THE LOCAL HÖLDER EXPONENT FOR THE DIMENSION OF INVARIANT SUBSETS OF THE CIRCLE

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Abstract. We consider for each t the set $K(t)$ of points of the circle whose forward orbit for the doubling map does not intersect $(0, t)$, and look at the dimension function $\eta(t) := \text{H.dim } K(t)$. We prove that at every bifurcation parameter t, the local Hölder exponent of the dimension function equals the value of the function $\eta(t)$ itself. A similar statement holds for general expanding maps of the circle: namely we consider the topological entropy of the map restricted to the survival set, and obtain bounds on its local Hölder exponent in terms of the value of the function.

The theory of open dynamical systems, also known as dynamical systems "with holes", was developed to model physical phenomena with escape of mass. One of the simplest models which can be analyzed rigorously is the case of expanding maps of the circle $\mathbb{S} := \mathbb{R}/\mathbb{Z}$ where the hole is an interval with a fixed point on its boundary.

More precisely, we shall fix an integer $d \geq 2$ once and for all, and consider $g(x) := dx \mod 1$ the map given by multiplication by d on the circle $\mathbb{S}$. For each $t \in [0, 1]$, let us define the set

$$K(t) := \{x \in \mathbb{R}/\mathbb{Z} \ : \ g^k(x) \notin (0, t) \quad \forall k \geq 0\}$$

of elements whose forward orbit under g does not intersect the interval $(0, t)$. For each t, the set $K(t)$ is compact and forward-invariant for g. One can see immediately that $K(0) = \mathbb{S}$ and $K(1) = \{0\}$; moreover, $K(t)$ is a decreasing family of sets, in the sense that $s < t$ implies $K(s) \supseteq K(t)$.

We shall consider the *dimension function*

$$\eta(t) := \text{H.dim } K(t)$$

which gives the Hausdorff dimension of the set $K(t)$ as a function of the parameter t.

The function $\eta(t)$ was introduced by Urbański [15], who proved that it is continuous, but not globally analytic. In fact, he showed that the dimension function is a "devil's staircase", i.e. it is locally constant almost everywhere (see Figure 1).

In order to describe the finer analytical properties of $\eta(t)$, we shall call *stable set* the set of parameters t for which the set-valued function $t \mapsto K(t)$ is locally constant at t, and the complement of the stable set will be called *bifurcation set* and denoted by $\mathcal{U}$. Clearly, the dimension function is locally constant on the stable set (actually, η is locally constant *only* on the stable set, see after Theorem 1).

We shall prove that on the bifurcation set the dimension function $\eta(t)$ has the following strong self-parametrizing property:

at every bifurcation parameter, the local Hölder exponent of the dimension function equals the value of the function itself.

To state the result precisely, let us define the *local Hölder exponent* of a function $f : I \to \mathbb{R}$ at a point $x \in I$ as the limit

$$\alpha(f, x) := \liminf_{x' \to x} \frac{\log |f(x) - f(x')|}{\log |x - x'|}.$$

The main theorem is the following.

Theorem 1. *Let $d \geq 2$. Then, for each parameter t in the bifurcation set, the local Hölder exponent of η at t equals $\eta(t)$, i.e.*

$$\alpha(\eta, t) = \eta(t). \tag{1}$$

As a corollary, the dimension function is always locally Hölder continuous except at $t = \frac{d-1}{d}$ (where $\eta(t) = 0$) and becomes more and more regular as t tends to 0. Actually η is known to be differentiable at $t = 0$ (see [8]). At $t = \frac{d-1}{d}$, the function is not Hölder continuous, and we shall show that its modulus of continuity is of order $\frac{\log \log(1/x)}{\log(1/x)}$ (see Proposition 15). Moreover, the intervals where the function is constant correspond precisely to the connected components of the stable set, and these are characterized in terms of *Lyndon words* (see below).

The dimension function η is directly related to other quantities which have been widely studied. First of all, if we denote $M_n := \{x \in \mathbb{S} \ : \ g^k(x) \notin (0, t) \text{ for } k = 0, \dots, n-1\}$ the set of points which do not fall into the hole under the first n iterates, one defines the *escape rate* γ to be

$$\gamma := \lim_{n \to \infty} -\frac{1}{n} \log |M_n|.$$

The escape rate is directly related to η by the formula

$$\eta = 1 - \frac{\gamma}{\log d}.$$

In particular, the asymptotic behaviour of γ in the "small hole" case is quite well-understood (see among others [8] and [3]); this gives an asymptotic expansion of $\eta(t)$ as $t \to 0$. Moreover, if we denote by $H(t)$ the topological entropy of the restriction of the map g to $K(t)$, one has the relation (see e.g. [16])

$$H(t) = \eta(t) \cdot \log d.$$

Actually Urbański [15] casts the problem in the following more general setting. Recall that a C^1 map $f : \mathbb{S} \to \mathbb{S}$ is called *expanding* if $\inf_{x \in \mathbb{S}} |f'(x)| >$

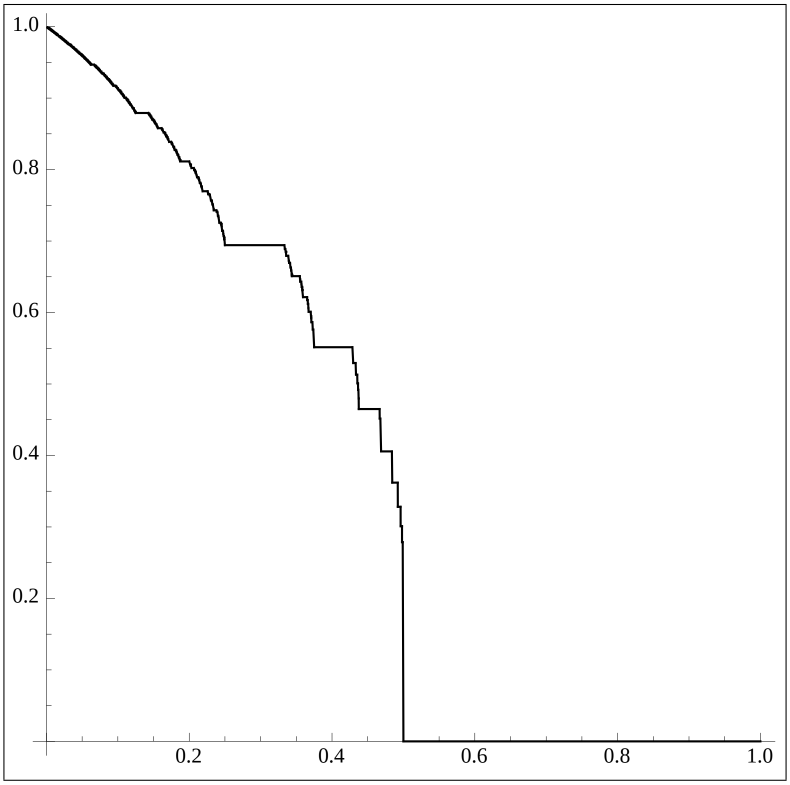

FIGURE 1. The dimension function $\eta(t)$ for $d = 2$.

