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Fractional convolution quadrature based on Generalized Adams Methods

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Abstract In this paper we present a product quadrature rule for Volterra integral equations with weakly singular kernels based on the Generalized Adams Methods. The formulas represent numerical solvers for Fractional Differential Equations, which inherit the linear stability properties already known for the integer order case. The numerical experiments confirm the valuable properties of this approach.

Keywords Fractional Differential Equations · Product Quadrature Rule · Generalized Adams Methods

Mathematics Subject Classification (2000) 65L06 · 65L05 · 65R20 · 65L20

1 Introduction

In this paper we are interested in the numerical solution of Fractional Differential Equations (FDEs) of the type

$$D^{\alpha}_{t_0} y(t) = f(t, y(t)), \quad t_0 < t \leq T, \quad 0 < \alpha < 1, \tag{1}$$

where $D^{\alpha}_{t_0} y(t)$ denotes the Caputo fractional derivatives defined by [7]

$$D^{\alpha}_{t_0} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{y'(u)}{(t-u)^{\alpha}} du. \tag{2}$$

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As is well known, the use of Caputo's definition allows one to treat the initial conditions at t_0 for FDEs in the same manner as for integer order differential equations, whereas this is not possible using the Riemann-Liouville approach (see e.g. [19] and [20] for a wide background). Setting $y(t_0) = y_0$ the solution of (1) exists and is unique under the hypothesis that f is continuous and fulfils a Lipschitz condition with respect to the second variable (see e.g. [10] for a proof).

As for the integer order case $\alpha = 1$, a classical approach for solving (1) is based on the discretization of the fractional derivative (2), which generalizes the well known Grunwald-Letnikov discretization (see [20]), leading to the so-called Fractional Backward Differentiation Formulas (FBDFs, [13,15]). Besides, since the solution of (1) can be written as

$$y(t) = y(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-u)^{\alpha-1} f(u, y(u)) du, \tag{3}$$

which represents a Volterra integral equation of the second kind with weakly singular kernel and constant forcing function, each quadrature scheme for the above integral leads to a numerical solver for (2). In this setting, the most studied approaches are the Fractional Linear Multistep Methods (FLMMs, [17]) and the so called Adams product quadrature rules ([9], [16]), in which the Adams formulas for ordinary differential equations (ODEs) are extended to (3). For a wide background about the most established techniques for solving Volterra equations we may refer to [8].

The aim of this paper is to extend the Generalized Adams Methods for ODEs (see [6]) in order to define product quadrature rules for the solution of (3). We call the resulting schemes *Fractional Generalized Adams Methods* (FGAMs). When used over an assigned uniform partition of the interval of integration $I = [0, T]$ (we have set $t_0 = 0$ for simplicity) given by

$$t_n = nh, \qquad n = 0, 1, \ldots, N, \qquad h = T/N, \tag{4}$$

these methods discretize (3) as follows

$$y_n = y_0 + h^\alpha \sum_{j=0}^{M} w_{n,j} f_j + h^\alpha \sum_{j=0}^{n+k_2} \omega_{n-j} f_j, \quad n = M+1, \ldots, N-k_2, \quad k_2 \geq 0, \tag{5}$$

where $y_n \approx y(t_n)$, $f_n = f(t_n, y_n)$, the weights $w_{n,j}$ and ω_n are independent of h, and M depends on the order of the method and on α. The discrete problem (5) is completed by assigning the values of the numerical solution over the first $M + 1$ and the last k_2 meshpoints. As usual for these type of methods, we call

$$S_n = y_0 + h^\alpha \sum_{j=0}^{M} w_{n,j} f_j, \qquad \Omega_n = h^\alpha \sum_{j=0}^{n+k_2} \omega_{n-j} f_j,$$

the *starting* and the *convolution* terms, respectively. For $k_2 = 0$ we recover the Adams product formulas studied in [16]. In this situation the resulting schemes suffer from the usual order barrier for A-stable methods. In particular, in [16] it is proved that the order of an A-stable convolution quadrature cannot exceed 2. Clearly, this result represents an extension of the famous second Dahlquist barrier for linear multistep methods (LMMs) for ordinary differential equations. As for

ODEs (see [1–3]), this barrier can be overtaken by considering "super-future" points that in our case consists in taking $k_2 > 0$.

We remark that with respect to FLMMs, in which the coefficients of the convolution term are given by the Taylor expansion of the α-power of the generating function of the underlying formula for ODEs (see [12,15]), an Adams type approach is local in principle, and hence a variable stepsize implementation can be considered.

The paper is organized as follows. In Section 2 we introduce the FGAMs, extending the definition of the Generalized Adams Methods to the fractional order case. In Section 3 we discuss the starting quadrature in order to ensure the consistency of the method with a given order. In Section 4 we study the convergence of the methods. In Section 5 we examine the linear stability properties, giving a characterization of the stability region which can be used to draw the boundary loci. Finally, a numerical experiment is reported in Section 6 and some conclusions are contained in Section 7.

2 Fractional Generalized Adams Methods

Following the notation used in [8], for each $t \in [0, T]$ let

$$J[\phi](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \phi(u) du, \qquad \phi(u) = f(u, y(u)). \tag{6}$$

In addition, for the assigned uniform partition (4), let

$$J^{(m)}[\phi](t) = \frac{1}{\Gamma(\alpha)} \int_{t_{m-1}}^{t_m} (t-u)^{\alpha-1} \phi(u) du, \qquad m = 1, \dots, N, \tag{7}$$

so that

$$J[\phi](t_n) = \sum\nolimits_{m=1}^{n} J^{(m)}[\phi](t_n), \qquad n = 1, \dots, N. \tag{8}$$

For a given $k > 0$, let (k_1, k_2) be a couple of nonnegative integers such that $k_1 + k_2 = k$. Denoting by Π_k the set of polynomials of degree $\leq k$, for $m = k_1, \dots, N - k_2$, with $N \geq k$, let $p_m \in \Pi_k$ be the polynomial which interpolates the function ϕ at $t_{m-k_1}, \dots, t_{m+k_2}$, that is,

$$p_m(t_{m+k_2-j}) = \phi(t_{m+k_2-j}) =: \phi_{m+k_2-j}, \quad j = 0, 1, \dots, k.$$

In this way, as for the standard Adams methods, we consider the local approximation

$$J^{(m)}[\phi](t_n) \approx J^{(m)}[p_m](t_n) =: \Omega_n^{(m)}[\phi], \qquad n \geq m, \tag{9}$$

as the basis formula for the numerical approximation of $J[\phi](t)$ and hence of (3).

Using the Newton representation of the interpolating polynomial p_m we can write

$$p_m(t) = \sum\nolimits_{j=0}^{k} \frac{\nabla^j \phi_{m+k_2}}{j! h^j} \overline{p}_{m,j}(t),$$
$$\overline{p}_{m,j}(t) = \prod\nolimits_{l=0}^{j-1} (t - t_{m+k_2-l}), \quad j = 0, 1, \dots, k.$$

In order to have an explicit expression for the coefficients of the approximation $\Omega_n^{(m)}[\phi]$, by (7) and (9) we need to evaluate

$$\frac{1}{j!h^j\Gamma(\alpha)}\int_{t_{m-1}}^{t_m}(t_n-u)^{\alpha-1}\overline{p}_{m,j}(u)du, \qquad j=0,1,\dots,k, \quad n\geq m.$$

Setting $u=t_m+\tau h$, the previous integral can be rewritten as

$$h^\alpha I_{n-m}^{(j)} := h^\alpha\frac{1}{\Gamma(\alpha)}\int_{-1}^{0}(n-m-\tau)^{\alpha-1}\binom{\tau-k_2+j-1}{j}d\tau$$

where $I_{n-m}^{(j)}$ is independent of h.

