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ON YAMABE-TYPE PROBLEMS ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Let (M, g) be an n-dimensional compact Riemannian manifold with boundary. We consider the Yamabe-type problem

$$\begin{cases} -\Delta_g u + au = 0 & \text{on } M, \\ \partial_\nu u + \frac{n-2}{2} bu = (n-2)u^{n/(n-2)\pm\varepsilon} & \text{on } \partial M, \end{cases}$$

where $a \in C^1(M)$, $b \in C^1(\partial M)$, ν is the outward pointing unit normal to ∂M, $\Delta_g u := \operatorname{div}_g \nabla_g u$, and ε is a small positive parameter. We build solutions which blow up at a point of the boundary as ε goes to zero. The blowing-up behavior is ruled by the function $b - H_g$, where H_g is the boundary mean curvature.

1. Introduction

Let (M, g) be a smooth, compact Riemannian manifold of dimension $n \geq 3$ with a boundary ∂M which is the union of a finite number of smooth closed compact submanifolds embedded in M.

A well-known problem in differential geometry is whether (M, g) is necessarily conformally equivalent to a manifold of constant scalar curvature whose boundary is minimal. When the boundary is empty this is called the Yamabe problem (see Yamabe [1960]), which has been completely solved by Aubin [1976], Schoen [1984] and Trudinger [1968]. Cherrier [1984] and Escobar [1992a; 1992b] studied the problem in the context of manifolds with boundary and gave an affirmative solution to the question in almost every case. The remaining cases were studied by Marques [2005; 2007], by Almaraz [2010] and by Brendle and Chen [2014].

Once the problem is solvable, a natural question about compactness of the full set of solutions arises. Concerning the Yamabe problem, it was first raised by Schoen in a topics course at Stanford University in 1988. A necessary condition is that the manifold is not conformally equivalent to the standard sphere $\mathbb{S}^n$, since the group of

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conformal transformations of the round sphere is not compact itself. The problem of compactness has been widely studied in recent years and has been completely solved by Brendle [2008], Brendle and Marques [2009] and Khuri, Marques and Schoen [Khuri et al. 2009].

In the presence of a boundary, a necessary condition is that M is not conformally equivalent to the standard ball $\mathbb{B}^n$. The problem when the boundary of the manifold is not empty has been studied by V. Felli and M. Ould Ahmedou [2003; 2005], Han and Li [1999] and Almaraz [2011a; 2011b]. In particular, Almaraz studied the compactness property in the case of scalar-flat metrics. Indeed, the zero scalar curvature case is particularly interesting because it leads one to study a linear equation in the interior with a critical Neumann-type nonlinear boundary condition

$$\begin{cases} -\Delta_g u + \dfrac{n-2}{4(n-1)} R_g u = 0 & \text{on } M,\ u>0 \text{ in } M, \\ \partial_\nu u + \dfrac{n-2}{2} H_g u = (n-2)u^{n/(n-2)} & \text{on } \partial M, \end{cases} \tag{1-1}$$

where ν is the outward pointing unit normal to ∂M, R_g is the scalar curvature of M with respect to g, and H_g is the boundary mean curvature with respect to g.

We note that in this case compactness of solutions is equivalent to establish a priori estimates for solutions to equation (1-1). Almaraz [2011b] proved that compactness holds for a generic metric g. On the other hand, in [Almaraz 2011a] it was proved that if the dimension of the manifold is $n \geq 25$, compactness does not hold because it is possible to build blowing-up solutions to (1-1) for a suitable metric g. We point out that the problem of compactness in dimension $n \leq 24$ is still not completely understood.

An interesting issue, closely related to the compactness property, is the stability problem. One can ask whether or not the compactness property is preserved under perturbations of the equation, which is equivalent to having or not having uniform a priori estimates for solutions of the perturbed problem. Let us consider the more general problem

$$\begin{cases} -\Delta_g u + a(x)u = 0 & \text{in } M,\ u>0 \text{ in } M, \\ \partial_\nu u + b(x)u = (n-2)u^{n/(n-2)} & \text{on } \partial M. \end{cases} \tag{1-2}$$

We say that the problem (1-2) is stable if for any sequences of C^1 functions $a_\varepsilon : M \to \mathbb{R}$ and $b_\varepsilon : \partial M \to \mathbb{R}$ converging in C^1 to functions $a : M \to \mathbb{R}$ and $b : \partial M \to \mathbb{R}$, for any sequence of exponents $p_\varepsilon := n/(n-2) \pm \varepsilon$ converging to the critical one $n/(n-2)$ and for any sequence of associated solutions u_ε bounded in $H^1(M)$ of the perturbed problems

$$\begin{cases} -\Delta_g u + a_\varepsilon(x)u = 0 & \text{in } M,\ u_\varepsilon>0 \text{ in } M, \\ \partial_\nu u + \dfrac{n-2}{2} b_\varepsilon(x)u = (n-2)u_\varepsilon^{n/(n-2)\pm\varepsilon} & \text{on } \partial M, \end{cases} \tag{1-3}$$

there is a subsequence u_{ε_k} which converges in C^2 to a solution to the limit problem (1-2). The stability of the Yamabe problem has been introduced and studied by Druet [2003; 2004] and by Druet and Hebey [2005a; 2005b]. Recently, Esposito, Pistoia and Vetois [Esposito et al. 2014], Micheletti, Pistoia and Vetois [Micheletti et al. 2009] and Esposito and Pistoia [2014] proved that a priori estimates fail for perturbations of the linear potential or of the exponent.

In this paper, we investigate the question of stability of the problem (1-2). It is clear that it is not stable if it is possible to build solutions u_ε to perturbed problems (1-3) which blow up at one or more points of the manifold as the parameter ε goes to zero. Here, we show that the behavior of the sequence u_ε is dictated by the difference

$$\varphi(q) = b(q) - H_g(q) \quad \text{for } q \in \partial M. \tag{1-4}$$

More precisely, we consider the problem

$$\begin{cases} -\Delta_g u + a(x)u = 0 & \text{on } M,\ u > 0 \text{ in } M, \\ \dfrac{\partial}{\partial \nu} u + \dfrac{n-2}{2} b(x)u = (n-2)u^{n/(n-2)\pm\varepsilon} & \text{on } \partial M. \end{cases} \tag{1-5}$$

We assume that $a \in C^1(M)$ and $b \in C^1(\partial M)$ are such that the linear operator $\mathcal{L}u := -\Delta_g u + au$ with Neumann boundary condition $\mathcal{B}u := \partial_\nu u + \frac{1}{2}(n-2)bu$ is coercive; namely, there exists a constant $c > 0$ such that

$$\int_M \left(|\nabla_g u|^2 + a(x)u^2\right) d\mu_g + \frac{n-2}{2}\int_{\partial M} b(x)u^2 \, d\sigma \geq c\|u\|^2_{H^1(M)}. \tag{1-6}$$

Here $\varepsilon > 0$ is a small parameter, $\Delta_g u := \operatorname{div}_g \nabla_g u$, and the space $H^1(M)$ is the closure of $C^\infty(M)$ with respect to the norm

$$\|u\|_{H^1} = \left(\int_M \left(|\nabla_g u|^2 + u^2\right) d\mu_g\right)^{1/2}.$$

The problem (1-5) turns out to be either slightly subcritical or slightly supercritical if the exponent in the nonlinearity is either $n/(n-2)-\varepsilon$ or $n/(n-2)+\varepsilon$, respectively. Let us state our main result.

Theorem 1. *Assume (1-6) and $n \geq 7$.*

(i) *If $q_0 \in \partial M$ is a strict local minimum point of the function φ defined in (1-4) with $\varphi(q_0) > 0$, then provided $\varepsilon > 0$ is small enough, there exists a solution u_ε of (1-5) in the slightly subcritical case such that u_ε blows up at a boundary point when $\varepsilon \to 0^+$.*

(ii) *If $q_0 \in \partial M$ is a strict local maximum point of the function φ defined in (1-4) with $\varphi(q_0) < 0$, then provided $\varepsilon < 0$ is small enough, there exists a solution u_ε of (1-5) in the slightly supercritical case such that u_ε blows up at a boundary point when $\varepsilon \to 0^+$.*

We say that u_ε blows up at a point q_0 of the boundary if there exists a family of points $q_\varepsilon \in \partial M$ such that $q_\varepsilon \to q_0$ as $\varepsilon \to 0$ and, for any neighborhood $U \subset M$ of q_0, we have that $\sup_{q\in U} u_\varepsilon(q) \to +\infty$ as $\varepsilon \to 0$.

Our result does not concern the stability of the geometric Yamabe problem (1-1). Indeed, the function φ in (1-4) turns out to be identically zero. It would be interesting to discover the function which rules the behavior of blowing-up sequences in this case. We expect that it depends on the trace-free second fundamental form as it is suggested by Almaraz [2011b], where a compactness result in the subcritical case is established.

The case of low dimension also remains open, where we expect that the function φ in (1-4) should be replaced by a function which depends on the Weyl tensor of the boundary, as suggested by Escobar [1992a; 1992b].

The proof of our result relies on a very well known Ljapunov–Schmidt procedure. In Section 2 we set up the problem, and in Section 3 we reduce the problem to a finite dimensional one, which is then studied in Section 4.

2. Setting of the problem

Let us rewrite problem (1-5) in a more convenient way.