1. Let $f : \mathbb{S} \to \mathbb{S}$ be an expanding map of the circle of degree d, and let us assume that 0 is a fixed point of f. He considers the survival set

$$K_f(t) := \{x \in \mathbb{S} \ : \ f^k(x) \notin (0, t) \quad \forall k \geq 0\}$$

and the topological entropy of f restricted to the survival set, i.e. the function

$$H(t) := h_{top}(f\mid_{K_f(t)}). \tag{2}$$

As it is well-known, in this case f is topologically conjugate to the map $g(x) := dx \mod 1$, and the conjugacy is Hölder continuous, thus we have the

Corollary 2. *Let f be an expanding C^1 map of the circle, of degree d, and let $H(t)$ be defined as above. Then H is locally Hölder continuous at points t for which $H(t) > 0$, and there are positive constants C_1, C_2 which depend only on f for which the local Hölder exponent satisfies*

$$\alpha(H, t) \geq C_1 H(t) \qquad \forall t \in [0, 1]$$

and for every t in the bifurcation set we also have

$$\alpha(H, t) \leq C_2 H(t).$$

As a consequence, H is not locally Hölder continuous at the point

$$t_\star := \sup\{t \ : \ H(t) > 0\}.$$

In a recent paper [1] O. Bandtlow and H.H. Rugh independently obtain similar results using thermodynamic formalism. In fact, they prove an inequality of type $\alpha(H,t) \geq CH(t)$ for more general expansive interval maps and more general holes.

The set $K(t)$ is also related to the problem of diophantine approximation (see [12]): in fact, if one considers the set

$$F_t := \{x \in \mathbb{S} \ : \ x - m/2^n \geq t/2^n \text{ for all but finitely many } m, n\}$$

of points which are not well-approximable (in a suitable sense) by dyadic numbers, then one has for any $t \in [0,1]$

$$\eta(t) = \text{H.dim } F_t.$$

One important tool for the proof of Theorem 1 is a formula, due to Urbański ([15], pg. 305) which allows to compute the value of $\eta(t)$ given the expansion in base d of $t \in \mathcal{U}$. In order to state the result precisely, note that each real number $t \in (0,1]$ admits exactly one expansion $t = .\epsilon_1\epsilon_2\dots$ in base d such that the sequence $(\epsilon_n)_{n\in\mathbb{N}}$ is not eventually 0. We shall call such expansion the *non-degenerate* expansion of t. Let $t \in \mathcal{U}$ be a bifurcation parameter, and let $t = .\epsilon_1\epsilon_2\dots$ be its non-degenerate expansion in base d. Then $\eta = \eta(t)$ the Hausdorff dimension of $K(t)$ is given by

$$\eta = -\frac{\log \lambda}{\log d} \tag{3}$$

where λ is a root of the equation $P_t(\lambda) = 1$, and $P_t(X)$ is the power series

$$P_t(X) := \sum_{k=1}^{\infty} (d - 1 - \epsilon_k) X^k. \tag{4}$$

Let us stress that this formula is only valid for $t \in \mathcal{U}$, see Remark 12.

The other main ingredient in our approach is an explicit characterization of the expansions in base d of the elements of $\mathcal{U}$. In particular, we will show (Proposition 5) that the connected components of the complement of $\mathcal{U}$ are naturally labelled by *Lyndon words*, i.e. finite words which are minimal for the lexicographic order among their cyclic permutations. This also answers the question of Nilsson ([12], Sec. 6), who asks for a characterization of the plateaux of the dimension function for $d > 2$; in the case $d = 2$, our characterization is essentially equivalent to that given in [12]. In fact, using this description of $\mathcal{U}$ we will recover in a self-contained, elementary way the main results of [15], using combinatorics on words rather than thermodynamic formalism (see Remark 17). Finally, the statement of Theorem 1 also holds for an alternative definition of local Hölder exponent (see Theorem 16).

Note that, without any reference to dynamics, one can ask whether there exist monotone, continuous functions $f : [0,1] \to [0,1]$ with the property that, for each $t \in [0,1]$ either f is locally constant at t, or the local Hölder exponent of f at t equals $f(t)$. The functions $\eta(t)$ for varying d provide infinitely many such examples (hence this property does not determine the function uniquely).

Moreover, functions with this property seem to appear also in relation to other families of dynamical systems. In particular, if one considers the function $h(\theta) := \frac{h_{top}(f_\theta)}{\log 2}$ which expresses the (normalized) topological entropy of a real quadratic polynomial f_θ as a function of its external angle θ (or equivalently, as a function of its kneading sequence) then it is also expected that the local Hölder exponent of $h(t)$ at t equals the value of the function $h(t)$ (see e.g. [4] and [6], Sec. 4). Note that in this case the *kneading series* of [10] plays the same role as the power series $P_t(X)$ in this paper. However, in this case the modulus of continuity at the smallest t where $h(t) = 0$ (the Feigenbaum parameter) is of order $\frac{1}{\log(1/x)}$ (see e.g. [13], Section 9.1). Another more complicated, non-monotone case where the local Hölder exponent is at least conjectured to equal the value of the function at every point is given by the (normalized) *core entropy* function for quadratic polynomials introduced by W. Thurston (see e.g. [14]).

The underlying phenomenon, of which Theorem 1 and Corollary 2 provide quantitative statements in a specific case, seems to be that systems with low entropy are less stable than systems with high entropy, in the sense that a small perturbation leads to a large variation in entropy. It would be of great interest to investigate to what extent is this phenomenon universal.

Word combinatorics and ordering

Let $d \geq 2$ be fixed once and for all. We define the *alphabet* as the set $\mathcal{A} := \{0, 1, \dots, d-1\}$, and (finite or infinite) sequences of elements of $\mathcal{A}$ will be called *words*. If $S, T \in \mathcal{A}^n$ are finite words of equal length, we write $S < T$ to denote the lexicographical order; moreover, we shall extend the order to a partial order on the set of all finite words in the following way. If $S = (a_1, \dots, a_n)$ and $T = (b_1, \dots, b_m)$ are finite words, we write $S \ll T$ (and read S is *strongly less* than T) if there exists an index $k \leq \min\{m, n\}$ such that $a_i = b_i$ for all $1 \leq i \leq k-1$ and $a_k < b_k$. For instance, 001 is strongly less than 01 but not strongly less than 00101.

