

# Minimal clusters of four planar regions with the same area

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## Abstract

We prove that the optimal way to enclose and separate four planar regions with equal area using the less possible perimeter requires all regions to be connected. Moreover, the topology of such optimal clusters is uniquely determined.

## 1 Introduction

We consider the problem of enclosing and separating  $N$  regions of  $\mathbb{R}^2$  with prescribed area and with the minimal possible interface length.

The case  $N = 1$  corresponds to the celebrated isoperimetric problem whose solution, the circle, was known since antiquity.

For  $N \geq 1$  first existence and partial regularity in  $\mathbb{R}^n$  was given by Almgren [?] while Taylor [?] describes the singularities for minimizers in  $\mathbb{R}^3$ . Existence and regularity of minimizers in  $\mathbb{R}^2$  was proved by Morgan [?] (see also [?]): the regions of a minimizer in  $\mathbb{R}^2$  are delimited by a finite number of circular arcs which meet in triples at their end-points (see Theorem ??).

Foisy et al. [?] proved that for  $N = 2$  in  $\mathbb{R}^2$  the two regions of any minimizer are delimited by three circular arcs joining in two points (standard double bubble) and are uniquely determined by their enclosed areas. Wichiramala [?] proved that for  $N = 3$  in  $\mathbb{R}^2$  the three regions of any minimizer are delimited by six circular arcs joining in four points. Such configuration (standard triple bubble) is uniquely determined by the given enclosed areas, as shown by Montesinos [?]. The case  $N = 4$  has been considered in [?] where some partial information on minimal clusters is obtained.

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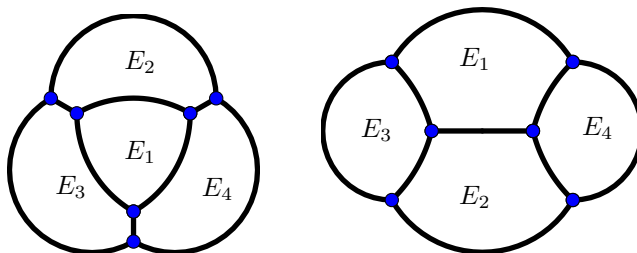


Figure 1: The *flower* (left hand side) and *sandwich* (right hand side) topologies.

The minimization problem can be stated also for  $N = \infty$  regions with equal areas (the *honeycomb conjecture*, see [?]): Hales [?] proved that the hexagonal grid is indeed the solution.

In obtaining the results with  $N = 2$  or  $N = 3$  planar regions, the main difficulty is to prove that each region of the minimizer is connected. In fact, in general, this is an open question (soap bubble conjecture, Conjecture ??, see Morgan and Sullivan [?]).

To investigate such a conjecture, in this paper (which originates from the Ph.D. Thesis [?] of the second author) we consider the case of  $N = 4$  regions in the plane. In Theorem ?? we prove that if the four planar regions have equal areas then the conjecture is true: the minimizing clusters must be connected. However, in this case, connectedness and stationarity is not enough to uniquely determine the topology of minimizers. In fact there are two nontrivial possible topologies: we call them the *flower* and the *sandwich* topologies (see Figure ??). We then exclude the flower topology, to conclude that minimizers have the sandwich type (Theorem ??).

We conjecture that the minimizer with equal areas is symmetric i.e.: the regions  $E_1$  and  $E_3$  are congruent to the regions  $E_2$  and  $E_4$  respectively (see Conjecture ?? for more details).

The problem of dividing the sphere in regions of equal areas has also been considered. See for example [?] where it is proven that the minimizer for four equal areas in the sphere is given by a geodesic tetrahedron.

The plan of the paper is as follows. In Section ?? we set up the notation and collect the known results that we need in the rest of the paper. In Section ?? we present some tools which apply to general planar clusters. In particular notice that Proposition ?? gives an estimate by below on the measure of each connected component of a minimal cluster. This estimate can be used to obtain an upper bound on the total number of connected components of a cluster as in Theorem ??.

In Section ?? we start the analysis of planar clusters with four equal areas. In particular we find a precise estimate on the length of the minimizers (Proposition ??), we prove that possible components of a disconnected region cannot be too small (Proposition ??) and cannot be too big (Proposition ??). This estimates enable us to prove that a minimizer can have at most six connected

components (Proposition ??). In Section ?? we exclude the clusters with six components. In Section ?? we exclude the clusters with five components and obtain the connectedness result Theorem ?. In Section ?? we consider all connected clusters (four components) and exclude the flower topology (Proposition ?, Theorem ?).

## 2 Notation and preliminary results

Let us denote with  $\mathbf{E} = (E_1, \dots, E_N)$  an  $N$ -uple of measurable subsets of  $\mathbb{R}^2$ . We will say that  $\mathbf{E}$  is an  $N$ -cluster if  $m(E_i \cap E_j) = 0$  for all  $i \neq j$  ( $m(\cdot)$  is the Lebesgue measure). The *external region*  $E_0$  is defined as

$$E_0 = \mathbb{R}^2 \setminus \bigcup_{i=1}^N E_i.$$

The sets  $E_0, E_1, \dots, E_N$  will be called the *regions* of the cluster  $\mathbf{E}$ .

We define the *measure* and the *perimeter* of a cluster by:

$$\mathbf{m}(\mathbf{E}) := (m(E_1), \dots, m(E_N)), \quad P(\mathbf{E}) := \frac{1}{2} \sum_{i=0}^N P(E_i)$$

where  $P(E_i)$  is the *perimeter* of the measurable set  $E_i$ . For regular sets  $E_i$  one has  $P(E_i) = \mathcal{H}^1(\partial E_i)$  which is the length of the boundary of  $E_i$ .

Given a measurable set  $E$  we say that  $C$  with  $m(C) > 0$  is a *component* of  $E$  if

$$m(E) = m(C) + m(E \setminus C) \quad \text{and} \quad P(E) = P(C) + P(E \setminus C)$$

(i.e. the decomposition  $E = C \cup (E \setminus C)$  does not add any boundary). We say that  $E$  is *connected* if it has no component  $C$  with  $0 < m(C) < m(E)$  ( $C = E$  is a trivial component). Notice that in our definitions a *component* does not need to be connected: in general a component can be a union of connected components. We say that a cluster  $\mathbf{E}$  is *connected* if each region  $E_i$ , for  $i = 1, \dots, N$ , is connected. We say that a cluster is *disconnected* if it is not connected (i.e. at least one region is not connected).

A component  $C$  of a region  $E_i$  of the cluster  $\mathbf{E}$  (with  $i \neq 0$ ) is said to be *external* if is adjacent to the external region  $E_0$  (formally  $P(C \cup E_0) < P(C) + P(E_0)$ ) otherwise it is said to be *internal*.

Given a vector of positive numbers  $\mathbf{a} \in \mathbb{R}_+^N$ ,  $\mathbf{a} = (a_1, \dots, a_N)$ ,  $a_i > 0$  we will define the family of *competitors* as the clusters with measure  $\mathbf{a}$ :

$$\mathcal{C}(\mathbf{a}) = \{\mathbf{E} : \mathbf{m}(\mathbf{E}) = \mathbf{a}\}$$

among these we will consider the following optimization problem:

$$p(\mathbf{a}) = \inf\{P(\mathbf{E}) : \mathbf{E} \in \mathcal{C}(\mathbf{a})\}$$

and the corresponding minimizers:

$$\mathcal{M}(\mathbf{a}) = \{\mathbf{E} \in \mathcal{C}(\mathbf{a}) : P(\mathbf{E}) = p(\mathbf{a})\}.$$

We will also consider the *weak* variants of this minimization problem:

$$\begin{aligned} \mathcal{C}^*(\mathbf{a}) &= \{\mathbf{E} : \mathbf{m}(\mathbf{E}) \geq \mathbf{a}\} \\ p^*(\mathbf{a}) &= \inf\{P(\mathbf{E}) : \mathbf{E} \in \mathcal{C}^*(\mathbf{a})\} \\ \mathcal{M}^*(\mathbf{a}) &= \{\mathbf{E} \in \mathcal{C}^*(\mathbf{a}) : P(\mathbf{E}) = p^*(\mathbf{a})\}. \end{aligned}$$

(the comparison between vectors of  $\mathbb{R}^N$  is understood componentwise).

**Definition 2.1** (regular cluster). We say that a planar  $N$ -cluster  $\mathbf{E}$  is *regular* when:

1. each region (including the external region  $E_0$ ) is (up to a negligible set) a closed set which is equal to the closure of its interior points (and in the following we will assume that the Lebesgue representant of the regions  $E_i$  is always a closed set);
2. each region, but the external one  $E_0$ , is bounded;
3. the boundary of the cluster, defined by

$$\partial\mathbf{E} = \bigcup_{k=1}^N \partial E_k$$

is the continuous embedding of a finite planar graph (i.e. there are a finite number of simple continuous curves which we will call *edges* which can only meet in their end-points which we will call *vertices* and the *faces* of the graph correspond to the connected components of the regions);

4. each *vertex* has order at least three (i.e. it coincides with at least three end-points of the edges).

Notice that the perimeter of a region  $E_i$  of a regular cluster  $\mathbf{E}$  is the sum of the length of the edges of  $E_i$ . Moreover since each edge belongs to the boundary of exactly two regions, we have

$$P(\mathbf{E}) = \frac{1}{2} \sum_{k=0}^N P(E_k) = \sum_{\sigma \text{ edge of } \mathbf{E}} \ell(\sigma) = \mathcal{H}^1(\partial\mathbf{E}).$$

**Definition 2.2** (stationary cluster). We say that a regular planar cluster  $\mathbf{E} = (E_1, \dots, E_N)$  is *stationary* if it satisfies the following conditions:

1. every edge is either a circular arc or a straight segment (which, in the following, we will identify with an arc of zero curvature);

2. in every *vertex* exactly three arcs meet, defining three equal angles of 120 degrees;
3. it is possible to associate a real number  $p_i$  (which we will call *pressure*) to each region  $E_i$  of the cluster, so that  $p_0 = 0$  and such that any arc between the regions  $E_i$  and  $E_j$  has curvature  $|p_i - p_j|$  (it is a straight segment if  $p_i = p_j$ ) and the region with higher pressure is towards the side where the the arc is convex.

In particular it follows that the sum of the signed curvatures of the three arcs meeting in a vertex is always zero.

**Theorem 2.3** (existence and regularity). *[?, ?] Given  $\mathbf{a} \in \mathbb{R}_+^N$  the family of clusters  $\mathcal{M}(\mathbf{a})$  is not empty and every minimal cluster  $\mathbf{E} \in \mathcal{M}(\mathbf{a})$  is regular and stationary.*

**Theorem 2.4** (existence and regularity, weak case). *[?] Given  $\mathbf{a} \in \mathbb{R}_+^N$  the family of clusters  $\mathcal{M}^*(\mathbf{a})$  is not empty and every minimal cluster  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  is regular and stationary.*

Weak minimizers have some additional properties which makes them a better ambient space for our investigation.

**Proposition 2.5** (properties of weak minimizers). *[?] Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$ ,  $\mathbf{a} \in \mathbb{R}_+^N$ . Then:*

1. the external region  $E_0$  is connected;
2. all the pressures  $p_i$  are nonnegative;
3. if  $m(E_i) > a_i$  then  $p_i = 0$ .

**Theorem 2.6** (pressure formula). *[?] Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  with  $\mathbf{a} \in \mathbb{R}_+^N$ . Then*

$$P(\mathbf{E}) = 2 \sum_{i=1}^N p_i m(E_i).$$

**Lemma 2.7** (turning angle). *[?] Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$ ,  $\mathbf{a} \in \mathbb{R}_+^N$  and let  $C$  be a connected component of some region  $E_i$  of  $\mathbf{E}$ . Let  $n$  be the number of edges of  $C$  and let  $L_j$  be the total length of the edges of  $C$  in common with the region  $E_j$  ( $L_j = 0$  if  $C$  and  $E_j$  have not edges in common). Then, if  $i \neq 0$ , it holds*

$$\frac{(6-n)\pi}{3} = \sum_{j=0}^N (p_i - p_j) L_j$$

where  $p_j$  is the pressure of the region  $E_j$ . For  $i = 0$  we have instead

$$\frac{(6+n)\pi}{3} = \sum_{j=1}^N (p_j - p_0) L_j.$$

**Proposition 2.8.** [?] [?] [?] Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  with  $\mathbf{a} \in \mathbb{R}_+^N$ . Let  $M$  be the total number of bounded connected components of the regions of  $\mathbf{E}$ .

1. Every bounded connected component is simply connected.
2. Two connected components of  $\mathbf{E}$  cannot share more than a single edge.
3. If  $N > 2$  then each connected component  $C$  of  $\mathbf{E}$  has at least three edges.
4. Each connected component of a region with  $k$  connected components has at most  $M + 1 - k$  edges, and if it is internal it has at most  $M - k$  edges.
5. The total number of edges is  $3(M - 1)$  and the total number of vertices is  $2(M - 1)$ .
6. If  $M \leq 6$  then  $\mathbf{E} \in \mathcal{M}(\mathbf{a})$  (i.e.  $\mathbf{E}$  is a strong minimizer).

