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SHARP LOWER BOUNDS FOR COULOMB ENERGY

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ABSTRACT. We prove L^p lower bounds for Coulomb energy for radially symmetric functions in $\dot{H}^s(\mathbb{R}^3)$ with $\frac{1}{2} < s < \frac{3}{2}$. In case $\frac{1}{2} < s \leq 1$ we show that the lower bounds are sharp.

In this paper we prove lower bounds for the Coulomb energy

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|}\, dxdy$$

if radial symmetry of φ is assumed.
In the general case, without restricting to radial functions, the upper bound for the Coulomb energy is given by the well known Hardy-Littlewood-Sobolev inequality while lower bounds have been proved only very recently. In particular if one can control suitable homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^3)$ the L^p lower bound for the Coulomb energy is given by the following inequalities

$$\|\varphi\|_{L^{2p}(\mathbb{R}^3)} \leq C(p,s)\|\varphi\|_{\dot{H}^s(\mathbb{R}^3)}^{\frac{\theta}{2-\theta}} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|}\, dxdy \right)^{\frac{1-\theta}{4-2\theta}} \tag{0.1}$$

with $\theta = \frac{6-5p}{3-2ps-2p}$. Here the parameters $s > 0$ and $1 < p \leq \infty$ satisfy

$$\begin{aligned}
&p \in \left[\frac{3}{3-2s}, \frac{1+2s}{1+s}\right] && \text{if } 0 < s < 1/4\,,\\
&p = \frac{3}{3-2s} = \frac{1+2s}{1+s} && \text{if } s = 1/4\,,\\
&p \in \left[\frac{1+2s}{1+s}, \frac{3}{3-2s}\right] && \text{if } 1/4 < s < 3/2\,,\\
&p \in \left[\frac{1+2s}{1+s}, \infty\right) && \text{if } s = 3/2\,,\\
&p \in \left[\frac{1+2s}{1+s}, \infty\right] && \text{if } s > 3/2\,.
\end{aligned}$$

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These bounds have been proved in [3] while the case $s = \frac{1}{2}$ has been first considered in [4]. These bounds follows from a suitable Gagliardo-Nirenberg inequality, see Theorem 2.44 of [1], together with the following well known identity

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} dxdy = c ||\varphi^2||^2_{\dot{H}^{-1}(\mathbb{R}^3)}.$$

We shall underline that in many physical applications involving Sobolev norms and Coulomb energy the radially symmetric assumption of φ is natural due to the rotational invariance of energy functionals (see e.g [8] in the context of stability of matter). Our purpose is to see if it is possible to control lower L^p norms if one assumes *radial symmetry* of φ.

In the sequel we use two theorems that are crucial for our improvement in case of radial symmetry. The first is the following pointwise decay for radial functions in $\dot{H}^s(\mathbb{R}^d) \cap L^q_a(\mathbb{R}^d)$, see [6], where $L^q_a(\mathbb{R}^d)$ is the weighted Lebesgue space with the norm

$$||u||_{L^q_a(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |x|^a |u|^q dx \right)^{\frac{1}{q}}$$

Theorem 0.1 (De Nápoli [6]). *Let φ be a radial function in $\dot{H}^s(\mathbb{R}^d) \cap L^q_a(\mathbb{R}^d)$ with $s > \frac{1}{2}$ and $-(d-1) < a < d(q-1)$, then*

$$|\varphi(x)| \leq C(d,s,q,a)|x|^{-\sigma} ||(-\Delta)^{\frac{s}{2}} \varphi||^{\theta}_{L^2(\mathbb{R}^d)} ||\varphi|||^{1-\theta}_{L^q_a(\mathbb{R}^d)}$$

where $\theta = \frac{2}{2sq+2-q}$, $\sigma = \frac{2as+2ds-a-2s}{2sq+2-q}$.