By construction, the approximation (9) leads to a numerical method which is *consistent of order* $p=k+1$, that is, if $\phi\in C^p([0,T])$ then

$$J^{(m)}[\phi](t_n)-\Omega_n^{(m)}[\phi]=h^{p+\alpha}\theta_{n-m}\phi^{(p)}(\xi_{n,m}), \quad \xi_{n,m}\in[t_{m-1},t_m], \tag{10}$$

where, recalling that $k=k_1+k_2$,

$$\theta_{n-m}=\frac{1}{\Gamma(\alpha)}\int_{-1}^{0}(n-m-\tau)^{\alpha-1}\binom{k_1+\tau}{p}d\tau, \quad n\geq m.$$

Applying the mean value theorem to the above expression one easily verifies that asymptotically

$$\theta_{n-m}\sim(n-m+1)^\alpha-(n-m)^\alpha\sim\alpha(n-m+1)^{\alpha-1}. \tag{11}$$

Taking the sum over all allowed subintervals, after some computations we obtain

$$\sum_{m=k_1}^{n}\Omega_n^{(m)}[\phi]=h^\alpha\sum_{j=0}^{k-1}\overline{w}_{n,j}\phi_j+h^\alpha\sum_{j=0}^{n+k_2}\omega_{n-j}\phi_j, \tag{12}$$

where

$$\Omega_n[\phi]:=h^\alpha\sum_{j=0}^{n+k_2}\omega_{n-j}\phi_j \tag{13}$$

is the convolution term, in which, by setting $I_l^{(j)}=0$ for $l<0$,

$$\omega_r=\sum_{j=0}^{k}\nabla^j I_{r+k_2}^{(j)}, \quad r\geq -k_2. \tag{14}$$

It is important to remark that the coefficients $\overline{w}_{n,j}$ in (12) appear as consequence of the definition (14). Indeed the corresponding sum plays the role of a correction term during the transitory phase.

In the sequel, for each $s\geq 1$, we make use of the following notation

$$\ell_s=\left\{\{a_r\}_{r\in\mathbb{N}}:\sum_{r=1}^{\infty}|a_r|^s<\infty\right\}. \tag{15}$$

The following result shows the asymptotic behavior of the coefficients ω_r and, by standard arguments in convolution quadrature (see [8, §6], [15]), it states that the method is *stable*. The proof can be obtained with a slight modification of [16, Lemma 4.1].

Proposition 1 *For $r \geq 1$,*

$$\omega_r = \frac{r^{\alpha-1}}{\Gamma(\alpha)} + v_r, \quad \{v_r\}_{r\in\mathbb{N}} \in \ell_1. \tag{16}$$

Remark 1 The coefficients $\overline{w}_{n,j}$ are suitable combinations of the terms $\nabla^j I^{(j)}_{n+k_2}$ for certain values of j and hence their asymptotic behavior is the same of the ω_n (see again [16, Lemma 4.1]).

In order to study the convergence properties, let us consider the quantity

$$\Delta_n[\phi] := J[\phi](t_n) - \Omega_n[\phi]. \tag{17}$$

Based on the definition given in [8, §6], the following result states that the convolution quadrature $\Omega_n[\phi]$ is *convergent of order p*. As in [15], we assume that if a function $\phi(t)$ is undefined for $t = 0$, we set $\phi(0) = 0$.

Proposition 2 *Let $\phi(t) = t^\lambda$, $\lambda > -1$. Then*

$$\Delta_n[\phi] = O\left(h^{\lambda+1}\right) + O(h^p),$$

for each n such that $t_n = nh \in [a, T]$, $a > 0$ fixed.

Proof Using the definitions of $J[\phi](t_n)$ and $\Omega_n[\phi]$ given in (6) and (13), respectively, and taking into account (7)-(8) and (12), we can write

$$\begin{aligned}\Delta_n\left[t^\lambda\right] = &\frac{1}{\Gamma(\alpha)}\int_0^{t_{k_1-1}} (t_n-u)^{\alpha-1}u^\lambda du\\ &+h^\alpha \sum\nolimits_{j=0}^{k-1} \overline{w}_{n,j} t_j^\lambda\\ &+\sum_{m=k_1}^{n}\left(J^{(m)}\left[t^\lambda\right](t_n) - \Omega_n^{(m)}\left[t^\lambda\right]\right).\end{aligned}$$

Now, defining $u = t_n x$, and using the mean value theorem we obtain

$$\frac{1}{\Gamma(\alpha)}\int_0^{t_{k_1-1}} (t_n-u)^{\alpha-1}u^\lambda du = \frac{t_n^{\alpha+\lambda}}{\Gamma(\alpha)}\int_0^{\frac{k_1-1}{n}} (1-x)^{\alpha-1}x^\lambda dx \sim h^{\lambda+1}t_n^{\alpha-1}. \tag{18}$$

Moreover, by Remark 1 we easily find that

$$h^\alpha \sum\nolimits_{j=0}^{k-1} \overline{w}_{n,j} t_j^\lambda \sim h^{\lambda+1} t_n^{\alpha-1}. \tag{19}$$

Finally, from (10)-(11), by denoting with $\lambda^{(p)} = \lambda(\lambda-1)\cdots(\lambda-p+1)$ we get

$$\begin{aligned}&\sum_{m=k_1}^{n}\left(J^{(m)}\left[t^\lambda\right](t_n) - \Omega_n^{(m)}\left[t^\lambda\right]\right) = \lambda^{(p)} h^{p+\alpha} \sum\nolimits_{m=k_1}^{n} \xi_{n,m}^{\lambda-p}\theta_{n-m}\\ &\sim \alpha\lambda^{(p)} h^{\alpha+\lambda} \sum\nolimits_{m=k_1}^{n} m^{\lambda-p}(n-m+1)^{\alpha-1}\\ &= \alpha\lambda^{(p)} t_n^{\alpha-1} h^{\lambda+1} \sum\nolimits_{m=k_1}^{n} m^{\lambda-p}\left(1-\frac{m+1}{n}\right)^{\alpha-1}\\ &\sim \alpha\lambda^{(p)} t_n^{\alpha-1} h^{\lambda+1}\left(n^{\lambda+1-p}+1\right) = \alpha\lambda^{(p)}\left(h^p t_n^{\alpha+\lambda-p} + h^{\lambda+1}t_n^{\alpha-1}\right).\end{aligned} \tag{20}$$

