symmetric image?... It's enough to do the symmetric of this one. So the solvers have constructed a figure with two robust angle bisectors that intersect perpendicularly [Figure 3].
- 9 *Stefano:* The only thing is that this [Figure 3] isn't a triangle!
- 10 *Giulio:* Therefore now we could do like this here [drawing the lines through the symmetric points and the two drawn vertices of the triangle].
- 11 *Interviewer:* Yes.
- 12 *Stefano:* It's that something atrocious comes out!
- 13 *Giulio:* And here... theoretically the point of intersection should be ... .the points... very small detail... hmmm
- ...
- 14 *Stefano:* No, we proved that this is equal to this [pointing to angles], and this is equal to this because they are bisectors... these two are equal so these are parallel.
- ...
- 15 *Stefano:* These two [referring to the two parallel lines] have a hole so it is not a triangle.

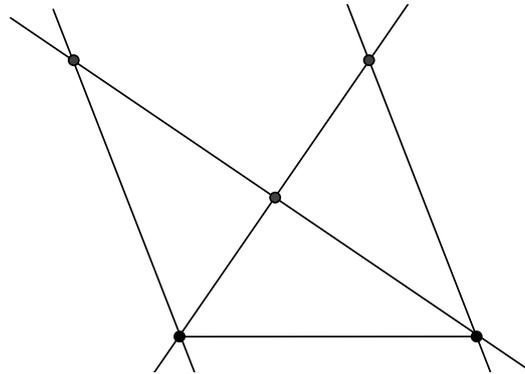


Figure 3. Solvers' figure

We interpret this episode as follows. The solvers use the DGS to construct two perpendicular lines and the symmetric image to construct the property of them being bisectors. Once the construction is completed, they discern properties that are consequences of these two robustly constructed properties, and consequently notice that “the figure must have two adjacent angles with two parallel sides” (Lines 9-15). As soon as they recognize “a hole” in the triangle-to-be (Line 15) the pseudo object exists: that is, a figure that has a “base” and two parallel sides, and that has the property “triangle” projected onto it. The appearance of this pseudo object reveals to the solvers the impossibility of accomplishing a correct robust construction and thus allows them to solve the problem.

### Excerpt 3. The Case of Tommaso and Simone

Tommaso and Simone proceeded by constructing a proper triangle and two of its bisectors. Then they marked an angle formed by the bisectors and start dragging one vertex of the triangle in the attempt to get the measure to say “90°” (Figure 4).

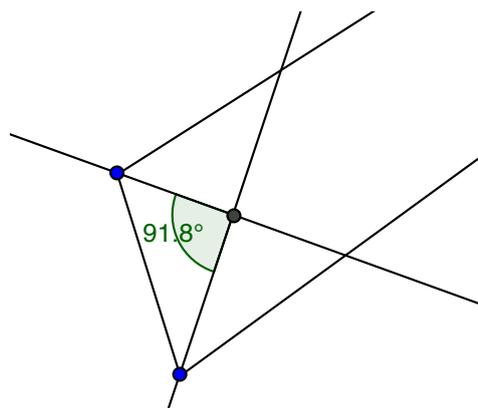


Figure 4. Tommaso and Simone' construction

- 1 *Simone:* It's endless!
- 2 *Simone:* 91.2 [reading the measure of the angle between the bisectors.]

- 3 *Simone:* Well, yes, in any case it will come out!
- 4 *Tommaso:* How do you know? maybe...
- 5 *Simone:* Well, of course! It's not like it can go on forever! At the end it will make it to be 90!
- ...
- 6 *Tommaso:* I don't think it is possible. The solvers seem unsure about the possibility of constructing such a triangle, but now seems to think it is not possible. They start reasoning differently.
- 7 *Simone:* Eh, it is impossible to construct it! Because... I only have these two bisectors.
- 8 *Interviewer:* Hmm.
- 9 *Simone:* How can I...
- 10 *Simone:* Since... the perpendicular bisectors... it means here there is a rhombus... or a square
- 11 *Simone:* If like here... [he draws a segment]... Here... there were... a rhombus... this would be 90, 90... or a square. And therefore... then... Eh, I mean, if this is like a rhombus, no? here there is 90 and here there is 90, and these are the bisectors.
- 12 *Simone:* And then... and then I bring these up [pointing to the vertical-looking sides of the triangle] and I find their point of... of intersection.

With respect to Stefano and Giulio, here the solvers choose a different pair of properties to construct robustly: (a) the triangle, and (b) the bisectors. They do not construct but (we assume) project the property perpendicular bisectors onto the figure. Nevertheless, they are not able to conceive a contradiction in it or in the new object they conceive —the rhombus. Hence this rhombus is a proto-pseudo object, and the solvers do not seem to make the transition to conceiving it as a pseudo object. It is significant that the solvers say “it has 4 right angles” (Line 11) pointing to the figure that even has a marked measure of one of the angles, and the measure says “91°!”. No contradiction among the properties of the “rhombus” is perceived and the solvers are not able to reach a conclusion. We advance the hypothesis that if they had been able to conceive the rhombus as a pseudo object, they would have been able to solve the problem geometrically. Instead, they resort to an algebraic explanation that they cannot coordinate with what they see on the screen. They seem to keep on believing that the triangle always has a third vertex “somewhere up high”.

When we compare Excerpt 2 and Excerpt 3, a determining difference, from a cognitive point of view, is that, in one case, the solvers conceive a pseudo object, and, in the other, they do not. This can be explained by the solvers' different

choice of the properties to construct robustly. The choice determines the type of guidance that the DGS can provide to reasoning by contradiction. In Excerpt 3, starting from the triangle and trying to obtain perpendicularity of the bisectors through dragging allows the solvers to use the DGS (only) as a sort of “amplified paper-and-pencil drawing” in that it allowed the exploration of many cases without having to redraw the figure. On the other hand, in Excerpt 2 the DGS generates two robust parallel lines as a consequence of the constructed properties, thus “guiding” the solvers in perceiving “a hole” in the triangle-to-be and thus such object as a pseudo object.

## CONCLUSION

We have introduced the concept of pseudo object and illustrated how it can contribute significantly to reasoning by contradiction in Euclidean geometry. In particular, in a DGS environment, construction and dragging strategies leading to degeneration of a pseudo object could guide to ascertainment of a geometric theorem or property. The hybrid nature of a pseudo object seems to be conducive to formulating an exchange of meaning between dynamic visual reasoning in DGS and theoretical reasoning in the Euclidean axiomatic system (in this case proof by contradiction). We have shown that there can be a strong subjective element in the process of producing a geometrical proof (or a convincing argument) via the solver’s conscious choices of construction and dragging in a DGS. We hope this paper will open up a window of discussion to view proof in dynamic geometry environment in ways that can enrich the formal deductive reasoning approach.

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This paper was originally published as Baccaglioni-Frank, A., Antonini, S., Leung, A., & Mariotti, M. A. (2011). Reasoning by contradiction in dynamic geometry. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 81-88). Ankara, Turkey: PME.

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Recibido: enero de 2012. Aceptado: abril de 2012  
Handle: \*\*