**Theorem 2.9** (removal of triangle components). [?] Let  $\mathbf{E} \in \mathcal{C}(\mathbf{a})$  be a stationary regular cluster and suppose that a connected component  $C$  of some region  $E_i$  has three edges. Consider the three arcs which arrive at the three vertices of  $C$  but are not edges of  $C$ . The circles containing these three arcs meet in a point  $P$  inside the component  $C$ .

Moreover the cluster  $\mathbf{E}'$  obtained from  $\mathbf{E}$  by removing the component  $C$  and prolonging the three edges, is itself a stationary regular cluster  $\mathbf{E}' \in \mathcal{C}(\mathbf{a}')$  with  $a'_i = a_i - m(C)$  (and the region  $E_i$  disappears if  $C$  was the only component of  $E_i$ ) and  $a'_j \geq a_j$  for all  $j \neq i$ . Also the pressures  $p'_j$  of the regions of  $\mathbf{E}'$  are equal to the pressure  $p_j$  of the corresponding regions of  $\mathbf{E}$  (if  $E_i$  disappears because  $C$  was the only component of  $E_i$ , the regions must be relabeled but again the pressures of the corresponding regions remain the same).

**Theorem 2.10** (double bubble monotonicity). [?] Given  $r_1 > 0$  and  $r_2 > 0$ , up to isometries, there exists a unique double bubble  $\mathbf{E}$  such that the external radii of the two regions  $E_1$  and  $E_2$  are  $r_1$  and  $r_2$  respectively. If we increase one radius (say)  $r_1$  then the area of the corresponding region  $E_1$  increases while the area of the other region  $E_2$  decreases. As a consequence there is a unique double bubble with prescribed areas.

**Conjecture 2.11** (soap bubble conjecture). [?] For all  $\mathbf{a} \in \mathbb{R}_+^N$  each  $\mathbf{E} \in \mathcal{M}(\mathbf{a})$  is connected.

The main aim of this paper is to prove that the conjecture holds in the case  $\mathbf{a} = (1, 1, 1, 1)$ .

### 3 Estimates on general clusters

**Lemma 3.1** (isoperimetric inequality for clusters). Given  $\mathbf{E} \in \mathcal{C}^*(\mathbf{a})$  one has

$$P(\mathbf{E}) \geq \sqrt{\pi} \left( \sqrt{\sum_{k=1}^N a_k} + \sum_{k=1}^N \sqrt{a_k} \right).$$

*Proof.* Given any  $\mathbf{E} \in \mathcal{C}^*(\mathbf{a})$ , by applying the isoperimetric inequality

$$P(\mathbf{E}) \geq 2\sqrt{\pi}\sqrt{\min\{m(\mathbf{E}), m(\mathbf{R}^2 \setminus \mathbf{E})\}}$$

one has:

$$P(\mathbf{E}) = \frac{1}{2} \sum_{k=0}^N P(E_k) \geq \sqrt{\pi} \left( \sqrt{\sum_{k=1}^N m(E_k)} + \sum_{k=1}^N \sqrt{m(E_k)} \right). \quad (1)$$

□

**Proposition 3.2** (variation I). *Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  and suppose that  $C_i$  is a component of the region  $E_i$ . Let  $\ell$  be the sum of the lengths of the edges of  $C_i$  in common with the region  $E_k \neq E_i$  ( $k = 0$  is also admitted). Then*

$$\ell \leq 2\sqrt{\pi}\sqrt{m(C_i)}.$$

*Proof.* Let  $B$  be any ball disjoint from  $\mathbf{E}$  with the same area as  $C_i$ , so that  $P(B) = 2\sqrt{\pi}\sqrt{m(C_i)}$ . Consider the cluster  $\mathbf{E}'$  obtained by  $\mathbf{E}$  by means of the following variations on the regions  $E_i$  and  $E_j$ :

$$E'_i = (E_i \setminus C_i) \cup B, \quad E'_j = E_j \cup C_i.$$

Clearly we have  $m(E'_i) = m(E_i)$  and  $m(E'_j) > m(E_j)$ . Hence  $\mathbf{E}' \in \mathcal{C}^*(\mathbf{a})$ . Moreover, since the edges of length  $\ell$  has been removed and the ball  $B$  has been added, by the minimality of  $\mathbf{E}$  we have:

$$0 \leq P(\mathbf{E}') - P(\mathbf{E}) = P(B) - \ell = 2\sqrt{\pi}\sqrt{m(C_i)} - \ell.$$

□

**Proposition 3.3** (variation II). *Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  and suppose that  $C_i$  is a component of the region  $E_i$  with  $0 < m(C_i) < m(E_i)$ . Let  $\ell_k$  be the sum of the lengths of the edges of  $C_i$  in common with the region  $E_k \neq E_i$  ( $k = 0$  is also admitted). Then*

$$\ell_k \leq \frac{m(C_i)}{|2a_i - m(C_i)|} P(\mathbf{E}). \quad (2)$$

Moreover, if we denote by  $r \leq N$  the number of regions which have an edge in common with  $C_i$ , for all  $\lambda \geq P(\mathbf{E})$  one has:

$$m(C_i) \geq \frac{16\pi a_i^2}{r^2 \lambda^2} \left( 1 - \frac{16\pi a_i}{r^2 \lambda^2} \right). \quad (3)$$

*Proof.* Let

$$t = \sqrt{\frac{m(E_i)}{m(E_i) - m(C_i)}} = \sqrt{1 + \frac{m(C_i)}{m(E_i) - m(C_i)}} \leq 1 + \frac{1}{2} \frac{m(C_i)}{m(E_i) - m(C_i)}$$

and consider a new cluster  $\mathbf{E}'$  whose regions are defined by  $E'_i = t(E_i \setminus C_i)$ ,  $E'_k = t(E_k \cup C_i)$  and  $E'_j = tE_j$  when  $j \notin \{i, k\}$ . Simply speaking, the cluster  $\mathbf{E}'$  has been obtained from  $\mathbf{E}$  by giving  $C_i$  to  $E_k$  and then rescaling of a factor  $t > 1$ .

Notice that  $t$  was defined so that

$$m(E'_i) = t^2(m(E_i) - m(C_i)) = m(E_i)$$

and clearly every other region does not decrease its measure since  $t > 1$ . So  $\mathbf{E}' \in \mathcal{C}^*(\mathbf{a})$  is a weak competitor to  $\mathbf{E}$ . On the other hand since in the cluster  $\mathbf{E}'$  all edges in common between the component  $tC_i$  and the region  $tE_k$  have been removed (and these edges have a total length of  $t\ell_k$ ) we have

$$P(\mathbf{E}') = t(P(\mathbf{E}) - \ell_k).$$

Since  $P(\mathbf{E}) \leq P(\mathbf{E}')$  one obtains:

$$\begin{aligned} P(\mathbf{E}) &\leq t(P(\mathbf{E}) - \ell_k) \leq \left(1 + \frac{m(C_i)}{2(m(E_i) - m(C_i))}\right) (P(\mathbf{E}) - \ell_k) \\ &= P(\mathbf{E}) + \frac{m(C_i)}{2(m(E_i) - m(C_i))} P(\mathbf{E}) - \frac{2m(E_i) - m(C_i)}{2(m(E_i) - m(C_i))} \ell_k \end{aligned}$$

which is equivalent to

$$\ell_k \leq \frac{m(C_i)}{2m(E_i) - m(C_i)} P(\mathbf{E}).$$

Using  $0 \leq a_i \leq m(E_i)$  and  $m(C_i) \leq m(E_i)$  one can easily check that

$$2m(E_i) - m(C_i) \geq |2a_i - m(C_i)|$$

so that (??) is proven.

Now if the component  $C_i$  has edges in common with at least  $r$  other regions, there is  $k$  such that  $\ell_k \geq P(C_i)/r$ . By also applying the isoperimetric inequality  $P(C_i) \geq 2\sqrt{\pi}\sqrt{m(C_i)}$  we obtain:

$$2\sqrt{\pi}\sqrt{m(C_i)} \leq r\ell_k \leq \frac{rm(C_i)}{|2a_i - m(C_i)|} P(\mathbf{E}) \leq \frac{r\lambda m(C_i)}{|2a_i - m(C_i)|}$$

if  $P(\mathbf{E}) \leq \lambda$  as in the statement of the Theorem being proved. Whence, by squaring and then dividing by  $m(C_i)$ , we obtain

$$4\pi \leq \frac{r^2\lambda^2 m(C_i)}{(2a_i - m(C_i))^2} = \frac{r^2\lambda^2 m(C_i)}{4a_i^2 - 4a_i m(C_i) + m^2(C_i)}$$

which is equivalent to the following quadratic inequality in  $m(C_i)$ :

$$m^2(C_i) - \left(4a_i + \frac{r^2\lambda^2}{4\pi}\right) m(C_i) + 4a_i^2 \leq 0.$$



The corresponding equation has two positive solutions, and  $m(C_i)$  is larger than the smaller of the two. So we obtain:

$$\begin{aligned} m(C_i) &\geq 2a_i + \frac{r^2\lambda^2}{8\pi} - \sqrt{\left(2a_i + \frac{r^2\lambda^2}{8\pi}\right)^2 - 4a_i^2} \\ &= 2a_i - \frac{r^2\lambda^2}{8\pi} \left( \sqrt{1 + \frac{32\pi a_i}{r^2\lambda^2}} - 1 \right). \end{aligned} \quad (4)$$

By using the inequality:

$$\sqrt{1+x} \leq 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16}$$

with  $x = \frac{32\pi a_i}{r^2\lambda^2}$ , after some straightforward simplifications, we obtain (??).  $\square$

The following result is not used in the rest of the paper, but might be interesting by itself.

**Theorem 3.4.** *Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  be an  $N$ -cluster with  $N \geq 3$  and suppose that  $C_i$  is a component of the region  $E_i$  with  $0 < m(C_i) < m(E_i)$  and suppose that  $r$  is the number of regions which are adjacent to  $C_i$ .*

Let

$$\|\mathbf{a}\|_{\frac{1}{2}} = \left( \sum_{j=1}^N \sqrt{a_j} \right)^2, \quad \|\mathbf{a}\|_{-1} = \left( \sum_{j=1}^N (a_j)^{-1} \right)^{-1}.$$

Then

$$m(C_i) \geq \frac{20}{9} \frac{a_i^2}{r^2 \|\mathbf{a}\|_{\frac{1}{2}}} \geq \frac{20}{9} \frac{a_i^2}{N^2 \|\mathbf{a}\|_{\frac{1}{2}}}. \quad (5)$$

In particular, the number  $M_i$  of connected components of  $E_i$  has the following bound

$$M_i \leq \frac{9}{20} N^2 \frac{\|\mathbf{a}\|_{\frac{1}{2}}}{a_i}$$

and hence the total number  $M$  of connected components of  $\mathbf{E}$  is bounded by

$$M \leq \frac{9}{20} N^2 \frac{\|\mathbf{a}\|_{\frac{1}{2}}}{\|\mathbf{a}\|_{-1}}.$$

*Proof.* Consider, as a competitor, a cluster  $\mathbf{E}'$  whose regions  $E'_i$  are disjoint balls with area  $a_i$  and let

$$\lambda = P(\mathbf{E}') = 2\sqrt{\pi} \sum_{j=1}^N \sqrt{a_j} = 2\sqrt{\pi} \sqrt{\|\mathbf{a}\|_{\frac{1}{2}}}$$

Since  $\mathbf{E}' \in \mathcal{C}^*(\mathbf{a})$  we have  $P(\mathbf{E}) \leq \lambda$ . Notice that  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  implies that  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a}^*)$  with  $\mathbf{a}^* = \mathbf{m}(\mathbf{E})$ , we can apply Proposition ?? with  $\mathbf{a}^*$  in place

of  $\mathbf{a}$  and with  $\lambda$  defined as above. So (??) holds with this value of  $\lambda$  and  $\mathbf{a}^*$  in place of  $\mathbf{a}$ .

Notice also that  $\lambda = P(\mathbf{E}') \geq P(\mathbf{E}) \geq P(E_i) \geq 2\sqrt{\pi}\sqrt{m(E_i)} = 2\sqrt{\pi}\sqrt{a_i^*}$ . Moreover  $r \geq 3$  since, by Proposition ??, we know that for  $N \geq 3$  every component has at least three edges. Hence we know that

$$1 - \frac{16\pi a^* i}{r^2 \lambda^2} \geq 1 - \frac{16\pi a_i^*}{9 \cdot 4\pi a_i^*} = \frac{5}{9}.$$

So (??) becomes (notice that  $r \leq N$ )

$$m(C_i) \geq \frac{16\pi (a_i^*)^2}{4\pi r^2 \|\mathbf{a}\|_{\frac{1}{2}}} \cdot \frac{5}{9} = \frac{20}{9} \cdot \frac{(a_i^*)^2}{r^2 \|\mathbf{a}\|_{\frac{1}{2}}} \geq \frac{20}{9} \cdot \frac{(a_i^*)^2}{N^2 \|\mathbf{a}\|_{\frac{1}{2}}}$$

and, noting that  $a_i^* = m(E_i) \geq a_i$ , (??) is proved.