Collecting (18), (19) and (20), we obtain the result. □

3 Starting quadrature

As stated in the Introduction we are interested in an approximation of $J[\phi](t_n)$ of the type

$$J\left[\phi\right](t_n) \approx S_n\left[\phi\right] + \Omega_n\left[\phi\right] =: J_n[\phi], \tag{21}$$

where $\Omega_n\left[\phi\right]$ is defined by (13), and $S_n\left[\phi\right]$ is the so-called *starting quadrature* defined by

$$S_n\left[\phi\right] = h^\alpha \sum_{j=0}^{M} w_{n,j}\phi_j,$$

for a certain M and suitable coefficients $w_{n,j}$. We denote by $E_n[\phi]$ the overall truncation (or quadrature) error associated to (21), given by

$$E_n[\phi] := J[\phi](t_n) - J_n[\phi] = \Delta_n[\phi] - S_n\left[\phi\right], \tag{22}$$

see (17). It is known that if $y(t)$ is the exact solution of (3) with $f(t,y)$ smooth enough, then $\phi(t) = f(t,y(t))$ might be generated by functions of the form $\phi_{\mu,\ell}(t) = t^{\mu+\ell\alpha}$ where μ and ℓ are nonnegative integers, [8, §6]. This means that $\phi(t)$ might contain non-smooth components in proximity of the origin. It follows that the starting quadrature must be chosen appropriately, in order to get a convolution quadrature for which $E_n[\phi] = O(h^p)$ uniformly for all $nh \geq a > 0$. This objective is gained by imposing

$$E_n[\phi_{\mu,\ell}] = 0, \qquad \text{for all } (\mu,\ell) \in M_p(\alpha) \tag{23}$$

where

$$M_p(\alpha) = \{(\mu,\ell) \in \mathbb{N}_0 \times \mathbb{N}_0 : \ell \leq \ell_p(\alpha), \mu \leq \mu_p(\alpha,\ell)\} \tag{24}$$

with $\mu_p(\alpha,\ell) = p - 1 - \ell\alpha$ and

$$\ell_p(\alpha) = \begin{cases} (p-1)/\alpha & \text{if } \alpha \text{ is irrational;} \\ \min(q-1,(p-1)/\alpha) & \text{if } \alpha = m/q \text{ with } m \text{ and } q \text{ coprime.} \end{cases}$$

After some computation, one verifies that the conditions in (23) are fulfilled if the starting weights solve the Vandermonde type system

$$\sum_{j=0}^{M} w_{n,j} j^{\mu+\ell\alpha} = \frac{\Gamma(\mu+\ell\alpha+1)n^{\mu+(\ell+1)\alpha}}{\Gamma(\mu+(\ell+1)\alpha+1)} - \sum_{j=0}^{n+k_2} \omega_{n-j} j^{\mu+\ell\alpha}. \tag{25}$$

In particular, if we set

$$M = \#M_p(\alpha) - 1, \tag{26}$$

where the symbol # denotes the cardinality, then the system (25) has a unique solution for each $n \geq M+1$. The resulting starting weights are independent of h and it can be proved that (see e.g. [8, §6])

$$w_{n,j} = O(n^{\alpha-1}). \tag{27}$$

The following theorem summarizes the basic properties of a FGAM which extend the ones of a standard Adams product quadrature formula and are in perfect agreement with those obtained for FLMMs in [17].

Theorem 1 *If the starting weights of a FGAM constructed on $p = k + 1$ points are generated by (25)-(26), then for any function $\phi(t) = \Phi(t, t^\alpha)$, with $\Phi(x_1, x_2)$ sufficiently differentiable, the truncation error is*

$$E_n[\phi] = O\left(h^p t_n^{\alpha+\bar{\beta}-p}\right), \tag{28}$$

where (see (24)),

$$\bar{\beta} = \min\left\{\mu + \ell\alpha \ s.t. \ (\mu, \ell) \in (\mathbb{N}_0 \times \mathbb{N}_0) \setminus M_p(\alpha)\right\} > p - 1. \tag{29}$$

4 Convergence analysis

If we denote by $e_n = y(t_n) - y_n$ the global error at $t = t_n$ and by $\mathcal{L}$ the Lipschitz constant of $f(t, y)$, then it is not difficult to verify that, see (5),

$$\|e_n\| \le h^\alpha \mathcal{L} \left(\sum_{j=M+1}^{\min(n+k_2, N-k_2)} |\omega_{n-j}| \, \|e_j\| \right) + g_n + \|E_n\|, \quad n = M+1, \ldots, N-k_2, \tag{30}$$

where E_n is the n-th truncation error of the convolution quadrature as defined by (22), and

$$g_n = h^\alpha \mathcal{L} \left(\sum_{j=0}^{M} |w_{n,j} - \omega_{n-j}| \, \|e_j\| \right) + h^\alpha \mathcal{L} \left(\sum_{j=N-k_2+1}^{n+k_2} |\omega_{n-j}| \, \|e_j\| \right),$$

collects the error contributes of the initial and final conditions. In particular, if we assume to know an approximation of the first $M + 1$ and the last k_2 values of the numerical solution with accuracy $O(h^{\bar{\beta}})$ and $O(h^{p-\alpha})$, respectively, then, by virtue of (16) and (27),

$$g_n = O(h^{\bar{\beta}+1} t_n^{\alpha-1}) + O(h^p) = O(h^p t_n^{\alpha+\bar{\beta}-p}), \tag{31}$$

where the last equality follows from (29).

For $k_2 = 0$ the error analysis can be accomplished using Gronwall type inequalities (see [18]), such as the one given in [8, Theorem 1.5.6]. The convergence properties of the corresponding method then follow directly from the stability and consistency properties. For $k_2 > 0$ the situation is rather different and we consider the following approach.

Let us collect into the vector $\mathbf{e}$ the norms of the global errors at the interior meshpoints, i.e.,

$$\mathbf{e} = \left(\|e_{M+1}\|, \ldots, \|e_{N-k_2}\|\right)^T$$

and define $\mathbf{g} = (g_{M+1}, \ldots, g_{N-k_2})^T$ and $\mathbf{E} = \left(\|E_{M+1}\|, \ldots, \|E_{N-k_2}\|\right)^T$. The system of inequalities in (30) can be rewritten in matrix form as

$$\mathbf{e} \le h^\alpha \mathcal{L} \, |\Omega| \, \mathbf{e} + \mathbf{g} + \mathbf{E}, \tag{32}$$

(here and below, for matrix arguments the inequalities and the absolute value have to be intended component by component), where

$$(\Omega)_{ij} = \begin{cases} \omega_{i-j} & \text{for } i-j \geq -k_2, \\ 0 & \text{otherwise.} \end{cases}$$

Let moreover

$$A \equiv A_{\mathcal{N}} := \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & \ddots & \ddots & \\ & & -1 & 1 \end{pmatrix} \in \mathbb{R}^{\mathcal{N}\times\mathcal{N}}, \tag{33}$$

with

$$\mathcal{N} = N - k_2 - M. \tag{34}$$

In order to determine an upper bound for the global errors at the interior points, we need the following preliminary results.

Lemma 1 *If $\alpha \in (0,1]$ then the matrix Ω can be written as*

$$\Omega = A^{-\alpha} + U$$

where U is a Toeplitz matrix with $\|U\|_\infty$ uniformly bounded with respect to $\mathcal{N}$.

