Classical solutions of the divergence equation with Dini-continuous datum

Luigi C. Berselli and Placido Longo
Dipartimento di Matematica
Università di Pisa
Via F. Buonarroti 1/c,
I-56127, Pisa, Italia.
email: luigi.carlo.berselli@unipi.it, placido.longo@unipi.it

Abstract

We consider the boundary value problem associated to the divergence operator with vanishing Dirichlet boundary conditions and we prove the existence of classical solutions under slight assumptions on the regularity of the datum.

1 Introduction

In this paper we deal with the existence of classical solutions for the boundary value problem

$$\begin{cases} \operatorname{div} u = F & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

namely we look for solutions $u : \Omega \to \mathbb{R}^n$, such that $u \in C^1(\Omega) \cap C^0(\overline{\Omega})$. Here $\Omega \subset \mathbb{R}^n$ is a smooth and bounded domain, while F is a given continuous function satisfying the compatibility condition $\int_\Omega F(x)\, dx = 0$. This is a classical problem in mathematical fluid mechanics, strictly connected with the Helmholtz decomposition and the div-curl lemma (see Kozono and Yanagisawa [19]). We recall that by dropping the boundary condition a solution of the first order system (1) can be readily obtained by taking the gradient of the Newtonian potential of F. These aspects are extensively covered in Galdi [14, Ch. 3] with special attention to the work of Bogovskiĭ [7], where the problem is solved in the Sobolev spaces $H_0^{1,p}(\Omega)$. Further developments may also be found in Borchers and Sohr [8]. For different approaches and

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results the reader could consider the books by Ladyžhenskaya [20] and Tartar [24], especially regarding the solution in the Hilbert case, while Amrouche and Girault [1] used the negative norm theory developed by Nečas [21].

Our approach follows closely the Bogovskiĭ's one, where the representation formula (2) below, in analogy with the "cubature" formulae of Sobolev, gives explicitly a special solution of the problem (1), which *per se* has infinitely many solutions. The formula (2) turns out to be extremely flexible in the applications to many different settings as, for instance, the recent results for weighted $L^{p(x)}$-spaces (see Huber [16]). Classical results in Hölder spaces have been shown in Kapitanskiĭ and Piletskas [17], as a corollary of a more general result, which seems to be obtained in a way different from ours. For the Hölder case see also the recent review in Csató, Dacorogna, and Kneuss [11]. In addition, we also note that the non-uniqueness feature of the first order system (1) allows some existence results with more regularity than expected from the usual theorems, as the striking results of Bourgain and Brezis [9], coming from a non-linear selection principle of the solutions to a linear problem (see the extensions to the Dirichlet problem in Bousquet, Mironescu, and Russ [10]).

Our interest for the problem is twofold: on one side, we want to investigate the results close to the limiting case $F \in L^\infty(\Omega) \cap C(\Omega)$, for which counterexamples to the existence of a solution are known (see Bourgain and Brezis [9] and Dacorogna, Fusco, and Tartar [12]); on the other side, we are interested in relaxing as much as possible the assumptions needed to get classical solutions. This is motivated by the aim of finding weaker sufficient assumptions which allow to construct classical solutions to fluid mechanics problems. Since the continuity of F is not enough to that purpose, we went back to the very old results by Dini [13] and Petrini [22] about the Poisson equation, and we consider the problem with the additional hypothesis that F is Dini-continuous (see below for a definition). Our proof follows closely an argument used by Korn to obtain a similar regularity result for the second derivative of the Newtonian potential (see Gilbarg and Trudinger [15, Ch. 4]) and exploits the property of the Dini continuity to "regularize" the singularity in the second derivative of the potential, outside the theory of Calderòn-Zygmund of singular integral operators. About the boundary condition, our proof is based on some "new" insight of the formula (in the sense that we made some simple observations on the Bogovskiĭ formula which we cannot find stated explicitly elsewhere in literature) still valid when the datum F cannot be approximated with compactly supported smooth data. Such an approximation seems to play a fundamental role in Sobolev or Orlicz spaces.

We wish to mention that the link between Dini continuity and existence of classical solutions in fluid mechanics started with Shapiro [23] in the steady case and found a very interesting application with the paper of Beirão da Veiga [2], where the 2D Euler equations for incompressible fluids are solved in a critical spaces for the vorticity. More recently, the same results have been also employed by Koch [18] and in [5] to study fine properties of the long-time behavior of the Euler equations. Finally, the interest for classical solutions of the Stokes system has been revived in the recent papers of Beirão da Veiga [3, 4], that provided a further motivation to our analysis of the divergence and curl operator, since they are among the building blocks of the theory. We also point out that the system (1) is not elliptic, hence the well-known results for elliptic equations and systems do not apply directly.

To conclude we also mention that the problem of classical solutions for the curl equation (again with Dirichlet condition) will be treated elsewhere [6] following the same method, by using similar (but more complicated) representation formula, known in that case.

2 Notation and preliminary results

In this section we recall the main definitions we will use, as well as some basic facts about the representation formula developed by Bogovskiĭ. The results of this section are well-known, but some of them on the role of the boundary condition are not explicitly available in the literature.

In the following we denote by $B(x, R) = \{y \in \mathbb{R}^n : |y - x| < R\}$, the ball of radius R centered at x, by $S^{n-1} = \{y \in \mathbb{R}^n : |y| = 1\}$ the unit sphere of $\mathbb{R}^n$, and by $|S^{n-1}|$ its $(n-1)$-dimensional measure.

We also denote by $C_D(\Omega)$ the space of the (uniformly) Dini-continuous functions F, *i.e.*, such that if one introduces the modulus of (uniform) continuity

$$\omega(F, \rho) = \sup_{\substack{x,y \in \Omega \\ |x-y| < \rho}} |F(x) - F(y)|,$$

the function $\omega(F, \rho)/\rho$ is integrable around 0^+. We equip the space of Dini continuous functions with the following norm

$$\|F\|_{C_D} = \max_{x \in \overline{\Omega}} |F(x)| + \int_0^{\mathrm{diam}(\Omega)} \frac{\omega(F, \rho)}{\rho}\, d\rho,$$

and it turns out to be a Banach space. We remark that, by the uniform continuity, any function in $C_D(\Omega)$ may be extended up to the boundary of Ω with the same modulus of continuity. We observe also that $C^{0,\alpha}(\Omega) \subset C_D(\Omega)$ for all $0 < \alpha \leq 1$ and recall that its relevance in partial differential equations comes from the result that, if $f \in C_D(\Omega)$, then the solution of the Poisson equation

$$\Delta u = f$$

with zero Dirichlet conditions satisfies $D^2 u \in C(\Omega)$ (see for instance Gilbarg and Trudinger [15, Pb. 4.2].