First of all, assumption (1-6) allows us to endow the Hilbert space $H := H^1(M)$ with the scalar product

$$\langle\langle u, v\rangle\rangle_H := \int_M (\nabla_g u \nabla_g v + a(x)uv)\, d\mu_g + \frac{n-2}{2}\int_{\partial M} b(x)uv\, d\sigma$$

and the induced norm $\|u\|_H^2 := \langle\langle u, u\rangle\rangle_H$. We define the exponent

$$s_\varepsilon = \begin{cases} \dfrac{2(n-1)}{n-2} & \text{in the subcritical case,} \\ \dfrac{2(n-1)}{n-2} + n\varepsilon & \text{in the supercritical case,} \end{cases}$$

and the Banach space $\mathcal{H} := H^1(M) \cap L^{s_\varepsilon}(\partial M)$ endowed with the norm $\|u\|_{\mathcal{H}} = \|u\|_H + |u|_{L^{s_\varepsilon}(\partial M)}$.

Notice that in the subcritical case $\mathcal{H}$ is identical to the Hilbert space H.

By trace theorems, we have the inclusion $W^{1,\tau}(M) \subset L^t(\partial M)$ for any t and τ satisfying $t \le \tau(n-1)/(n-\tau)$.

We consider $i : H^1(M) \to L^{2(n-1)/(n-2)}(\partial M)$ and its adjoint with respect to $\langle\langle\cdot,\cdot\rangle\rangle_H$, namely

$$i^* : L^{2(n-1)/n}(\partial M) \to H^1(M)$$

defined by

$$\langle\langle \varphi, i^*(g)\rangle\rangle_H = \int_{\partial M} \varphi g\, d\sigma \quad \text{for all } \varphi \in H^1,$$

so that $u = i^*(g)$ is the weak solution of the problem

$$\begin{cases} -\Delta_g u + a(x)u = 0 & \text{on } M, \\ \dfrac{\partial}{\partial \nu}u + \dfrac{n-2}{2}b(x)u = g & \text{on } \partial M. \end{cases} \tag{2-1}$$

We recall that by [Nittka 2011], if $u \in H^1(M)$ is a solution of (2-1), then for $2n/(n+2) \le q \le n/2$ and $r > 0$ we have

$$\|u\|_{L^{(n-1)q/(n-2q)}(\partial M)} = \|i^*(g)\|_{L^{(n-1)q/(n-2q)}(\partial M)} \le \|g\|_{L^{(n-1)q/(n-q)+r}(\partial M)}. \tag{2-2}$$

By this result, we can choose q and r such that

$$\frac{(n-1)q}{n-2q} = \frac{2(n-1)}{n-2} + n\varepsilon \quad \text{and} \quad \frac{(n-1)q}{n-q} + r = \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon}, \tag{2-3}$$

that is,

$$q = \frac{2n + n^2\left(\frac{n-2}{n-1}\right)\varepsilon}{n+2+2n\left(\frac{n-2}{n-1}\right)\varepsilon} \quad \text{and} \quad r = \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\varepsilon} - \frac{2(n-1)+n(n-2)\varepsilon}{n+(n-2)\left(\frac{n}{n-1}\right)\varepsilon}.$$

So, if $u \in L^{2(n-1)/(n-2)+n\varepsilon}(\partial M)$, then

$$|u|^{\frac{n}{n-2}+\varepsilon} \in L^{\frac{2(n-1)+n(n-2)\varepsilon}{n+\varepsilon(n-2)}}(\partial M)$$

and, in light of (2-2), also $i^*(|u|^{n/(n-2)+\varepsilon}) \in L^{2(n-1)/(n-2)+n\varepsilon}(\partial M)$.

Finally, we rewrite problem (1-5) — both in the subcritical and the supercritical case — as

$$u = i^*(f_\varepsilon(u)), \quad u \in \mathcal{H}, \tag{2-4}$$

where the nonlinearity $f_\varepsilon(u)$ is defined as $f_\varepsilon(u) := (n-2)(u^+)^{n/(n-2)+\varepsilon}$ in the supercritical case or $f_\varepsilon(u) := (n-2)(u^+)^{n/(n-2)-\varepsilon}$ in the subcritical case. Here $u^+(x) := \max\{0, u(x)\}$. By assumption (1-6), a solution to problem (2-4) is strictly positive and actually is a solution to problem (1-5). Therefore, we are led to build solutions to problem (2-4) which blow-up at a boundary point as ε goes to zero.

The main ingredient to cook up our solutions are the standard bubbles

$$U_{\delta,\xi}(x,t) := \frac{\delta^{(n-2)/2}}{((\delta+t)^2+|x-\xi|^2)^{(n-2)/2}}, \quad (x,t) \in \mathbb{R}^{n-1}\times\mathbb{R}_+,\ \delta>0,\ \xi\in\mathbb{R}^{n-1},$$

which are all the solutions to the limit problem

$$\begin{cases} -\Delta U = 0 & \text{on } \mathbb{R}^{n-1}\times\mathbb{R}_+, \\ \partial_\nu U = (n-2)U^{n/(n-2)} & \text{on } \mathbb{R}^{n-1}\times\{t=0\}. \end{cases} \tag{2-5}$$

We set $U_\delta(x,t) := U_{\delta,0}(x,t)$.

We also need to introduce the linear problem

$$\begin{cases} -\Delta V = 0 & \text{on } \mathbb{R}^{n-1} \times \mathbb{R}_+, \\ \partial_\nu V = n U_1^{2/(n-2)} V & \text{on } \mathbb{R}^{n-1} \times \{t = 0\}. \end{cases} \tag{2-6}$$

In [Almaraz 2011b] it has been proved that the n-dimensional space of solutions of (2-6) is generated by the functions

$$V_i = \frac{\partial U_1}{\partial x_i} = (2-n)\frac{x_i}{((1+t)^2 + |x|^2)^{n/2}} \quad \text{for } i = i, \ldots, n-1,$$

$$V_0 = \frac{\partial U_\delta}{\partial \delta}\Big|_{\delta=1} = \frac{n-2}{2}\left(\frac{1}{(1+t)^2+|x|^2}\right)^{n/2}\big(t^2 + |x|^2 - 1\big).$$

Next, for a point $q \in \partial M$ and the $(n-1)$-dimensional unitary ball $B^{n-1}(0, R)$ in $\mathbb{R}^{n-1}$, we introduce the Fermi coordinates $\psi_q^\partial : B^{n-1}(0, R) \times [0, R) \to M$. We read the bubble on the manifold as the function

$$W_{\delta,q}(\xi) = U_\delta\big((\psi_q^\partial)^{-1}\xi\big)\chi\big((\psi_q^\partial)^{-1}\xi\big),$$

and the functions V_i on the manifold as the functions

$$Z^i_{\delta,q}(\xi) = \frac{1}{\delta^{(n-2)/2}} V_i\Big(\frac{1}{\delta}(\psi_q^\partial)^{-1}\xi\Big)\chi\big((\psi_q^\partial)^{-1}\xi\big) \quad \text{for } i = 0, \ldots, n-1,$$

where $\chi(x, t) = \tilde{\chi}(|x|)\tilde{\chi}(t)$, for $\tilde{\chi}$ a smooth cut off function, $\tilde{\chi}(s) \equiv 1$ for $0 \le s < R/2$ and $\tilde{\chi}(s) \equiv 0$ for $s \ge R$. Then, it is necessary to split the Hilbert space H into the sum of the orthogonal spaces

$$K_{\delta,q} = \operatorname{Span}\big\langle Z^0_{\delta,q}, \ldots, Z^{n-1}_{\delta,q}\big\rangle$$

and

$$K^\perp_{\delta,q} = \big\{\varphi \in H^1(M) \mid \langle\langle \varphi, Z^i_{\delta,q}\rangle\rangle_H = 0 \text{ for all } i = 0, \ldots, n-1\big\}.$$

Finally, we can look for a solution to problem (2-4) in the form

$$u_\varepsilon(x) = W_{\delta,q}(x) + \phi(x)$$

where the blow-up point q is in ∂M, the blowing-up rate δ satisfies

$$\delta := d\varepsilon \quad \text{for some } d > 0 \tag{2-7}$$

and the remainder term ϕ belongs to the infinite dimensional space $K^\perp_{\delta,q} \cap \mathcal{H}$ of codimension n. We are led to solve the system

$$\Pi^\perp_{\delta,q}\big\{W_{\delta,q}(x) + \phi(x) - i^*\big(f_\varepsilon(W_{\delta,q}(x) + \phi(x))\big)\big\} = 0, \tag{2-8}$$

$$\Pi_{\delta,q}\big\{W_{\delta,q}(x) + \phi(x) - i^*\big(f_\varepsilon(W_{\delta,q}(x) + \phi(x))\big)\big\} = 0, \tag{2-9}$$

$\Pi^\perp_{\delta,q}$ and $\Pi_{\delta,q}$ being the projections on $K^\perp_{\delta,q}$ and $K_{\delta,q}$, respectively.