Proof It is known that if $\alpha \in (0,1]$ then for each $r \geq 1$

$$\frac{r^{\alpha-1}}{\Gamma(\alpha)} = (-1)^r \binom{-\alpha}{r} + \hat{v}_r, \qquad \{\hat{v}_r\}_{r\in\mathbb{N}} \in \ell_1, \tag{35}$$

(see e.g. [11, p. 47]). Therefore, by Proposition 1 one gets

$$\begin{aligned} \omega_{i-j} = \frac{(i-j)^{\alpha-1}}{\Gamma(\alpha)} + v_{i-j} &= (-1)^{i-j}\binom{-\alpha}{i-j} + \hat{v}_{i-j} + v_{i-j} \\ &:= (-1)^{i-j}\binom{-\alpha}{i-j} + u_{i-j} \\ &= \left(A^{-\alpha}\right)_{ij} + u_{i-j}, \qquad i > j, \end{aligned}$$

where $\{u_r\}_{r\in\mathbb{N}} \in \ell_1$. □

Since $A^{-\alpha} \geq O$, from the previous lemma one immediately deduces that $|\Omega| \leq A^{-\alpha} + |U|$. Therefore, by (32), setting

$$Z := I - h^\alpha \mathcal{L}(A^{-\alpha} + |U|), \tag{36}$$

one obtains

$$Z\mathbf{e} \leq \mathbf{g} + \mathbf{E}. \tag{37}$$

Lemma 2 *If h is sufficiently small then the matrix Z defined in (36) is nonsingular, $Z^{-1} \geq O$ and $\left\|Z^{-1}\right\|_\infty$ is uniformly bounded with respect to $\mathcal{N}$.*

Proof If h is sufficiently small, the matrix Z can be factorized as

$$Z = Z_1 Z_2, \quad Z_1 = I - h^\alpha \mathcal{L} A^{-\alpha}, \quad Z_2 = I - h^\alpha \mathcal{L} Z_1^{-1} |U|. \tag{38}$$

Now, in order to demonstrate the results of the lemma, we will prove that Z_1 and Z_2 are both M-matrices and the infinity matrix norm of their inverses are both uniformly bounded with respect to $\mathcal{N}$. Let us begin with the matrix Z_1. First of all, it is not difficult to verify that if $h^\alpha \mathcal{L} < 1$ then Z_1 is an M-matrix. Secondly, since $A^{-\alpha} - I$ is nilpotent of degree $\mathcal{N}$, we can write

$$\begin{aligned} Z_1^{-1} &= \frac{1}{1 - h^\alpha \mathcal{L}} \left(I - \frac{h^\alpha \mathcal{L}}{1 - h^\alpha \mathcal{L}} \left(A^{-\alpha} - I \right) \right)^{-1} \\ &= \frac{1}{1 - h^\alpha \mathcal{L}} \sum\nolimits_{j=0}^{\mathcal{N}-1} \left(\frac{h^\alpha \mathcal{L}}{1 - h^\alpha \mathcal{L}} \right)^j \left(A^{-\alpha} - I \right)^j. \end{aligned}$$

It follows that, for each $\mu > \mathcal{L}$ and h such that

$$0 < \frac{\mathcal{L}}{1 - h^\alpha \mathcal{L}} \leq \mu,$$

we have

$$\begin{aligned} \left\| Z_1^{-1} \right\|_\infty &\leq \frac{\mu}{\mathcal{L}} \sum\nolimits_{j=0}^{\mathcal{N}-1} (\mu h^\alpha)^j \left\| \left(A^{-\alpha} - I \right)^j \right\|_\infty \\ &\leq \frac{\mu}{\mathcal{L}} \sum\nolimits_{j=0}^{\mathcal{N}-1} (\mu h^\alpha)^j \left\| A^{-j\alpha} \right\|_\infty, \end{aligned} \tag{39}$$

where in the last inequality we have used the fact that, for each $j \geq 0$,

$$O \leq \left(A^{-\alpha} - I \right)^j \leq A^{-j\alpha}.$$

Now

$$A^{-j\alpha} = \sum_{l=0}^{\mathcal{N}-1} \binom{j\alpha}{l} \left(A^{-1} - I \right)^l,$$

and $A^{-1} - I \geq O$ is a strictly lower triangular Toeplitz matrix. Then, by [6, Lemma 4.4.2] and the Chu-Vandermonde identity (see, e.g., [5, p. 59-60]),

$$\begin{aligned} \left\| A^{-j\alpha} \right\|_\infty &= \sum_{l=0}^{\mathcal{N}-1} \binom{j\alpha}{l} \left\| \left(A^{-1} - I \right)^l \right\|_\infty \\ &= \sum_{l=0}^{\mathcal{N}-1} \binom{j\alpha}{l} \binom{\mathcal{N}-1}{l} \\ &= \binom{\mathcal{N}-1+j\alpha}{\mathcal{N}-1} = \frac{\Gamma(\mathcal{N}+j\alpha)}{\Gamma(j\alpha+1)\Gamma(\mathcal{N})}. \end{aligned}$$

Substituting the last expression in (39) we obtain

$$\left\| Z_1^{-1} \right\|_\infty \leq \frac{\mu}{\mathcal{L}} \sum\nolimits_{j=0}^{\mathcal{N}-1} \frac{\mu^j}{\Gamma(j\alpha+1)} \frac{\Gamma(\mathcal{N}+j\alpha)}{\Gamma(\mathcal{N})} h^{j\alpha},$$

and by Stirling's formula we have

$$\frac{\Gamma(\mathcal{N}+j\alpha)}{\Gamma(\mathcal{N})}h^{j\alpha} \leq C\,(1+\alpha)^{j\alpha}\,T^{j\alpha},$$

where C is a constant depending on α. Therefore

$$\left\|Z_1^{-1}\right\|_\infty \leq C\frac{\mu}{\mathcal{L}}E_\alpha\left(\mu(1+\alpha)^\alpha T^\alpha\right), \tag{40}$$

where E_α denotes the one-parameter Mittag-Leffler function (see e.g. [20, §1.2]) so that $\left\|Z_1^{-1}\right\|_\infty$ is uniformly bounded.

Let us now consider the matrix Z_2 in (38) and its inverse. By Lemma 1 and (40), for each $\theta < 1$ we can define h_0 such that for $h \leq h_0$

$$h^\alpha \mathcal{L}\left\|Z_1^{-1}\,|U|\right\|_\infty \leq \theta < 1,$$

so that Z_2 is an M-matrix and

$$\left\|Z_2^{-1}\right\|_\infty \leq \sum_{j=0}^{\infty}\left(h^\alpha\mathcal{L}\left\|Z_1^{-1}\,|U|\right\|_\infty\right)^j \leq (1-\theta)^{-1},$$

namely, $\left\|Z_2^{-1}\right\|_\infty$ is uniformly bounded. □

Theorem 2 *There exists $h_0 > 0$ such that for $h \leq h_0$ the global error of the FGAM with starting weights generated by (25)-(26) fulfils the inequality*

$$\|\mathbf{e}\|_\infty \leq K\left(\|\mathbf{g}\|_\infty + \|\mathbf{E}\|_\infty\right) = O\left(h^{\min(\bar\beta+\alpha,p)}\right)$$

where K is a suitable constant independent of h, and $\bar\beta$ is given in (29).

Proof The statement is a consequence of (28), (31), (37) and Lemma 2. □

5 Linear stability

The linear stability properties of the Generalized Adams Methods when applied to ODEs are well known in the literature (see e.g. [6]). Dealing with FDEs, in this section we provide and justify a definition of the stability region of a FGAM showing also the boundary loci for some values of k and α.