2.1 Bogovskiĭ's formula and its variants

The aim of this section is to provide a representation formula for a solution of the divergence problem, due to Bogovskiĭ [7], as well as several useful variants and consequences.

Unless differently specified (namely, in the last section), the following hypotheses will be tacitly assumed throughout all the paper. Let B denote the open unit ball of $\mathbb{R}^n$, $n \geq 2$, centered at the origin. The scalar function $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\psi \subseteq B$, and it is not vanishing identically. We will denote by $\partial_j \psi$ the partial derivative of ψ with respect to its j^{th} argument. The domain Ω will be a bounded open subset of $\mathbb{R}^n$, star-shaped with respect to any point of $\overline{B}$.

The main results to be proved are the following Theorems.

Theorem 1. *Assume the previous hypotheses. Let $q > n$ and let $F \in L^q(\Omega)$. Then:*

i) The Bogovskiĭ's formula

$$v(x) = \int_\Omega F(y) \left[\frac{x-y}{|x-y|^n} \int_{|x-y|}^{+\infty} \psi\left(y + \xi \frac{x-y}{|x-y|} \right) \xi^{n-1} d\xi \right] dy, \qquad (2)$$

defines for any $x \in \mathbb{R}^n$ (and not only almost everywhere) a function $v : \mathbb{R}^n \to \mathbb{R}^n$;

ii) $\quad v(x) = 0 \quad \forall\, x \in \mathbb{R}^n \backslash \Omega$;

iii) For any $q > n$

$$|\, v(x) \,| \leq c \, \|F\|_{L^q(\Omega)} \quad \forall x \in \mathbb{R}^n,$$

where c depends only on n, ψ, $\operatorname{diam}\Omega$, and q;

iv)

$$v(x) = \int_\Omega F(y) \left[(x-y) \int_1^\infty \psi\left(y + \alpha(x-y) \right) \alpha^{n-1} d\alpha \right] dy;$$

v)

$$v(x) = \int_\Omega F(y) \left[\frac{x-y}{|x-y|^n} \int_0^\infty \psi\left(x + r \frac{x-y}{|x-y|} \right) (|x-y| + r)^{n-1} dr \right] dy;$$

vi)

$$v(x) = \int_{x-\Omega} F(x-z) \, \frac{z}{|z|^n} \int_0^\infty \psi\left(x + r\frac{z}{|z|} \right) (|z| + r)^{n-1} \, dr \, dz,$$

where $x - \Omega = \{z \in \mathbb{R}^n : \exists\, y \in \Omega \text{ such that } z = x - y\}$.

Important consequences of the Theorem 1 are the next regularity results for the "potential" v in the interior, as well as at the boundary.

Theorem 2. *Under the same hypotheses of Theorem 1, if $F \in C_0^\infty(\Omega)$ then $v \in C_0^\infty(\Omega)$.*

Theorem 3. *Under the same assumptions of Theorem 1, it follows that $v \in C^0(\mathbb{R}^n) \subseteq C^0(\overline{\Omega})$.*

The proofs require several lemmas, and we start with an estimate for the kernel that appears in the Bogovskiĭ's formula (2).

Definition 4. *Let ψ be as above. Then we set*

$$N(x,y) := \frac{x-y}{|x-y|^n} \int_{|x-y|}^{+\infty} \psi\left(y + \xi \frac{x-y}{|x-y|}\right) \xi^{n-1} d\xi,$$

and we remark that we can rewrite the Bogovskiĭ's formula as follows

$$v(x) = \int_\Omega N(x,y)\, F(y)\, dy.$$

Lemma 5. *There exists a constant $c > 0$, depending only on n, ψ, and* $\operatorname{diam}\Omega$*, such that*

$$|N(x,y)| \le c|x-y|^{1-n} \quad \forall\, x,y \in \mathbb{R}^n : \; x \neq y.$$

Proof. Since the smooth function ψ vanishes outside B, then it follows that $\psi\left(y + \xi \frac{x-y}{|x-y|}\right) = 0$, when $\left|y + \xi \frac{x-y}{|x-y|}\right| \ge 1$. Since $\left|y + \xi \frac{x-y}{|x-y|}\right| \ge |\xi - |y||$, we have that $\psi\big(y + \xi \frac{x-y}{|x-y|}\big)$ is zero for $\xi \in \mathbb{R}^+$ such that $\xi > 1 + \operatorname{diam}\,\Omega$, and therefore

$$\left|\int_{|x-y|}^{+\infty} \psi\left(y + \xi \frac{x-y}{|x-y|}\right) \xi^{n-1} d\xi\right| \le (1 + \operatorname{diam}\,\Omega)^n \cdot \max_{x \in \mathbb{R}^n} |\psi(x)| = c,$$

and the lemma follows. □

The following simple remarks have important consequences in the study of the support of v.