Now suppose that  $C_i$  be the component of  $E_i$  with smaller area. Then  $a_i^* = m(E_i) \geq M_i \cdot m(C_i)$  and we have

$$M_i \leq \frac{a_i^*}{m(C_i)} \leq \frac{a_i^*}{\frac{20}{9} \frac{(a_i^*)^2}{N^2 \|\mathbf{a}\|_{\frac{1}{2}}}} = \frac{9}{20} \cdot \frac{N^2 \|\mathbf{a}\|_{\frac{1}{2}}}{a_i^*} \leq \frac{9}{20} \cdot \frac{N^2 \|\mathbf{a}\|_{\frac{1}{2}}}{a_i}$$

and summing up for  $i = 1, \dots, N$  we obtain:

$$M = \sum_{i=1}^N M_i \leq \frac{9}{20} N^2 \|\mathbf{a}\|_{\frac{1}{2}} \sum_{i=1}^N \frac{1}{a_i} = \frac{9}{20} N^2 \frac{\|\mathbf{a}\|_{\frac{1}{2}}}{\|\mathbf{a}\|_{-1}}.$$

□

**Proposition 3.5.** *Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  and let  $C$  be a connected component of some region  $E_i$ . Let  $n$  be the number of edges of  $C$ . Then we have the following estimate on the pressure of the region  $E_i$ :*

$$p_i \geq \frac{(6-n)\pi}{3P(C)} + \left(1 - \frac{\ell}{P(C)}\right) p_{\min} \geq \frac{(6-n)\pi}{3P(C)}$$

where  $\ell$  is the length of the external edge of  $C$  ( $\ell = 0$  if  $C$  is internal) and  $p_{\min}$  is the lowest pressure of the bounded regions which are adjacent to  $C$ .

*Proof.* By Lemma ?? we have

$$\begin{aligned} \frac{(6-n)\pi}{3} &= \sum_j (p_i - p_j) L_j = p_i \sum_j L_j - \sum_{j \neq 0} p_j L_j \\ &\leq p_i \sum_j L_j - p_{\min} \sum_{j \neq 0} L_j = p_i P(C) - p_{\min} (P(C) - \ell) \end{aligned}$$

where the sum in  $j$  is extended to the regions  $E_j$  which are adjacent to  $C$ . The first estimate of the statement follows.

To get the second estimate recall that  $p_{\min} \geq 0$  in view of Proposition ??. □

**Proposition 3.6** (variation III). *Let  $\mathbf{E} \in \mathcal{M}^*(\mathbf{a})$  be a cluster and let  $B$  and  $C$  be two different components of the same bounded region  $E_i$  of  $\mathbf{E}$ . Let  $p_i$  be the pressure of  $E_i$ . Suppose that  $B$  is external and let  $L$  be the length of the external arc of  $B$  and  $n$  be the number of different regions which are adjacent to  $C$ . Then*

$$p_i \geq \frac{P(C)}{n m(C)} - \frac{2}{L} \geq \frac{2\sqrt{\pi}}{n\sqrt{m(C)}} - \frac{2}{L}.$$

*Proof.* Suppose  $i = 1$  and consider all the regions which are adjacent to  $C$ . Suppose that  $E_2$  is the region whose edges in common with  $C$  have largest total length. Let  $\ell$  be such total length in common between  $C$  and  $E_2$ : we have that  $n\ell \geq P(C)$ .

Let  $\gamma$  be the external edge of  $B$  and let  $v$  and  $w$  be its vertices. The arc  $\gamma$  has radius  $R = 1/p_1$ , length  $L$  and spans an angle  $\theta = L/R$ . Given  $h > 0$  we are going to modify  $B$  by increasing the radius  $R$  up to  $R+h$ . Just consider the two radii in  $v$  and  $w$ : extend them of a length  $h$  and join them with a parallel arc of radius  $R+h$ . Let  $D$  be the strip between these two parallel arcs. We have  $m(D) = ((R+h)^2 - R^2)\theta/2 = Lh + Lh^2/(2R) \geq Lh$ . It is easy to see that  $D \subseteq E_0$  (since all the external arcs are convex and meet at angles of 120 degrees). Fix  $h = m(C)/L$  and consider the following variation:

$$E'_1 = (E_1 \setminus C) \cup D, \quad E'_2 = E_2 \cup C.$$

If we let  $\mathbf{E}' = (E'_1, E'_2, E_3, \dots, E_N)$  we notice that  $m(E'_1) \geq m(E_1)$  (since  $m(D) \geq Lh = m(C)$ ) so  $\mathbf{E}' \in \mathcal{C}^*(\mathbf{a})$ . Moreover, in computing the perimeter of  $\mathbf{E}'$  the edges in common between  $C$  and  $E_2$  have been removed so we gain  $\ell$  while the arc of length  $L$  has increased to length  $2h + L(R+h)/R$  and so we have, by the minimality of  $\mathbf{E}$ :

$$0 \leq P(\mathbf{E}') - P(\mathbf{E}) \leq -\ell + 2h + L\frac{R+h}{R} - L = m(C) \left( \frac{1}{R} + \frac{2}{L} \right) - \ell$$

To obtain the statement just remember that  $1/R = p_1$  and remember that  $\ell \geq P(C)/n$ .  $\square$

**Lemma 3.7.** *Let  $\mathbf{E} \in \mathcal{C}(a_1, a_2)$  be a connected stationary cluster (a double bubble) with  $a_1 \geq a_2$ . Then the pressures  $p_1, p_2$  satisfy the following relations*

$$\frac{k_8}{\sqrt{a_1}} \leq p_1 \leq p_2 \leq \frac{k_8}{\sqrt{a_2}}$$

with

$$k_8 := \sqrt{\frac{2\pi}{3} + \frac{\sqrt{3}}{4}}, \quad 1.5897 < k_8 < 1.5898.$$

*Proof.* By Theorem ?? we know that the external radii  $r_1, r_2$  and areas  $a_1, a_2$  of a double bubble are in one-to-one correspondence. Moreover we know that when  $r_1 = r_2$  we have  $a_1 = a_2$  because the resulting double bubble is symmetric

and a direct computation gives  $a_1 = a_2 = k_8^2 r^2$  (with  $r = r_1 = r_2$ ). Hence, by the monotonicity proven in Theorem ??, since we have  $a_1 \geq a_2$  by assumption, we know that  $r_1 \geq r_2$  and hence  $p_1 \leq p_2$  (remember that  $p_i = 1/r_i$ ). Hence monotonicity gives also:

$$a_1 \geq k_8^2 r_1^2, \quad a_2 \leq k_8^2 r_2^2$$

whence

$$p_1 = \frac{1}{r_1} \geq \frac{k_8}{\sqrt{a_1}}, \quad p_2 = \frac{1}{r_2} \leq \frac{k_8}{\sqrt{a_2}}.$$

□

**Lemma 3.8** (reduction to double-bubble). *Let  $\mathbf{E} = (E_1, \dots, E_N)$  be a stationary cluster which is reducible to a double bubble  $(E'_i, E'_j)$  by subsequent removal of triangular components where  $E'_i \supseteq E_i$ ,  $E'_j \supseteq E_j$ ,  $E'_i \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_j)$  and  $E'_j \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_i)$ . Let  $\mathbf{a} = \mathbf{m}(\mathbf{E})$  and  $a = \sum_{k=1}^N a_k$ .*

*Then*

$$\frac{k_8}{\sqrt{\max\{a - a_i, a - a_j\}}} \leq \min\{p_i, p_j\} \leq \max\{p_i, p_j\} \leq \frac{k_8}{\sqrt{\min\{a_i, a_j\}}}.$$

*Proof.* By Theorem ?? we know that the pressures of the double bubble are equal to the corresponding pressures of the cluster  $\mathbf{E}$ . Also notice that, for  $k = i, j$  one has  $m(E'_k) \geq m(E_k) = a_k$  ( $k = i, j$ ), while  $m(E'_i) \leq m(\mathbb{R}^2 \setminus (E_0 \cup E_j)) = a - a_j$  and  $m(E'_j) \leq m(\mathbb{R}^2 \setminus (E_0 \cup E_i)) = a - a_i$  so, by Theorem ?? we obtain the desired result. □

**Lemma 3.9** (perimeter of triple bubble). *One has*

$$p^*(1, 1, 1) = 6\sqrt{\frac{\pi}{2} + \frac{1}{\sqrt{3}}} \geq k_{10} := 8.7939$$

*Proof.* From [?] we know that each  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1) = \mathcal{M}(1, 1, 1)$  is a standard triple bubble where each region  $E_i$  is composed by the union of an half-circle and an isosceles triangle with two angles of 30 degrees. If  $r$  is the radius of the half circles, the area of each region turns out to be

$$1 = \left(\frac{\pi}{2} + \frac{1}{\sqrt{3}}\right) r^2$$

while the perimeter is given by

$$P(\mathbf{E}) = (3\pi + 2\sqrt{3})r = 6\left(\frac{\pi}{2} + \frac{1}{\sqrt{3}}\right)r = 6\sqrt{\frac{\pi}{2} + \frac{1}{\sqrt{3}}}.$$

□

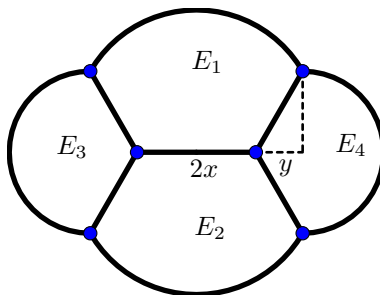


Figure 2: The competitor cluster defined in Proposition ??.

## 4 Estimates on $\mathcal{M}^*(1, 1, 1, 1)$

**Proposition 4.1** (the competitor). *We have  $p^*(1, 1, 1, 1) \leq k_0 := 11.1962$ .*

*Proof.* Let  $x := 0.2707$ ,  $y := 0.394$  and  $R = 2(x + y)/\sqrt{3}$ . Consider the cluster represented in Figure ??. The area of the regions with four edges is given by:

$$m(E_1) = m(E_2) = (2x + y)y\sqrt{3} + \frac{\pi}{3}R^2 - \frac{\sqrt{3}}{4}R^2 > 1$$

while the area of the regions with three edges is:

$$m(E_3) = m(E_4) = \sqrt{3}y^2 + \frac{\pi}{2}(y\sqrt{3})^2 > 1.$$

So  $\mathbf{E} \in \mathcal{C}^*(1, 1, 1, 1)$ . And we have

$$P(\mathbf{E}) = 2\frac{2\pi}{3}R + 2\pi\sqrt{3}y + 2x + 8y \geq k_0.$$

□

**Proposition 4.2.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  and suppose that  $C$  is a component of some region. Then:*

$$m(C) \geq k_2 := 0.0244.$$

*Moreover, if the number of regions which have an edge in common with  $C$  is not larger than 3 one has*

$$m(C) \geq k_6 := 0.0425.$$

*Proof.* We can apply Proposition ?? with  $a_i = 1$ ,  $r \leq 4$ ,  $P(\mathbf{E}) \leq k_0$  so  $rP(\mathbf{E}) \leq \lambda := 4k_0$ . We obtain:

$$m(C) \geq \frac{\pi}{k_0^2} \left( 1 - \frac{\pi}{k_0^2} \right) \geq k_2.$$

And with  $r \leq 3$  we would have

$$m(C) \geq \frac{16\pi}{9k_0^2} \left(1 - \frac{16\pi}{9k_0^2}\right) \geq k_6.$$

□

**Proposition 4.3.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  be such that the region  $E_1$  can be decomposed in two parts  $E_1 = E'_1 \cup C_1$  with*

$$m(E_1) = m(E'_1) + m(C_1), \quad m(E'_1) \geq m(C_1), \quad P(E_1) = P(E'_1) + P(C_1)$$

then

$$m(C_1) \leq k_1 := 0.1605, \quad P(C_1) \leq k_7 := 1.4199$$

*Proof.* Let  $m = m(C_1)$ . By Lemma ??, one has

$$P(\mathbf{E}) \geq \sqrt{\pi} \left( \sqrt{4} + \sqrt{m} + \sqrt{m(E'_1)} + 3\sqrt{1} \right) = \sqrt{\pi} (\sqrt{m} + \sqrt{m(E'_1)} + 5)$$

whence

$$\sqrt{m} + \sqrt{m(E'_1)} \leq \frac{P(\mathbf{E})}{\sqrt{\pi}} - 5 \leq \frac{k_0}{\sqrt{\pi}} - 5 \leq c_1 := 1.3168$$

On one hand we have assumed that  $m(E'_1) \geq m(C_1) = m$ , so  $2\sqrt{m} \leq c_1 < \sqrt{2}$  which gives  $m \leq 1/2$ . On the other hand we know that  $m(E'_1) = m(E_1) - m \geq 1 - m$ , whence

$$\sqrt{m} + \sqrt{1 - m} \leq c_1.$$

Now let  $f(x) = \sqrt{x} + \sqrt{1 - x}$ . By computing the sign of  $f'(x)$  we easily notice that  $f(x)$  is increasing for  $x \in [0, 1/2]$ . By direct computation one checks that  $f(k_1) > c_1$  (in fact  $k_1$ , which is defined in the statement of the theorem being proved, has been chosen to satisfy this relation). Since we know that  $f(m) \leq c_1$  and  $m \leq 1/2$  we conclude that  $m = m(C_1) < k_1$ . To get the estimate on the perimeter, we use again the isoperimetric inequality:

$$\begin{aligned} P(C_1) &= 2P(\mathbf{E}) - (P(E'_1) + P(E_0) + \sum_{i=2}^4 P(E_i)) \\ &\leq 2k_0 - 2\sqrt{\pi}(\sqrt{1 - m(C_1)} + \sqrt{4} + 3\sqrt{1}) \\ &\leq 2k_0 - 2\sqrt{\pi}(\sqrt{1 - k_1} + 5) \leq k_7. \end{aligned}$$

□

**Definition 4.4** (big/small components). Let  $\mathbf{E}$  be a regular  $N$ -cluster. We say that a component  $C$  of a region  $E_i$  is *small* if  $m(C) \leq m(E_i)/2$ . Otherwise we say that  $C$  is *big*. Notice that at most one connected component of each region can be big.