Let us consider the usual scalar test problem

$$D_0^\alpha\, y(t) = \lambda y(t), \quad \lambda \in \mathbb{C}, \qquad y(0) = y_0, \tag{41}$$

whose exact solution is given by

$$y(t) = E_\alpha(\lambda t^\alpha)y_0.$$

It is known that $y(t) \to 0$ as $t \to +\infty$ whenever $\lambda \in S_\alpha$ where

$$\mathcal{S}_\alpha = \{\mu \in \mathbb{C} : |\pi - \arg(\mu)| < (1-\alpha/2)\,\pi\}. \tag{42}$$

Applying the method (5) to (41) we obtain

$$y_n = S_n + q \sum_{j=0}^{n+k_2} \omega_{n-j} y_j, \tag{43}$$

where $q = h^\alpha \lambda$ and

$$S_n = y_0 + q \sum_{j=0}^{M} w_{n,j} y_j.$$

For a classical implicit convolution quadrature, that is $k_2 = 0$, the stability region of a method is defined as

$$\mathcal{D} \equiv \{q \in \mathbb{C} : \, y_n = y_n(q) \to 0 \quad \text{as} \quad n \to +\infty\},$$

and the method is said to be A-stable if $\mathcal{S}_\alpha \subseteq \mathcal{D}$. A characterization of this region is given in the following result, which is a reformulation of Theorem 2.1 in [16].

Theorem 3 *Let $k_2 = 0$. If $w_{n,j} \sim n^{\alpha-1}$ and $\omega_n = \frac{n^{\alpha-1}}{\Gamma(\alpha)} + u_n$, with $\{u_n\}_{n\in\mathbb{N}} \in \ell_1$, then*

$$\mathcal{D} = \{q \in \mathbb{C} : \, q \neq 1/\omega(z), \quad \forall \ |z| \leq 1\},$$

where $\omega(z) = \sum_{n=0}^{\infty} \omega_n z^n$ is the generating function of the convolution term.

From the hypothesis on the convolution weights and (35) one deduces that

$$\omega(z) = (1-z)^{-\alpha} + \hat{\omega}(z), \text{ where } |\hat{\omega}(z)| \text{ is bounded for } |z| \leq 1.$$

This implies that $g(z) = (1-z)^\alpha \omega(z)$ is bounded for each $|z| \leq 1$ too. Therefore we can express the stability region in the following equivalent form

$$\mathcal{D} = \{q \in \mathbb{C} : \, \chi(z,q) := (1-z)^\alpha - q g(z) \neq 0, \quad \forall \ |z| \leq 1\}. \tag{44}$$

More generally, namely when $k_2 \geq 0$, the generating function of the convolution weights of a FGAM is the power series $\omega(z) = \sum_{n=0}^{\infty} \omega_{n-k_2} z^n$. However, by virtue of Proposition 1, the corresponding $g(z) = (1-z)^\alpha \omega(z)$ is also bounded for $|z| \leq 1$.

Definition 1 A function $\chi(z)$ is said to be *of type* (m_1, m_2) if $\chi(z)$ is bounded for $|z| \leq 1$ and has exactly m_1 and m_2 zeros inside and on the boundary of the unit circle, respectively.

Definition 2 The region $\mathcal{D}_{k_2}$ of the complex plane defined by

$$\mathcal{D}_{k_2} = \left\{q \in \mathbb{C} : \, \chi(z,q) := z^{k_2}(1-z)^\alpha - q g(z) \text{ is of type } (k_2, 0)\right\} \tag{45}$$

is called the *stability region* of a FGAM used with k_2 final conditions.

Clearly, for $k_2 = 0$, $\mathcal{D}_{k_2}$ reduces to $\mathcal{D}$, the region of stability of a classical Adams method, see (44).

In what follows we demonstrate that if $q \in \mathcal{D}_{k_2}$ then the numerical solution of the FGAM simulates the behavior of the exact solution. Setting

$$(1-z)^\alpha = \sum\nolimits_{n=0}^{\infty} \gamma_n z^n, \quad \gamma_n = (-1)^n \binom{\alpha}{n}, \tag{46}$$
$$g(z) = \sum_{n=0}^{\infty} g_{n-k_2} z^n,$$

there exist $\{\eta_{n,j}\}_{n \geq M+1}$, $j = 0, \ldots, M + k_2$, such that (43) can be rewritten as

$$\sum_{j=0}^{n} \gamma_{n-j} y_j - q \sum_{j=0}^{n+k_2} g_{n-j} y_j = \gamma_{0n} y_0 + q \sum_{j=0}^{M+k_2} \eta_{n,j} y_j \tag{47}$$

with

$$\gamma_{0n} = \sum_{j=0}^{n} \gamma_j, \quad n = M+1, \ldots, N-k_2. \tag{48}$$

Remark 2 For later reference we observe that $\{g_n\} \in \ell_1$; moreover, by (46)-(48) and (35), used with $-\alpha$ replaced by α, we get

$$\gamma_n \sim n^{-\alpha-1}, \qquad \gamma_{0n} \sim n^{-\alpha}, \tag{49}$$

and it is possible to prove that $\{\eta_{n,j}\} \in \ell_1$ for each $j = 0, 1, \ldots, M + k_2$.

Formula (47) is equivalent to

$$\sum_{j=M+1}^{n} \gamma_{n-j} y_j - q \sum_{j=M+1}^{\min(n+k_2, N-k_2)} g_{n-j} y_j - q \sum_{j=M+1}^{M+k_2} \eta_{n,j} y_j = b_n + \tilde{b}_n, \tag{50}$$

where

$$b_n = \gamma_{0n} y_0 - \sum_{j=0}^{M} \left[\gamma_{n-j} - q\,(g_{n-j} + \eta_{n,j})\right] y_j, \tag{51}$$
$$\tilde{b}_n = q \sum_{j=N-k_2+1}^{n+k_2} g_{n-j} y_j.$$

We can rewrite formula (50) in matrix form as (see (33)-(34))

$$\left(A_{\mathcal{N}}^{\alpha} - q\,(G_{\mathcal{N}} + R_{\mathcal{N}})\right) \mathbf{y}_{\mathcal{N}} = \mathbf{b}_{\mathcal{N}} + \widetilde{\mathbf{b}}_{\mathcal{N}}, \tag{52}$$

where $\mathbf{y}_{\mathcal{N}} = (y_{M+1}, \ldots, y_{N-k_2})^T$,

$$(A_{\mathcal{N}}^{\alpha})_{ij} = \begin{cases} \gamma_{i-j} & \text{for } i \geq j \\ 0 & \text{otherwise} \end{cases}, \qquad (G_{\mathcal{N}})_{ij} = \begin{cases} g_{i-j} & \text{for } i \geq j - k_2 \\ 0 & \text{otherwise} \end{cases},$$
$$(R_{\mathcal{N}})_{ij} = \begin{cases} \eta_{i+M,j+M} & \text{for } j \leq k_2 \\ 0 & \text{otherwise} \end{cases}, \tag{53}$$
$$\mathbf{b}_{\mathcal{N}} = (b_{M+1}, \ldots, b_{N-k_2})^T, \qquad \tilde{\mathbf{b}}_{\mathcal{N}} = (\tilde{b}_{M+1}, \ldots, \tilde{b}_{N-k_2})^T.$$