Definition 3. *Let us define a finite word S to be* Lyndon *if it is strongly less than all its proper suffixes; that is, if for each decomposition $S = XY$ in two non-empty words one has*

$$S \ll Y.$$

For instance, 011 is Lyndon because $011 \ll 11$ and $011 \ll 1$, but 01101 is not Lyndon, because 01 is a suffix of 01101 but 01101 is not strongly less than 01. Note that words of one letter are Lyndon by definition.

We shall call $d-rational$ a rational number which admits a finite expansion in base d, and denote $\mathbb{Q}_{(d)}$ the set of d-rational numbers contained in the interval $(0, 1)$.

Definition 4. *A d-rational number $r \in (0, 1)$ is called* Lyndon *(for a given base d) if it admits a finite expansion $r = .\epsilon_1 \dots \epsilon_m$ in base d such that the word $S = \epsilon_1 \dots \epsilon_m$ is Lyndon (note that such expansion is unique, because the Lyndon property implies $\epsilon_m \neq 0$).*

We shall denote $\mathbb{Q}_{Lyn}$ the set of Lyndon rational numbers in $(0, 1)$. Note that the one letter word 0 is the only Lyndon word which does not correspond to a Lyndon rational number.

Finally, if $S = \epsilon_1 \dots \epsilon_m$ is a finite word and $x \in [0, 1]$, we shall denote by

$$S \cdot x := \sum_{i=1}^{m} \epsilon_i d^{-i} + x d^{-m} \tag{5}$$

the number whose expansion in base d is S followed by the expansion of x. The affine map $x \mapsto S \cdot x$ is an inverse to g^m and is uniformly contracting with derivative d^{-m}.

Lyndon words appear in several contexts in combinatorics: for a reference, see e.g. [9], pg. 64. Another equivalent definition given in the literature is that a word S is Lyndon if it is the smallest among all its cyclic permutations, i.e. if one has

$$S < YX$$

whenever $S = XY$ is a decomposition of S in two non-empty words (for the equivalence, cf. Lemma 9).

The bifurcation set

Let us start by considering the function $t \mapsto K(t)$ as a function into sets. We shall call a parameter $t \in [0, 1]$ *stable* if the function $t \mapsto K(t)$ is locally constant at t: that is, if there exists $\epsilon > 0$ such that the equality

$$K(t') = K(t)$$

holds for each $t' \in [t - \epsilon, t + \epsilon]$. We call such a set of parameters the *stable set*. A parameter which is not stable will be called a *bifurcation parameter*, and the set of all bifurcation parameters will be called *bifurcation set* and denoted by $\mathcal{U}$.

The set $\mathcal{U}$ is closed with no interior, and has the following characterization:

$$\mathcal{U} = \{t \in [0, 1] \ : \ g^k(t) \notin (0, t) \ \forall k \geq 0\} \tag{6}$$

(for a proof, see Lemma 6.) The main goal of this section is to characterize all connected components of the complement of $\mathcal{U}$; we shall see that they are naturally labeled by Lyndon rational numbers.

Let us define for each d-rational $r \in \mathbb{Q}_{(d)}$ the interval

$$I_r := (.\epsilon_1 \dots \epsilon_m, .\overline{\epsilon_1 \dots \epsilon_m})$$

with left endpoint r and right endpoint the rational number with periodic base-d expansion $\epsilon_1 \dots \epsilon_m$. For instance, in the case $d = 2$, if $r = 1/4 = .01$, then $.\overline{01} = 1/3$ hence $I_{1/4} = (1/4, 1/3)$. Note also $I_{1/2} = (1/2, 1)$.

Proposition 5. *The connected components of the complement of $\mathcal{U}$ are parametrized by Lyndon rational numbers. Indeed, we have the identities*

$$[0,1] \setminus \mathcal{U} = \bigsqcup_{r \in \mathbb{Q}_{Lyn}} I_r = \bigcup_{r \in \mathbb{Q}_{(d)}} I_r.$$

The proposition will follow from the following lemmata.

Lemma 6. *Let $t \in [0, 1)$. Then the following are equivalent:*

(1) *the element t belongs to $K(t)$;*
(2) *t is a bifurcation parameter.*

Proof of Lemma 6. If $t \in K(t)$, then for each $t' > t$ the element t belongs to $K(t) \setminus K(t')$, proving (2).

If instead $t \notin K(t)$, then let $t' := \inf\{x \in [t, 1] \ : \ x \in K(t)\} > t$ (the inequality is strict since $K(t)$ is closed). We claim that $K(t') = K(t)$; indeed, if $x \in K(t)$ then for each k we have $g^k(x) \in K(t) \subset [t', 1]$, hence $x \in K(t')$. Moreover, let us show that there exists $\epsilon > 0$ such that for $t' \in (t - \epsilon, t)$ we have $K(t') = K(t)$. If not, then there is a sequence of parameters $t_n \to t$, $t_n < t$ and a sequence of elements $x_n \in K(t_n) \setminus K(t)$. By taking forward images of x_n, we then get a sequence $y_n = g^{k_n}(x_n) \in K(t_n) \cap [t_n, t]$: this implies that $y_n \to t$ and

$$g^k(y_n) \in [t_n, 1]$$

for each k and n: thus, since g is continuous on $\mathbb{S}$, we have $g^k(t) \in [t, 1]$, which contradicts the fact that $t \notin K(t)$. $\square$

Lemma 7. *For each d-rational $r \in \mathbb{Q}_{(d)}$, the interval I_r is contained in the stable set $[0,1] \setminus \mathcal{U}$.*

Proof. Indeed, if $r = .\epsilon_1 \dots \epsilon_m$, then the map g^m is uniformly expanding of derivative d^m, it has $\overline{r} = .\overline{\epsilon_1 \dots \epsilon_m}$ as its fixed point and maps $(r, \overline{r})$ onto $(0, \overline{r})$. Thus, if $x \in (r, \overline{r})$, then $|g^m(x) - \overline{r}| > |x - \overline{r}|$, hence $g^m(x) \in (0, x)$ and $x \notin \mathcal{U}$. $\square$

Lemma 8. *Let $x \notin \mathcal{U}$. Then x belongs to some interval I_r with $\partial I_r \subseteq \mathcal{U}$.*

Proof. Let $x \notin \mathcal{U}$, and $k \geq 1$ be the minimum value such that

$$g^k(x) \in (0, x). \tag{7}$$

Let $x = .\epsilon_1 \epsilon_2 \dots$ be the non-degenerate expansion of x, denote $S_k := \epsilon_1 \dots \epsilon_k$ its truncation and write $r := S_k \cdot 0 = .\epsilon_1 \dots \epsilon_k$. Note that the map g^k is an orientation-preserving bijection from $J_k := S_k \cdot (0, 1]$ onto $(0, 1]$ with derivative d^k, and $\overline{r} := .\overline{\epsilon_1 \dots \epsilon_k}$ is its fixed point. Now note that by construction x belongs to J_k; moreover, if $x \geq \overline{r}$, then $g^k(x) = d^k(x - \overline{r}) + \overline{r} \geq x$, contradicting eq. (7). Thus, x belongs to $I_r := (r, \overline{r})$, proving the first part of the claim.