Lemma 6. *If $x \notin \Omega$ and $\psi\left(y + \xi \frac{x-y}{|x-y|}\right) \neq 0$ holds true for some $\xi > |x-y|$, then $y \notin \Omega$.*

Proof. In fact, since ψ is not zero, if follows that

$$\left|y + \xi \frac{x-y}{|x-y|}\right| \le 1.$$

Moreover, $x = y + |x-y|\, \frac{x-y}{|x-y|}$ and hence x belongs to the segment of endpoints y and $y + \xi \frac{x-y}{|x-y|}$, for any $\xi > |x-y|$. If it were $y \in \Omega$, by the hypotheses on Ω the entire segment would lay in it, and that contradicts the assumption on x. □

An immediate consequence of the last lemma is the following result

Lemma 7. *The above function $N(x,y)$ verifies*

$$N(x,y) \equiv 0 \quad \forall\, x \notin \Omega, \text{ and } \forall\, y \in \Omega$$

We can now give the proof of Theorem 1

Proof of Theorem 1. By the above lemma, the vector $v(x)$ vanishes outside Ω, and then *ii)* follows from *i)*.
To prove *i)* and *iii)*, fix $q > n$. Let p be such that $1/p + 1/q = 1$, and then $p \in [1, \frac{n}{n-1}[$. Fix also any $x \in \Omega$. Since Ω is bounded, by Lemma 5 it turns out that $N(x,\cdot) \in L^p(\Omega)$ and by Hölder's inequality it follows that

$$\begin{aligned} |v(x)| &\le \int_\Omega |F(y)|\ |N(x,y)|\, dy \le c \int_\Omega |F(y)|\ |x-y|^{1-n}\, dy \\ &\le c\|F\|_{L^q(\Omega)} \left(\int_\Omega |x-y|^{p(1-n)}\, dy \right)^{1/p}. \end{aligned}$$

Since $\Omega \subseteq B(x, \text{diam } \Omega)$

$$\int_\Omega |x-y|^{p(1-n)}\, dy \le \int_{B(x,\, \text{diam } \Omega)} |x-y|^{p(1-n)}\, dy = \int_{B(0,\, \text{diam } \Omega)} |z|^{p(1-n)}\, dz,$$

and so *i)* and *ii)* follow.

Finally, by setting $g(z) = |z|^{1-n}$, we have

$$|v(x)| \le c\|g\|_{L^p(B(0,\, \text{diam } \Omega))} \|F\|_{L^q(\Omega)},$$

where the right side is independent of x. Since, by *ii)*, v vanishes outside Ω, it follows immediately *iii)*.

To get *iv)*, it is enough to put $\xi = \alpha|x-y|$ in the initial formula.
By putting $r = \xi - |x-y|$, instead, it follows *v)*.
Finally, by introducing $z = x - y$ in *v)* it follows *vi)*. □

Remark 8. *It is useful to remark explicitly that the Bogovskiĭ's "potential" v vanishes at the boundary for any $F \in L^q(\Omega)$, without any other assumption than those made on Ω and ψ in Theorem 1. It is relevant to observe that it does not come by approximating F by $C_0^\infty(\Omega)$ functions and by taking limits, but it is a property which descends directly from the formula for a large class of functions. In our case that is especially useful in that a given function in $C_D(\Omega)$ cannot be approximated in uniform norm by regular function with compact support, unless it vanishes at the boundary.*

Proof of Theorem 2. From the formula in Theorem 1 *vi)*, it follows that

$$\begin{aligned} v(x) &= \int_{x-\Omega} F(x-z)\, \frac{z}{|z|^n} \int_0^\infty \psi\left(x + r\frac{z}{|z|}\right)\ (|z|+r)^{n-1}\, dr\, dz \\ &= \int_{x-\text{supp}\, F} F(x-z)\, \frac{z}{|z|^n} \int_0^{1+\text{diam }\Omega} \psi\left(x + r\frac{z}{|z|}\right)\ (|z|+r)^{n-1}\, dr\, dz. \end{aligned}$$

Since ψ and F, as well as all their derivatives of any order, are bounded on $\mathbb{R}^n$, the integrand is bounded by a multiple of the function $|z|^{1-n}$, which is integrable on Ω. By differentiating inside the integral, it follows $v \in C^\infty(\Omega)$.

Finally, in order to get $\operatorname{supp} v \subset \Omega$, let

$$E = \{z \in \Omega : \ z = (1-\lambda)z_1 + \lambda z_2 \quad z_1 \in \operatorname{supp} F, \ z_2 \in \overline{B}, \ \lambda \in [0,1]\},$$

which is a compact subset of Ω by the hypotheses, and fix any $x \in \Omega \backslash E$. Now, it will be shown that $y + r(x-y) \notin \overline{B}$ for any $y \in \operatorname{supp} F$ and any $r > 1$. In fact, as it has been already seen in the proof of Lemma 6, x belongs to the segment of endpoints y and $y + r(x-y)$ for any $r \geq 1$. Thus, if $y + r(x-y) \in \overline{B}$, it would follow $x \in E$, that contradicts the assumption on x.

Hence, $\psi(y + r(x-y)) \equiv 0 \quad \forall x \in \Omega \backslash E$ and, by the formula in Theorem 1 *iv)*, it follows the theorem. □

Proof of Theorem 3. Let $\{F_k\} \subset C_0^\infty(\Omega)$ such that $F_k \to F$ in $L^q(\Omega)$ for some $q > n$, set to zero outside Ω, and let v_k and v the corresponding values obtained by the Bogovskiĭ's formula. By Theorem 1 *ii)* and *iii)*, it follows that v_k converge uniformly to v in $\mathbb{R}^n$. Since, by Theorem 2, $\{v_k\} \subset C_0^\infty(\mathbb{R}^n)$, the theorem follows immediately. □

2.2 Further properties of the kernel

In this section we prove some properties of the kernel appearing in the Bogovskiĭ's formulas which turn to be useful in the following. Next lemma provides an identity about the derivatives of the kernel as it appears in the second formula (Theorem 1 *iv)*).

Lemma 9. *For any fixed $x, y \in \Omega, \ x \neq y$ let*

$$N_i(x,y) := (x_i - y_i) \int_1^\infty \psi(y + \alpha(x-y))\, \alpha^{n-1} d\alpha.$$

Then, it follows that

$$\partial_{x_j} N_i(x,y) = (x_i - y_i) \int_1^\infty (\partial_j \psi)(y + \alpha(x-y))\, \alpha^{n-1} d\alpha - \partial_{y_j} N_i(x,y).$$

Proof. Since $|y + \alpha(x-y)| \geq 1$ for $\alpha \geq (1 + |y|)/|x-y|$ the integrand is bounded on a compact subset of $\mathbb{R}$. By differentiating under the sign of integral, it follows that

$$\begin{aligned} \partial_{x_j} N_i(x,y) = \delta_{ij} & \int_1^\infty \psi(y + \alpha(x-y))\, \alpha^{n-1} d\alpha \\ & + (x_i - y_i) \int_1^\infty (\partial_j \psi)(y + \alpha(x-y)) \alpha\, \alpha^{n-1}\, d\alpha, \end{aligned}$$

while

$$\partial_{y_j} N_i(x,y) = -\delta_{ij} \int_1^\infty \psi(y+\alpha(x-y))\, \alpha^{n-1} d\alpha \\ + (x_i - y_i) \int_1^\infty (\partial_j \psi)(y+\alpha(x-y))\,(1-\alpha)\alpha^{n-1}\, d\alpha,$$

and hence the lemma. □

The following result provides the fundamental estimates on $\partial_{x_j} N_i(x,y)$, which allow to exploit the Calderòn-Zygmund theory to obtain the original Bogovskiĭ's results about the $H_0^{1,p}(\Omega)$ regularity of v, and the Dini continuity hypothesis to prove the results below.