3. The finite dimensional reduction

In this section we perform the finite dimensional reduction. We rewrite the auxiliary equation (2-8) in the equivalent form

$$L(\phi) = N(\phi) + R, \tag{3-1}$$

where $L = L_{\delta,q} : K^{\perp}_{\delta,q} \cap \mathcal{H} \to K^{\perp}_{\delta,q} \cap \mathcal{H}$ is the linear operator

$$L(\phi) = \Pi^{\perp}_{\delta,q}\{\phi(x) - i^*(f'_\varepsilon(W_{\delta,q})[\phi])\},$$

$N(\phi)$ is the nonlinear term

$$N(\phi) = \Pi^{\perp}_{\delta,q}\{i^*\big(f_\varepsilon(W_{\delta,q}(x)+\phi(x))\big) - i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*(f'_\varepsilon(W_{\delta,q})[\phi])\} \tag{3-2}$$

and the error term R is defined by

$$R = \Pi^{\perp}_{\delta,q}\{i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\}. \tag{3-3}$$

3.1. *The invertibility of the linear operator L.*

Lemma 2. *For $a, b \in \mathbb{R}$ with $0 < a < b$, there exists a positive constant $C_0 = C_0(a, b)$ such that, for ε small, for any $q \in \partial M$, for any $d \in [a, b]$ and for any $\phi \in K^{\perp}_{\delta,q} \cap \mathcal{H}$, we have*

$$\|L_{\delta,q}(\phi)\|_{\mathcal{H}} \geq C_0 \|\phi\|_{\mathcal{H}}.$$

Proof. We argue by contradiction. Suppose that there exist two sequences of real numbers $\varepsilon_m \to 0$ and $d_m \in [a, b]$, a sequence of points $q_m \in \partial M$ and a sequence of functions $\phi_{\varepsilon_m d_m, q_m} \in K^{\perp}_{\varepsilon_m d_m, q_m} \cap \mathcal{H}$ such that

$$\|\phi_{\varepsilon_m d_m, q_m}\|_{\mathcal{H}} = 1 \quad \text{and} \quad \|L_{\varepsilon_m d_m, q_m}(\phi_{\varepsilon_m d_m, q_m})\|_{\mathcal{H}} \to 0 \text{ as } m \to +\infty.$$

For the sake of simplicity, we set $\delta_m = \varepsilon_m d_m$ and define

$$\tilde{\phi}_m := \delta_m^{(n-2)/2} \phi_{\delta_m, q_m}(\psi^{\partial}_{q_m}(\delta_m \eta))\chi(\delta_m \eta) \quad \text{for } \eta = (z, t) \in \mathbb{R}^n_+,\ z \in \mathbb{R}^{n-1},\ t \geq 0.$$

Since $\|\phi_{\varepsilon_m d_m, q_m}\|_H \leq 1$, by a change of variables we easily get that $\{\tilde{\phi}_m\}_m$ is bounded in $D^{1,2}(\mathbb{R}^n_+)$ (but not in $H^1(\mathbb{R}^n_+)$). Therefore, there exists $\tilde{\phi} \in D^{1,2}(\mathbb{R}^n_+)$ such that $\tilde{\phi}_m \rightharpoonup \tilde{\phi}$ almost everywhere, weakly in $D^{1,2}(\mathbb{R}^n_+)$, in $L^{2n/(n-2)}(\mathbb{R}^n_+)$ and strongly in $L^{2(n-1)/(n-2)}_{\text{loc}}(\partial\mathbb{R}^n_+)$.

Since $\phi_{\delta_m, q_m} \in K^{\perp}_{\delta_m, q_m}$, and taking (2-6) into account, for $i = 0, \dots, n-1$ we get

$$o(1) = \int_{\mathbb{R}^n_+} \nabla\tilde{\phi} \nabla V_i \, dz \, dt = n \int_{\mathbb{R}^{n-1}} U_1^{2/(n-2)}(z, 0) V_i(z, 0) \tilde{\phi}(z, 0) \, dz. \tag{3-4}$$

Indeed, by a change of variables we have

$$\begin{aligned}
0 &= \langle\!\langle \phi_{\delta_m,q_m}, Z^i_{\delta_m,q_m} \rangle\!\rangle_H \\
&= \int_M \big(\nabla_g \phi_{\delta_m,q_m} \nabla_g Z^i_{\delta_m,q_m} + a(x) \phi_{\delta_m,q_m} Z^i_{\delta_m,q_m} \big)\, d\mu_g \\
&\qquad + \frac{n-2}{2} \int_{\partial M} b(x) \phi_{\delta_m,q_m} Z^i_{\delta_m,q_m}\, d\sigma \\
&= \int_{\mathbb{R}^n_+} |g_{q_m}(\delta\eta)|^{1/2} \delta^{(n-2)/2} g^{\alpha\beta}_{q_m}(\delta\eta) \frac{\partial}{\partial\eta_\alpha} V_i(\eta) \chi(\delta\eta) \frac{\partial}{\partial\eta_\alpha} \phi_{\delta_m,q_m}(\psi^\partial_{q_m}(\delta_m\eta))\, \delta\eta \\
&\qquad + \int_{\mathbb{R}^n_+} |g_{q_m}(\delta\eta)|^{1/2} \delta^{(n+2)/2} a(\psi^\partial_{q_m}(\delta\eta)) V_i(\eta) \phi_{\delta_m,q_m}(\psi^\partial_{q_m}(\delta_m\eta))\, \delta\eta \\
&\qquad + \int_{\partial\mathbb{R}^n_+} |g_{q_m}(\delta z, 0)|^{1/2} \delta^{n/2} b(\psi^\partial_{q_m}(\delta\eta)) \phi_{\delta_m,q_m}(\psi^\partial_{q_m}(\delta_m z, 0)) V_i(\delta_m z, 0)\, dz \\
&= \int_{\mathbb{R}^n_+} \nabla V_i(\eta) \nabla \tilde{\phi}_m(\eta) + \delta^2 a(q_m) V_i(\eta) \tilde{\phi}_m(\eta)\, \delta\eta \\
&\qquad + \delta \int_{\partial\mathbb{R}^n_+} b(q_m) V_i(z,0) \tilde{\phi}_m(z,0)\, \delta\eta + O(\delta) \\
&= \int_{\mathbb{R}^n_+} \nabla V_i(\eta) \nabla \tilde{\phi}_m(\eta) + O(\delta) = \int_{\mathbb{R}^n_+} \nabla V_i(\eta) \nabla \tilde{\phi}(\eta) + o(1),
\end{aligned}$$

By definition of L_{δ_m,q_m} we have

$$\phi_{\delta_m,q_m} - i^*(f'_\varepsilon(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]) - L_{\delta_m,q_m}(\phi_{\delta_m,q_m}) = \sum_{i=0}^{n-1} c^i_m Z^i_{\delta_m,q_m}. \tag{3-5}$$

We want to prove that, for all $i = 0, \ldots, n-1$, $c^i_m \to 0$ as $m \to \infty$. Multiplying (3-5) by $Z^j_{\delta_m,q_m}$ we obtain, by definition of i^*,

$$\begin{aligned}
\sum_{i=0}^{n-1} c^i_m \langle\!\langle Z^i_{\delta_m,q_m}, Z^j_{\delta_m,q_m} \rangle\!\rangle_H &= \langle\!\langle i^*(f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]), Z^j_{\delta_m,q_m} \rangle\!\rangle_H \\
&= \int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}] Z^j_{\delta_m,q_m}\, d\sigma.
\end{aligned}$$

Moreover, by multiplying (3-5) by ϕ_{δ_m,q_m} we obtain that

$$\|\phi_{\delta_m,q_m}\|_H - \int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m}) \phi^2_{\delta_m,q_m}\, d\sigma \to 0.$$

Thus $(f'_{\varepsilon_m}(W_{\delta_m,q_m}))^{1/2} \phi_{\delta_m,q_m}$ is bounded and weakly convergent in $L^2(\partial M)$. With this consideration we easily get

$$\begin{aligned}\int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]Z^j_{\delta_m,q_m}\,d\sigma\\ &= \int_{\partial M}(f'_{\varepsilon_m}(W_{\delta_m,q_m}))^{1/2}\phi_{\delta_m,q_m}(f'_{\varepsilon_m}(W_{\delta_m,q_m}))^{1/2}Z^j_{\delta_m,q_m}\,d\sigma\\ &= n\int_{\mathbb{R}^{n-1}} U_1^{2/(n-2)}(z,0)\tilde{\phi}(z,0)V_i(z,0)\,dz+o(1)=o(1),\end{aligned}$$

once we take (3-4) into account.

Now, it is easy to prove that

$$\langle\!\langle Z^i_{\delta_m,q_m}, Z^j_{\delta_m,q_m}\rangle\!\rangle_H = C\delta_{ij}+o(1),$$

hence we can conclude that $c^i_m \to 0$ as $m\to\infty$ for each $i=0,\dots,n-1$. This, combined with (3-5) and using $\|L_{\varepsilon_m d_m,q_m}(\phi_{\varepsilon_m d_m,q_m})\|_{\mathcal{H}}\to 0$, gives us that

$$\left\|\phi_{\delta_m,q_m}-i^*(f'_\varepsilon(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}])\right\|_{\mathcal{H}} = \sum_{i=0}^{n-1} c^i_m\|Z^i\|_{\mathcal{H}}+o(1)=o(1). \tag{3-6}$$

Choose a smooth function $\varphi\in C_0^\infty(\mathbb{R}^n_+)$ and define

$$\varphi_m(x)=\frac{1}{\delta_m^{(n-2)/2}}\varphi\Big(\frac{1}{\delta_m}\big(\psi^\partial_{q_m}\big)^{-1}(x)\Big)\chi\big(\big(\psi^\partial_{q_m}\big)^{-1}(x)\big)\quad\text{for } x\in M.$$