**Corollary 4.5.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$ . Then each region  $E_i$  has exactly one big connected component  $E'_i$ . Furthermore  $m(E'_i) \geq 1 - k_1$ , where  $k_1$  is the constant introduced in Proposition ??.*

*Proof.* Let  $E_i^1, \dots, E_i^M$  be the connected components of the region  $E_i$ . Suppose by contradiction that all  $E_i^j$  are small:  $m(E_i^j) \leq m(E_i)/2$ ,  $j = 1, \dots, M$ . Let  $K$  be the smallest index such that

$$\sum_{j=1}^K m(E_i^j) > k_1. \quad (6)$$

We claim that

$$\sum_{j=1}^K m(E_i^j) < m(E_i) - k_1. \quad (7)$$

Otherwise we would have (notice that  $k_1 < 1/4$ )

$$\begin{aligned} \sum_{j=1}^{K-1} m(E_i^j) &= \sum_{j=1}^K m(E_i^j) - m(E_i^K) \geq m(E_i) - k_1 - m(E_i^K) \\ &\geq m(E_i) - k_1 - \frac{m(E_i)}{2} \geq \frac{m(E_i)}{2} - k_1 \\ &\geq \frac{1}{2} - k_1 > k_1 \end{aligned}$$

which is a contradiction since  $K$  was the minimal index satisfying the inequality (??). So, if we define

$$E'_i = \bigcup_{j=1}^K E_i^j, \quad E''_i = E_i \setminus E'_i$$

we have (by (??) and (??))

$$m(E'_i) > k_1, \quad m(E''_i) > k_1.$$

This is now a contradiction with Proposition ??, since the smaller of the two components  $E'_i, E''_i$  should have a measure smaller than  $k_1$ .

Finally if  $E'_i$  is the big connected component of the region  $E_i$ , applying Proposition ?? with  $C_i = E_i \setminus E'_i$ , we find  $m(E_i) \geq 1 - k_1$ .  $\square$

**Corollary 4.6.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$ . Then at most one of the big components is internal.*

*Proof.* Suppose by contradictions that two big components  $E_i^1$  and  $E_j^1$  are internal. Then by the isoperimetric inequality:

$$\begin{aligned} P(\mathbf{E}) &\geq P(E_i^1 \cup E_j^2) + P(E_0) \\ &\geq 2\sqrt{\pi} \left( \sqrt{m(E_i^1) + m(E_j^2)} + \sqrt{m(E_1) + m(E_2) + m(E_3) + m(E_4)} \right) \\ &\geq 2\sqrt{\pi} \left( \sqrt{2(1 - k_1)} + \sqrt{4} \right) \geq 11.6831 > k_0 \geq p^*(1, 1, 1, 1). \end{aligned}$$

Which is a contradiction.  $\square$

**Proposition 4.7.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  be such that both regions  $E_i$  and  $E_j$  are disconnected ( $i \neq j$ ). Then every small component  $C$  of either  $E_i$  or  $E_j$  satisfies:*

$$m(C) \leq k_3 := 0.0408, \quad P(C) \leq k_9 := 0.7154.$$

*Proof.* Without loss of generality we might suppose that  $i = 1, j = 2$ . Let  $E'_1$  be the larger small component of  $E_1$  and let  $E'_2$  be the larger small component of  $E_2$ . Suppose moreover that  $m := m(E'_1) \geq m(E'_2)$ . Then we have

$$\begin{aligned} m(E_1 \setminus E'_1) &\geq 1 - m, & m(E'_1) &= m, \\ m(E_2 \setminus E'_2) &\geq 1 - m, & m(E'_2) &\geq k_2. \end{aligned}$$

So, from the isoperimetric inequality:

$$\frac{P(\mathbf{E})}{\sqrt{\pi}} \geq \sqrt{m(\mathbb{R}^2 \setminus E_0)} + \sum_{i=1}^2 \sqrt{m(E_i \setminus E'_i)} + \sum_{i=1}^2 \sqrt{m(E'_i)} + \sum_{j=3}^4 \sqrt{m(E_j)}$$

we obtain:

$$\begin{aligned} \frac{P(\mathbf{E})}{\sqrt{\pi}} &\geq \sqrt{4} + \sqrt{1 - m} + \sqrt{m} + \sqrt{1 - m} + \sqrt{k_2} + 2\sqrt{1} \\ &= 4 + 2\sqrt{1 - m} + \sqrt{m} + \sqrt{k_2}. \end{aligned}$$

If we set  $f(x) = 2\sqrt{1 - x} + \sqrt{x}$  and remember that  $P(\mathbf{E}) \leq k_0$  (Proposition ??) we obtain

$$f(m) \leq \frac{k_0}{\sqrt{\pi}} - 4 - \sqrt{k_2} \leq c_2 := 2.1606$$

We have:

$$f'(x) = -(1 - x)^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = -\frac{1}{2}(1 - x)^{-\frac{3}{2}} - \frac{1}{4}x^{-\frac{3}{2}}.$$

By direct computation one checks that  $f'(k_1) > 0.1565 > 0$  and since  $f'' < 0$  we know that  $f$  is strictly increasing on  $[0, k_1]$ . By direct computation one checks  $k_3$  was chosen so that  $f(k_3) > c_2$ . If, by contradiction,  $m > k_3$  since  $m \in [k_2, k_1]$  (by Proposition ?? and Proposition ??) we would have  $f(m) > f(k_3) > c_2$  against (?). So  $m < k_3$ .

Since  $m$  was the measure of the largest small component we obtain the first estimate:  $m(C) \leq m \leq k_3$ .

To prove the estimate on the perimeter  $P(C)$  suppose now that  $C = E'_1$  (not it will not matter if  $E'_1$  is larger or smaller than  $E'_2$ ). Recall that (Proposition ??)

$$m(E'_1) \geq k_2, \quad m(E'_2) \geq k_2$$

and the previous estimate gives:

$$m(E_1 \setminus E'_1) \geq 1 - k_3, \quad m(E_2 \setminus E'_2) \geq 1 - k_3.$$



Hence, using the isoperimetric inequality we have

$$\begin{aligned} 2P(\mathbf{E}) &= P(E'_1) + P(E'_2) + \sum_{i=1}^2 P(E_i \setminus E'_i) + \sum_{i=3}^4 P(E_i) + P(\mathbb{R}^2 \setminus E_0) \\ &\geq P(E'_1) + 2\sqrt{\pi} \left( \sqrt{k_2} + 2\sqrt{1-k_3} + 2\sqrt{1} + \sqrt{4} \right) \end{aligned}$$

whence, recalling also that  $P(\mathbf{E}) \leq k_0$ :

$$P(E'_1) \leq 2k_0 - 2\sqrt{\pi}(\sqrt{k_2} + 2\sqrt{1-k_3} + 4) \leq k_9.$$

□

**Proposition 4.8.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  be such that the region  $E_i$  has at least three components. Then every small component  $C$  of  $E_i$  satisfies:*

$$m(C) \leq k_4 := 0.0411.$$

*Proof.* Without loss of generality we might suppose that  $i = 1$ . Notice that, by Corollary ??, there are at least two small components of  $E_1$ . Let  $E'_1$  be the larger small component of  $E_1$  and  $E''_1$  be another small component of  $E_1$ . Let  $m := m(E'_1) \geq m(E''_1)$ . Then we have

$$m(E_1 \setminus (E'_1 \cup E''_1)) \geq 1 - m - m, \quad m(E'_1) = m, \quad m(E''_1) \geq k_2.$$

So, from the isoperimetric inequality:

$$\begin{aligned} \frac{P(\mathbf{E})}{\sqrt{\pi}} &\geq \sqrt{m(\mathbb{R}^2 \setminus E_0)} + \sqrt{m(E_i \setminus (E'_i \cup E''_i))} \\ &\quad + \sqrt{m(E'_i)} + \sqrt{m(E''_i)} + \sum_{j=2}^4 \sqrt{m(E_j)} \end{aligned}$$

we obtain:

$$\begin{aligned} \frac{P(\mathbf{E})}{\sqrt{\pi}} &\geq \sqrt{4} + \sqrt{1-2m} + \sqrt{m} + \sqrt{k_2} + 3\sqrt{1} \\ &= 5 + \sqrt{1-2m} + \sqrt{m} + \sqrt{k_2}. \end{aligned} \tag{8}$$

If we set  $f(x) = \sqrt{1-2x} + \sqrt{x}$  and remember that  $P(\mathbf{E}) \leq k_0$  (Proposition ??) we obtain

$$f(m) \leq \frac{k_0}{\sqrt{\pi}} - 5 - \sqrt{k_2} \leq c_3 := 1.1606$$

We have:

$$f'(x) = -(1-2x)^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = -(1-2x)^{-\frac{3}{2}} - \frac{1}{4}x^{-\frac{3}{2}}.$$

By direct computation one checks that  $f'(k_1) > 0.0344 > 0$  and since  $f'' < 0$  we know that  $f$  is strictly increasing on  $[0, k_1]$ . By direct computation one checks that  $k_4$  has been chosen so that  $f(k_4) > c_3$ . If, by contradiction,  $m > k_4$  since  $m \in [k_2, k_1]$  (by Proposition ?? and Proposition ??) we would have  $f(m) > f(k_4) > c_3$  against (?). So  $m \leq k_4$ .

□

**Proposition 4.9.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$ . Then the total number of small components is not larger than two.*

*Proof.* Suppose by contradiction that the cluster  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  has at least three small components  $C_1, C_2, C_3$ . Suppose  $m := m(C_1) \geq m(C_2) \geq m(C_3)$ . Let  $C = C_1 \cup C_2 \cup C_3$  and let  $E'_i = E_i \setminus C$  for  $i = 1, \dots, 4$ .

From the isoperimetric inequality:

$$\frac{P(\mathbf{E})}{\sqrt{\pi}} \geq \sqrt{m(\mathbb{R}^2 \setminus E_0)} + \sum_{i=1}^4 \sqrt{m(E'_i)} + \sum_{i=1}^3 \sqrt{m(C_i)}.$$

Now consider the quantity

$$A = \sum_{i=1}^4 \sqrt{m(E'_i)}$$

to get an estimate of  $A$  from below we use the estimates  $k_2 \leq m(C_i) \leq m$  but we have to distinguish three different cases:

1. if the small components all belong to the same region we have  $A \geq \sqrt{1-3m} + 3\sqrt{1}$ ;
2. if only two of the small components belong to the same region:  $A \geq \sqrt{1-2m} + \sqrt{1-m} + 2\sqrt{1}$ ;
3. if the three small components belong to three different regions:  $A \geq 3\sqrt{1-m} + \sqrt{1}$ .

With a straightforward algebraic manipulation one can check that for all  $x \in [0, 1/3]$  one has

$$3\sqrt{1-x} + 1 \geq \sqrt{1-2x} + \sqrt{1-x} + 2 \geq \sqrt{1-3x} + 3$$

so that in every case it holds  $A \geq \sqrt{1-3m} + 3$ .

Hence

$$\begin{aligned} \frac{P(\mathbf{E}')}{\sqrt{\pi}} &\geq \sqrt{4} + \sqrt{1-3m} + 3 + \sqrt{m} + 2\sqrt{k_2} \\ &= \sqrt{1-3m} + \sqrt{m} + 5 + 2\sqrt{k_2} \end{aligned} \tag{9}$$

If we set  $f(x) = \sqrt{1-3x} + \sqrt{x}$  and remember that  $P(\mathbf{E}') = P(\mathbf{E}) \leq k_0$  (Proposition ??) we obtain

$$f(m) \leq \frac{k_0}{\sqrt{\pi}} - 5 - 2\sqrt{k_2} \leq k_5 := 1.0044$$

We have:

$$f'(x) = -\frac{3}{2}(1-3x)^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}, \quad f''(x) = -\frac{9}{4}(1-x)^{-\frac{3}{2}} - \frac{1}{4}x^{-\frac{3}{2}}.$$

By direct computation one checks that  $f(k_1) > 1.1206 > k_5$  and  $f(k_2) > 1.1189 > k_5$ . And since  $f'' < 0$  we know that  $f$  is concave and hence  $f(x) > k_5$  if  $x \in [k_2, k_1]$ . Since  $f(m) \leq k_5$  and we already know that  $m \geq k_2$  (Proposition ??) we conclude that  $m > k_1$ , which is a contradiction.  $\square$

**Corollary 4.10.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$ . Then there are at most six bounded connected components. Four connected components are big and at most two are small (see Definition ??).*

*If the small components are exactly two, they have measure between  $k_2$  and  $k_4$ , they are external, and they have edges in common with all the other regions. If the two small components belong to the same region they both have four edges, while if they belong to different regions they might have four or five edges.*

*If there is only one small component it has measure not larger than  $k_1$ .*

*Proof.* By Proposition ?? there are at most two small components, so the total number of bounded connected components is at most six.