Remark 0.1. The strategy of the proof of Theorem 0.1 is based on Fourier representation for radial functions in $\mathbb{R}^d$ (identifying the function with its profile)

$$\varphi(x) = (2\pi)^{\frac{d}{2}} |x|^{-\frac{d-2}{2}} \int_0^\infty J_{\frac{d-2}{2}}(|x|\rho) \hat{\varphi}(\rho) \rho^{\frac{d}{2}} d\rho$$

where $J_{\frac{d-2}{2}}$ is the Bessel function of order $\frac{d-2}{2}$. The argument is similar to the one developed in [5] for the pointwise decay of radial function in $\dot{H}^s(\mathbb{R}^d)$, i.e to split φ into low and high frequency parts. The pointwise decay of high frequency part of φ will be controlled by the boundess of Sobolev norm while the decay of low frequency part by the boundness of the weighted Lebesgue norm.

The second theorem is the following lower bound for the Coulomb energy by Ruiz, see [9].

Theorem 0.2 (Ruiz [9]). *Given $\alpha > \frac{1}{2}$, there exists $c = c(\alpha) > 0$ such that for any measurable $\varphi : \mathbb{R}^d \to \mathbb{R}$ we have*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|^{d-2}} dxdy \geq c \left(\int_{\mathbb{R}^d} \frac{|\varphi(x)|^2}{|x|^{\frac{d-2}{2}} (1 + |\log|x||)^\alpha} dx \right)^2.$$

Let us define

$$\mathcal{E}^s = \{\varphi \in \dot{H}^s_{rad}(\mathbb{R}^3) \ \ s.t \iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dxdy < \infty\}$$

with

$$||\varphi||_{\mathcal{E}^s} = \left(||\varphi||^2_{\dot{H}^s(\mathbb{R}^3)} + \left(\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|} dxdy \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}.$$

Following the argument of Ruiz [9] it is easy to show that $||\cdot||_{\mathcal{E}^s}$ is a norm and $C_0^\infty(\mathbb{R}^3)$ is dense in $\mathcal{E}^s$. In [9] Ruiz proved that for $\mathcal{E}^1$ the following continuous embedding

$$\mathcal{E}^1 \hookrightarrow L^p \quad p \in (\frac{18}{7}, 6].$$

The result by Ruiz follows from two steps: first, Theorem 0.2 proves that $\mathcal{E}^1 \subset \dot{H}^1_{rad}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, V(x)dx)$ where $V(x) = \frac{1}{(1+|x|)^\gamma}$ with $\gamma > \frac{1}{2}$, second, a weighted Sobolev embedding for radial function proved by Su, Wang and Wilem [10] gives the inclusion

$$\dot{H}^1_{rad}(\mathbb{R}^3) \cap L^2(\mathbb{R}^3, V(x)dx) \subset L^q(\mathbb{R}^3) \quad q \in [\frac{2(4+\gamma)}{4-\gamma}, 6]$$

The aim of our paper is to find continuous embeddings and hence better lower bounds for the Coulomb energy assuming radial symmetry when $\frac{1}{2} < s < \frac{3}{2}$. As a particular case we recover $p = \frac{18}{7}$ as end-point exponent when $s = 1$.

Theorem 0.3. *$\mathcal{E}^s \hookrightarrow L^p(\mathbb{R}^3)$ continuously for*

$$p \in \left(\frac{16s+2}{6s+1}, \frac{6}{3-2s} \right] \qquad \text{if } 1/2 < s < 3/2.$$

The above result is sharp when $\frac{1}{2} < s \leq 1$ as showed by the following

Theorem 0.4. *Let $\frac{1}{2} < s \leq 1$, then the space $\mathcal{E}^s$ is not embedded in L^p for $p < \frac{16s+2}{6s+1}$.*

From the continuous embedding for $\mathcal{E}^s$ it is elementary to derive the scaling invariant lower bounds for the Coulomb energy given by (0.1) for $p \in \left(\frac{16s+2}{6s+1}, \frac{6}{3-2s}\right]$ and $\frac{1}{2} < s < \frac{3}{2}$. Moreover we prove that the best constants of the lower bounds are achieved among radially symmetric functions.