Theorem 10. *For any $i,j = 1,\dots,n$ there exist functions K_{ij} and G_{ij} such that*

$$\partial_{x_j} N_i(x,y) = K_{ij}(x, x-y) + G_{ij}(x,y)$$

where $K_{ij}(x,\cdot)$ is a Calderòn-Zygmund singular kernel and G_{ij} is a weakly singular kernel in the sense that, if one sets

$$k_{ij}(x,z) \equiv |z|^n K_{ij}(x,z),$$

then, there exist constants $c = c(\psi, n)$ and $M = M(\psi,\, n,\, \mathrm{diam}\ \Omega)$ such that:

i) $k_{ij}(x,tz) \;=\; k_{ij}(x,z) \quad \forall x \in \Omega,\ \forall z \neq 0,\ \forall t > 0$;

ii) $\|k_{ij}(x,z)\|_{L^\infty(\Omega \times S^{n-1})}$ *is finite;*

iii) $\int_{|z|=1} k_{ij}(x,z)\, dz = 0 \quad \forall x \in \Omega$;

iv) $|G_{ij}(x,y)| \le c|x-y|^{1-n}$;

v) $|\partial_{x_j} N_i(x,y)| \le M|x-y|^{-n} \quad \forall x \in \Omega \quad \forall y \in \mathbb{R}^n \backslash \{x\}$.

Proof. Fix $x, y \in \Omega,\ x \neq y$. By introducing $r = \alpha|x-y| - |x-y|$ it follows that

$$\begin{aligned} \partial_{x_j} N_i(x,y) &= \partial_{x_j} \left[(x_i - y_i) \int_1^\infty \psi(y+\alpha(x-y))\alpha^{n-1} d\alpha \right] \\ &= \delta_{ij} \int_1^\infty \psi(y+\alpha(x-y))\alpha^{n-1} d\alpha + (x_i - y_i) \int_1^\infty \partial_j \psi(y+\alpha(x-y))\alpha^n d\alpha \\ &= \frac{\delta_{ij}}{|x-y|^n} \int_0^\infty \psi\left(x + r\frac{x-y}{|x-y|}\right)(r+|x-y|)^{n-1} dr + \\ &\quad + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty \partial_j \psi\left(x + r\frac{x-y}{|x-y|}\right)(r+|x-y|)^n\, dr. \end{aligned}$$

By using the binomial expansion inside the integrals, the last sum may be written as follows

$$K_{ij}(x, x-y) + G_{ij}(x,y),$$

where

$$K_{ij}(x, x-y) = \frac{\delta_{ij}}{|x-y|^n} \int_0^\infty \psi\left(x + r\frac{x-y}{|x-y|}\right) r^{n-1}\, dr + \frac{x_i - y_i}{|x-y|^{n+1}} \int_0^\infty \partial_j \psi\left(x + r\frac{x-y}{|x-y|}\right) r^n\, dr,$$

involves only the terms not containing any strictly positive power of $|x-y|$, while all of the others, grouped as G, contain at least a factor $|x-y|$ coming from the expansion; therefore, G_{ij} verifies the estimate in *iv)*.

The homogeneity property in *i)* follows immediately from the previous expression of K_{ij}.

Furthermore, $z \in S^{n-1}$ implies

$$\begin{aligned} |k_{ij}(x,z)| &\le \left|\int_0^\infty \psi\,(x + r\,z)\, r^{n-1} dr\right| + \left|\int_0^\infty \partial_j\psi\,(x + r\,z)\, r^n dr\right| \\ &\le \|\psi\|_{L^\infty(\mathbb{R}^n)} (\text{diam }\Omega)^n + \|\partial_j \psi\|_{L^\infty(\mathbb{R}^n)} (\text{diam }\Omega)^{n+1}, \end{aligned}$$

and *ii)* follows.

Finally,

$$\begin{aligned} &\int_{|z|=1} k_{ij}(x,z)\, dz = \\ &\quad = \delta_{ij} \int_{|z|=1} \int_0^\infty \psi\,(x + rz)\, r^{n-1} dr + \int_{|z|=1} z_i \int_0^\infty \partial_j \psi\,(x + rz)\, r^n\, dr \\ &\quad = \int_{\mathbb{R}^n} \left[\delta_{ij} \psi(x+y) + y_i \partial_j \psi(x+y)\right]\, dy. \end{aligned}$$

After an integration by parts, the last integral turns out to be zero, and *iii)* is proved.

Finally, *ii)* and *iv)* imply immediately *v)*, on the bounded set Ω. □

2.3 The approximating functions for the solution

The main tool we applied in this paper is an old aged argument exploited by Korn (see, e.g., Gilbarg and Trudinger [15, Ch. 4]) in the study of the existence of classical solutions of the Poisson equation, based on a suitable "cutoff" of the singularity present in the second derivatives of the Newtonian potential, that provides a way to approximate the solution v by regular functions.

To this end we introduce the function $\eta \in C^\infty(\mathbb{R}^+)$ such that $\eta(t) \equiv 0$ on $[0,1]$, $\eta(t) \equiv 1$ if $t \ge 2$, and $|\eta'(t)| \le 2 \quad \forall t \in \mathbb{R}^+$.