If we have two small components they can either belong to the same region, and then by Proposition ?? each small component has measure not larger than  $k_4$ . Or, the two components belong to different regions and then by Proposition ?? each small component has measure not larger than  $k_3 < k_4$ . Every small component which is adjacent only to three other regions would have measure larger than  $k_6$  by Proposition ?? and since  $k_6 > k_4$  this is impossible. So every small component must have edges in common with all the other four regions, included the external one: so they have at least four edges and are external. If the two components belong to two different regions they can have four or five edges (the two small component might have an edge in common). If the two components belong to the same region, each other region is connected and hence they cannot have more than four edges (each edge is adjacent to a different component).

If there is only one small component we can only apply Proposition ?? to get the estimate with the constant  $k_1$ .  $\square$

## 5 Clusters with six components

In this section we will consider possible minimizers  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  with exactly six bounded components and we will exclude that they exist.

The following Corollary assures that we have  $m(E_i) = 1$  for  $i = 1, \dots, 4$ . This will be used in the following without further notice.

**Corollary 5.1.**  $\mathcal{M}^*(1, 1, 1, 1) = \mathcal{M}(1, 1, 1, 1)$ .

*Proof.* Given any  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  by Corollary ?? we know that  $\mathbf{E}$  has no more than six bounded components. By Proposition ?? we conclude that  $\mathbf{E} \in \mathcal{M}(1, 1, 1, 1)$ , hence  $\mathcal{M}^*(1, 1, 1, 1) \subseteq \mathcal{M}(1, 1, 1, 1)$ . Since  $\mathcal{M}^*(1, 1, 1, 1)$  is not empty (Theorem ??) we obtain  $p^*(1, 1, 1, 1) = P(\mathbf{E}) = p(1, 1, 1, 1)$ .

On the other hand, given  $\mathbf{E}' \in \mathcal{M}(1, 1, 1, 1)$  we have  $\mathbf{E}' \in \mathcal{C}^*(1, 1, 1, 1)$  and since  $P(\mathbf{E}') = p(1, 1, 1, 1) = p^*(1, 1, 1, 1)$  we conclude that  $\mathbf{E}' \in \mathcal{M}^*(1, 1, 1, 1)$ .  $\square$

**Corollary 5.2.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$ . Then we exclude that one region  $E_i$  can have three components.*

*Proof.* Suppose by contradiction that the region  $E_1$  is composed by three components: one big and two small (recall that, by Corollary ??, each region has one big component). By Proposition ?? we know that every component has at least three edges. By Corollary ??, a small component has four edges, so, the two small components have exactly four vertices and the region  $E_1$  has at least  $3 + 4 + 4 = 11$  vertices. But the total number of bounded connected components is  $M = 6$  and by Proposition ?? the number of vertices should be  $v = 2(M - 1) = 10$ . This is a contradiction.  $\square$

**Proposition 5.3.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$ . Then we exclude that two different regions are disconnected.*

*Proof.* By contradiction suppose that  $C_1$  and  $C_2$  are small components of  $E_1$  and  $E_2$  respectively and let  $E'_1 = E_1 \setminus C_1$  and  $E'_2 = E_2 \setminus C_2$  be the two big components.

Recall that, by Corollary ??, the small components  $C_1$  and  $C_2$  have four or five edges.

If the component  $C_i$  ( $i = 1, 2$ ) has five edges, by Proposition ?? and Proposition ??, one finds that

$$p_i \geq \frac{\pi}{3P(C)} \geq \frac{\pi}{3k_9} > 1.4637 > \frac{k_0}{8} \quad (10)$$

On the other hand if  $C_i$  has only four edges, one finds:

$$p_i \geq \frac{2\pi}{3P(C)} \geq \frac{2\pi}{3k_9} > \frac{k_0}{4}.$$

Remember that, by Theorem ?? and Proposition ??, we have

$$p_1 + p_2 + p_3 + p_4 = \frac{P(\mathbf{E})}{2} \leq \frac{k_0}{2}.$$

Without loss of generality we might and shall suppose that  $p_1 \geq p_2$ .

Notice that  $p_1$  and  $p_2$  are both larger than the average and, in particular,  $p_2$  is not the lowest pressure:  $p_2 > \min\{p_3, p_4\}$ . If both regions  $C_1$  and  $C_2$  had four edges, we would find  $p_1 + p_2 > k_0/2$  which is a contradiction. Hence we know that  $C_1$  has four or five edges and  $C_2$  has five edges (if  $C_i$  has four edges  $p_i$  is the higher pressure).

*Step 1:* we claim that at most one component is internal. By Corollary ?? we know that the small components are external and by Corollary ?? we know that at most one big component is internal. The claim follows.

*Step 2:* we claim that  $E'_2$  is external and has three or four edges.

Notice that since at most one component is internal, and we have a total of 6 bounded components, the external region  $E_0$  has either 5 or 6 vertices. On the other hand the big component  $E'_2$  has at least 3 vertices and the small component  $C_2$  has 5 vertices. Two of the vertices of  $C_2$  are in common with the vertices of  $E_0$  and, if  $E'_2$  were internal, all its vertices would be distinct from the vertices of  $E_0$  and, of course, from the vertices of  $C_2$ . So we find at least  $3+3+5=11$  distinct vertices of the cluster  $\mathbf{E}$  while we know (Proposition ??) that  $\mathbf{E}$  has exactly 10 vertices.

The same contradiction holds in the case that  $E'_2$  has more than four vertices since also in this case at least three of them would be internal.

*Step 3:* we claim that  $E'_1$  and  $E'_2$  are adjacent. Let  $\ell_1$  and  $\ell_2$  be the lengths of the external edges of  $E'_1$  and  $E'_2$  respectively ( $\ell_i = 0$  if  $E'_i$  is internal). Suppose by contradiction that  $E'_1$  and  $E'_2$  have no common edge. Then

$$k_0 \geq P(\mathbf{E}) \geq P(E'_1) + P(E'_2) + P(E_0) - (\ell_1 + \ell_2)$$

and by applying the isoperimetric inequality and the estimates  $m(E'_i) \geq 1 - k_3$  we obtain:

$$k_0 \geq 2\sqrt{\pi}(2\sqrt{1-k_3} + \sqrt{4}) - (\ell_1 + \ell_2)$$

whence

$$\frac{\ell_1 + \ell_2}{2} \geq 2\sqrt{\pi}(\sqrt{1-k_3} + 1) - \frac{k_0}{2} > c_4 := 1.4186.$$

If we let  $\ell_i$  be the largest between  $\ell_1$  and  $\ell_2$  we have  $\ell_i > c_4$  and from Proposition ?? we obtain the following estimate on the pressure of the corresponding region  $E_i$  (remember that every component of  $\mathbf{E}$  is adjacent to at most four different regions):

$$p_i \geq \frac{\sqrt{\pi}}{2\sqrt{m(C_i)}} - \frac{2}{\ell_i} \geq \frac{\sqrt{\pi}}{2\sqrt{k_3}} - \frac{2}{c_4} > 2.9776 > \frac{k_0}{4}. \quad (11)$$

Remember that  $p_1 + p_2 + p_3 + p_4 \leq k_0/2$  so  $p_i$  is the highest pressure (actually  $i = 1$  since we decided that  $p_1 \geq p_2$ ). Then let  $n \geq 3$  be the number of edges of  $E'_i$  and let  $L_{i,j}$  be the total length of the edges in common between  $E'_i$  and  $E_j$  (so that  $L_{i,0} = \ell_i$ ):

$$\pi \geq \frac{(6-n)\pi}{3} = \sum_j (p_i - p_j)L_{i,j} \geq p_i \ell_i$$

whence:

$$p_i \leq \frac{\pi}{\ell_i} \leq \frac{\pi}{c_4} < 2.2146$$

which is in contradiction with with (??).

*Step 4:* if a connected region  $E_i$  ( $i = 3, 4$ ) is internal, it is adjacent to both  $E'_1$  and  $E'_2$ .

The proof is the same as in the previous Step. Just take  $E_i$  in place of  $E'_2$  and  $E'_1$  or  $E'_2$  in place of  $E'_1$ . Notice that  $\ell_2 = 0$  so that  $\ell_i = \ell_1$  and the proof completes in exactly the same way (the estimates are actually stronger).

*Step 5:* we claim that if one of  $E_3$  or  $E_4$  is internal and the other one is external with only three edges, then  $E_3$  and  $E_4$  must be adjacent. We proceed in a similar way as the step before. Suppose by contradiction that  $E_3$  is internal and not adjacent to  $E_4$ .

So  $E_3$  is only adjacent to the components of  $E_1$  and  $E_2$  and it has at most four edges, so, by Lemma ??, we have

$$0 < \frac{(6-4)\pi}{3} \leq \sum_{i=1}^2 (p_3 - p_i) L_{3,i}.$$

We deduce that  $p_3 \geq p_2$  since otherwise (being  $p_1 \geq p_2$ ) the right hand side of the previous equation would be negative. So  $p_3 \geq p_2 \geq k_0/8$ .

Now, let  $\ell_i$  be the length of the external edges of  $E'_i$  (recall that only one component can be internal hence  $E'_i$  is external and  $\ell_i > 0$ ). We have

$$k_0 \geq P(\mathbf{E}) \geq P(E'_1 \cup E'_2 \cup E_3) + P(E_0) - (\ell_1 + \ell_2)$$

whence, by applying the isoperimetric inequality,

$$\frac{\ell_1 + \ell_2}{2} \geq \sqrt{\pi}(\sqrt{2(1-k_3)} + 1 + \sqrt{4}) - \frac{k_0}{2} > c_5 := 0.9747.$$

Now if  $\ell_i$  is the maximum between  $\ell_1$  and  $\ell_2$  we know that  $\ell_i > c_5$ . By Proposition ?? (since any component can be adjacent to at most 4 different regions), we have

$$p_i \geq \frac{2\sqrt{\pi}}{4\sqrt{m(C_i)}} - \frac{2}{\ell_i} \geq \frac{\sqrt{\pi}}{2\sqrt{k_3}} - \frac{2}{c_5} > 2.3355 > \frac{3}{16}k_0.$$

So  $p_1 > 3k_0/16$  (since  $p_1$  has been chosen to be the maximum between  $p_1$  and  $p_2$ ).

Now we work on  $E_4$  which is external with  $m = 3$  edges. Remember that  $p_2$  cannot be the lowest pressure and since  $p_1 \geq p_2$  and  $p_3 \geq p_2$  we deduce that  $p_4$  is the lowest pressure. Hence, by Lemma ??

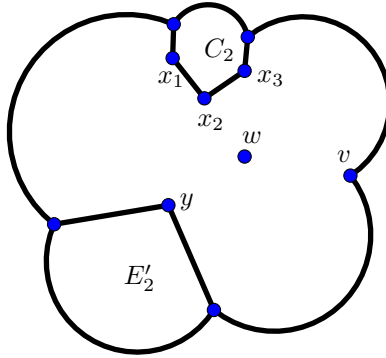
$$\pi = \frac{(6-m)\pi}{3} = \sum_j (p_4 - p_j) L_{4,j} \leq p_4 L_{4,0}$$

and by Proposition ??

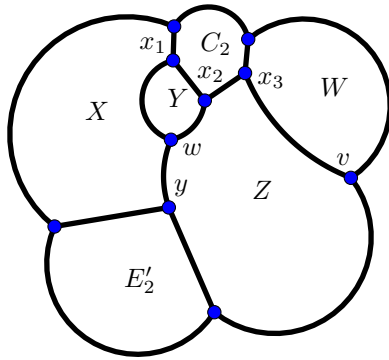
$$p_4 \geq \frac{\pi}{L_{4,0}} \geq \frac{\pi}{2\sqrt{\pi}\sqrt{m(E_4)}} = \frac{\sqrt{\pi}}{2} > 0.8862 > \frac{k_0}{16}$$

So, we have found that

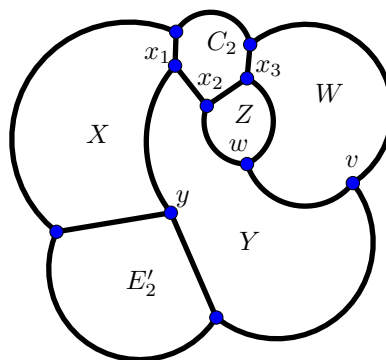
$$P(\mathbf{E}) = 2(p_1 + p_2 + p_3 + p_4) > 2 \left( \frac{3k_0}{16} + \frac{k_0}{8} + \frac{k_0}{8} + \frac{k_0}{16} \right) = k_0$$



(a) Case (i), incomplete diagram



(b)



(c)

Figure 3: Diagrams used in the proof of Proposition ??, Step 6, case (i).