We have that $\|\varphi_m\|_H$ is bounded and, by (3-6), that

$$\begin{aligned}&\langle\!\langle\phi_{\delta_m,q_m},\varphi_m\rangle\!\rangle_H\\ &\quad=\int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\varphi_m\,d\sigma+\langle\!\langle\phi_{\delta_m,q_m}-i^*(f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]),\varphi_m\rangle\!\rangle_H\\ &\quad=\int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\varphi_m\,d\sigma+o(1)\\ &\quad=(n\pm\varepsilon_m(n-2))\int_{\mathbb{R}^{n-1}}\frac{1}{\delta_m^{\pm\varepsilon n/(n-2)}}U_1^{2/(n-2)\pm\varepsilon_m}(z,0)\tilde{\phi}_m(z,0)\varphi\,dz+o(1)\\ &\quad=n\int_{\mathbb{R}^{n-1}}U_1^{2/(n-2)}(z,0)\tilde{\phi}(z,0)\varphi(z,0)\,dz+o(1),\end{aligned}$$

by the strong $L^{2(n-1)/(n-2)}_{\mathrm{loc}}(\partial\mathbb{R}^n_+)$ convergence of $\tilde{\phi}_m$. On the other hand,

$$\langle\!\langle\phi_{\delta_m,q_m},\varphi_m\rangle\!\rangle_H=\int_{\mathbb{R}^n_+}\nabla\tilde{\phi}\nabla\varphi\,\delta\eta+o(1),$$

so $\tilde{\phi}$ is a weak solution of (2-5) and we conclude that

$$\tilde{\phi}\in\operatorname{Span}\{V_0,V_1,\dots,V_n\}.$$

This, combined with (3-4), gives that $\tilde{\phi}=0$. Proceeding as before we have

$$\begin{aligned}
&\langle\langle \phi_{\delta_m,q_m}, \phi_{\delta_m,q_m}\rangle\rangle_H \\
&= \int_{\partial M} f'_{\varepsilon_m}(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}]\phi_{\delta_m,q_m}\,d\sigma + o(1) \\
&= (n\pm\varepsilon_m(n-2))\int_{\mathbb{R}^{n-1}} \frac{1}{\delta_m^{\pm\varepsilon n/(n-2)}} U_1^{2/(n-2)\pm\varepsilon_m}(z,0)\tilde{\phi}_m^2(z,0)\varphi\,dz + o(1) = o(1).
\end{aligned}$$

In a similar way, by (3-6) we have

$$|\phi_{\delta_m,q_m}|_{L^{s_\varepsilon}} = \left| i^*(f'_\varepsilon(W_{\delta_m,q_m})[\phi_{\delta_m,q_m}])\right|_{L^{s_\varepsilon}} + o(1) = o(1),$$

which gives $\|\phi_{\delta_m,q_m}\|_{\mathcal{H}} \to 0$, which is a contradiction. □

3.2. *The estimate of the error term R.*

Lemma 3. *For $a, b\in\mathbb{R}$ with $0<a<b$, there exists a positive constant $C_1=C_1(a,b)$ such that, for ε small, for any $q\in\partial M$ and for any $d\in[a,b]$ we have*

$$\|R_{\varepsilon,\delta,q}\|_{\mathcal{H}} \le C_1\varepsilon|\ln\varepsilon|$$

Proof. We estimate

$$\begin{aligned}
&\left\| i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\right\|_H \\
&\qquad \le \left\| i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*\big(f_0(W_{\delta,q}(x))\big)\right\|_H + \left\| i^*\big(f_0(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\right\|_H.
\end{aligned}$$

By definition of i^* there exists Γ which solves the equation

$$\begin{cases} -\Delta_g\Gamma + a(x)\Gamma = 0 & \text{on } M, \\ \dfrac{\partial}{\partial\nu}\Gamma + \dfrac{n-2}{2}b(x)\Gamma = f_0(W_{\delta,q}) & \text{on } \partial M, \end{cases} \tag{3-7}$$

so, by (3-7), we have

$$\begin{aligned}
&\left\| i^*\big(f_0(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\right\|_H \\
&\qquad = \|\Gamma(x) - W_{\delta,q}(x)\|_H^2 \\
&\qquad = \int_M [-\Delta_g(\Gamma - W_{\delta,q}) + a(\Gamma - W_{\delta,q})](\Gamma - W_{\delta,q})\,d\mu_g \\
&\qquad\qquad + \int_{\partial M}\left[\frac{\partial}{\partial\nu}(\Gamma - W_{\delta,q}) + \frac{(n-2)}{2}b(x)(\Gamma - W_{\delta,q})\right](\Gamma - W_{\delta,q})\,d\mu_g \\
&\qquad = \int_M [\Delta_g W_{\delta,q} - aW_{\delta,q}](\Gamma - W_{\delta,q})\,d\mu_g \\
&\qquad\qquad + \int_{\partial M}\left[f_0(W_{\delta,q}) - \frac{\partial}{\partial\nu}W_{\delta,q}\right](\Gamma - W_{\delta,q})\,d\mu_g \\
&\qquad\qquad - \frac{n-2}{2}\int_{\partial M} b(x)W_{\delta,q}(\Gamma - W_{\delta,q})\,d\mu_g := I_1 + I_2 + I_3.
\end{aligned}$$

We obtain

$$I_1 = \|\Gamma - W_{\delta,q}\|_H O(\delta). \tag{3-8}$$

In fact,

$$\begin{aligned} I_1 &\le |\Delta_g W_{\delta,q} - aW_{\delta,q}|_{L^{2n/(n+2)}(M)} |\Gamma - W_{\delta,q}|_{L^{2n/(n-2)}(M)} \\ &\le |\Delta_g W_{\delta,q} - aW_{\delta,q}|_{L^{2n/(n+2)}(M)} \|\Gamma - W_{\delta,q}\|_H. \end{aligned}$$

We easily have that $|W_{\delta,q}|_{L^{2n/(n+2)}} = O(\delta^2)$. For the other term we have, in coordinates,

$$\Delta_g W_{\delta,q} = \Delta[U_\delta\chi] + (g^{ab} - \delta_{ab})\partial_{ab}[U_\delta\chi] - g^{ab}\Gamma_{ab}^k \partial_k [U_\delta\chi], \tag{3-9}$$

Γ_{ab}^k being the Christoffel symbols. Using the expansion of the metric g^{ab} given by (4-2) and (4-3) we have that

$$\begin{aligned} \left|(g^{ab} - \delta_{ab})\partial_{ab}[U_\delta\chi]\right|_{L^{2n/(n+2)}(M)} &= O(\delta), \\ \left|g^{ab}\Gamma_{ab}^k \partial_k[U_\delta\chi]\right|_{L^{2n/(n+2)}(M)} &= O(\delta^2). \end{aligned} \tag{3-10}$$

Since U_δ is a harmonic function we deduce

$$|\Delta[U_\delta\chi]|_{L^{2n/(n+2)}(M)} = |U_\delta\Delta\chi + 2\nabla U_\delta\nabla\chi|_{L^{2n/(n+2)}(M)} = O(\delta^2). \tag{3-11}$$

For the second integral I_2 we have

$$I_2 = \|\Gamma - W_{\delta,q}\|_H O(\delta^2), \tag{3-12}$$

since

$$\begin{aligned} I_2 &\le \left| f_0(W_{\delta,q}) - \frac{\partial}{\partial\nu} W_{\delta,q}\right|_{L^{2(n-1)/n}(\partial M)} |\Gamma - W_{\delta,q}|_{L^{2(n-1)/n-2}(\partial M)} \\ &\le C\left| f_0(W_{\delta,q}) - \frac{\partial}{\partial\nu} W_{\delta,q}\right|_{L^{2(n-1)/n}(\partial M)} \|\Gamma - W_{\delta,q}\|_H, \end{aligned}$$

and, using the boundary condition for (2-5), we have

$$\begin{aligned} &\left| f_0(W_{\delta,q}) - \frac{\partial}{\partial\nu} W_{\delta,q}\right|_{L^{2(n-1)/n}(\partial M)} \\ &= \frac{1}{\delta^{n/2}}\Bigg(\int_{\mathbb{R}^{n-1}} |g(\delta z,0)|^{1/2}\Big[(n-2)U^{n/(n-2)}(z,0)\chi^{n/(n-2)}(\delta z,0) \\ &\qquad\qquad - \chi(\delta z,0)\frac{\partial U}{\partial t}(z,0)\Big]^{\frac{2(n-1)}{n}} \delta^{n-1}\,dz\Bigg)^{\frac{n}{2(n-1)}} \\ &\le C\Bigg(\int_{\mathbb{R}^{n-1}} \big[(n-2)U^{n/(n-2)}(z,0)[\chi^{n/(n-2)}(\delta z,0) \\ &\qquad\qquad - \chi(\delta z,0)]\big]^{\frac{2(n-1)}{n}}\,dz\Bigg)^{\frac{n}{2(n-1)}} = O(\delta^2). \end{aligned} \tag{3-13}$$

Lastly,

$$I_3 \le |W_{\delta,q}|_{L^{2(n-1)/n}(\partial M)} |\Gamma - W_{\delta,q}|_{L^{2(n-1)/(n-2)}(\partial M)} = \|\Gamma - W_{\delta,q}\|_H O(\delta). \tag{3-14}$$

By (3-8), (3-12) and (3-14) we conclude that

$$\left\| i^*\big(f_0(W_{\delta,q}(x))\big) - W_{\delta,q}(x) \right\|_H = \|\Gamma(x) - W_{\delta,q}(x)\|_H = O(\delta).$$

To conclude the proof we estimate the term $\left\| i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*\big(f_0(W_{\delta,q}(x))\big) \right\|_H$. We have, by the properties of i^*, that

$$\begin{aligned}
&\left\| i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*\big(f_0(W_{\delta,q}(x))\big) \right\|_H \\
&\le \left| W_{\delta,q}(x)^{n/(n-2)\pm\varepsilon} - W_{\delta,q}^{n/(n-2)}(x) \right|_{L^{2(n-1)/n}(\partial M)} \\
&= \left(\int_{\mathbb{R}^{n-1}} \left[\left(\frac{1}{\delta^{\pm\varepsilon(n-2)/2}} U^{\pm\varepsilon}(z,0) - 1 \right) U^{n/(n-2)}(z,0) \right]^{\frac{2(n-1)}{n}} dz \right)^{\frac{n}{2(n-1)}} + O(\delta^2).
\end{aligned}$$