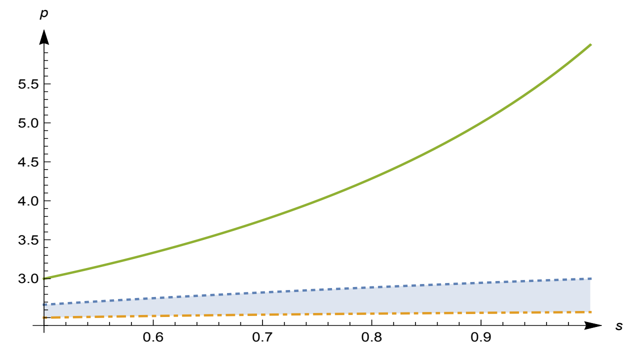

FIGURE 1. L^p lower bound for Coulomb energy: without radial symmetry the lower bound is $\frac{2+4s}{1+s}$ (dotted) and for the radially symmetric case is $\frac{16s+2}{6s+1}$ (dash-dotted). The bold line plots the Sobolev embedding exponent for $\frac{1}{2} < s \leq 1$.

Corollary 0.1. *Let φ be radially symmetric, then the following scaling invariant inequality holds*

$$\|\varphi\|_{L^{2p}(\mathbb{R}^3)} \leq C(p,s)\|\varphi\|^{\frac{\theta}{2-\theta}}_{\dot{H}^s(\mathbb{R}^3)} \left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|}\, dxdy \right)^{\frac{1-\theta}{4-2\theta}} \tag{0.2}$$

with $\theta = \frac{6-5p}{3-2ps-2p}$. Here the parameters s and p satisfy

$$p \in (\frac{8s+1}{6s+1}, \frac{3}{3-2s}] \quad \textit{if } 1/2 < s < 3/2.$$

Assume, moreover, that $p \neq \frac{3}{3-2s}$, then the best constant in (0.2) *is achieved in the set of radially symmetric functions.*

In Figure 1 the behavior of p as a function of s is plotted.

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1. Proof of Theorem 0.3

Proposition 1.1. *Let $\gamma > \frac{1}{2}$ and $\frac{\gamma}{2} < s < \frac{3}{2}$ then exists $c(\gamma, s) > 0$ such that for any $\varphi \in \mathcal{E}^s$*

$$\left(\int_{\mathbb{R}^3} |x|^{-\gamma}|\varphi|^2 dx \right) \leq c(\gamma,s)||\varphi||^2_{\mathcal{E}^s}.$$

Proof. By elementary computation we notice that if $2s > \gamma$

$$\left(\int_{\mathbb{R}^3} |x|^{-\gamma}|\varphi|^2 dx\right) \leq R^{2s-\gamma}\left(\int_{B(0,R)} \frac{|\varphi|^2}{|x|^{2s}}dx\right)+\frac{(1+R)^\gamma}{R^\gamma}\left(\int_{B(0,R)^c} \frac{|\varphi|^2}{(1+|x|)^\gamma}dx\right).$$

On the other hand, if $0 < s < \frac{3}{2}$, by Pitt inequality [2],

$$c_s||\varphi||^2_{\dot{H}^s(\mathbb{R}^3)} = c_s\left(\int_{\mathbb{R}^3} |\hat{\varphi}|^2|\xi|^{2s}d\xi\right) \geq \left(\int_{\mathbb{R}^3} \frac{|\varphi|^2}{|x|^{2s}}d\xi\right)$$

where $c_s = \pi^{2s}\left[\frac{\Gamma(\frac{3-2s}{4})}{\Gamma(\frac{3+2s}{4})}\right]^2$ and by Ruiz's Theorem 0.2

$$\left(\int_{\mathbb{R}^3} \frac{|\varphi|^2}{(1+|x|)^\gamma}dx\right) \leq c(\gamma)\left(\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|}dxdy\right)^{\frac{1}{2}}.$$

□

Proof of Theorem 0.3.