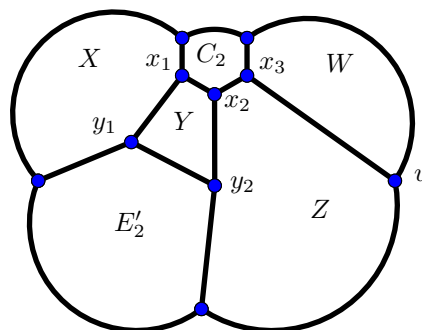
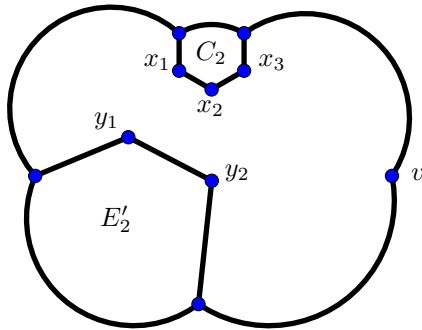


Figure 4: Diagram used in the proof of Proposition ??, Step 6, case (ii).

which contradicts the minimality of  $\mathbf{E}$ . The claim is proved.

*Step 6:* we claim that  $E_0$  has not five edges. Suppose by contradiction that  $E_0$  has exactly five edges and consider two possible cases:  $E'_2$  has either (i) three or (ii) four edges.

If  $E'_2$  has three edges the region  $E_2 = C_2 \cup E'_2$  has 8 distinct vertices (since  $C_2$  has five vertices). Three vertices of  $C_2$  (let us call them  $x_1, x_2$  and  $x_3$ ) are not vertices of  $E_0$ , and one vertex of  $E'_2$  (let us call it  $y$ ) is not a vertex of  $E_0$ . On the other hand  $E_0$  has five vertices, and four of them are shared by  $C_2$  and  $E'_2$ . We denote by  $v$  the remaining vertex. Up to now we have considered 9 vertices in total, since the cluster  $\mathbf{E}$  has exactly 10 vertices, there is an additional vertex  $w$  belonging to neither  $E_0$  nor  $E_2$ . The situation is depicted in Figure ???. We see that 11 edges have been already identified, so 4 edges are missing.

Consider the three edges which meet in the vertex  $w$ . At least two of them should connect  $w$  to the vertices  $x_k$  of  $C_2$ . In fact if only one edge connects  $w$  to  $C_2$  the other two edges of  $w$  should go to  $v$  and  $y$  and hence the two remaining vertices of  $C_2$  should be joined together which is not admitted (we would obtain a two sided component). Not all three edges of  $w$  can join the three free vertices of  $C_2$  because otherwise we would obtain two three-sided internal components. But we know that at most one component can be internal. So, exactly two edges join  $w$  with two vertices of  $C_2$ . The two vertices of  $C_2$  must be consecutive, otherwise the third vertex  $x_2$  could not be connected to anything (the edge would be closed in the loop:  $w, x_3, x_2, x_1$ ). We have two possibilities: the two vertices are either  $x_1$  and  $x_2$  or  $x_2$  and  $x_3$  (the order of the vertices is given by the Figure, where  $x_1$  is "closer" to the component  $E'_2$ ).

In the first case ( $x_1$  and  $x_2$  are joined to  $w$ ) the third edge in  $w$  cannot go to  $x_3$  (already excluded) and cannot go to  $v$  because otherwise the edge from  $x_3$  to  $y$  would cross the already defined edges. So the diagram is completed by an edge joining  $w$  with  $y$  and an edge joining  $v$  with  $x_3$ . The resulting diagram is depicted in Figure ???. We know that  $C_1$  is external and has four or five edges: the only possibility is  $X = C_1$ . Then  $E'_1$  must be adjacent to  $E'_2$  so it must be  $Z = E'_1$ : however  $E'_1$  cannot be adjacent to  $C_1$  and we get a contradiction.

In the second case ( $x_2$  and  $x_3$  are joined to  $w$ ) we can complete the diagram in a unique way, by adding an edge from  $w$  to  $v$  and an edge from  $y$  to  $x_1$  as represented in Figure ???. In this case we have  $X = E'_1$  since  $E'_1$  must be adjacent to  $E'_2$  but cannot have six edges. So  $W = C_1$  because  $C_1$  is external and not adjacent to  $E'_1$ . So  $Y$  and  $Z$  are the two connected regions  $E_3$  and  $E_4$ . However in *Step 4* we proved that the connected region, if internal, must be adjacent to both  $E'_1$  and  $E'_2$  which is not the case for the component  $Z$ . So this configuration must be excluded, too.

So, the case when  $E'_2$  has only three edges has been completed and excluded. Suppose now (ii) that  $E'_2$  has four edges. In this case no additional vertex must be added, and we are in the situation depicted in Figure ???. Let  $x_1, x_2$  and  $x_3$  be the free vertices of  $C_2$  and  $v$  be the free vertex of  $E_0$ , as before. Let  $y_1$  and  $y_2$  be the two free vertices of  $E'_2$ . There are three edges missing in the diagram and there is only one possibility (since the edges from  $C_2$  cannot go back to  $C_2$  and they cannot cross each other):  $x_1$  is joined to  $y_1$ ,  $x_2$  to  $y_2$  and  $x_3$  to  $v$ . The



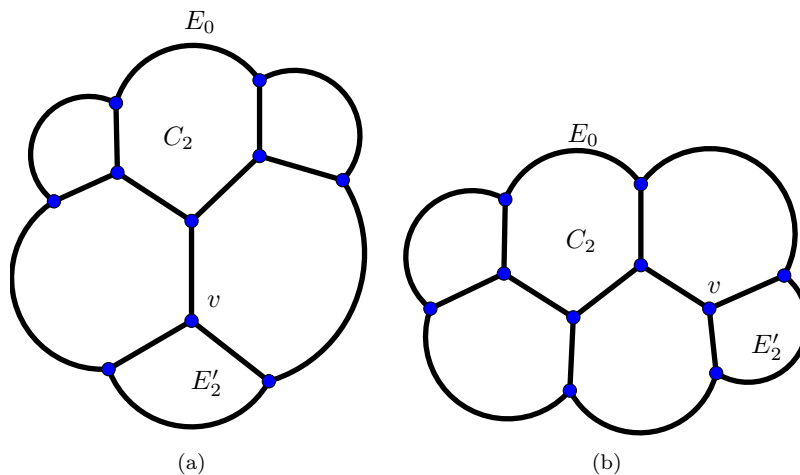


Figure 5: Diagrams used in the proof of Proposition ?? Step 7.

component  $C_1$  is external with four or five edges, hence  $C_1$  is either  $X$  or  $Z$ . The component  $E'_1$  is adjacent to  $E'_2$  but cannot be adjacent to  $C_1$  hence  $E'_1$  is either  $X$  or  $Z$ . So  $Y$  and  $W$  are the two connected regions  $E_3$  and  $E_4$ : say  $Y = E_3$  and  $W = E_4$ .

But now we notice that  $E_3$  is internal and  $E_4$  is external with only three edges, hence by Step 5 they should be adjacent, which is not the case.

*Step 7: conclusion.* We know now that  $E_0$  has six edges. Recall that  $C_2$  is external and has five vertices, two of which are shared with  $E_0$  while  $E'_2$  has at least three vertices (all distinct from  $C_2$ ) two of which are shared with the vertices of  $E_0$ . So we have identified 6 vertices of  $E_0$  and at least  $3 + 1 = 4$  internal vertices of  $E_2 = C_2 \cup E'_2$ . We know that the cluster has 10 vertices in total, so we have identified all of them. In particular we conclude that  $E'_2$  has three vertices. Let  $x_1, x_2$  and  $x_3$  be the three internal vertices of  $C_2$  and let  $v$  be the internal vertex of  $E'_2$ .

If we look at the edges, we have already identified the six edges of  $E_0$ , other four are the internal edges of  $C_2$  and other two are the internal edges of  $E'_2$ . To reach the total of 15 edges, we need to place other three edges. No edge can join two points of  $C_2$  (otherwise a two sided component would rise). So the three missing edges start from the three internal points of  $C_2$ . One of them goes to the internal vertex of  $E'_2$  and the other two go to the two free vertices of  $E_0$ .

There are now two possibilities: either (a) the vertex  $v$  is connected to the middle of the three internal vertices of  $C_2$  or (b) it is connected to one lateral vertex (see Figure ??)

We can easily exclude case (a) because the component  $C_1$  must be one of the two five-sided components ( $C_1$  has either four or five edges and there are no components with four edges) while  $E'_1$  must be adjacent to  $E'_2$  and hence must be the other component with five edges. But this is a contradiction since  $C_1$

cannot be adjacent to  $E'_1$ .

So we remain with the configuration of case (b). The region with three edges adjacent to  $C_2$  is not  $C_1$  (because  $C_1$  has four or five edges) and it cannot be  $E'_1$  because  $E'_1$  must be adjacent to  $E'_2$ . Hence we conclude that it is one of  $E_3$  and  $E_4$ . Let us say it is  $E_3$ . Then  $E_4$  must be the region with five edges, because otherwise  $C_1$  and  $E'_1$  would be adjacent to each other. So  $C_1$  has four edges and hence  $p_1 \geq k_0/4$  is the region with higher pressure and  $p_2 \geq k_0/8$  is the second higher pressure while  $p_3 + p_4 \leq k_0/8$ .

We know that  $E_3$  has three edges,  $E_4$  has five edges and both  $E_3$  and  $E_4$  are external. Let  $L_{j,k}$  be the total length of the edges between  $E_j$  and  $E_k$ . Applying Proposition ?? we obtain, for  $j = 3, 4$ :

$$L_{j,0} \leq 2\sqrt{\pi}\sqrt{m(E_j)} = 2\sqrt{\pi} \quad (12)$$

Since  $p_1$  and  $p_2$  are the largest pressures and  $E_3$  is not adjacent to  $E_4$  we have, for  $j = 3, 4$

$$L_{j,0} p_j \geq \sum_{k=1}^4 L_{j,k}(p_j - p_k)$$

hence, by Lemma ??

$$L_{3,0} p_3 \geq \pi, \quad L_{4,0} p_4 \geq \frac{\pi}{3} \quad (13)$$

and putting together with (??) we obtain

$$p_3 \geq \frac{\pi}{L_{3,0}} \geq \frac{\sqrt{\pi}}{2}, \quad p_4 \geq \frac{\pi}{3L_{4,0}} \geq \frac{\sqrt{\pi}}{6}.$$

Now we are going to improve the estimates on  $p_1$  and  $p_2$ . First notice that if we denote by  $\ell_i$  the length of the external edge of  $C_i$  we have, by Proposition ?? (notice that  $m(C_i) < k_3 < 1$ ),

$$\ell_i \leq \frac{m(C_i)}{|2 - m(C_i)|} P(\mathbf{E}) \leq \frac{m(C_i)}{2 - k_3} k_0$$

while, by the isoperimetric inequality, we have

$$P(C_i) \geq 2\sqrt{\pi}\sqrt{m(C_i)}.$$

Now, applying Proposition ?? to the component  $C_i$  with  $i = 1, 2$ , which has

$n_i = i + 3$  edges, we have

$$\begin{aligned}
p_i &\geq \frac{(6 - n_i)\pi}{3P(C_i)} + p_{\min} \left( 1 - \frac{\ell_i}{P(C_i)} \right) \\
&\geq \frac{(3 - i)\pi}{3k_9} + \frac{\sqrt{\pi}}{6} \left( 1 - \frac{m(C_i)k_0}{(2 - k_3)2\sqrt{\pi}\sqrt{m(C_i)}} \right) \\
&= \frac{(3 - i)\pi}{3k_9} + \frac{\sqrt{\pi}}{6} \left( 1 - \frac{\sqrt{m(C_i)}k_0}{2\sqrt{\pi}(2 - k_3)} \right) \\
&\geq \frac{(3 - i)\pi}{3k_9} + \frac{\sqrt{\pi}}{6} \left( 1 - \frac{\sqrt{k_3}k_0}{2\sqrt{\pi}(2 - k_3)} \right) \geq \frac{(3 - i)\pi}{3k_9} + c_7
\end{aligned}$$

with  $c_7 := 0.1992$ . By using (??)

$$\begin{aligned}
P(\mathbf{E}) &= 2(p_1 + p_2 + p_3 + p_4) \geq 2 \left( \frac{2\pi}{3k_9} + c_7 + \frac{\pi}{3k_9} + c_7 + \frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{6} \right) \\
&= \frac{2\pi}{k_9} + 4c_7 + \frac{4}{3}\sqrt{\pi} \geq 11.9428 > k_0
\end{aligned}$$

which is a contradiction. □

## 6 Clusters with five components

In this section we consider a weak minimizer  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  with five bounded components. Only one region is disconnected: we will assume the region is  $E_1$  and we denote with  $E'_1$  and  $C_1$  respectively, its big and small connected components.