To estimate the last integral, we first recall two Taylor expansions with respect to ε:

$$U^{\pm\varepsilon} = 1 \pm \varepsilon \ln U + \frac{1}{2}\varepsilon^2 \ln^2 U + o(\varepsilon^2), \tag{3-15}$$

$$\delta^{\mp\varepsilon(n-2)/2} = 1 \mp \varepsilon \frac{n-2}{2} \ln \delta + \varepsilon^2 \frac{(n-2)^2}{8} \ln^2 \delta + o(\varepsilon^2 \ln^2 \delta). \tag{3-16}$$

In light of (3-15) and (3-16) we have

$$\begin{aligned}
&\left\| i^*(f_\varepsilon(W_{\delta,q})) - i^*(f_0(W_{\delta,q})) \right\|_H \\
&\le \left(\int_{\mathbb{R}^{n-1}} \left| \left(\mp \frac{n-2}{2} \varepsilon \ln \delta \pm \varepsilon \ln U(z,0) + O(\varepsilon^2) \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. + O(\varepsilon^2 \ln \delta) \right) U^{n/(n-2)}(z,0) \right|^{\frac{2(n-1)}{n}} dz \right)^{\frac{n}{2(n-1)}} + O(\delta^2) \\
&= \frac{n-2}{2} \varepsilon \ln \delta \left| U(z,0) \right|^{n/(n-2)}_{L^{2(n-1)/(n-2)}(\mathbb{R}^{n-1})} \\
&\qquad\qquad + \varepsilon \left(\int_{\mathbb{R}^{n-1}} U^{2(n-1)/(n-2)}(z,0) \ln U(z,0)\, dz \right)^{\frac{n}{2(n-1)}} \\
&\qquad\qquad\qquad + O(\varepsilon^2) + O(\varepsilon^2 |\ln \delta|) + O(\delta^2) \\
&= O(\varepsilon) + O(\varepsilon |\ln \delta|) + O(\delta^2).
\end{aligned} \tag{3-17}$$

Choosing $\delta = d\varepsilon$ concludes the proof of Lemma 3 for the subcritical case.

For the supercritical case, we have to control $|R_{\varepsilon,\delta,q}|_{L^{s_\varepsilon}(\partial M)}$. As in the previous case we consider

$$|R_{\varepsilon,\delta,q}|_{L^{s_\varepsilon}(\partial M)} \le \big|i^*\big(f_\varepsilon(W_{\delta,q}(x))\big) - i^*\big(f_0(W_{\delta,q}(x))\big)\big|_{L^{s_\varepsilon}(\partial M)} + \big|i^*\big(f_0(W_{\delta,q}(x))\big) - W_{\delta,q}(x)\big|_{L^{s_\varepsilon}(\partial M)}.$$

As before, set $\Gamma = i^*(f_0(W_{\delta,q}(x))$. Since Γ solves (3-7), $\Gamma - W_{\delta,q}$ solves

$$\begin{cases} -\Delta_g(\Gamma - W_{\delta,q}) + a(x)(\Gamma - W_{\delta,q}) = -\Delta_g W_{\delta,q} + a(x) W_{\delta,q} & \text{on } M,\\ \dfrac{\partial}{\partial\nu}(\Gamma - W_{\delta,q}) + \dfrac{n-2}{2} b(x)(\Gamma - W_{\delta,q}) & \\ \qquad\qquad = f_0(\Gamma) + \dfrac{\partial}{\partial\nu} W_{\delta,q} + \dfrac{n-2}{2} b(x) W_{\delta,q} & \text{on } \partial M. \end{cases}$$

We choose q as in (2-3), and $r = \varepsilon$. Thus, by Theorem 3.14 in [Nittka 2011], we have

$$|\Gamma - W_{\delta,q}|_{L^{s_\varepsilon}(\partial M)} \le |-\Delta_g W_{\delta,q} + a(x) W_{\delta,q}|_{L^{q+\varepsilon}(M)} + \Big| f_0(\Gamma) + \frac{\partial}{\partial\nu} W_{\delta,q} + \frac{n-2}{2} b(x) W_{\delta,q}\Big|_{L^{(n-1)q/(n-q)+\varepsilon}(\partial M)}.$$

We remark that

$$q = \frac{2n + n^2\big(\frac{n-2}{n-1}\big)\varepsilon}{n+2+2n\big(\frac{n-2}{n-1}\big)\varepsilon} = \frac{2n}{n+2} + O^+(\varepsilon) \quad \text{with } 0 < O^+(\varepsilon) < C\varepsilon$$

for some positive constant C. By direct computation we have

$$|a(x) W_{\delta,q}|_{L^{q+\varepsilon}(M)} \le C\delta^{2-O^+(\varepsilon)},$$
$$|b(x) W_{\delta,q}|_{L^{(n-1)q/(n-q)+\varepsilon}(\partial M)} \le C\delta^{1-O^+(\varepsilon)}.$$

Moreover, proceeding as in (3-9), (3-10), (3-11) and (3-13) we get

$$|\Delta_g W_{\delta,q}|_{L^{q+\varepsilon}(M)} \le C\delta^{2-O^+(\varepsilon)},$$
$$\Big| f_0(\Gamma) + \frac{\partial}{\partial\nu} W_{\delta,q}\Big|_{L^{(n-1)q/(n-q)+\varepsilon}(\partial M)} \le C\delta^{1-O^+(\varepsilon)}.$$

Since $i^*(f_\varepsilon(W_{\delta,q}))$ solves (1-5), and $i^*(f_\varepsilon|u|^{n/(n-2)+\varepsilon}(W_{\delta,q}))$ solves (1-5), we again use Theorem 3.14 in [Nittka 2011]. Taking (3-15) and (3-16) into account, we finally get

$$\begin{aligned}
(3\text{-}18)\quad & \left|i^*(f_\varepsilon(W_{\delta,q})) - i^*(f_0(W_{\delta,q}))\right|_{L^{s_\varepsilon}(\partial M)} \\
&\le |f_\varepsilon(W_{\delta,q}) - f_0(W_{\delta,q})|_{L^{2(n-1)/n+O^+(\varepsilon)}(\partial M)} \\
&\le \delta^{-O^+(\varepsilon)}\biggl(\int_{\mathbb{R}^{n-1}}\Bigl[\Bigl(\frac{1}{\delta^{\varepsilon(n-2)/2}}U^\varepsilon(z,0)-1\Bigr) \\
&\qquad \cdot U^{n/(n-2)}(z,0)\Bigr]^{\frac{2(n-1)}{n}+O^+(\varepsilon)} dz\biggr)^{\frac{1}{2(n-1)/n+O^+(\varepsilon)}} + O(\delta^2) \\
&= \delta^{-O^+(\varepsilon)}(O(\varepsilon|\ln\delta|)+O(\varepsilon)) + O(\delta^2).
\end{aligned}$$

Now, choosing $\delta = d\varepsilon$, we can conclude the proof, since

$$\delta^{-O^+(\varepsilon)} = 1 + O^+(\varepsilon)|\ln(\varepsilon d)| = 1 + O^+(\varepsilon|\ln\varepsilon|) = O(1). \qquad \square$$

3.3. ***Solving (2-8): the remainder term ϕ.***

Proposition 4. *For $a, b \in \mathbb{R}$ with $0 < a < b$, there exists a positive constant $C = C(a,b)$ such that, for ε small, for any $q \in \partial M$ and for any $d \in [a,b]$ there exists a unique $\phi_{\delta,q}$ which solves (2-8). This solution satisfies*

$$\|\phi_{\delta,q}\|_{\mathcal{H}} \le C\varepsilon|\ln\varepsilon|.$$

Moreover the map $q \mapsto \phi_{\delta,q}$ is a $C^1(\partial M, \mathcal{H})$ map.

Proof. First of all, we point out that N is a contraction mapping. We remark that the conjugate exponent of s_ε is

$$s_\varepsilon' = \begin{cases} \dfrac{2(n-1)}{n} & \text{in the subcritical case,} \\[2ex] \dfrac{2(n-1)+\varepsilon n(n-2)}{n+\varepsilon n(n-2)} & \text{in the supercritical case.} \end{cases}$$

By the properties of i^* and using the expansion of $f_\varepsilon(W_{\delta,q}+\phi_1)$ centered in $W_{\delta,q}+\phi_2$ we have

$$\begin{aligned}
\|N(\phi_1)-N(\phi_2)\|_{\mathcal{H}} &\le \bigl\| f_\varepsilon(W_{\delta,q}+\phi_1) - f_\varepsilon(W_{\delta,q}+\phi_2) - f_\varepsilon'(W_{\delta,q})[\phi_1-\phi_2]\bigr\|_{L^{s_\varepsilon'}(\partial M)} \\
&\le \bigl\|\bigl(f_\varepsilon'(W_{\delta,q}+\theta\phi_1+(1-\theta)\phi_2) - f_\varepsilon'(W_{\delta,q})\bigr)[\phi_1-\phi_2]\bigr\|_{L^{s_\varepsilon'}(\partial M)}
\end{aligned}$$

and, since $|\phi_1-\phi_2|^{s_\varepsilon'} \in L^{s_\varepsilon/s_\varepsilon'}(\partial M)$ and $|f_\varepsilon'(\cdot)|^{s_\varepsilon'} \in L^{(s_\varepsilon/s_\varepsilon')'}(\partial M)$ as $f_\varepsilon'(\cdot) \in L^{s_\varepsilon}(\partial M)$, we have

$$\begin{aligned}
&\|N(\phi_1)-N(\phi_2)\|_{\mathcal{H}} \\
&\qquad \le \bigl\|\bigl(f_\varepsilon'(W_{\delta,q}+\theta\phi_1+(1-\theta)\phi_2) - f_\varepsilon'(W_{\delta,q})\bigr)\bigr\|_{L^{s_\varepsilon}(\partial M)} \|\phi_1-\phi_2\|_{L^{s_\varepsilon}(\partial M)} \\
&\qquad = \gamma\|\phi_1-\phi_2\|_{\mathcal{H}},
\end{aligned}$$

where

$$\gamma = \left\| \left(f_\varepsilon'(W_{\delta,q} + \theta\phi_1 + (1-\theta)\phi_2) - f_\varepsilon'(W_{\delta,q}) \right) \right\|_{L^{s_\varepsilon}(\partial M)} < 1$$

provided $\|\phi_1\|_{\mathcal{H}}$ and $\|\phi_2\|_{\mathcal{H}}$ are sufficiently small.