By Theorem 0.2 we have that for any $\alpha > \frac{1}{2}$, $\mathcal{E}^s \subset L^2(\mathbb{R}^3, V(x)dx)$ where $V(x) = (\frac{1}{1+|x|})^\gamma$, $\gamma > \frac{1}{2}$. Let us call $p^* = \frac{2+4s}{1+s}$, the end-point exponent for (0.1). Hölder inequality assures that for $p < p^*$

$$\int_{B(0,1)} |\varphi|^p dx < \mu(B(0,1))^{\frac{p^*-p}{p}}\left(\int_{B(0,1)} |\varphi|^{p^*}dx\right)^{\frac{p}{p^*}} \leq$$

$$\leq C\|\varphi\|^{(\frac{\theta}{2-\theta})p}_{\dot{H}^s(\mathbb{R}^3)}\left(\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\varphi(x)|^2|\varphi(y)|^2}{|x-y|}\,dxdy\right)^{(\frac{1-\theta}{4-2\theta})p}$$

with $\theta = \frac{6-\frac{5}{2}p^*}{3-p^*s-p^*}$. On the other hand by Proposition 1.1 and the radial decay given by Theorem 0.1 choosing $a = -\gamma$, $q = 2$ and $d = 3$,

$$\int_{B(0,1)^c} |\varphi|^p dx = \int_{B(0,1)^c} |\varphi(x)|^2|\varphi(x)|^{p-2}dx \leq \tag{1.1}$$

$$\leq C||\varphi||^{\theta(p-2)}_{\dot{H}^s(\mathbb{R}^3)}||\varphi|||^{(1-\theta)(p-2)}_{L^2_{-\gamma}(\mathbb{R}^3)}\int_{B(0,1)^c} |x|^{-\sigma(p-2)}|\varphi(x)|^2dx$$

where $\theta = \frac{1}{2s}$, $\sigma = \frac{-2\gamma s+4s+\gamma}{4s}$. Now

$$\lim_{\gamma\to\frac{1}{2}} \frac{-2\gamma s+4s+\gamma}{4s}(p-2) = (\frac{3s+\frac{1}{2}}{4s})(p-2)$$

and this implies again by Proposition 1.1 that

$$\int_{B(0,1)^c} |\varphi|^p dx < +\infty$$

provided that $(\frac{3s+\frac{1}{2}}{4s})(p-2) > \frac{1}{2}$, i.e if $p > \frac{16s+2}{6s+1}$. □

2. Proof of Theorem 0.4

The proof of Theorem 0.4 is obtained constructing a counterexample, i.e a function u such that

$$\begin{aligned} &\|u\|^2_{\dot{H}^s(\mathbb{R}^3)} \simeq 1 \\ &\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy \simeq 1 \\ &||u||^p_{L^p(\mathbb{R}^3)} \to +\infty \end{aligned} \tag{2.1}$$

Proof of Theorem 0.4.

The case $s = 1$ has been proved by Ruiz [9]. Set $u : \mathbb{R}^3 \to \mathbb{R}^+$

$$u(x) = \begin{cases} \varepsilon \dfrac{S - \big||x| - R\big|}{S} & \text{for } \big||x| - R\big| < S \\ 0 & \text{elsewhere} \end{cases} \tag{2.2}$$

where $R > S >> 1 >> \varepsilon > 0$ will be precised in the sequel.

We recall, by Ruiz [9, Section 4], that

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy \le C\varepsilon^4 S^2 R^3$$

and

$$||u||^p_{L^p(\mathbb{R}^3)} \ge C\varepsilon^p S R^2.$$

Moreover we have

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^3)} \le C\frac{\varepsilon^2 R^2}{S^{2s-1}}. \tag{2.3}$$

The proof of (2.3) is not difficult and it will postponed to Lemma 2.1.