**Proposition 6.1.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  be a cluster with 5 connected components. Then, up to a relabeling of the components, the topology of  $\mathbf{E}$  is one of the cases represented in Figure ??.*

*Proof.* Suppose that  $E_1$  is the only disconnected region and let  $E'_1$  and  $C_1$  respectively be the big and small connected components of  $E_1$ . By Proposition ?? we know that  $\partial\mathbf{E}$  is composed by 12 edges and 8 vertices moreover both  $E'_1$  and  $C_1$  may have at most 4 edges if they are external and 3 edges if they are internal.

*Step 1.* Suppose that both  $E'_1$  and  $C_1$  have four edges (and hence they are external). All the 8 vertices of the cluster are vertices of either  $E'_1$  or  $C'_1$  and both  $E'_1$  and  $C_1$  have an external edge with two external vertices. The external region  $E_0$  has four edges.

The remaining two internal vertices of  $E'_1$  must be connected with the two internal vertices of  $C_1$  (remember that we cannot have two edges with the same end points, because two-sided components are not allowed). Hence the cluster is of type (A) in Figure ??.

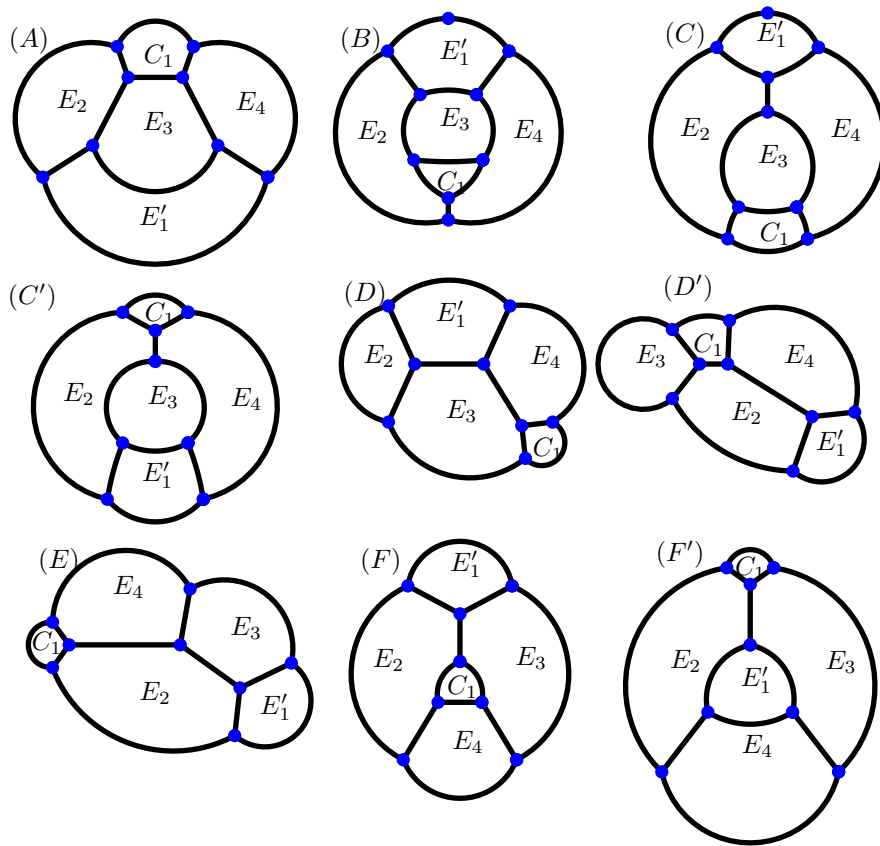


Figure 6: Classification of clusters with five components, Proposition ??.

*Step 2.* Suppose that  $E'_1$  has 4 edges (hence it is external) and suppose  $C_1$  is external with 3 edges. In this case we need to add an additional vertex  $v$ .

If  $v$  is external then the external region  $E_0$  has five edges. The vertex  $v$  must be connected to an internal vertex of  $E'_1$  while the other internal vertex of  $E'_1$  must be connected to the internal vertex of  $C_1$ . The resulting topology is (D).

If, instead, the additional vertex  $v$  is internal, it must be connected to the two internal vertices of  $E'_1$  and to the internal vertex of  $C_1$ . Hence we are in case (C').

*Step 3.* Suppose  $E'_1$  has 4 edges (hence it is external) and suppose  $C_1$  is internal with 3 edges. Since the external region must have at least three edges, there is an additional external vertex  $v$  and  $E_0$  has three edges. One of the three vertices of  $C_1$  must be connected to the vertex  $v$  while the other two vertices of  $C_1$  must be connected to the two internal vertices of  $E'_1$ . The resulting topology is (B).

*Step 4.* Suppose  $E'_1$  has 3 edges and is external while  $C_1$  has four edges (and hence is external). We repeat the same reasoning of Step 2 with  $E'_1$  and  $C_1$  exchanged and we obtain cases (D') and (C).

*Step 5.* Suppose  $E'_1$  has 3 edges and is internal while  $C_1$  has four edges (and hence is external). We repeat the same reasoning of Step 3 and obtain case (B) with  $E'_1$  and  $C_1$  exchanged. But in this case we would have two big internal components:  $E'_1$  and  $E_3$  and this is impossible in view of Corollary ??.

*Step 6.* Suppose that both  $E'_1$  and  $C_1$  have three edges and are external. There are two additional vertices  $v, w$  which are not vertices of  $E'_1$  or  $C_1$ . Since the external region  $E_0$  has at most 5 edges (there are only 5 bounded components) one of the two vertices, say  $v$ , is internal. The other vertex  $w$  cannot be internal, because otherwise  $v$  and  $w$  need to be joined by two different edges, which is not possible. The internal vertex  $v$  must be connected to  $w$  and to the two internal vertices of  $E'_1$  and  $C_1$ . The resulting topology is (E).

*Step 7.* Suppose that both  $E'_1$  and  $C_1$  have three edges and suppose  $E'_1$  is external and  $C_1$  is internal. We need to place two additional vertices  $v$  and  $w$ . Certainly one among  $v$  and  $w$  is external, since  $E_0$  has at least three edges. In case both  $v$  and  $w$  are external  $E_0$  has four edges.

If two of the three vertices of  $C_1$  are connected to the same vertex, we would obtain an additional three sided component (say it is  $E_2$ ). Hence we notice we have three components with three edges:  $E'_1$ ,  $C_1$  and  $E_2$ . Let  $n_0$ ,  $n_3$  and  $n_4$  be the number of edges of  $E_0$ ,  $E_3$  and  $E_4$ . By Euler's formula we have  $24 = 3 \times 3 + n_0 + n_3 + n_4 \leq 9 + 4 + n_3 + n_4$ , which means that  $\max\{n_3, n_4\} \geq \frac{11}{2}$ , i.e  $\max\{n_3, n_4\} \geq 6$  (notice that  $n_3$  and  $n_4$  are integers), which is impossible by Proposition ?? (each component can only have one edge in common with each other component).

So the three vertices of  $C_1$  are connected to  $v$ ,  $w$  and to the internal vertex of  $E'_1$ . Necessarily  $v$  and  $w$  are also connected to the external vertices of  $E'_1$  hence they are both external and  $E_0$  has 4 edges. The resulting cluster is of type (F).

*Step 8.* Suppose that both  $E'_1$  and  $C_1$  have three edges and suppose that  $E'_1$  is internal and  $C_1$  is external. We obtain the same classification of Step 7 but

with  $E'_1$  and  $C_1$  exchanged. We obtain case (F').

*Step 9.* Suppose that both  $E'_1$  and  $C'_1$  have three edges and are both internal. This is impossible because the external region would only have two edges, which is excluded.  $\square$

**Proposition 6.2.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$ . Then  $\mathbf{E}$  cannot have the topologies (B), (C), (C'), (D), (D'), (E), (F) of Figure ??.*

*Proof.* Notice that in each case it is possible (by subsequently removing triangular components) to reduce the cluster  $\mathbf{E}$  to a double bubble  $(E''_1, E''_2)$  where  $E''_1 \supseteq E'_1$  and  $E''_2 \supseteq E_2$ .

So, by applying Lemma ?? we obtain at once

$$p_1 \leq \frac{k_8}{\sqrt{\min\{m(E'_1), m(E_2)\}}} = \frac{k_8}{\sqrt{1 - m(C_1)}} \leq \frac{k_8}{\sqrt{1 - k_1}} \leq 1.7352. \quad (14)$$

In the case when  $C_1$  has only three edges (i.e. cases (B), (C'), (D), (E), and (F)) we can apply Proposition ?? and then Proposition ?? to obtain

$$p_1 \geq \frac{(6-3)\pi}{3P(C_1)} \geq \frac{\pi}{k_7} \geq 2.2125$$

and this is in contradiction with (??).

In both cases (C) and (D') we can reduce the triangular components to find a double bubble  $(E''_2, E''_4)$  with  $E''_2 \supseteq E_2$  and  $E''_4 \supseteq E_4$ . Moreover  $E''_2 \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_4)$  and  $E''_4 \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_2)$  so that  $m(E''_2) \leq 3$  and  $m(E''_4) \leq 3$ . So, by using Lemma ?? we obtain

$$\min\{p_2, p_4\} \geq \frac{k_8}{\sqrt{\min\{4 - m(E_4), 4 - m(E_2)\}}} = \frac{k_8}{\sqrt{3}}.$$

In case (D') we can find another reduction to a double bubble  $(E''_2, E''_3)$  and, as before, we find

$$\min\{p_2, p_3\} \geq \frac{k_8}{\sqrt{3}}$$

so that, in this case,  $\min\{p_2, p_3, p_4\} \geq k_8/\sqrt{3}$ .

In case (C) we apply Proposition ?? to the component  $C_1$  to obtain:

$$p_1 \geq \frac{(6-4)\pi}{3P(C_1)} \geq \frac{2\pi}{3k_7} \geq 1.4750 > 0.9179 \geq \frac{k_8}{\sqrt{3}}$$

and then we apply the same Proposition ?? to  $E_3$  to obtain (notice that we consider  $\ell = 0$  since  $E_3$  is internal):

$$p_3 \geq \frac{(6-3)\pi}{3P(E_3)} + \min\{p_1, p_2, p_4\} \geq \min\{p_1, p_2, p_4\} \geq \frac{k_8}{\sqrt{3}}.$$

So, in both cases C and D', we obtain

$$\min\{p_2, p_3, p_4\} \geq \frac{k_8}{\sqrt{3}}.$$

Now we need to estimate the length  $\ell$  of the external edge of  $C_1$ . By Proposition ?? we have (notice that  $m(C_1) < k_1 < 1$ ),

$$\ell \leq \frac{m(C_1) \cdot P(\mathbf{E})}{|2 - m(C_1)|} \leq \frac{m(C_1)k_0}{2 - k_1}$$

while, by Proposition ??, we have

$$P(C_1) \leq k_7.$$

By applying Proposition ??, and using the previous estimates, we get

$$\begin{aligned} p_1 &\geq \frac{(6-4)\pi}{3P(C_1)} + \min\{p_2, p_3, p_4\} \left(1 - \ell \cdot \frac{1}{P(C_1)}\right) \\ &\geq \frac{2\pi}{3k_7} + \frac{k_8}{\sqrt{3}} \left(1 - \frac{m(C_1)k_0}{2 - k_1} \cdot \frac{1}{2\sqrt{\pi}\sqrt{m(C_1)}}\right) \\ &\geq \frac{2\pi}{3k_7} + \frac{k_8}{\sqrt{3}} \left(1 - \frac{\sqrt{k_1}k_0}{2\sqrt{\pi}(2 - k_1)}\right) \geq 1.7615 \end{aligned}$$

which, again, is in contradiction with (??).  $\square$

**Proposition 6.3.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  has 5 components. Then we exclude that  $\mathbf{E}$  has the topology ( $F'$ ) of Figure ??.*

*Proof.* By removing the triangular components we are able to reduce the cluster  $\mathbf{E}$  to a double bubble  $(E''_2, E''_3)$  with  $E''_2 \supseteq E_2$  and  $E''_3 \supseteq E_3$ . Notice that  $E''_2 \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_3)$  and  $E''_3 \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_2)$  so that  $m(E''_2) \leq 3$  and  $m(E''_3) \leq 3$ . So, by Lemma ??, we obtain

$$\min\{p_2, p_3\} \geq \frac{k_8}{\sqrt{3}}.$$

We repeat the same argument with  $E_4$  in place of  $E_3$  to obtain  $\min\{p_2, p_4\} \geq \frac{k_8}{\sqrt{3}}$  so that

$$\min\{p_2, p_3, p_4\} \geq \frac{k_8}{\sqrt{3}}.$$

Now we estimate the length  $\ell$  of the external edge of  $C_1$  by using Proposition ??:

$$\ell \leq \frac{m(C_1)}{|2 - m(C_1)|} \cdot P(\mathbf{E})$$

i.e. (notice that  $m(C_1) < k_1 < 1$ )

$$\frac{\ell}{P(C_1)} \leq \frac{\ell}{2\sqrt{\pi}\sqrt{m(C_1)}} \leq \frac{\sqrt{m(C_1)}P(\mathbf{E})}{2\sqrt{\pi}(2 - m(C_1))} \leq \frac{\sqrt{k_1}k_0}{2\sqrt{\pi}(2 - k_1)}$$

and we apply Proposition ?? to obtain

$$\begin{aligned} p_1 &\geq \frac{(6-3)\pi}{3P(C_1)} + \min\{p_2, p_3\} \left(1 - \frac{\ell}{P(C_1)}\right) \\ &\geq \frac{\pi}{k_7} + \frac{k_8}{\sqrt{3}} \left(1 - \frac{\sqrt{k_1}k_0}{(2-k_1)2\sqrt{\pi}}\right) \geq c_8 := 2.4990. \end{aligned}$$