In the same way we can prove that $\|N(\phi)\|_{\mathcal{H}} \le \gamma \|\phi\|_{\mathcal{H}}$ with $\gamma < 1$ if $\|\phi\|_{\mathcal{H}}$ is sufficiently small.

Next, by Lemmas 2 and 3 we have

$$\|L^{-1}(N(\phi) + R_{\varepsilon,\delta,q})\|_{\mathcal{H}} \le C(\gamma \|\phi\|_{\mathcal{H}} + \varepsilon |\ln \varepsilon|),$$

where $C = \max\{C_0, C_0C_1\} > 0$, for the constants C_0, C_1 which appear in Lemmas 2 and 3. Notice that, given $C > 0$, it is possible (up to a choice of $\|\phi\|_{\mathcal{H}}$ sufficiently small) to choose $0 < C\gamma < \frac{1}{2}$.

Now, if $\|\phi\|_{\mathcal{H}} \le 2C\varepsilon|\ln \varepsilon|$, then the map

$$T(\phi) := L^{-1}(N(\phi) + R_{\varepsilon,\delta,q})$$

is a contraction from the ball $\|\phi\|_{\mathcal{H}} \le 2C\varepsilon|\ln\varepsilon|$ in itself, so, by the fixed point theorem, there exists a unique $\phi_{\delta,q}$ with $\|\phi_{\delta,q}\|_{\mathcal{H}} \le 2C\varepsilon|\ln\varepsilon|$ solving (3-1), and hence (2-8). The regularity of the map $q \mapsto \phi_{\delta,q}$ can be proven via the implicit function theorem. □

4. The reduced problem

Problem (1-5) has a variational structure. Weak solutions to (1-5) are critical points of the energy functional $J_\varepsilon : \mathcal{H} \to \mathbb{R}$ given by

$$J_\varepsilon(u) = \frac{1}{2}\int_M (|\nabla u|^2 + a(x)u^2)\, d\mu_g + \frac{n-2}{4}\int_{\partial M} b(x)u^2\, d\sigma - \frac{(n-2)^2}{2n-2 \pm \varepsilon(n-2)} \int_{\partial M} u^{(2n-2)/(n-2)\pm\varepsilon}\, d\sigma.$$

Let us introduce the reduced energy $I_\varepsilon : (0, +\infty) \times \partial M \to \mathbb{R}$ by

$$I_\varepsilon(d, q) := J_\varepsilon(W_{\varepsilon d,q} + \phi_{\varepsilon d,q}), \tag{4-1}$$

where the remainder term $\phi_{\varepsilon d,q}$ has been found in Proposition 4.

4.1. ***The reduced energy.*** Here we use the following expansion for the metric tensor on M:

$$g^{ij}(y) = \delta_{ij} + 2h_{ij}(0)y_n + O(|y|^2) \quad \text{for } i, j = 1, \dots, n-1, \tag{4-2}$$

$$g^{in}(y) = \delta_{in} \quad \text{for } i = 1, \dots, n-1, \tag{4-3}$$

$$\sqrt{g}(y) = 1 - (n-1)H(0)y_n + O(|y|^2), \tag{4-4}$$

where $(y_1,\dots,y_n)$ are the Fermi coordinates and, by definition of h_{ij},

$$H=\frac{1}{n-1}\sum_i^{n-1} h_{ii}. \tag{4-5}$$

We also recall that on ∂M the Fermi coordinates coincide with the exponential ones, so we have that

$$\sqrt{g}(y_1,\dots,y_{n-1},0)=1+O(|y|^2). \tag{4-6}$$

To improve the readability of this paper, hereafter we write $z=(z_1,\dots,z_{n-1})$ to indicate the first $n-1$ Fermi coordinates and t to indicate the last one, so $(y_1,\dots,y_{n-1},y_n)=(z,t)$. Moreover, indices i,j conventionally refer to sums from 1 to $n-1$, while l,m usually refer to sums from 1 to n.

Proposition 5. (i) *If $(d_0,q_0)\in(0,+\infty)\times\partial M$ is a critical point for the reduced energy I_ε defined in (4-1), then $W_{\varepsilon d_0,q_0}+\phi_{\varepsilon d_0,q_0}\in\mathcal{H}$ solves problem (1-5).*

(ii) *It holds true that*

$$\begin{cases} I_\varepsilon(d,q)=c_n(\varepsilon)+\varepsilon[\alpha_n d\varphi(q)-\beta_n\ln d]+o(\varepsilon) & \text{in the subcritical case},\\ I_\varepsilon(d,q)=c_n(\varepsilon)+\varepsilon[\alpha_n d\varphi(q)+\beta_n\ln d]+o(\varepsilon) & \text{in the supercritical case},\end{cases}$$

C^0-uniformly with respect to d in compact subsets of $(0,+\infty)$ and $q\in\partial M$. Here $c_n(\varepsilon)$ is a constant which only depends on ε and n, α_n and β_n are positive constants which only depend on n, and $\varphi(q)=h(q)-H_g(q)$ is the function defined in (1-4).

Proof. (i) Set $q:=q(y)=\psi^\partial_{q_0}(y)$. Since (d_0,q_0) is a critical point, we have, for any $h\in 1,\dots,n-1$,

$$\begin{aligned}0&=\frac{\partial}{\partial y_h}I_\varepsilon\big(d,\psi^\partial_{q_0}(y)\big)\Big|_{y=0}\\
&=\Big\langle\!\Big\langle W_{\varepsilon d,q(y)}+\phi_{\varepsilon d,q(y)}-i^*(f_\varepsilon(W_{\varepsilon d,q(y)}+\phi_{\varepsilon d,q(y)})),\\
&\qquad\qquad\frac{\partial}{\partial y_h}W_{\varepsilon d,q(y)}+\frac{\partial}{\partial y_h}\phi_{\varepsilon d,q(y)}\Big\rangle\!\Big\rangle_H\Big|_{y=0}\\
&=\sum_{i=0}^{n-1}c^i_\varepsilon\Big\langle\!\Big\langle Z^i_{\varepsilon d,q(y)},\frac{\partial}{\partial y_h}W_{\varepsilon d,q(y)}+\frac{\partial}{\partial y_h}\phi_{\varepsilon d,q(y)}\Big\rangle\!\Big\rangle_H\Big|_{y=0}\\
&=\sum_{i=0}^{n-1}c^i_\varepsilon\Big\langle\!\Big\langle Z^i_{\varepsilon d,q(y)},\frac{\partial}{\partial y_h}W_{\varepsilon d,q(y)}\Big\rangle\!\Big\rangle_H\Big|_{y=0}-\sum_{i=0}^{n-1}c^i_l\Big\langle\!\Big\langle\frac{\partial}{\partial y_h}Z^i_{\varepsilon d,q(y)},\phi_{\varepsilon d,q(y)}\Big\rangle\!\Big\rangle_H\Big|_{y=0},\end{aligned}$$

using that $\phi_{\varepsilon d,q(y)}$ is a solution of (2-8) and that

$$\Big\langle\!\Big\langle Z^i_{\varepsilon d,q(y)},\frac{\partial}{\partial y_h}\phi_{\varepsilon d,q(y)}\Big\rangle\!\Big\rangle_H=-\Big\langle\!\Big\langle\frac{\partial}{\partial y_h}Z^i_{\varepsilon d,q(y)},\phi_{\varepsilon d,q(y)}\Big\rangle\!\Big\rangle_H$$

since $\phi_{\varepsilon d,q(y)} \in K^{\perp}_{\varepsilon d,q(y)}$ for all y. Now it is enough to observe that

$$\Big\langle\!\Big\langle \frac{\partial}{\partial y_h} Z^i_{\varepsilon d,q(y)}, \phi_{\varepsilon d,q(y)} \Big\rangle\!\Big\rangle_H \le \Big\| \frac{\partial}{\partial y_h} Z^i_{\varepsilon d,q(y)} \Big\|_H \|\phi_{\varepsilon d,q(y)}\|_H = o(1),$$
$$\Big\langle\!\Big\langle Z^i_{\varepsilon d,q(y)}, \frac{\partial}{\partial y_h} W_{\varepsilon d,q(y)} \Big\rangle\!\Big\rangle_H = \frac{1}{\varepsilon d} \langle\!\langle Z^i_{\varepsilon d,q(y)}, Z^h_{\varepsilon d,q(y)} \rangle\!\rangle_H = \frac{1}{\varepsilon d}\delta^{ih} + o(1),$$

to conclude that

$$0 = \frac{1}{\varepsilon d} \sum_{i=0}^{n-1} c^i_\varepsilon (\delta^{ih} + o(1)),$$

and so $c^i_\varepsilon = 0$ for all $i = 0, \dots, n-1$. This concludes the proof of (i).

(ii) We prove (ii) in two steps.