In order to have $\|u\|^2_{\dot{H}^s(\mathbb{R}^3)} \simeq 1$ we choose $S = \varepsilon^{\frac{2}{2s-1}} R^{\frac{2}{2s-1}}$. At this point we have

$$\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy \le C\varepsilon^4 \varepsilon^{\frac{4}{2s-1}} R^{\frac{4}{2s-1}} R^3 = C\varepsilon^{\frac{8s}{2s-1}} R^{\frac{6s+1}{2s-1}}$$

and we choose $R = \varepsilon^{-\frac{8s}{6s+1}}$ to have the Coulomb norm bounded, so

$$S = \varepsilon^{\frac{2}{2s-1}} R^{\frac{2}{2s-1}} = \varepsilon^{\frac{2}{2s-1}} \varepsilon^{-\frac{8s}{6s+1}\frac{2}{2s-1}} = \varepsilon^{\frac{2-4s}{(6s+1)(2s-1)}} = \varepsilon^{-\frac{2}{6s+1}}.$$

We remark that, since $s > 1/2$, then $8s > 2$ and $R > S$, as required in the definition of $u(x)$.

Concluding, we have

$$||u||^p_{L^p(\mathbb{R}^3)} \geq C\varepsilon^p S R^2 \simeq \varepsilon^{p-\frac{16s+2}{6s+1}}$$

that diverges for $p < \frac{16s+2}{6s+1}$ when $\varepsilon \to 0$. The claim follows immediately. $\square$

Lemma 2.1. *Let u be defined in (2.2). Then*

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^3)} \leq C\frac{\varepsilon^2 R^2}{S^{2s-1}}.$$

Proof. We want to compute the $\dot{H}^s$ norm of u for $s < 1$, that is

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^3)} = C(s) \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy.$$

where $C(s) = 2^{2s-1}\pi^{-\frac{3}{2}}\frac{\Gamma(\frac{3+2s}{2})}{\Gamma(-s)}$.

We observe that $u(x) - u(y) \neq 0$ in the following five subsets of $\mathbb{R}^3 \times \mathbb{R}^3$:

$$\begin{aligned}
A_1 &= \{R-S \leq |y| \leq R+S,\ |x| \leq R-S\}\\
A_2 &= \{R-S \leq |y| \leq R+S,\ |x| \geq R+S\}\\
A_3 &= \{R-S \leq |x| \leq R+S,\ |y| \leq R-S\}\\
A_4 &= \{R-S \leq |x| \leq R+S,\ |y| \geq R+S\}\\
A_5 &= \{R-S \leq |x| \leq R+S,\ R-S \leq |y| \leq R+S\}
\end{aligned}$$

and, by symmetry, we obtain

$$\begin{aligned}
\frac{\|u\|^2_{\dot{H}^s(\mathbb{R}^3)}}{C(s)} &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy = 2\iint_{A_1} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy\\
&\quad + 2\iint_{A_2} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy + \iint_{A_5} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy\\
&\leq 2 \int_{R-S\leq|y|\leq R+S} \int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|u\|^2_{\dot H^s(\mathbb{R}^3)} \lesssim & \int\limits_{R-S\le|y|\le R+S} \int\limits_{|x-y|\le S} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \\
&+ \int\limits_{R-S\le|y|\le R+S} \int\limits_{|x-y|\ge S} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}} dxdy \\
\lesssim & \int\limits_{R-S\le|y|\le R+S} \int\limits_{|x-y|\le S} \frac{\varepsilon^2}{S^2} \frac{|x-y|^2}{|x-y|^{3+2s}} dxdy \\
&+ \int\limits_{R-S\le|y|\le R+S} \int\limits_{|x-y|\ge S} \frac{\varepsilon^2}{|x-y|^{3+2s}} dxdy
\end{aligned}
$$

using that $|u(x)-u(y)| \le \sup_\xi |\nabla u(\xi)||x-y| \le \frac{\varepsilon}{S}|x-y|$ in the first term and that $|u(x)| \le \varepsilon$ in the second term. At this point, with the change of variable $t = x-y$ we get