We claim moreover that for each $h = \{1, \dots, k-1\}$ we have

$$g^h(J_k) \subseteq (\overline{r}, 1). \tag{8}$$

which implies that both r and $\overline{r}$ belong to $\mathcal{U}$, thus $\partial I_r \subseteq \mathcal{U}$ as required. To prove eq. (8), let us pick $y \in J_k$; if $y \geq \overline{r}$, then

$$g^h(y) = g^h(x) + d^h(y - x) > x + (y - x) = y \geq \overline{r}. \tag{9}$$

Now, if there exists $y \in J_k \cap (0, \overline{r})$ such that $g^h(y) < \overline{r}$, then by the intermediate value theorem there must exist $z \in J_k \cap (0, \overline{r})$ such that $g^h(z) = \overline{r}$, hence $g^k(z) = g^{k-h}(\overline{r}) \geq \overline{r}$ by the previous observation (eq. (9) with $k-h$ instead of h). However, this is contradictory because $g^k(z) \in g^k(J_k \cap (0, \overline{r})) = (0, \overline{r})$. □

Lemma 9. *Let $S = \epsilon_1 \dots \epsilon_m \in \mathcal{A}^m$ be a word with $\epsilon_m \neq 0$, and $r = .\epsilon_1 \dots \epsilon_m$ the associated d-rational. Then the following are equivalent:*

1. *∂I_r belongs to $\mathcal{U}$;*
2. *S is a Lyndon word.*

Proof. If S is Lyndon, then for each $h \in \{1, \dots, m-1\}$ we have

$$g^h(r) = .\epsilon_{h+1} \dots \epsilon_m \overline{0} > .\epsilon_1 \dots \epsilon_m \overline{0} = r$$

and similarly $g^h(\overline{r}) > \overline{r}$, thus the endpoints of I_r belong to $\mathcal{U}$. Conversely, if $r \in \mathcal{U}$ then for each $h \in \{1, \dots, m-1\}$

$$.\epsilon_1 \dots \epsilon_m \overline{0} = r \leq g^h(r) = .\epsilon_{h+1} \dots \epsilon_m \overline{0}$$

hence $\epsilon_1 \dots \epsilon_m \ll \epsilon_{h+1} \dots \epsilon_m$ unless $\epsilon_{h+1} \dots \epsilon_m$ is a prefix of $\epsilon_1 \dots \epsilon_m$ and $\epsilon_{m-h+1} \dots \epsilon_m$ is all zeros, which is not possible since $\epsilon_m \neq 0$ by hypothesis. □

Proof of Proposition 5. It is easy to prove that the complement of $\mathcal{U}$ is open; namely, if $t \notin \mathcal{U}$, then by Lemma 6 (1) there exists $k \geq 0$ such that $g^k(t) \in (0, t)$, and such condition is open in t. From Lemma 8, Lemma 9 and Lemma 7 respectively we have the chain of inclusions

$$[0,1] \setminus \mathcal{U} \subseteq \bigcup_{\partial I_r \subseteq \mathcal{U}} I_r \subseteq \bigcup_{r \in \mathbb{Q}_{(d)}} I_r \subseteq [0,1] \setminus \mathcal{U}$$

thus equality must hold. Note also that two intervals I_r whose endpoints lie in $\mathcal{U}$ may not overlap, hence their union must be disjoint. Moreover, by Lemma 9 the set of rationals r for which $\partial I_r \subseteq \mathcal{U}$ coincides with the set $\mathbb{Q}_{Lyn}$ of Lyndon rationals, hence we get

$$[0,1] \setminus \mathcal{U} = \bigsqcup_{r \in \mathbb{Q}_{Lyn}} I_r = \bigcup_{r \in \mathbb{Q}_{(d)}} I_r.$$

As a consequence, the complement of $\mathcal{U}$ contains a right neighborhood of any d-rational, hence $\mathcal{U}$ has no interior. □

Structure and dimension of K(t)

In this chapter we show that the set $K(t)$ has a countable Markov partition which we can easily describe, and can be used to compute the Hausdorff dimension of $K(t)$, thus giving an alternative proof of Urbański's formula ([15], pg. 305).

Proposition 10. *Let $d \geq 2$, and $t \in \mathcal{U}$ be a bifurcation parameter with non-degenerate base-d expansion $t = .\epsilon_1\epsilon_2 \dots$. Then $\eta = \eta(t)$ the Hausdorff dimension of $K(t)$ is given by*

$$\eta = -\frac{\log \lambda}{\log d}$$

where λ is a root of the equation

$$P_t(\lambda) = 1 \tag{10}$$

and $P_t(X)$ is the power series

$$P_t(X) := \sum_{k=1}^{\infty} (d - 1 - \epsilon_k) X^k.$$

Note that the series $P_t(X)$ always converges inside the unit disk and, by the intermediate value theorem, equation (10) has exactly one root in the interval $(0, 1]$. Whenever t has a purely periodic expansion of period p, the series $P_t(X)$ becomes a rational function and $\lambda = d^{-\eta(t)}$ is the root of a polynomial of degree p.

As an example, in the case $d = 2$, if $t = .\overline{001} = 1/7$, then

$$P_t(X) = X + X^2 + X^4 + X^5 + \dots = \frac{X + X^2}{1 - X^3}$$

so $\lambda = 2^{-\eta(t)}$ is a root of $P_t(\lambda) = 1$, i.e. satisfies $\lambda^3 + \lambda^2 + \lambda - 1 = 0$.

Let $t \in (0, 1]$, and $t = .\epsilon_1\epsilon_2 \dots$ be its non-degenerate expansion in base d. For each $k \geq 1$ and $s \in \mathcal{A}$, define the word

$$S_{k,s}(t) := \epsilon_1 \dots \epsilon_{k-1} s$$

and consider the set of words

$$\Sigma(t) := \{S_{k,s}(t) \ : \ \epsilon_k < s\}.$$

The following proposition characterizes precisely the elements which belong to $K(t)$ in terms of the set $\Sigma(t)$.