By Lemma ?? applied to the component  $E'_1$  we have

$$\begin{aligned} \pi &= \frac{(6-3)\pi}{3} = \sum_{j=0}^4 (p_1 - p_j)L_j \geq (p_1 - \max\{p_0, p_2, p_3, p_4\})P(E'_1) \\ &= (p_1 - \max\{p_2, p_3, p_4\})2\sqrt{\pi}\sqrt{1-k_1} \end{aligned}$$

so that

$$\max\{p_2, p_3, p_4\} \geq p_1 - \frac{\sqrt{\pi}}{2\sqrt{1-k_1}}$$

Hence

$$\begin{aligned} P(\mathbf{E}) &= 2(p_1 + p_2 + p_3 + p_4) \geq 2(p_1 + \max\{p_2, p_3, p_4\} + 2\min\{p_2, p_3, p_4\}) \\ &\geq 4c_8 - 2 \cdot \frac{\sqrt{\pi}}{2\sqrt{1-k_1}} + 4 \cdot \frac{k_8}{\sqrt{3}} \geq 11.5561 \geq k_0 \end{aligned}$$

which is a contradiction.  $\square$

**Proposition 6.4.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  be a cluster with 5 components. Then we exclude that  $\mathbf{E}$  has the topology (A) depicted in Figure ??.*

*Proof.* First of all notice that

$$\begin{aligned} 2k_0 &\geq 2P(\mathbf{E}) = P(E'_1) + P(C_1) + P(E_2) + P(E_4) + P(E_0) + P(E_3) \\ &\geq 2\sqrt{\pi} \left( \sqrt{1-k_1} + \sqrt{k_2} + 2\sqrt{1} + \sqrt{4} \right) + P(E_3) \end{aligned}$$

so that

$$P(E_3) \leq 2k_0 - 2\sqrt{\pi} \left( \sqrt{1-k_1} + \sqrt{k_2} + 4 \right) \leq c_9 := 4.4111.$$

Now let  $\ell_j$  be the total length of the external edges of the region  $E_j$  ( $j = 1, 2, 4$ ). If we remove  $E_1$  from  $\mathbf{E}$  we obtain a 3-cluster  $\mathbf{E}' = (E_2, E_3, E_4)$  with  $\mathbf{E}' \in \mathcal{C}^*(1, 1, 1)$ . Hence, by Lemma ?? we have  $P(\mathbf{E}') \geq k_{10}$ . Moreover

$$\ell_1 = P(\mathbf{E}) - P(\mathbf{E}') \leq k_0 - k_{10}.$$

We can repeat the same argument for  $\ell_2$  and  $\ell_4$  to obtain

$$\max\{\ell_1, \ell_2, \ell_4\} \leq k_0 - k_{10}. \quad (15)$$



By Proposition ?? we have (notice that we let  $\ell = 0$  since  $E_3$  is internal)

$$p_3 \geq \frac{(6-4)\pi}{3P(E_3)} + \min\{p_1, p_2, p_4\} > \min\{p_1, p_2, p_4\}. \quad (16)$$

The same proposition applied to the component  $C_1$  gives

$$p_1 \geq \frac{(6-4)\pi}{3P(C_1)} \geq \frac{2\pi}{3k_7} \geq 1.4750.$$

Since

$$\begin{aligned} k_0 &\geq P(\mathbf{E}) = 2(p_1 + p_2 + p_3 + p_4) \geq 2p_1 + 6 \min\{p_2, p_3, p_4\} \\ &\geq \frac{4\pi}{3k_7} + 6 \min\{p_2, p_3, p_4\}, \end{aligned}$$

we obtain

$$\min\{p_2, p_3, p_4\} \leq \frac{k_0}{6} - \frac{2\pi}{9k_7} \leq 1.3744,$$

so that

$$p_1 > \min\{p_2, p_3, p_4\}. \quad (17)$$

Putting together (??) and (??) we can say that the minimum among  $p_1, p_2, p_3, p_4$  is either  $p_2$  or  $p_4$ . Without loss of generality we can assume that such a minimum is  $p_2$ .

Hence, applying Lemma ?? to the region  $E_2$  we obtain

$$\frac{(6-4)\pi}{3} = \sum_{i=0}^4 (p_2 - p_i)L_i \leq p_2\ell_2$$

where  $L_i$  is the total length of the edges between  $E_2$  and  $E_i$  (so that  $L_0 = \ell_2$ ) and we used the estimate  $p_2 - p_i \leq 0$  for  $i \neq 0$ . So, using (??)

$$\min\{p_1, p_2, p_3, p_4\} = p_2 \geq \frac{2\pi}{3\ell_2} \geq \frac{2\pi}{3(k_0 - k_{10})}.$$

Now, use again Proposition ?? on the region  $E_3$  to obtain

$$p_3 \geq \frac{(6-4)\pi}{3P(E_3)} + \min\{p_1, p_2, p_4\} \geq \frac{2\pi}{3c_9} + \frac{2\pi}{3(k_0 - k_{10})} \geq c_{10} := 1.3466. \quad (18)$$

Finally we apply Lemma ?? to the region  $E_0$  to obtain

$$\frac{(6+4)\pi}{3} = p_1\ell_1 + p_2\ell_2 + p_4\ell_4 \leq \max\{\ell_1, \ell_2, \ell_4\}(p_1 + p_2 + p_4)$$

hence, using also (??)

$$p_1 + p_2 + p_4 \geq \frac{10\pi}{3(k_0 - k_{10})}.$$

So, using also (??), we have

$$P(\mathbf{E}) = 2p_3 + 2(p_1 + p_2 + p_4) \geq 2c_{10} + \frac{20\pi}{3(k_0 - k_{10})} \geq 11.4116 > k_0$$

which is a contradiction.  $\square$

**Theorem 6.5.** *Let  $\mathbf{E} \in \mathcal{M}(1, 1, 1, 1)$ . Then  $\mathbf{E}$  is connected.*

*Proof.* By Corollary ?? we know that  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$ .

By Corollary ?? and by Proposition ?? we know that each region  $E_i$  has exactly one big component and the total number of small components is not larger than two.

If the cluster has exactly two small components, with Corollary ??, we exclude that they belong to the same region and with Proposition ?? we exclude that they belong to two different regions.

Finally, from Proposition ?? and Propositions ??, ?? and ?? we exclude that the cluster has exactly one small connected component (which means five connected components in total).  $\square$

## 7 Connected clusters (four components)

**Proposition 7.1.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  be a connected cluster. Then  $\mathbf{E}$  has one of the two topologies depicted in Figure ??.*

*Proof.* Since every region is connected, by Proposition ?? every region (comprising  $E_0$ ) has three or four edges and the cluster has a total of nine edges and six vertices. Let  $x$  be the number of regions (bounded or unbounded) with four edges and let  $y$  the number of regions (bounded or unbounded) with three edges. We have one unbounded region  $E_0$  and four bounded regions, hence:  $x + y = 5$ . Moreover summing up all the edges of all the regions we would count each edge twice, hence we have:  $4x + 3y = 18$ . Solving the system of two equations gives  $x = 3$ ,  $y = 2$  hence we have three regions with four edges and two regions with three edges.

If the unbounded region  $E_0$  has three edges (note that there is a total of six vertices), there is one internal region and three external regions. The internal region can only have three edges (because it is not adjacent to  $E_0$ ) and we are in the first case of the statement.

If the unbounded region  $E_0$  has four edges, all the bounded regions are external: two of them have three edges and two have four edges. The regions with four edges are adjacent to all other regions hence the regions with three edges don't touch each other. We are in the second case of the statement.  $\square$

**Proposition 7.2.** *Let  $\mathbf{E} \in \mathcal{M}^*(1, 1, 1, 1)$  be a connected cluster. Then  $\mathbf{E}$  has not the flower topology.*

*Proof.* Suppose by contradiction that  $\mathbf{E}$  has the flower topology and let  $E_1$  be the internal three sided region.

First of all notice that

$$k_0 \geq P(\mathbf{E}) \geq P(E_0) + P(E_1) \geq 2\sqrt{\pi}\sqrt{4} + P(E_1)$$

so that

$$P(E_1) \leq k_0 - 4\sqrt{\pi} \leq c_{11} := 4.1064.$$

Now let  $\ell_2$  be the length of the external edge of the region  $E_2$ . If we remove  $E_2$  from  $\mathbf{E}$  we obtain a 3-cluster  $\mathbf{E}' = (E_1, E_3, E_4)$  with  $\mathbf{E}' \in \mathcal{C}^*(1, 1, 1)$ . Hence, by Lemma ?? we have  $P(\mathbf{E}') \geq k_{10}$ . Moreover

$$\ell_2 = P(\mathbf{E}) - P(\mathbf{E}') \leq k_0 - k_{10}.$$

We can repeat the same argument for the lengths  $\ell_3$  and  $\ell_4$  of the external edges of  $E_3$  and  $E_4$ , to obtain

$$\max\{\ell_2, \ell_3, \ell_4\} \leq k_0 - k_{10}. \quad (19)$$

By removing the triangular components we are able to reduce the cluster  $\mathbf{E}$  to a double bubble  $(E_2'', E_3'')$  with  $E_2'' \supseteq E_2$  and  $E_3'' \supseteq E_3$ . Notice that  $E_2'' \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_3)$  and  $E_3'' \subseteq \mathbb{R}^2 \setminus (E_0 \cup E_2)$  so that  $m(E_2'') \leq 3$  and  $m(E_3'') \leq 3$ . So, by Lemma ??, we obtain

$$\min\{p_2, p_3\} \geq \frac{k_8}{\sqrt{3}}.$$

We repeat the same argument with  $E_4$  in place of  $E_3$  to obtain  $\min\{p_2, p_4\} \geq \frac{k_8}{\sqrt{3}}$  so that

$$\min\{p_2, p_3, p_4\} \geq \frac{k_8}{\sqrt{3}}.$$

Now, use again Proposition ?? on the region  $E_1$  to obtain (notice that we let  $\ell = 0$  since  $E_1$  is internal)

$$p_1 \geq \frac{(6-3)\pi}{3P(E_1)} + \min\{p_2, p_3, p_4\} \geq \frac{\pi}{c_{11}} + \frac{k_8}{\sqrt{3}} \geq c_{12} := 1.6829. \quad (20)$$

Finally we apply Lemma ?? to the region  $E_0$  to obtain

$$\frac{(6+3)\pi}{3} = p_2\ell_2 + p_3\ell_3 + p_4\ell_4 \leq \max\{\ell_1, \ell_2, \ell_4\} \cdot (p_2 + p_3 + p_4)$$

hence, using also (??)

$$p_2 + p_3 + p_4 \geq \frac{3\pi}{k_0 - k_{10}}.$$

So, using also (??), we have

$$P(\mathbf{E}) = 2p_1 + 2(p_2 + p_3 + p_4) \geq 2c_{12} + \frac{6\pi}{k_0 - k_{10}} \geq 11.2124 > k_0$$

which is a contradiction.  $\square$

**Theorem 7.3.** *Let  $E \in \mathcal{M}(1, 1, 1, 1)$ . Then  $E$  has the sandwich topology as in Figure ??.*

*Proof.* By Theorem ?? we know that  $E$  is connected so by Proposition ?? we know that  $E$  can either have the flower or the sandwich topology. With Proposition ?? we exclude the flower topology and the result follows.  $\square$

**Conjecture 7.4.** *Up to isometries there is a unique stationary cluster  $E \in \mathcal{C}(1, 1, 1, 1)$  with the sandwich topology. In such a cluster the regions  $E_1$  and  $E_3$  are isometric to  $E_2$  and  $E_4$  (respectively).*

A stationary sandwich cluster with given external radii  $r_1, r_2, r_3, r_4$  can be uniquely constructed by taking the double bubble with external radii  $r_1, r_2$  and then “growing” a triangular region in each vertex so that it reaches the prescribed external radius. In fact both the area and the radii of a growing triangular region are strictly increasing, see [?, ?]

Not only that, since when the triangular region grows, the area of the quadrangular regions decreases, if  $r_1$  and  $r_2$  are fixed it is possible to find  $r_3 = r_4$  so that the area of the triangular regions become equal to the area of the smaller quadrangular region. If  $r_1 = r_2$  we then find that all four regions have equal area and with a rescaling we obtain a stationary symmetric sandwich cluster in  $\mathcal{C}(1, 1, 1, 1)$ .

We believe that when  $r_1 > r_2$  then the previous construction would yield a cluster where  $m(E_1) > m(E_2) = m(E_3) = m(E_4)$ . This would exclude that asymmetric sandwich cluster can have all equal areas.

We have some numerical computations [?] which confirm this.

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