Definition 11. *For any $q > n$, $F \in L^q(\Omega)$ and $\epsilon > 0$ let us set*

$$\begin{aligned} v^\epsilon(x) &= \int_\Omega F(y)\ (x-y)\ \left[\int_1^\infty \psi(y + \alpha(x-y))\ \alpha^{n-1} d\alpha\right]\ \eta\left(\frac{|x-y|}{\epsilon}\right)\, dy \\ &= \int_\Omega F(y)\ N(x,y)\ \eta\left(\frac{|x-y|}{\epsilon}\right)\, dy. \end{aligned}$$

We remark that if $|x-y|<\epsilon$ the integrand is zero and therefore the integrand belongs to $C^\infty(\mathbb{R}^n)$ and it is bounded by Lemma 5, while if $|x-y|\geq\epsilon$ the set of α such that $|y+\alpha(x-y)|\leq 1$, where ψ could be not null, is bounded as well. It follows immediately that v^ϵ is well-defined and belongs to $C^\infty(\mathbb{R}^n)$. Moreover, under the same hypotheses of Theorem 2 and following the same argument, one proves that it belongs actually to $C_0^\infty(\Omega)$.

Theorem 12. *Assume the same hypotheses of Theorem 1, and let η and v^ϵ as above. Then:*

i) If, in addition, $F\in L^\infty(\Omega)$, then

$$v^\epsilon \to v \text{ uniformly for } x\in\mathbb{R}^n;$$

ii) If, moreover, $\int_{\mathbb{R}^n}\psi(x)\,dx=1$ and $F\in C^0(\overline{\Omega})$, then

$$\lim_{\epsilon\to 0}\operatorname{div} v^\epsilon(x) = -\psi(x)\int_\Omega F(y)\,dy + F(x) \qquad \forall\, x\in\Omega.$$

Proof. To prove *i)*, let us fix any $x\in\Omega$. Remark that, by the Lemma 5, for any $\epsilon<\operatorname{dist}(x,\partial\Omega)$

$$\begin{aligned}|v_i^\epsilon(x)-v_i(x)| &\leq \int_\Omega |\,F(y)N_i(x,y)|\,\left|\eta\left(\frac{|x-y|}{\epsilon}\right)-1\right|\,dy\\ &\leq c\|F\|_{L^\infty(\Omega)}\int_{|x-y|<\epsilon}\frac{1}{|x-y|^{n-1}}\,dy \leq c\|F\||_{L^\infty(\Omega)}\int_{|z|<\epsilon}|z|^{1-n}\,dz,\end{aligned}$$

and by the absolute continuity of the last integral, it follows than it tends to zero independently of $x\in\Omega$. To complete the proof of *i)* it is enough to remark that by Lemma 7 $v^\epsilon(x)=v(x)=0\quad \forall x\notin\Omega$.

To prove *ii)*, by differentiating v_i^ϵ at any $x\in\Omega$ it follows that

$$\begin{aligned}\partial_{x_j}v_i^\epsilon(x) &= \int_\Omega F(y)\delta_{ij}\left[\int_1^\infty \psi(y+\alpha(x-y))\,\alpha^{n-1}d\alpha\right]dy\\ &+\int_\Omega F(y)(x_i-y_i)\left[\int_1^\infty(\partial_j\psi(y+\alpha(x-y))\,\alpha^n d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)dy\\ &+\int_\Omega F(y)(x_i-y_i)\left[\int_1^\infty\psi(y+\alpha(x-y))\,\alpha^{n-1}d\alpha\right]\eta'\left(\frac{|x-y|}{\epsilon}\right)\frac{x_j-y_j}{|x-y|}\epsilon^{-1}\,dy\\ &=\int_\Omega F(y)\delta_{ij}\left[\int_1^\infty\psi(y+\alpha(x-y))\,\alpha^{n-1}d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)dy\\ &+\int_\Omega F(y)(x_i-y_i)\left[\int_1^\infty(\partial_j\psi(y+\alpha(x-y))\,\alpha^n d\alpha\right]\eta\left(\frac{|x-y|}{\epsilon}\right)dy\\ &+\int_\Omega F(y)\frac{(x_i-y_i)(x_j-y_j)}{|x-y|}\left[\int_1^\infty\psi(y+\alpha(x-y))\,\alpha^{n-1}d\alpha\right]\eta'\left(\frac{|x-y|}{\epsilon}\right)\epsilon^{-1}dy,\end{aligned}$$

and therefore

$$
\begin{aligned}
\operatorname{div} v^\epsilon(x) &= n \int_\Omega F(y) \left[\int_1^\infty \psi(y+\alpha(x-y))\, \alpha^{n-1} d\alpha\right] \eta\left(\frac{|x-y|}{\epsilon}\right) dy \\
&\quad + \sum_{i=1}^n \int_\Omega F(y)(x_i - y_i) \left[\int_1^\infty (\partial_j \psi(y+\alpha(x-y))\, \alpha^n d\alpha\right] \eta\left(\frac{|x-y|}{\epsilon}\right) dy \\
&\quad + \int_\Omega F(y)|x-y| \left[\int_1^\infty \psi(y+\alpha(x-y))\, \alpha^{n-1} d\alpha\right] \eta'\left(\frac{|x-y|}{\epsilon}\right) \epsilon^{-1} dy \\
&\quad =: A + B + C.
\end{aligned}
$$

Now,

$$
\begin{aligned}
A + B &= \int_\Omega F(y) \eta\left(\frac{|x-y|}{\epsilon}\right) \quad \times \\
&\quad \times \int_1^\infty \left[\psi(y+\alpha(x-y))\, n\alpha^{n-1} + \alpha^n \sum_{i=1}^n \partial_{x_i}\psi(y+\alpha(x-y))(x_i-y_i)\right] d\alpha\, dy \\
&= \int_\Omega F(y)\, \eta\left(\frac{|x-y|}{\epsilon}\right) \int_1^\infty \frac{d}{d\alpha}\left[\psi(y+\alpha(x-y))\alpha^n\right] d\alpha\, dy \\
&= -\psi(x) \int_\Omega F(y)\, \eta\left(\frac{|x-y|}{\epsilon}\right) dy \quad \longrightarrow \quad -\psi(x) \int_\Omega F(y)\, dy,
\end{aligned}
$$

as ϵ tends to zero. Moreover, by setting into C first $\alpha = \xi/|x-y|$, next $\xi = r + |x-y|$, and finally $z = \epsilon^{-1}(x-y)$ one obtains