Step 1. *We prove that for ε small enough and for any $q \in \partial M$,*

$$|J_\varepsilon(W_{\delta,q} + \phi_{\delta,q}) - J_\varepsilon(W_{\delta,q})| \le \|\phi_{\delta,q}\|^2_{\mathcal{H}} + C\varepsilon|\ln\varepsilon|\, \|\phi_{\delta,q}\|_{\mathcal{H}} = o(\varepsilon).$$

We have

$$\begin{aligned}
&|J_\varepsilon(W_{\delta,q} + \phi_{\delta,q}) - J_\varepsilon(W_{\delta,q})| \\
&\quad = \left| \int_M [-\Delta_g W_{\delta,q} + a(x) W_{\delta,q}] \phi_{\delta,q}\, d\mu_g \right| + \frac{1}{2}\|\phi_{\delta,q}\|^2_H \\
&\qquad + \left| \int_{\partial M} \Big[\frac{\partial}{\partial \nu} W_{\delta,q} + \frac{n-2}{2} b(x) W_{\delta,q} - f_0(W_{\delta,q}) \Big] \phi_{\delta,q}\, d\sigma \right| \\
&\qquad + \left| \int_{\partial M} \big[f_0(W_{\delta,q}) - f_\varepsilon(W_{\delta,q}) \big] \phi_{\delta,q}\, d\sigma \right| \\
&\qquad + \Bigg| \int_{\partial M} \frac{(n-2)^2}{2n-2\pm\varepsilon(n-2)} \big[(W_{\delta,q} + \phi_{\delta,q})^{(2n-2)/(n-2)\pm\varepsilon} - W_{\delta,q}^{(2n-2)/(n-2)\pm\varepsilon} \big] \\
&\qquad\qquad - f_\varepsilon(W_{\delta,q}) \phi_{\delta,q}\, d\sigma \Bigg|.
\end{aligned}$$

With the same estimate of I_1 in Lemma 3 we obtain that

$$\left| \int_M [-\Delta_g W_{\delta,q} + a(x) W_{\delta,q}] \phi_{\delta,q}\, d\mu_g \right| = O(\delta) \|\phi_{\delta,q}\|_H,$$

and in light of the estimate of I_2 and I_3 in Lemma 3 we get

$$\left| \int_{\partial M} \Big[\frac{\partial}{\partial \nu} W_{\delta,q} + \frac{n-2}{2} b(x) W_{\delta,q} - f_0(W_{\delta,q}) \Big] \phi_{\delta,q}\, d\sigma \right| = O(\delta) \|\phi_{\delta,q}\|_H.$$

In the subcritical case, following the computation in (3-17) we obtain

$$\begin{aligned}\left|\int_{\partial M}[f_0(W_{\delta,q})-f_\varepsilon(W_{\delta,q})]\phi_{\delta,q}\,d\sigma\right| &\le C|f_0(W_{\delta,q})-f_\varepsilon(W_{\delta,q})|_{L^{2(n-1)/n}(\partial M)}|\phi_{\delta,q}|_{L^{2(n-1)/(n-2)}(\partial M)}\\ &=[O(\varepsilon)+O(\varepsilon\ln\delta)]\|\phi_{\delta,q}\|_H=O(\varepsilon|\ln\varepsilon|)\|\phi_{\delta,q}\|_H,\end{aligned}$$

and in a similar way, for the supercritical case, in light of (3-18) we get

$$\begin{aligned}&\left|\int_{\partial M}[f_0(W_{\delta,q})-f_\varepsilon(W_{\delta,q})]\phi_{\delta,q}\,d\sigma\right|\\ &\quad\le C|f_0(W_{\delta,q})-f_\varepsilon(W_{\delta,q})|_{L^{2(n-1)/n+O^+(\varepsilon)}(\partial M)}|\phi_{\delta,q}|_{L^{2(n-1)/(n-2)-O^+(\varepsilon)}(\partial M)}\\ &\quad\le\big(\delta^{-O^+(\varepsilon)}(O(\varepsilon\ln\delta)+O(\varepsilon))+O(\delta^2)\big)\|\phi_{\delta,q}\|_H=O(\varepsilon|\ln\varepsilon|)\|\phi_{\delta,q}\|_H.\end{aligned}$$

Finally, by the Taylor expansion formula, for some $\theta\in(0,1)$ we immediately have

$$\begin{aligned}&\left|\int_{\partial M}\frac{(n-2)^2}{2n-2\pm\varepsilon(n-2)}\left[\big(W_{\delta,q}+\phi_{\delta,q}\big)^{\frac{2n-2}{n-2}\pm\varepsilon}-W_{\delta,q}^{\frac{2n-2}{n-2}\pm\varepsilon}\right]-f_\varepsilon(W_{\delta,q})\phi_{\delta,q}\,d\sigma\right|\\ &\qquad=\left|\frac{n\pm\varepsilon(n-2)}{2}\int_{\partial M}\big(W_{\delta,q}+\theta\phi_{\delta,q}\big)^{\frac{2}{n-2}\pm\varepsilon}\phi_{\delta,q}^2\,d\sigma\right|\\ &\qquad\le C\left[\int_{\partial M}\left|W_{\delta,q}+\theta\phi_{\delta,q}\right|^{\left(\frac{2}{n-2}\pm\varepsilon\right)\frac{s_\varepsilon}{s_\varepsilon-2}}d\sigma\right]^{\frac{s_\varepsilon-2}{s_\varepsilon}}\left[\int_{\partial M}|\phi_{\delta,q}|^{s_\varepsilon}\,d\sigma\right]^{\frac{2}{s_\varepsilon}}\\ &\qquad\le C|W_{\delta,q}+\theta\phi_{\delta,q}|_{L^{s_\varepsilon}}^{s_\varepsilon-2}\|\phi_{\delta,q}\|_{\mathcal H}^2\le C\|\phi_{\delta,q}\|_{\mathcal H}^2.\end{aligned}$$

Choosing $\delta=d\varepsilon$, and recalling that, by Proposition 4, $\|\phi_{\delta,q}\|_{\mathcal H}=O(\varepsilon|\ln\varepsilon|)$ concludes the proof.

Step 2. *We prove that*

$$J_\varepsilon(W_{\delta,q})=C(\varepsilon)+\varepsilon\Big(d\frac{n-2}{4}[b(q)-H(q)]\pm\ln d\frac{(n-2)^3(n-3)}{4(n-2)(2n-2)}\Big)\omega_{n-1}I_{n-2}^{n-2}+o(\varepsilon)$$

C^0*-uniformly with respect to d in compact subsets of* $(0,+\infty)$ *and* $q\in\partial M$, *where*

$$\begin{aligned}C(\varepsilon)=\frac12\int_{\mathbb R^n_+}|\nabla U(y)|^2\,dy &-\frac{(n-2)^2}{2n-2}\int_{\mathbb R^{n-1}}U^{\frac{2n-2}{n-2}}(z,0)\,dz\pm\varepsilon\frac{(n-2)^3}{2n-2}\int_{\mathbb R^{n-1}}U^{\frac{2n-2}{n-2}}(z,0)\,dz\\ &\mp\varepsilon\frac{(n-2)^2}{2n-2}\int_{\mathbb R^{n-1}}U^{\frac{2n-2}{n-2}}(z,0)\ln U(z,0)\,dz\\ &\mp\varepsilon|\ln\varepsilon|\frac{(n-2)^3}{2(2n-2)}\int_{\mathbb R^{n-1}}U^{\frac{2n-2}{n-2}}(z,0)\,dz,\end{aligned}$$

and

$$I_{n-2}^{n-2} = \int_0^\infty \frac{s^{n-2}}{(1+s^2)^{n-2}}\,dz,$$

and ω_{n-1} *is the volume of the* $(n-1)$*-dimensional unit ball.*

We compute each term separately. First, we have, by a change of variables and by (4-2), (4-3) and (4-4),

$$\begin{aligned}\int_M |\nabla W_{\delta,q}|^2\,d\mu_g &= \sum_{l,m=1}^{n}\int_{\mathbb{R}^n_+} g^{lm}(\delta y)\frac{\partial}{\partial y_l}U(y)\frac{\partial}{\partial y_m}U(y)\sqrt{g}(\delta y)\,dy + o(\delta)\\ &= \int_{\mathbb{R}^n_+}|\nabla U(y)|^2\,dy - \delta(n-1)H(q)\int_{\mathbb{R}^n_+} y_n|\nabla U(y)|^2\,dy\\ &\qquad + 2\delta\sum_{i,j=1}^{n-1}\int_{\mathbb{R}^n_+} y_n h_{ij}(q)\frac{\partial}{\partial y_i}U(y)\frac{\partial}{\partial y_j}U(y)\,dy + o(\delta).\end{aligned}$$

By a symmetry argument we can simplify the last integral to obtain, in a more compact form,

$$\begin{aligned}\frac{1}{2}\int_M |\nabla W_{\delta,q}|^2\,d\mu_g &= \frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2 - \delta\frac{(n-1)H(q)}{2}\int_{\mathbb{R}^n_+} y_n|\nabla U|^2\\ &\qquad + \delta\sum_{i=1}^{n-1} h_{ii}(q)\int_{\mathbb{R}^n_+} y_n\Big(\frac{\partial U}{\partial y_i}(y)\Big)^2 + o(\delta).\end{aligned}$$

Since $\dfrac{\partial U}{\partial y_i} = \dfrac{\partial U}{\partial y_l}$ for all $i, l = 1, \dots, n-1$, by (4-9) we get

$$\begin{aligned}\sum_{i=1}^{n-1} h_{ii}(q)\int_{\mathbb{R}^n_+} y_n\Big(\frac{\partial U}{\partial y_i}(y)\Big)^2\,dy &= \frac{1}{n-1}\sum_{i=1}^{n-1} h_{ii}(q)\int_{\mathbb{R}^n_+} y_n\sum_{l=1}^{n-1}\Big(\frac{\partial U}{\partial y_l}(y)\Big)^2\,dy\\ &= \frac{H(q)}{4}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz,\end{aligned}$$

and in light of (4-7) we conclude that

$$\frac{1}{2}\int_M |\nabla W_{\delta,q}|^2\,d\mu_g = \frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2 - \delta\frac{(n-2)H(q)}{4}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz + o(\delta).$$