$$
\begin{aligned}
\|u\|^2_{\dot H^s(\mathbb{R}^3)} \lesssim & \varepsilon^2 \int\limits_{R-S\le|y|\le R+S} \left[\int\limits_{|t|\le S} \frac{1}{S^2}\frac{1}{|t|^{1+2s}} dt + \int\limits_{|t|\ge S} \frac{1}{|t|^{3+2s}} dt \right] dy \\
\lesssim & \varepsilon^2 \int\limits_{R-S\le|y|\le R+S} \left[\int_0^S \frac{1}{S^2}\frac{r^2}{r^{1+2s}} dr + \int_S^\infty \frac{r^2}{r^{3+2s}} dr \right] dy \\
\lesssim & \varepsilon^2 R^2 S \left[\int_0^S \frac{1}{S^2} r^{1-2s} dr + \int_S^\infty r^{-1-2s} dr \right] \simeq \varepsilon^2 R^2 S \left[\frac{S^{2-2s}}{S^2} + S^{-2s} \right] \\
\simeq & \frac{\varepsilon^2 R^2}{S^{2s-1}}
\end{aligned}
$$

□

Proof of Corollary (0.1).

From Theorem 0.3 it follows that

$$
||\varphi_\lambda||^2_{L^p(\mathbb{R}^3)} \le C \left(||\varphi_\lambda||^2_{\dot H^s(\mathbb{R}^3)} + \left(\iint\limits_{\mathbb{R}^3\times\mathbb{R}^3} \frac{|\varphi_\lambda(x)|^2 |\varphi_\lambda(y)|^2}{|x-y|} dxdy \right)^{\frac{1}{2}} \right)
$$

if $p \in \left(\frac{16s+2}{6s+1}, \quad \frac{6}{3-2s}\right]$ and if $1/2 < s < 3/2$. Now let us consider the following scaling

$$\varphi_\lambda = \lambda^{\frac{3}{p}} \varphi(\lambda x),$$

such that $||\varphi_\lambda||_{L^p(\mathbb{R}^3)} = ||\varphi||_{L^p(\mathbb{R}^3)}$ for all $\lambda > 0$. By elementary computation one gets

$$||\varphi||^2_{L^p(\mathbb{R}^3)} \leq C \left(\lambda^{\frac{6}{p}-(3-2s)} ||\varphi||^2_{\dot{H}^s(\mathbb{R}^3)} + \lambda^{\frac{6}{p}-\frac{5}{2}} \left(\iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi(x)|^2 |\varphi(y)|^2}{|x-y|} dxdy \right)^{\frac{1}{2}} \right).$$

and minimizing R.H.S we get the desired inequality.
The argument to show the existence of maximizers is identical to the one used to show Theorem 2.2 in [3]. We just give a sketch of the proof for reader convenience. Let us fix p in the set $(\frac{16s+2}{6s+1}, \frac{6}{3-2s})$. By homogeneity and scaling we can assume that an optimizing sequence $\varphi_n \in \mathcal{E}^s$ satisfies

$$\|\varphi_n\|_{\dot{H}^s} = \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{|\varphi_n(x)|^2 \ |\varphi_n(y)|^2}{|x-y|} \, dx \, dy = 1$$

and

$$\|\varphi_n\|_{L^p} = C(p,s) + o(1) \, .$$

Thanks to inequality (0.2) we can find uniform upper bound on $\|\varphi_n\|_{L^{p_1}}$ and $\|\varphi_n\|_{L^{p_2}}$ for some $p_1 < p < p_2$. Therefore, by the well known pqr-Lemma (see [7])

$$\inf_n |\{|\varphi_n| > \eta\}| > 0 \, .$$

Now by Lieb's compactness lemma in $\dot{H}^s$, see Lemma 2.1 in [3], there exists $\varphi \neq 0$ such that $\varphi_n \rightharpoonup \varphi \in \dot{H}^s(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$. Finally, by the non-local Brezis–Lieb lemma for the Coulomb term (see Lemma 2.2 in [3]), and by the Hilbert structure of $\dot{H}^s(\mathbb{R}^3)$, we prove the existence of a maximizer. □

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