It is important to underline the fact that $\mathbf{b}_{\mathcal{N}}$ depends only on the initial values $\{y_0, \ldots, y_M\}$ while $\tilde{\mathbf{b}}_{\mathcal{N}}$ depends only on the final values $\{y_{N-k_2+1}, \ldots, y_N\}$. In addition, $\tilde{\mathbf{b}}_{\mathcal{N}}$ has (at most) only the last k_2 entries different from zero. We observe that $\mathrm{rank}(R_{\mathcal{N}}) = k_2$ independently of $\mathcal{N}$. Therefore, $-qR_{\mathcal{N}}$ can be considered as a perturbation of

$$T_{\mathcal{N}}(q) := A^{\alpha}_{\mathcal{N}} - q\, G_{\mathcal{N}}, \tag{54}$$

when $\mathcal{N}$ is sufficiently large. As for the ODEs, the stability properties of the numerical solution are determined by the properties of the operator (see (15))

$$T_{\infty}(q) := A^{\alpha}_{\infty} - q\, G_{\infty} : \ell_s \to \ell_s.$$

Proposition 3 *If $q \in \mathcal{D}_{k_2}$ then $T_{\infty}(q)$ is invertible with bounded inverse (continuously invertible).*

Proof The result follows from the Wiener-Hopf factorization of $T_{\infty}(q)$. Indeed, denoting by $\zeta_1, \ldots, \zeta_{k_2}$ the roots inside the unit circle of the function $\chi(z, q)$ defined in (45), we have

$$T_{\infty}(q) = UL, \tag{55}$$

where

$$U = \prod_{j=1}^{k_2} U_j, \quad U_j = \begin{pmatrix} 1 & -\zeta_j & & \\ & 1 & -\zeta_j & \\ & & \ddots & \ddots \\ & & & \ddots \end{pmatrix} = I - \zeta_j H, \tag{56}$$

H being the shift matrix. In this way

$$U_j^{-1} = \sum_{n=0}^{\infty} \zeta_j^n H^n \tag{57}$$

is bounded. For the matrix

$$L = \begin{pmatrix} l_0 & & & \\ l_1 & l_0 & & \\ l_2 & l_1 & l_0 & \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \tag{58}$$

setting $l(z) = \sum_{n=0}^{\infty} l_n z^n$, the generating function associated to L, we know that $l(z) \neq 0$ for $|z| \leq 1$ and $\{l_n\} \in \ell_1$, so that, for the Wiener inversion theorem, L is invertible with bounded inverse. □

From this result one deduces that if $q \in \mathcal{D}_{k_2}$ and $\mathcal{N}$ is sufficiently large then the matrix $T_{\mathcal{N}}(q)$ in (54) is nonsingular. Hereafter, for simplicity, we restrict the analysis to the case $k_2 = 1$ (the more general case can be treated following the approach proposed in [4]). Before proceeding we need the following two preliminary results.

Lemma 3 *If $k_2 = 1$, $q \in \mathcal{D}_{k_2}$ and $\mathcal{N}$ is sufficiently large then*

$$T_{\mathcal{N}}^{-1}(q)\, \mathbf{e}_{\mathcal{N}} \sim (\zeta_1^{\mathcal{N}-1}, \ldots, \zeta_1, 1)^T,$$

where $\mathbf{e}_{\mathcal{N}}$ is the last vector of the canonical basis in $\mathbb{R}^{\mathcal{N}}$.

Proof For reader's convenience, it is postponed to subsection 5.1.

Lemma 4 *Let $k_2 = 1$, $q \in \mathcal{D}_{k_2}$ and $\mathbf{c}_\infty \in \ell_s$. Then there exists $\mathbf{u}_\infty \in \ell_s$ such that, for $\mathcal{N}$ sufficiently large, and denoting by $\mathbf{c}_\mathcal{N}$ and $\mathbf{u}_\mathcal{N}$ the vectors of the first $\mathcal{N}$ components of $\mathbf{c}_\infty$ and $\mathbf{u}_\infty$, respectively,*

$$T_\mathcal{N}(q)^{-1}\mathbf{c}_\mathcal{N} - \mathbf{u}_\mathcal{N} \sim (\zeta_1^{\mathcal{N}-1}, \dots, \zeta_1, 1)^T.$$

Proof Let $\mathbf{u}_\infty \in \ell_s$ be the unique solution of $T_\infty(q)\mathbf{u}_\infty = \mathbf{c}_\infty$. The first $\mathcal{N}$ equations of the previous system can be written as $T_\mathcal{N}(q)\mathbf{u}_\mathcal{N} = \mathbf{c}_\mathcal{N} + \delta\mathbf{c}_\mathcal{N}$, where $\delta\mathbf{c}_\mathcal{N}$ has (at most) only the last entry different from zero (recall that $k_2 = 1$). The statement is then an immediate consequence of Lemma 3. □

We can now discuss the qualitative behavior of the numerical solution provided by a FGAM.

Theorem 4 *If $k_2 = 1$, $q \in \mathcal{D}_{k_2}$ and $\mathcal{N}$ is sufficiently large, then there exists $\mathbf{x}_\infty \in \ell_s$, with $s > 1/\alpha$, such that, denoting by $\mathbf{x}_\mathcal{N}$ the vector of its first $\mathcal{N}$ components, the solution $\mathbf{y}_\mathcal{N}$ of the linear system (52) verifies*

$$\mathbf{y}_\mathcal{N} - \mathbf{x}_\mathcal{N} \sim \begin{pmatrix} \zeta_1^{\mathcal{N}-1} \\ \vdots \\ \zeta_1 \\ 1 \end{pmatrix}.$$

Proof Under the considered hypotheses, the system (52) can be reformulated as

$$\left(I_\mathcal{N} - q\, T_\mathcal{N}^{-1}(q) R_\mathcal{N}\right) \mathbf{y}_\mathcal{N} = T_\mathcal{N}^{-1}(q) \left(\mathbf{b}_\mathcal{N} + \widetilde{\mathbf{b}}_\mathcal{N}\right) =: \mathbf{v}_\mathcal{N}. \tag{59}$$

We observe that if $k_2 = 1$ the entries of $R_\mathcal{N}$ and, consequently, of $-q\, T_\mathcal{N}^{-1}(q) R_\mathcal{N}$ are all equal to zero except for the ones in the first column (see (53)). If we set

$$\mathbf{r}_\mathcal{N} := -q\, T_\mathcal{N}^{-1}(q) R_\mathcal{N} \mathbf{e}_1,$$

then the solution of (59) is given by

$$\mathbf{y}_\mathcal{N} = \mathbf{v}_\mathcal{N} - \frac{\mathbf{e}_1^T \mathbf{v}_\mathcal{N}}{1 + \mathbf{e}_1^T \mathbf{r}_\mathcal{N}}\, \mathbf{r}_\mathcal{N}.$$