Proposition 11. *Let $t \in \mathcal{U}$ be a bifurcation parameter. Then we have the identity*

$$K(t) = \{t\} \cup \bigsqcup_{S \in \Sigma(t)} S \cdot K(t). \tag{11}$$

That is, an element belongs to $K(t)$ if and only if its (non-degenerate) expansion in base d is a concatenation of words in $\Sigma(t)$.

Proof. Let $x \in K(t)$. Then by definition $x \in [t,1]$, hence either $x = t$ or the expansion of t starts with $S_{k,s}$, where k is the first digit for which the expansions of t and x differ, and s is the k^{th} digit of x, which mst be larger than the k^{th} digit of t. Hence, $x = S_{k,s} \cdot y$ with $y \in [0,1]$, and since $K(t)$ is forward invariant then also $y = g^k(x)$ belongs to $K(t)$, so $x \in S_{k,s} \cdot K(t)$.

Conversely, let $x = S \cdot y$ with $S \in \Sigma(t)$ and $y \in K(t)$. We have to prove that $g^h(x) \in [t,1]$ for each $h \geq 0$. If $h \geq k$, then $g^h(x) = g^{h-k}(y) \in [t,1]$ and the claim is proven. On the other hand, fix $h \in \{0,\dots,k-1\}$ and compare the expansions of $g^h(x)$ and $g^h(t)$. Since the expansion of $g^h(x)$ begins with $.\epsilon_{h+1}\dots\epsilon_{k-1}s$ and the expansion of $g^h(t)$ begins with $.\epsilon_{h+1}\dots\epsilon_{k-1}\epsilon_k$ and $s > \epsilon_k$, then we have

$$g^h(x) \in [g^h(t), 1] \subseteq [t,1]$$

where in the last inequality we used that t belongs to $\mathcal{U}$, and the claim is proven. □

Proof of Proposition 10. Consider the set $\widetilde{K}(t) := \{x \in K(t) \ : \ g^k(x) \neq t \quad \forall k \geq 0\}$. Since $K(t)$ and $\widetilde{K}(t)$ differ by a countable set of preimages of t, their Hausdorff dimension is the same; moreover, by Proposition 11 we have

$$\widetilde{K}(t) = \bigsqcup_{S \in \Sigma(t)} S \cdot \widetilde{K}(t).$$

The set $\widetilde{K}(t)$ is thus the attractor of a countable iterated function system; each map $x \mapsto S_{k,s} \cdot x$ is an affine map of derivative d^{-k}, and moreover, by construction all the images $S_{k,s} \cdot \widetilde{K}(t)$ are disjoint and satisfy the open set condition ([5], [11]), hence the Hausdorff dimension η of $\widetilde{K}(t)$ is determined implicitly by the formula

$$1 = \sum_{S_{k,s} \in \Sigma(t)} d^{-k\eta}$$

which, since by definition of $\Sigma(t)$ for each k there are $d - 1 - \epsilon_k$ values of s, can also be written as

$$1 = \sum_{k=1}^{\infty} (d - 1 - \epsilon_k) d^{-k\eta}$$

thus taking $X = d^{-\eta}$ yields the claim. □

Remark 12. *Note that the hypothesis $t \in \mathcal{U}$ in Proposition 10 is essential. Actually, one can define for any $t \in [0,1]$ the function $\zeta(t) = -\log\lambda/\log d$, where λ is the unique real, positive root of the equation $P_t(X) = 1$. Then the function $\zeta(t)$ is no longer continuous, but for any $t \in [0,1]$ one has the relation (see Figure 2)*

$$\eta(t) = \min\{\zeta(s),\ 0 \leq s \leq t\}.$$

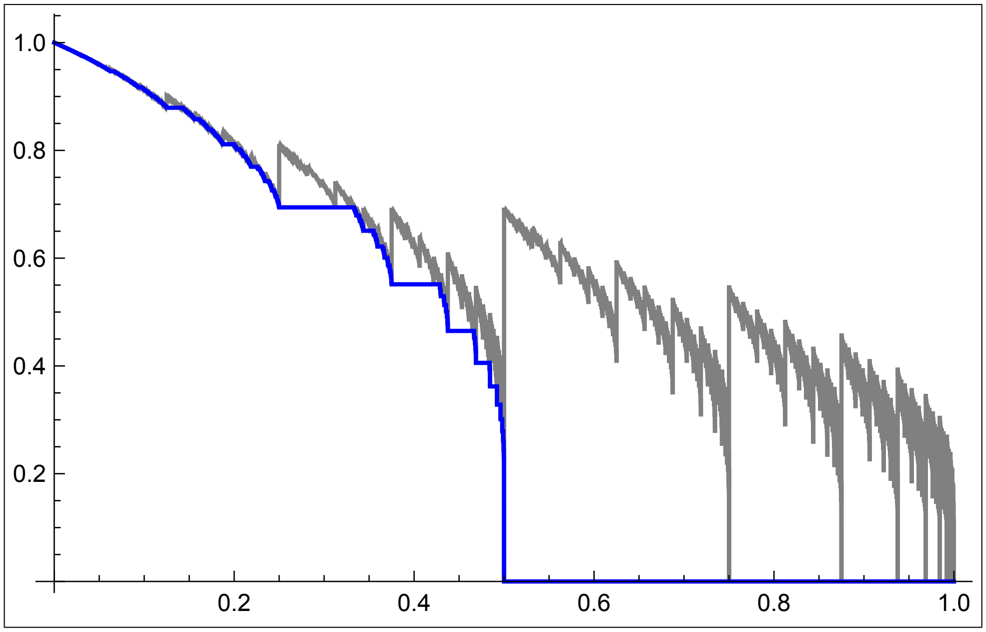

FIGURE 2. The functions $\zeta(t)$ and $\eta(t)$ for $d = 2$.

THE LOCAL HÖLDER EXPONENT

Let us recall that a function $f : I \to \mathbb{R}$ on an interval I is called *Hölder continuous* of exponent α if there exists a constant $C > 0$ such that for each $x, y \in I$ one has

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}.$$

Given $t \in I$, we define the *local Hölder exponent of f at t* to be

$$\alpha(f, t) := \liminf_{t' \to t} \frac{\log |f(t) - f(t')|}{\log |t - t'|}.$$

The goal of this section is to establish Theorem 1, namely that

$$\alpha(\eta, t) = \eta(t)$$

for any $t \in \mathcal{U}$: let us start with some preliminary remarks.