By a change of variables, we immediately obtain

$$\frac{1}{2}\int_M a(x)|W_{\delta,q}|^2\,d\mu_g = \frac{\delta^2}{2}\int_{\mathbb{R}^n_+} a(x)U^2(y)\sqrt{g}(\delta y)\,dy + o(\delta^2) = O(\delta^2).$$

Coming to the boundary integral, we get, by a change of variables, by (4-6), and by expanding b,

$$\begin{aligned}\frac{n-2}{4}\int_{\partial M} b(z)|W_{\delta,q}|^2\,d\sigma &= \delta\,\frac{n-2}{4}\int_{\mathbb{R}^{n-1}} b(\delta z)U^2(z,0)\sqrt{g}(\delta z)\,dz + O(\delta^2)\\ &= \delta b(q)\,\frac{n-2}{4}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz + O(\delta^2).\end{aligned}$$

Introducing the abbreviation $U_n(z) = U^{(2n-2)/(n-2)}(z,0)$, by (3-15), (3-16) and (4-6), we have

$$\begin{aligned}&\int_{\partial M} |W_{\delta,q}|^{(2n-2)/(n-2)\pm\varepsilon}\,d\sigma\\ &\qquad= \int_{\mathbb{R}^{n-1}} \delta^{\mp\varepsilon(n-2)/2}U_n(z)U^{\pm\varepsilon}(z,0)\sqrt{g}(\delta z)\,dz + o(\delta)\\ &\qquad= \int_{\mathbb{R}^{n-1}} U_n(z)\,dz \pm \varepsilon\int_{\mathbb{R}^{n-1}} U_n(z)\ln U(z,0)\,dz \mp \frac{n-2}{2}\,\varepsilon\ln\delta\int_{\mathbb{R}^{n-1}} U_n(z)\,dz\\ &\qquad\qquad + o(\delta) + O(\varepsilon^2) + O(\varepsilon^2\ln\delta),\end{aligned}$$

and, since $\dfrac{(n-2)^2}{2n-2\pm\varepsilon(n-2)} = \dfrac{(n-2)^2}{2n-2} \mp \varepsilon\,\dfrac{(n-2)^3}{2n-2}$, we get

$$\begin{aligned}&-\frac{(n-2)^2}{2n-2\pm\varepsilon(n-2)}\int_{\partial M} |W_{\delta,q}|^{(2n-2)/(n-2)-\varepsilon}\,d\sigma\\ &\qquad= -\frac{(n-2)^2}{2n-2}\int_{\mathbb{R}^{n-1}} U_n(z)\,dz \pm \varepsilon\,\frac{(n-2)^3}{2n-2}\int_{\mathbb{R}^{n-1}} U_n(z)\,dz\\ &\qquad\qquad\mp \varepsilon\,\frac{(n-2)^2}{2n-2}\int_{\mathbb{R}^{n-1}} U_n(z)\ln U(z,0)\,dz \pm \frac{(n-2)^3}{2(2n-2)}\varepsilon\ln\delta\int_{\mathbb{R}^{n-1}} U_n(z)\,dz\\ &\qquad\qquad + o(\delta) + O(\varepsilon^2) + O(\varepsilon^2\ln\delta).\end{aligned}$$

Notice that, with the choice $\delta = d\varepsilon$ it holds that $o(\delta) + O(\varepsilon^2) + O(\varepsilon^2\ln\delta) = o(\varepsilon)$ and $\varepsilon\ln\delta = \varepsilon\ln d - \varepsilon|\ln\varepsilon|$. At this point we have

$$\begin{aligned}J_\varepsilon(W_{\delta,q}) = C(\varepsilon) + \varepsilon d\,\frac{n-2}{4}\,[b(q) - H(q)]\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz&\\ \pm \varepsilon\,\frac{(n-2)^3}{2(2n-2)}\ln d\int_{\mathbb{R}^{n-1}} U_n(z)\,dz + o(\varepsilon|\ln\varepsilon|).&\end{aligned}$$

To conclude, observe that

$$\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz = \omega_{n-1}I_{n-2}^{n-2} \quad\text{and}\quad \int_{\mathbb{R}^{n-1}} U_n(z)\,dz = \omega_{n-1}I_{n-1}^{n-2},$$

where

$$I_\beta^\alpha = \int_0^\infty \frac{s^\alpha}{(1+s^2)^\beta}\,ds.$$

The conclusion follows after we observe that $I_{n-1}^{n-2} = \dfrac{n-3}{2(n-2)} I_{n-2}^{n-2}$ (for a proof, see [Almaraz 2011b, Lemma 9.4(b)]). □

4.2. *Proof of Theorem 1.* Let us introduce

$$\hat{I}(d,q) = \alpha_n d\varphi(q) - \beta_n \ln d.$$

If q_0 is a local minimizer of $\varphi(q)$ with $\varphi(q_0) > 0$, set $d_0 = \beta_n/(\alpha_n\varphi(q_0)) > 0$. Thus the pair (d_0, q_0) is a critical point for $\hat{I}$. Moreover, since there exists a neighborhood B such that $\varphi(q) > \varphi(q_0)$ on ∂B, it is possible to find a neighborhood $\widetilde{B} \subset [a,b] \times \partial M$, $(d_0, q_0) \in \widetilde{B}$ such that $\hat{I}(d,q) > \hat{I}(d_0,q_0)$ for $(d,q) \in \partial\widetilde{B}$. Since, in the subcritical case, by (i) of Proposition 5 we have

$$I_\varepsilon(d,q) = c_n(\varepsilon) + \varepsilon\hat{I}(d,q) + o(\varepsilon),$$

we get that for ε sufficiently small there is a $(d^*, q^*) \in \widetilde{B}$ such that $W_{\varepsilon d^*,q^*} + \phi_{\varepsilon d^*,q^*}$ is a critical point for I_ε. Then, by (i) of Proposition 5, $W_{\varepsilon d^*,q^*} + \phi_{\varepsilon d^*,q^*} \in \mathcal{H}$ is a solution for problem (1-5) in the subcritical case.

The proof for the supercritical case follows in a similar way. □

4.3. *Some technicalities.* *If U is a solution of (2-5), then the following hold:*

$$\int_{\mathbb{R}^n_+} t|\nabla U|^2\,dz\,dt = \frac{1}{2}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz, \tag{4-7}$$

$$\int_{\mathbb{R}^n_+} t|\nabla U|^2\,dz\,dt = 2\int_{\mathbb{R}^n_+} t|\partial_t U|^2\,dz\,dt, \tag{4-8}$$

$$\int_{\mathbb{R}^n_+} t\sum_{i=1}^{n-1}|\partial_{z_i} U|^2\,dz\,dt = \frac{1}{4}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz. \tag{4-9}$$

Proof. To simplify the notation, we set

$$\eta = (z,t) \in \mathbb{R}^n_+ \quad \text{where } z \in \mathbb{R}^{n-1} \text{ and } t \ge 0.$$

The first estimate can be obtained by integration by parts, taking into account that $\Delta U = 0$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}^n_+} \eta_n|\nabla U|^2\,\delta\eta &= -\sum_{l=1}^n \int_{\mathbb{R}^n_+} U\partial_l[\eta_n\partial_l U]\,\delta\eta = -\int_{\mathbb{R}^n_+} U\partial_n U\,\delta\eta - \int_{\mathbb{R}^n_+} \eta_n U\Delta U\,\delta\eta \\ &= -\frac{1}{2}\int_{\mathbb{R}^n_+} \partial_n[U^2]\,\delta\eta = \frac{1}{2}\int_{\mathbb{R}^{n-1}} U^2(z,0)\,dz.\end{aligned}$$

To obtain (4-8), we proceed in a similar way: since $\Delta U = 0$ we have

$$\begin{aligned}0 &= -\int_{\mathbb{R}^n_+} \Delta U \eta_n^2 \partial_n U \,\delta\eta = \sum_{l=1}^{n} \int_{\mathbb{R}^n_+} \partial_l U \partial_l [\eta_n^2 \partial_n U]\,\delta\eta \\ &= \int_{\mathbb{R}^n_+} 2\eta_n |\partial_n U|^2 \,\delta\eta + \sum_{l=1}^{n} \int_{\mathbb{R}^n_+} \eta_n^2 \partial_l U \partial_{ln}^2 U \,\delta\eta \\ &= \int_{\mathbb{R}^n_+} 2\eta_n |\partial_n U|^2 \,\delta\eta + \frac{1}{2}\int_{\mathbb{R}^n_+} \eta_n^2 \partial_n |\nabla U|^2 \,\delta\eta \\ &= \int_{\mathbb{R}^n_+} 2\eta_n |\partial_t U|^2 \,\delta\eta - \int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 \,\delta\eta,\end{aligned}$$

so (4-8) is proved. Now (4-9) is a direct consequence of the first two equalities. In fact, by (4-8) we have

$$\begin{aligned}\int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 \,\delta\eta &= \int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 \,\delta\eta + \int_{\mathbb{R}^n_+} \eta_n |\partial_n U|^2 \,\delta\eta \\ &= \int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 \,\delta\eta + \frac{1}{2}\int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 \,\delta\eta.\end{aligned}$$

Thus,

$$\int_{\mathbb{R}^n_+} \eta_n \sum_{i=1}^{n-1} |\partial_i U|^2 \,\delta\eta = \frac{1}{2}\int_{\mathbb{R}^n_+} \eta_n |\nabla U|^2 \,\delta\eta,$$

and in light of (4-7) we get the proof. □

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