Therefore, in order to determine the behavior of $\mathbf{y}_\mathcal{N}$, we need to study the behavior of $\mathbf{v}_\mathcal{N}$ and $\mathbf{r}_\mathcal{N}$. Let us start by considering $\mathbf{v}_\mathcal{N}$ defined in (59). From (51) and Remark 2 we deduce that $\mathbf{b}_\mathcal{N}$ is the vector of the first $\mathcal{N}$ components of $\mathbf{b}_\infty \in \ell_s$ for each $s > 1/\alpha$. On the other hand, if $k_2 = 1$, $\widetilde{\mathbf{b}}_\mathcal{N}$ has (at most) only the last entry different from zero. By applying Lemma 4 and Lemma 3 it follows that

$$\mathbf{v}_\mathcal{N} - \mathbf{u}_\mathcal{N} \sim (\zeta_1^{\mathcal{N}-1}, \dots, \zeta_1, 1)^T,$$

where $\mathbf{u}_\mathcal{N}$ is the vector of the first $\mathcal{N}$ components of $\mathbf{u}_\infty \in \ell_s$. Concerning the behavior of $\mathbf{r}_\mathcal{N}$, from Lemma 4 and Remark 2 one gets

$$\mathbf{r}_\mathcal{N} - \mathbf{a}_\mathcal{N} \sim (\zeta_1^{\mathcal{N}-1}, \dots, \zeta_1, 1)^T,$$

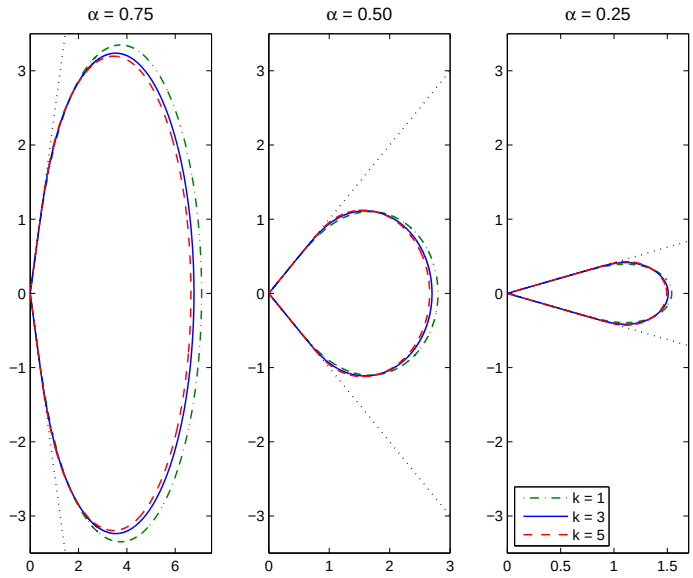

Fig. 1 Boundary loci of the FGAMs with $k = 1, 3, 5$.

where $\mathbf{a}_{\mathcal{N}}$ is the vector of the first $\mathcal{N}$ components of $\mathbf{a}_\infty \in \ell_1 \subseteq \ell_s$. Finally,

$$\frac{\mathbf{e}_1^T \mathbf{v}_{\mathcal{N}}}{1 + \mathbf{e}_1^T \mathbf{r}_{\mathcal{N}}} = \frac{\mathbf{e}_1^T \mathbf{u}_{\mathcal{N}}}{1 + \mathbf{e}_1^T \mathbf{a}_{\mathcal{N}}} \left(1 + O\left(\zeta_1^{\mathcal{N}-1}\right)\right).$$

Setting

$$\mathbf{x}_{\mathcal{N}} := \mathbf{u}_{\mathcal{N}} - \frac{\mathbf{e}_1^T \mathbf{u}_{\mathcal{N}}}{1 + \mathbf{e}_1^T \mathbf{a}_{\mathcal{N}}} \mathbf{a}_{\mathcal{N}},$$

one deduces that $\mathbf{x}_\infty \in \ell_s$ and then the statement follows. □

The boundary loci of the FGAM for different values of α, $k = 1, \ldots, 6$ and

$$k_2 = \left\lfloor \frac{k}{2} \right\rfloor$$

are reported in Figure 1 and 2, where the dotted lines mark the boundary of $\mathcal{S}_\alpha$ defined by (42). We remark that the chosen value of k_2 coincides with the one considered for the Generalized Adams Methods for ODEs (see [6]). As one can see, the presented methods appear to be always A-stable independently of k and α.

5.1 Proof of Lemma 3

If $k_2 = 1$, after some computations, from (55) one deduces that

$$T_{\mathcal{N}}(q) = L_{\mathcal{N}} U_{\mathcal{N}} + \mathbf{w}_{\mathcal{N}} \mathbf{e}_1^T,$$

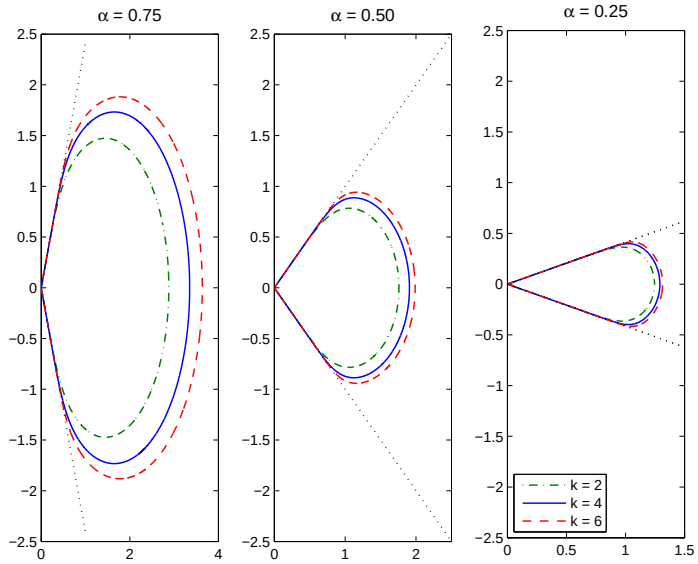

Fig. 2 Boundary loci of the FGAMs with $k = 2, 4, 6$.

where $L_{\mathcal{N}}$ and $U_{\mathcal{N}}$ are the principal submatrices of size $\mathcal{N}$ of L and U defined in (58) and (56), respectively, $\mathbf{e}_1$ is the first vector of the canonical basis in $\mathbb{R}^{\mathcal{N}}$, and

$$\mathbf{w}_{\mathcal{N}} = -\zeta_1(l_1, l_2, \ldots, l_{\mathcal{N}})^T.$$

We observe that $\|\mathbf{w}_{\mathcal{N}}\|_1$ is uniformly bounded with respect to $\mathcal{N}$ since, as stated in the sentence below (58), $\{l_n\} \in \ell_1$.
By using the Sherman-Morrison formula, we obtain

$$T_{\mathcal{N}}^{-1}(q) = (L_{\mathcal{N}}U_{\mathcal{N}})^{-1} - \frac{(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{w}_{\mathcal{N}}\mathbf{e}_1^T(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}}{1 + \mathbf{e}_1^T(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{w}_{\mathcal{N}}}.$$