Let us first note that for $t \in (\frac{d-1}{d}, 1]$ the set $K(t) = \{0\}$ is one point so $\eta(t) = 0$ and there is nothing to prove, so we shall assume $t \in [0, \frac{d-1}{d}]$. In this case, let us note that $P_t(X) = sX + \sum_{k=2}^{\infty}(d - 1 - \epsilon_k)X^k$ with $s \geq 1$, hence the function $P_t(x)$ is strictly increasing on $[0, 1)$. Moreover, $P_t(0) = 0$ and $\lim_{x\to 1^-} P_t(x) > 1$ unless $t = \frac{d-1}{d}$ (in which case $P_t(x) = x$), thus for each $t \in [0, \frac{d-1}{d}]$, the equation $P_t(x) = 1$ has a unique solution $\lambda \in (0, 1]$, which we will denote $\lambda(t)$. Note also that for each $x \in (0, 1)$ we have

$$1 \leq P_t'(x) \leq \frac{d}{(1 - x)^2} \tag{12}$$

hence $\lambda(t)$ is always a simple root of $P_t(X)$.

If $t \in [0,1]$, we shall denote $\epsilon_k(t)$ the k^{th} digit of the non-degenerate expansion of t. Moreover, if $t_1, t_2 \in [0,1]$, let us define $m(t_1,t_2)$ to be the length of the longest common prefix in the expansions of t_1 and t_2; namely,

$$m(t_1,t_2) := \sup\{k \geq 0 \ : \ \epsilon_h(t_1) = \epsilon_h(t_2) \quad \forall h \in \{1,\dots,k\}\}.$$

Lemma 13. *For each $t_0 > 0$, there exists a constant $C_1 > 0$ such that for each $t_1, t_2 \in \mathcal{U} \cap [t_0, 1]$ one has*

$$C_1 d^{-m(t_1,t_2)} \leq |t_1 - t_2| \leq d^{-m(t_1,t_2)}. \tag{13}$$

Proof. Let $m := m(t_1,t_2)$; the upper bound is given by $|t_1 - t_2| = d^{-m}|g^m(t_1) - g^m(t_2)| \leq d^{-m}$. To get the lower bound, first note that, since $K(t_0) \subseteq [t_0, 1]$ and $K(t_0)$ is forward invariant by g, we have

$$K(t_0) \subseteq g^{-1}(K(t_0)) \subseteq g^{-1}([t_0,1]) = \bigcup_{k=0}^{d-1} I_k$$

where $I_k := \left[\frac{t_0+k}{d}, \frac{1+k}{d}\right]$. Now, by definition of m, the two points $u_1 := g^m(t_1)$ and $u_2 := g^m(t_2)$ belong to two different intervals I_k, thus $|t_1 - t_2| = d^{-m}|u_1 - u_2| \geq \frac{t_0}{d}$, which gives the lower bound with $C_1 := \frac{t_0}{d}$. □

We are now ready to prove the main theorem stated in the introduction.

Proof of Theorem 1. Monotonicity of $\eta(t)$ is immediate from the definition, while continuity follows from Rouché's theorem. Indeed, let $t \in \mathcal{U} \cap [0, \frac{d-1}{d}]$ and suppose $\lambda = \lambda(t) < 1$. Then, $\lambda(t)$ is a simple root of $P_t(X)$, and $P_{t'}(X)$ converges uniformly on compact sets to $P_t(X)$ as $t' \to t$, hence the root $\lambda(t')$ converges to $\lambda(t)$. Suppose now $\lambda(t) = 1$, which implies $t = \frac{d-1}{d}$. Fix $\delta \in (0,1)$. Then $P_t(X) - 1 = X - 1$ has no roots in the strip $S_\delta = \{x + iy \ : \ 0 \leq x \leq 1 - \delta, |y| \leq \delta\}$, hence by Rouché's theorem $P_{t'}(X)$ also has no roots in S_δ for t' sufficiently close to t, hence $\lambda(t') \geq \delta$. This proves $\lambda(t') \to 1$ as $t' \to t$. Since $\eta(t) = -\frac{\log \lambda(t)}{\log d}$, then continuity of $\lambda(t)$ implies continuity of $\eta(t)$.

Let us now estimate the modulus of continuity of $\eta(t)$. First note that, since $\eta(t) = -\frac{\log \lambda(t)}{\log d}$ and the function $h(x) := -\frac{\log x}{\log d}$ is bi-Lipschitz on $[1/d, 1]$, it is equivalent to prove the claim for $\lambda(t)$. Let $t_1, t_2 \in \mathcal{U} \cap [0, \frac{d-1}{d}]$, and to simplify notation, we denote $\lambda_1 := \lambda(t_1)$, $\lambda_2 := \lambda(t_2)$, and also $P_1(X) := P_{t_1}(X)$, $P_2(X) := P_{t_2}(X)$ and suppose $\lambda_1, \lambda_2 < 1$.

Now, using that $P_2(\lambda_2) = P_1(\lambda_1) = 1$ and applying Lagrange's theorem we have

$$P_1(\lambda_1) - P_2(\lambda_1) = P_2(\lambda_2) - P_2(\lambda_1) = P_2'(\xi)(\lambda_2 - \lambda_1) \tag{14}$$

for some $\xi \in [\lambda_1, \lambda_2]$. On the other hand, by writing out the power series we get

$$P_1(\lambda_1) - P_2(\lambda_1) = \lambda_1^{m+1} R(t_1, t_2) \tag{15}$$

where $R(t_1,t_2) := (\epsilon_{m+1}(t_2) - \epsilon_{m+1}(t_1)) + \sum_{j=m+2}^{\infty}(\epsilon_j(t_2) - \epsilon_j(t_1))\lambda_1^{j-m-1}$ and $m = m(t_1,t_2)$. By comparing the two previous equations we get

$$|\lambda_1 - \lambda_2| = \lambda_1^{m+1}\frac{|R(t_1,t_2)|}{P_2'(\xi)} \leq \lambda_1^{m+1}\frac{d}{1-\lambda_1}$$

hence by combining it with the upper bound for $|t_1 - t_2|$ given by eq. (13) we have the following upper bound for the modulus of continuity: for each $t \in \mathcal{U} \cap (0, \frac{d-1}{d})$, there exists $C_2 > 0$ such that one has

$$|\lambda_1 - \lambda_2| \leq C_2|t_1 - t_2|^{\frac{-\log\lambda_1}{\log d}} = C_2|t_1-t_2|^{\eta(t_1)} \tag{16}$$

for each $t_1, t_2 \in \mathcal{U}$ sufficiently close to t. Since $\lambda(t)$ is constant on the complement of $\mathcal{U}$, the above upper bound actually works for *any* t_1, t_2 close to t, thus proving

$$\alpha(\eta, t) \geq \eta(t)$$

for each $t \in \mathcal{U}$.