$$
\begin{aligned}
C &= \int_\Omega \frac{F(y)}{|x-y|^{n-1}}\, \eta'\left(\frac{|x-y|}{\epsilon}\right) \epsilon^{-1} \left[\int_{|x-y|}^\infty \psi\left(y + \xi \frac{x-y}{|x-y|}\right) \xi^{n-1}\, d\xi\right] dy \\
&= \int_{\epsilon<|x-y|<2\epsilon} \frac{F(y)}{|x-y|^{n-1}}\, \eta'\left(\frac{|x-y|}{\epsilon}\right) \epsilon^{-1} \times \\
&\qquad \times \left[\int_0^\infty \psi\left(x + r\frac{x-y}{|x-y|}\right) (r + |x-y|)^{n-1}\, dr\right] dy \\
&= \int_{1<|z|<2} \frac{F(x-\epsilon z)}{|z|^{n-1}}\, \eta'(|z|) \left[\int_0^\infty \psi\left(x + r\frac{z}{|z|}\right) (r+\epsilon|z|)^{n-1}\, dr\right] dz.
\end{aligned}
$$

Now we claim that, as ϵ goes to 0, the last term tends to

$$
F(x) \int_{1<|z|<2} \frac{1}{|z|^{n-1}} \eta'(|z|) \left[\int_0^\infty \psi\left(x + r\frac{z}{|z|}\right) r^{n-1}\, dr\right] dz, \qquad (\bigstar)
$$

In fact, since $\psi\left(x + r\frac{z}{|z|}\right)$ vanishes when $r > 1 + \operatorname{diam}\Omega$, the inner integral is bounded by $\max|\psi|\,(1+\operatorname{diam}\Omega)^n =: M$. Hence, we get

$$
\begin{aligned}
&\left|\int_{1<|z|<2} \frac{F(x-\epsilon\, z) - F(x)}{|z|^{n-1}}\, \eta'(|z|) \left[\int_0^\infty \psi\left(x + r\frac{z}{|z|}\right)(r+\epsilon|z|)^{n-1}\, dr\right] dz\right| \\
&\quad \le 2M \int_{1<|z|<2} \frac{|F(x-\epsilon\, z) - F(x)|}{|z|^{n-1}}\, dz \le 2M \int_{1<|z|<2} \frac{\omega(\epsilon\,|z|)}{|z|^{n-1}}\, dz,
\end{aligned}
$$

where ω is the modulus of continuity of F. By the uniform continuity of F on Ω and the Lebesgue theorem on dominated convergence, the last integral vanishes as ϵ goes to zero and therefore the claim is proved.

Finally, by introducing in the limit ($\bigstar$) the radial and angular coordinates $\rho = |z|$ and $u = z/|z|$, one gets

$$\int_1^2 \eta'(\rho)\, d\rho \int_{S^{n-1}} \int_0^\infty \psi(x+ru)\, r^{n-1}\, dr\, du = (\eta(2) - \eta(1)) \int_{\mathbb{R}^n} \psi(w)\, dw = 1.$$

Therefore, $C \longrightarrow F(x)$ as ϵ tends to 0 and the lemma follows. $\square$

The following theorem, an immediate corollary of the previous one, is the cornerstone of the resolution of the divergence problem.

Theorem 13. *Assume the same hypotheses of the previous theorem. Moreover, let* $\int_{\mathbb{R}^n} \psi(y)\, dy = 1$, $F \in C^0(\overline{\Omega})$ *and* $\int_\Omega F(x)\, dx = 0$. *Then, as* ϵ *goes to zero,*

$$\operatorname{div} v^\epsilon(x) \to F(x) \qquad \forall x \in \Omega.$$

The next lemma, an immediate consequence of Lemma 9 and the opposite sign in the derivatives of $\eta(|x-y|/\epsilon)$, will be useful in proving the subsequent representation formula.

Lemma 14.

$$\partial_{x_j} \left[N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right)\right] = -\partial_{y_j} \left[N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right)\right] + \\ + \eta\left(\frac{|x-y|}{\epsilon}\right) (x_i - y_i) \int_1^\infty \partial_j \psi(y + \alpha(x-y))\, \alpha^{n-1} d\alpha.$$

As usual in potential theory, getting a representation formula for the derivatives of the function v_i is a crucial goal. We will obtain it through a limit of the derivatives of its "regular approximation" v^ϵ. Thus, let us start by differentiating the formula

$$v_i^\epsilon(x) = \int_\Omega F(y) N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right) dy = \int_{B_R} F(y) N_i(x,y)\, \eta\left(\frac{|x-y|}{\epsilon}\right) dy,$$

where $F \in L^\infty(\Omega)$ is extended by zero outside Ω and B_R is a ball of radius large enough such that $\Omega \subset\subset B_R$. By the previous lemma and Bogovskiĭ's formula in Theorem 1 *iv)* it follows that

$$\begin{aligned}
\partial_{x_j} v_i^\epsilon(x) &= \int_{B_R} F(y)\, \partial_{x_j} \left[N_i(x,y) \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\
&= \int_{B_R} [F(y)-F(x)]\, \partial_{x_j} \left[N_i(x,y) \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\
&\quad + F(x) \int_{B_R} \partial_{x_j} \left[N_i(x,y) \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\
&= \int_{B_R} [F(y)-F(x)]\, \partial_{x_j} \left[N_i(x,y) \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy \\
&\quad + F(x) \int_{B_R} \eta\left(\frac{|x-y|}{\epsilon}\right) (x_i - y_i) \int_1^\infty \partial_j \psi(y+\alpha(x-y))\, \alpha^{n-1} d\alpha\, dy \\
&\quad - F(x) \int_{B_R} \partial_{y_j} \left[N_i(x,y) \eta\left(\frac{|x-y|}{\epsilon}\right)\right] dy.
\end{aligned}$$

Since $\eta\left(\frac{|x-y|}{\epsilon}\right) = 1$ if $\epsilon < \mathrm{dist}(\partial B_R, \overline{\Omega})$, by the Gauss-Green formula the last integral is equal to $\int_{\partial B_R} N_i(x,y)\nu_j(y) d\sigma_y$.

The previous computation suggests to put forward a conjecture about the limit as ϵ goes to zero, which will be proved in the next theorem, that is the main result of the paper.