Therefore, by observing that, see (58), $L_{\mathcal{N}}^{-1}\mathbf{e}_{\mathcal{N}} = (1/l_0)\mathbf{e}_{\mathcal{N}}$, one has

$$\begin{aligned} T_{\mathcal{N}}^{-1}(q)\mathbf{e}_{\mathcal{N}} &\sim (L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{e}_{\mathcal{N}} - \frac{\mathbf{e}_1^T(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{e}_{\mathcal{N}}}{1 + \mathbf{e}_1^T(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{w}_{\mathcal{N}}}(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{w}_{\mathcal{N}} \\ &= \frac{1}{l_0}\left(U_{\mathcal{N}}^{-1}\mathbf{e}_{\mathcal{N}} - \frac{\mathbf{e}_1^T U_{\mathcal{N}}^{-1}\mathbf{e}_{\mathcal{N}}}{1 + \mathbf{e}_1^T(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{w}_{\mathcal{N}}}(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{w}_{\mathcal{N}}\right). \end{aligned}$$

The statement then follows from the facts that, see (56)-(57),

$$U_{\mathcal{N}}^{-1}\mathbf{e}_{\mathcal{N}} = (\zeta_1^{\mathcal{N}-1}, \ldots, \zeta_1, 1)^T,$$

and, recalling that $\|\mathbf{w}_{\mathcal{N}}\|_1$ is uniformly bounded with respect to $\mathcal{N}$, there exists $\theta > 0$ independent of $\mathcal{N}$ such that

$$\frac{\left\|(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{w}_{\mathcal{N}}\right\|_1}{\left|1 + \mathbf{e}_1^T(L_{\mathcal{N}}U_{\mathcal{N}})^{-1}\mathbf{w}_{\mathcal{N}}\right|}\left(\mathbf{e}_1^T U_{\mathcal{N}}^{-1}\mathbf{e}_{\mathcal{N}}\right) < \theta\,\zeta_1^{\mathcal{N}-1}. \quad \square$$

6 Boundary values and numerical illustrations

The effective use of the FGAMs in (5) requires the definition of suitable strategies for recovering the boundary values. To this aim, in order to achieve an approximation of the starting values $y(t_n), n = 1, 2, \ldots, M$, with a certain accuracy we have considered a collocation approach. It is known that in a neighborhood of the origin the true solution may have the expansion

$$y(t) = \sum_{(\mu,\ell)\in\mathbb{N}_0\times\mathbb{N}_0} c_{\mu\ell}\, t^{\mu+\ell\alpha}.$$

Let us define the set, see (24),

$$\mathcal{B} := \{\beta \in \mathbb{R} : \beta = \mu + \ell\alpha \text{ for } (\mu,\ell) \in M_p(\alpha)\}.$$

We denote by $\beta_m, m = 1, \ldots, M+1$ the elements of $\mathcal{B}$ and we assume that $\beta_m < \beta_{m+1}, m = 1, \ldots, M$ (note that $\beta_1 = 0$). Moreover, let us define the function

$$\varphi(t) = \sum_{m=1}^{M+1} a_m\, t^{\beta_m}.$$

Imposing the conditions, see (1),

$$\varphi(0) = y_0, \qquad D_0^\alpha \varphi(nh) = f(nh, \varphi(nh)), \quad 1 \le n \le M,$$

and then defining

$$y_n = \varphi(nh), \quad 1 \le n \le M,$$

we have that, by construction,

$$|y_n - y(t_n)| = O(h^{\bar\beta}), \quad 1 \le n \le M,$$

where $\bar\beta$ is defined in (29).

Regarding the approximation of $y(t_n), n = N - k_2 + 1, \ldots, N$, these values are computed implicitly through the application of a set of appropriate discretization formulas. Such formulas are derived from the local approximations of

$$J^{(m)}[\phi](t_n) \approx J^{(m)}\left[\hat p\right](t_n) =: \Omega_n^{(m)}\left[\phi\right], \quad m = N - k_2 + 1, \ldots, N,$$

$\hat p \in \Pi_k$ being such that

$$\hat p(t_{N-j}) = \phi(t_{N-j}), \qquad j = 0, 1, \ldots, k,$$

and by using a procedure similar to the one described in Section 2.

As a numerical illustration of the schemes proposed, we have considered their application for solving the initial value problem

$$D_0^\alpha\, y(t) = \begin{pmatrix} 1-2\mu & 1-\mu \\ 2\mu-2 & \mu-2 \end{pmatrix} y(t), \qquad y(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \tag{60}$$

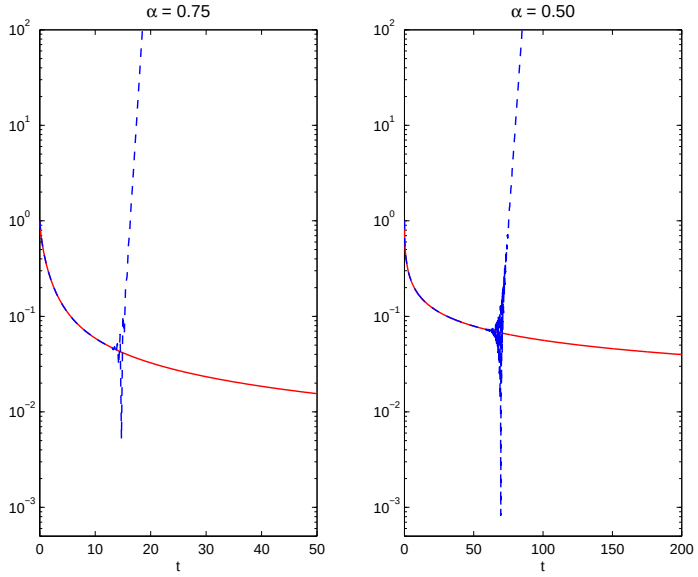

Fig. 3 Absolute value of the first component of the numerical solution of (60) provided by the FGAM (solid line) and the Adams product quadrature rule (dashed line) of order 4.

whose exact solution is

$$y(t) = \begin{pmatrix} 1 \\ -2 \end{pmatrix} E_\alpha(-t^\alpha),$$

independently of $\mu > 0$. When μ is large, the problem is stiff, since the eigenvalues of the Jacobian matrix are $-\mu$ and -1. With the chosen initial value, however, the stiff mode is not present in the continuous solution. As usual, the application of methods with inappropriate stability properties, determines a severe restriction on the choice of the stepsize. We have solved the problem with $\alpha = 0.75$ and $\alpha = 0.50$. In both cases we have set $\mu = 2500$. We have used the FGAM and the classical Adams product quadrature rule [16] of orders 4 with $h = 1/4$. In Figure 3, the approximations of the first component of the solution obtained by applying the FGAM (solid line) and the Adams product quadrature rule (dashed line) just mentioned have been reported. The graphics corresponding to the second component are similar for both methods and hence not reported.

As one can see, the property of stability of the FGAM allows one to get good approximations of solutions of initial value problems even when stiff modes are present and the stepsize used is rather large.

7 Conclusions

The extension to the fractional order case of the Generalized Adams Methods described in [6] has been investigated. As expected, the convergence properties of the FGAMs are in perfect agreement with the ones of the standard Adams

methods for FDEs. At the same time, as shown in Section 6, the linear stability properties make the generalization attractive for stiff problems since the barrier established in [16] is overtaken at the price of the computation of approximations of the endpoint values of the solution. This approach can also be used for the construction of reliable codes with automatic stepsize selection by using a block implementation of the schemes, like the Generalized Adams Methods for ODEs [14]. This topic will be the subject of future investigation.

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