For the lower bound, let us pick $t \in \mathcal{U}$. Now, by Lemma 14 there exists a sequence $t_n \to t$ with $t_n \neq t$ such that either for each k and each n we have $\epsilon_k(t) \leq \epsilon_k(t_n)$, or we have the reverse inequality for each k and each n; in both cases, $R(t,t_n)$ is a power series in $\lambda(t)$ all of whose coefficients are integers and have the same sign, hence $|R(t,t_n)| \geq 1$ and

$$|\lambda(t) - \lambda(t_n)| = \lambda(t)^{m+1}\frac{|R(t,t_n)|}{P_2'(\xi)} \geq \lambda(t)^{m+1} \cdot \frac{(1-\xi)^2}{d} \geq C_3|t-t_n|^{\eta(t)}$$

where $C_3 := \lambda(t)\inf_{\tau\in[t,t_n]}\frac{(1-\lambda(\tau))^2}{d}$, proving the lower bound $\alpha(\eta,t) \leq \eta(t)$. □

Lemma 14. *Let $t \in \mathcal{U}$. If t is not a d-rational, then there exists a sequence (t_n) of elements of $\mathcal{U}$ such that $t_n \to t$, $t_n > t$ for any n, and*

$$\epsilon_k(t) \leq \epsilon_k(t_n) \qquad \forall k, n.$$

If t is a d-rational, then there exists a sequence (t_n) of elements of $\mathcal{U}$ such that $t_n \to t$, $t_n < t$ for any n, and

$$\epsilon_k(t_n) \leq \epsilon_k(t) \qquad \forall k, n.$$

Proof. Let $t \in \mathcal{U}$ not a d-rational, and let $t = .\epsilon_1\epsilon_2\dots$ be its (non-degenerate) expansion in base d. For each n, let us define

$$t_n := .\epsilon_1 \dots \epsilon_n (d-1)^{\infty}.$$

By construction $t_n > t$, $t_n \to t$, and $\epsilon_k(t_n) \geq \epsilon_k(t)$ for each k. We need to check that $t_n \in \mathcal{U}$. Let us consider $g^r(t_n)$ and compare it to t_n. If $r \geq n$, then $g^r(t_n) = 0 \notin (0, t_n)$ as required. If $r < n$ instead, then we have $g^r(t_n) = .\epsilon_{r+1}\dots\epsilon_n(d-1)^{\infty}$. Since $t \in \mathcal{U}$, then $g^r(t) = .\epsilon_{r+1}\dots\epsilon_n\dots \geq t = .\epsilon_1\dots\epsilon_n\dots$, hence, if you set $S := \epsilon_1\dots\epsilon_n$ and $S_0 := \epsilon_{r+1}\dots\epsilon_n$, either $S_0 \gg S$ or S_0 is a prefix of S. In the first case $g^r(t_n) \geq t_n$ as required; in the second case, $S_0(d-1)^{n-r} \geq S$, so $g^r(t_n) = .S_0(d-1)^{\infty} \geq .S(d-1)^{\infty} = t_n$, as required.

Let us now deal with the case where t is a d-rational, and let $t = .\epsilon_1 \dots \epsilon_k (d-1)^\infty$ be its non-degenerate expansion, which we can take so that $\epsilon_1 \neq d-1$ and $\epsilon_k \neq d-1$. We claim that the number t_n with base-d expansion

$$t_n = .\overline{\epsilon_1 \dots \epsilon_k (d-1)^n}$$

satisfies the claim. Clearly, $t_n < t$ and $t_n \to t$, while $\epsilon_k(t_n) \leq \epsilon_k(t)$ for any k. We need to prove that $t_n \in \mathcal{U}$. Given r, consider $g^r(t_n)$: either the first digit of $g^r(t_n)$ is $(d-1)$, which implies $g^r(t_n) \notin (0, t_n)$ as $\epsilon_1 \neq d-1$, or $g^r(t_n)$ is of the form $g^r(t_n) = .\epsilon_{r+1} \dots \epsilon_k (d-1)^n \dots$. Then, if $S := \epsilon_1 \dots \epsilon_n$ and $S_0 := \epsilon_{r+1} \dots \epsilon_k$, we have either $S \ll S_0$, or S_0 is a prefix of S. If S_0 is a prefix of S, then one can write $S = S_0 S_1$ where S_1 is some non-empty word, and either $S_1 \ll (d-1)^n$ or S_1 is of the form $(d-1)^a$ for some $a \geq 1$, which contradicts the fact that $\epsilon_k \neq d-1$. □

Now we shall show that the function η is not Hölder continuous at $t_* = 1 - 1/d$ (which is the smallest t such that $\eta(t) = 0$). In fact the modulus of continuity of η at t_* is given by the function

$$\omega(x) := \frac{\log \log (1/x)}{\log(1/x)}$$

as shown in the following Proposition.

Proposition 15. *We have the limit*

$$\lim_{t \to t_*^-} \frac{\eta(t) - \eta(t_*)}{\omega(t_* - t)} = 1.$$

Proof. To begin we shall give a precise estimate of $\eta(t_n)$ where $t_n := t_* - 1/d^n$. It is easy to check that $t_n \in \mathcal{U}$ for all $n \geq 2$ and $P_{t_n}(X) = X + X^n$. In order to locate the unique positive solution λ_n of the equation $P_{t_n}(X) = 1$ let us observe that for any fixed $\alpha > 0$ one has

$$P_{t_n}\left(1 - \alpha \frac{\log n}{n}\right) = 1 - \alpha \frac{\log n}{n} + \frac{1}{n^\alpha}\left[1 + O\left(\frac{\log^2 n}{n}\right)\right] \qquad \text{as } n \to \infty.$$

Therefore, using the above formula for $\alpha = 1$ we get that there is n_0 such that $P_{t_n}(1 - \frac{\log n}{n}) < 1 \ \forall n \geq n_0$. On the other hand, for any $\alpha < 1$ there is $n_1 = n_1(\alpha)$ such that $P_{t_n}(1 - \alpha \frac{\log n}{n}) > 1 \ \forall n \geq n_1$. This means that as $n \to \infty$ we have $\lambda_n = 1 - \frac{\log n}{n}[1 + o(1)]$ and

$$\eta(t_n) = -\frac{\log \lambda_n}{\log d} = \frac{\log n}{n \log d}[1 + o(1)].$$

Recalling that $\log(t_* - t_n) = -n \log d$ we see that the modulus of continuity of η at t_* cannot be smaller than ω, indeed:

$$\lim_{n \to +\infty} \frac{\eta(t_n) - \eta(t_*)}{\omega(t_* - t_n)} = 1$$

On the other hand, if $t_n \leq t \leq t_{n+1}$ then, using the monotonicity of $\eta(t)$ and $\omega(t)$ and the fact that $\frac{\omega(t_*-t_n)}{\omega(t_*-t_{n+1})} \to 1$, we get

$$\lim_{t\to t_*^-} \frac{\eta(t)-\eta(t_*)}{\omega(t_*-t)} = 1.$$

□

Let us now turn to the proof of Corollary 2 stated in the introduction, namely the extension of these results to general expanding circle maps.