Theorem 15. *Assume all the hypotheses of the Theorem 1, and let η and v^ϵ as above. Furthermore, let ψ be such that $\int_{\mathbb{R}^n} \psi = 1$, $F \in C_D(\Omega)$, and*

$$\begin{aligned}
u_j^i(x) &= \int_{B_R} [F(y)-F(x)]\, \partial_{x_j} N_i(x,y)\, dy \\
&\quad + F(x) \int_{B_R} (x_i - y_i) \int_1^\infty \partial_j \psi(y+\alpha(x-y))\, \alpha^{n-1} d\alpha\, dy - \\
&\quad - F(x) \int_{\partial B_R} N_i(x,y) \nu_j(y) d\sigma_y.
\end{aligned}$$

Then:

- *i)* $u_j^i(x)$ *is well-defined for any* $x \in \Omega$*;*
- *ii)* $\partial_{x_j} v_i^\epsilon$ *converges uniformly to* u_j^i *on any* $\Omega' \subset\subset \Omega$*;*
- *iii)* $\partial_{x_j} v \equiv u_j^i$ *on* Ω*;*
- *iv)* $v \in C^1(\Omega)$.

Proof. To prove *i)*, fix any $x \in \Omega$. Remark that, after its extension by zero outside Ω, $F \in L^\infty(\mathbb{R}^n)$. For any $\epsilon < \operatorname{dist}(x, \partial\,\Omega)$ one has

$$\begin{aligned}
&\int_{B_R} |F(y) - F(x)| \left|\partial_{x_j} N_i(x,y)\right| \, dy \\
&= \int_{B(x,\epsilon)} |F(y) - F(x)| \left|\partial_{x_j} N_i(x,y)\right| \, dy \\
&\qquad + \int_{\{|x-y| \geq \epsilon\} \cap B_R} |F(y) - F(x)| \left|\partial_{x_j} N_i(x,y)\right| \, dy \\
&=: A + C.
\end{aligned}$$

Since $B(x,\epsilon) \subset \Omega$, by Theorem 10 *v)* it follows that

$$\begin{aligned}
A &\leq \int_{B(x,\epsilon)} \frac{|F(y) - F(x)|}{|y-x|} |y-x| \, |\partial_{x_j} N_i(x,y)| \, dy \\
&\leq \int_{B(x,\epsilon)} \frac{\omega(|y-x|)}{|y-x|} \, \frac{M}{|y-x|^{n-1}} \, dy,
\end{aligned}$$

where ω is the modulus of continuity of F in Ω. By introducing the radial and angular coordinates, the last integral becomes

$$M|S^{n-1}| \int_0^\epsilon \frac{\omega(F,\rho)}{\rho} \, d\rho,$$

and, by the Dini continuity hypothesis on F, it is finite.

Furthermore, since both F and $\partial_{x_j} N_i(x,y)$ are bounded on $\{|x-y| \geq \epsilon\}$, the term C is finite as well.

Finally, since $\partial_j \psi \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp} \partial_j \psi \subset B$, it follows that

$$\int_\Omega (x_i - y_i) \int_1^\infty \partial_j \psi(y + \alpha(x-y)) \, \alpha^{n-1} d\alpha \, dy,$$

is the value of Bogovskiĭ's formula corresponding to the bounded function $F \equiv 1$, evaluated by using $\partial_j \psi$ instead of ψ. By Theorem 1 *iii)*, it is globally bounded, and *i)* follows.

To prove *ii)*, fix any $\Omega' \subset\subset \Omega$. Thus, for any $x \in \Omega'$ and $\epsilon > 0$ such that

$2\epsilon < \mathrm{dist}(\overline{\Omega'}, \partial\Omega)$, it follows that

$$
\begin{aligned}
&|\partial_{x_j} v_i^\epsilon(x) - u_j^i(x)| \leq \\
&\quad \leq \left| \int_{B_R} [F(y) - F(x)]\, \partial_{x_j} \left\{ N_i(x,y) \;\left[\eta\left(\frac{|x-y|}{\epsilon} \right) - 1 \right] \right\} dy \;\right| \\
&\quad + \left| F(x) \int_{B_R} (x_i - y_i) \int_1^\infty \partial_{x_j} \psi(y + \alpha(x-y)) \;\left[\eta\left(\frac{|x-y|}{\epsilon} \right) - 1 \right] \; \alpha^{n-1} d\alpha \, dy \;\right| \\
&\quad \leq \int_{B(x,2\epsilon)} |F(x) - F(y)| \; |\partial_{x_j} N_i(x,y)| \, dy + \\
&\quad + \int_{B(x,2\epsilon)} |F(y) - F(x)| \; |N_i(x,y)| \; \left| \eta'\left(\frac{|x-y|}{\epsilon} \right) \frac{x_j - y_j}{|x-y|} \; \epsilon^{-1} \right| \; dy \\
&\quad + \int_{B(x,2\epsilon)} |F(x)| \; |x_i - y_i| \int_1^\infty |\partial_{x_j} \psi(y + \alpha(x-y)) \,| \; \alpha^{n-1} d\alpha \, dy \,| \\
&\quad =: D + E + H.
\end{aligned}
$$

As above, by Theorem 10 *v)* it follows that

$$
D \leq M \int_{B(x,2\epsilon)} \frac{|F(x) - F(y)|}{|y-x|^n} \, dy \leq M \; |S^{n-1}| \; \int_{\rho < 2\epsilon} \frac{\omega(F,\rho)}{\rho} \, d\rho.
$$

By the Dini continuity of F and the consequent absolute continuity of the integral, the last term vanishes as ϵ goes to zero, independently of $x \in \Omega'$.