Proof of Corollary 2. Since f is an expanding C^1 map, then it is conjugate to the linear map $g(x) := dx \mod 1$, and the conjugacy is Hölder continuous, i.e. there exists a Hölder continuous homeomorphism $\varphi : \mathbb{S} \to \mathbb{S}$ with Hölder continuous inverse such that $\varphi \circ f = g \circ \varphi$ (see e.g. [2], Section II.2). Since topological entropy is invariant by conjugacy, we have

$$h_{top}(f\mid_{K_f(t)}) = h_{top}(g\mid_{K(\tau)})$$

where $\tau = \varphi(t)$. Then since φ and its inverse are Hölder continuous, there exists $C > 0$ such that for any t we have

$$C^{-1} \leq \liminf_{t'\to t} \frac{\log|\tau'-\tau|}{\log|t'-t|} \leq \limsup_{t'\to t} \frac{\log|\tau'-\tau|}{\log|t'-t|} \leq C$$

where $\tau' = \varphi(t')$. Let us denote $H_f(t) = h_{top}(f\mid_{K_f(t)})$ and $H_g(\tau) = h_{top}(g\mid_{K(\tau)})$. Then, putting the estimates together, we get

$$\alpha(H_f,t) = \liminf_{t'\to t} \frac{\log|H_f(t')-H_f(t)|}{\log|t'-t|} \geq C \liminf_{\tau'\to\tau} \frac{\log|H_g(\tau')-H_g(\tau)|}{\log|\tau'-\tau|} =$$
$$= C H_g(\tau) = C H_f(t)$$

and similarly for the lower bound, so we get for any $t \in \mathcal{U}' = \varphi(\mathcal{U})$ the bounds

$$C^{-1}H_f(t) \leq \alpha(H_f,t) \leq C H_f(t).$$

As a consequence, if $t_\star := \sup\{t \ : \ H_f(t) > 0\}$, we have $H_f(t_\star) = 0$, so $\alpha(H_f,t_\star) = 0$ hence H_f is not locally Hölder continuous at $t_\star$. □

Note that an alternative way to define the local Hölder exponent of f at t is as the supremum of all values s for which f is Hölder continuous of exponent s on some neighborhood of t, i.e. as

$$\widetilde{\alpha}(f,t) := \sup\left\{ s \ : \ \lim_{\epsilon\to 0} \sup_{\substack{x,y\in B(t,\epsilon)\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^s} < \infty \right\}$$

where $B(t,\epsilon)$ is the open ball of radius ϵ and center t. Note that $\widetilde{\alpha}(f,t) > 0$ if and only if f is locally Hölder continuous at t. While in general $\widetilde{\alpha}(f,t) \leq \alpha(f,t)$, the two quantities need not be the same; however, in our case the same argument as in the proof of Theorem 1 shows:

Theorem 16. *For each $t \in \mathcal{U}$ we have the equality*

$$\widetilde{\alpha}(f,t) = \alpha(f,t) = \eta(t).$$

Proof. Since $\widetilde{\alpha}(\eta,t) \leq \alpha(\eta,t) = \eta(t)$, we only have to check that $\widetilde{\alpha}(\eta,t) \geq \eta(t)$. Since the claim is trivial if $\eta(t) = 0$, we can assume $t \in [0, \frac{d-1}{d})$. Then, by equation (16), there exists a constant C_2 and a neighborhood V of t such that for any t_1, t_2 in V

$$|\lambda(t_1) - \lambda(t_2)| \leq C_2 |t_1 - t_2|^{\eta(t_1)},$$

which implies that for any $\epsilon > 0$ we have

$$\widetilde{\alpha}(\eta,t) \geq \eta(t+\epsilon).$$

Since $\eta(t+\epsilon) \to \eta(t)$ as $\epsilon \to 0$, it follows that $\widetilde{\alpha}(\eta,t) \geq \eta(t)$, as required. □

Remark 17. *Using the characterization of $\mathcal{U}$ we can also give an elementary proof of the following result of Urbański (*[15]*, Theorem 2):*

$$\text{H.dim } K(t) = \text{H.dim } (\mathcal{U} \cap [t,1]) \qquad \forall t \in [0,1].$$

Proof. Indeed, if we denote by $\mathcal{P} \subset \mathcal{U}$ the set of elements in $\mathcal{U}$ with a purely periodic base-d expansion, it is easy to check that $\eta(\mathcal{P})$ is dense in $[0,1]$. Therefore, since both the function $\eta(t)$ and $\widetilde{\eta}(t) := \text{H.dim } (\mathcal{U} \cap [t,1])$ are decreasing, to prove our claim it is enough to check that the equality $\eta(t) = \widetilde{\eta}(t)$ holds for all $t \in \mathcal{P}$.

Since $\mathcal{U} \cap [t,1] \subseteq K(t)$ for any t (see eq.(6)), the inequality $\eta(t) \geq \widetilde{\eta}(t)$ is straightforward, so we only have to prove $\eta(t) \leq \widetilde{\eta}(t)$. If $t_0 \in \mathcal{P}$ then $t_0 = .\overline{\epsilon_1 ... \epsilon_m}$ is a fixed point of the affine contraction associated with the Lyndon word $W := \epsilon_1 ... \epsilon_m$ (see equation (5)). Given any $t_1 > t_0$ we can fix $\ell \geq 1$ such that $W^\ell \cdot 1 < t_1$, and define the set

$$S := W^\ell \cdot K(t_1);$$

it is easy to check that $S \subset \mathcal{U} \cap [t_0, 1]$ so $\text{H.dim } S \leq \text{H.dim } (\mathcal{U} \cap [t_0,1])$. Moreover, since S is an affine copy of $K(t_1)$ we have $\text{H.dim } S = \text{H.dim } K(t_1)$.

Thus we have proved that $\widetilde{\eta}(t_0) \geq \eta(t_1)$ for all $t_1 > t_0$, so our claim follows taking the limit for $t_1 \to t_0$ and using the continuity of $\eta(t)$. □

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