In order to estimate the second term E remark that, by Theorem 1 *v)* and the hypothesis on η'

$$
\begin{aligned}
E \leq & \int_{\epsilon \leq |x-y| \leq 2\epsilon} |F(x) - F(y)| \left| \eta'\left(\frac{|x-y|}{\epsilon} \right) \right| \frac{|x_j - y_j|}{|x-y|} \epsilon^{-1} \times \\
& \times \frac{|x_i - y_i|}{|x-y|^n} \int_0^\infty \left| \psi\left(x + r\frac{x-y}{|x-y|} \right) \right| (|x-y| + r)^{n-1} dr \, dy \\
\leq & \; 4 \int_{\epsilon \leq |x-y| \leq 2\epsilon} |F(x) - F(y)| \; \frac{1}{|x-y|^n} \times \\
& \quad \times \int_0^\infty \left| \psi\left(x + r\frac{x-y}{|x-y|} \right) \right| (|x-y| + r)^{n-1} dr \, dy.
\end{aligned}
$$

By introducing the variable $y = x + \rho u$, since

$$
\begin{aligned}
\int_0^\infty |\psi(x + ru)| (\rho + r)^{n-1} \, dr = & \int_0^{1+|x|} |\psi(x + ru)| \, (\rho + r)^{n-1} \, dr \\
& \leq \max_{\mathbb{R}^n} |\psi| \; (1 + \mathrm{diam}\; \Omega + 2\epsilon)^{n-1},
\end{aligned}
$$

it follows as above that the last term is bounded by a multiple of $\int_\epsilon^{2\epsilon} \frac{\omega(F,\rho)}{\rho} \, d\rho$ and, again by the absolute continuity of the integral, E vanishes as ϵ goes to zero, independently of $x \in \Omega'$.

Finally, by using $\partial_j\psi$ instead of ψ as in the proof of the previous *i)*, from Theorem 1 *iii)* it follows that for any $q > n$ and suitable constants c', c''

$$|H| \le c' \max_{\overline{\Omega}} |F(x)| \left\| \eta\left(\frac{|x-y|}{\epsilon}\right) - 1 \right\|_{L^q(\Omega)} \le$$

$$\le c'' \left\| \eta\left(\frac{|x-y|}{\epsilon}\right) - 1 \right\|_{L^q(B(x,\,\mathrm{diam}\ \Omega))}.$$

Since the last norm vanishes as ϵ goes to zero, for any $q > n$ and independently of $x \in \Omega$, *ii)* follows.

From *ii)*, by the classical theorem on a convergent sequence of functions whose derivatives converge uniformly, it follows *iii)*, while *iv)* follows immediately from *ii)*, *iii)*, since $v^\epsilon \in C^\infty(\mathbb{R}^n)$. $\square$

All the previous results lead to the following theorem.

Theorem 16. *Let $B = B(x_0, R)$ be an open ball in $\mathbb{R}^n, n \ge 2$, and let ψ be any function in $C_0^\infty(\mathbb{R}^n)$, verifying $\mathrm{supp}\ \psi \subseteq B$ and $\int_{\mathbb{R}^n} \psi = 1$. Let Ω be a bounded open subset of $\mathbb{R}^n$, star-shaped with respect to every point of $\overline{B}$. Then, for any $F \in C_D(\Omega)$ verifying $\int_\Omega F = 0$, the Bogovskiĭ's formula*

$$v(x) = \int_\Omega F(y) \left[\frac{x-y}{|x-y|^n} \int_{|x-y|}^{+\infty} \psi\left(y + \xi \frac{x-y}{|x-y|}\right) \xi^{n-1} d\xi \right] dy,$$

defines a solution $v \in C^1(\Omega) \cap C^0(\mathbb{R}^n)$ of the problem

$$\begin{cases} \mathrm{div}\, v(x) \;=\; F(x) & \textit{in } \Omega, \\ v \equiv 0 & \textit{on } \complement\Omega. \end{cases}$$

Proof. At first observe that, if $B = B(0,1)$, the theorem follows immediately from Theorem 1 *ii)*, Theorem 3, Theorem 13 and Theorem 15.

Otherwise, let us set $z = (x - x_0)/R$, $\widetilde{\Omega} = \{(x - x_0)/R : x \in \Omega\}$, $\widetilde{F}(z) = F(x_0 + Rz)$ and remark that $\widetilde{\Omega}$ and $\widetilde{F}$ fulfil the previous hypotheses with respect to $B = B(0,1)$. Now, let $w(z)$ be the solution of

$$\begin{cases} \mathrm{div}\, w(z) \;=\; \widetilde{F}(z) & \text{in } \widetilde{\Omega}, \\ w \equiv 0 & \text{on } \complement\widetilde{\Omega}, \end{cases}$$

whose existence follows by the initial observation, and remark that

$$\mathrm{div}\left[R\, w\left(\frac{x-x_0}{R}\right)\right] = (\mathrm{div}\, w)\left(\frac{x-x_0}{R}\right) = \widetilde{F}\left(\frac{x-x_0}{R}\right) = F(x).$$

Therefore

$$v(x) = R\, w\left(\frac{x-x_0}{R}\right),$$

is the requested solution, and the proof is completed. $\square$

3 Existence of classical solutions for the divergence problem in more general domains.

The aim of this brief section is to relax the very strong geometric restriction on the domain Ω requested in the above result, although at the price to renounce the simplicity of a single Bogovskiĭ's representation formula for the solution of the divergence problem.

The next theorem provides the existence of a classical solution in a wider class of domains including, for instance, those with a smooth boundary. To this aim, we start to prove a suitable "partition of unity" lemma.

Lemma 17. *Let Ω be a bounded subset of $\mathbb{R}^n$ with a locally Lipschitz boundary. Then, there exists an open covering $\mathcal{G} = \{G_1, \dots, G_m, G_{m+1}, \dots, G_{m+p}\}$ of $\overline{\Omega}$ such that, if one sets $\Omega_i := \Omega \cap G_i$ it follows that:*

- *Ω_i is star-shaped with respect to every point of an open ball B_i, with $\overline{B} \subset \Omega$ for any $i = 1, \dots, m+p$;*
- *$\partial\Omega \subset \cup_1^m G_i$;*
- *G_i is an open ball with closure in Ω for any $i = m+1, \dots, m+p$;*
- *$\Omega = \cup_1^{m+p} \Omega_i$.*

Furthermore, for any fixed $F \in C_D(\Omega)$ with $\int_\Omega F = 0$, there exist $F_i \in C_D(\Omega)$ such that:

i) $F_i \equiv 0 \quad$ on $\Omega \backslash \Omega_i$ for any $i = 1, \dots, m+p$

ii) $F \equiv \sum_1^{m+p} F_i$ on Ω

iii) $\int_\Omega F_i = 0$ for any $i = 1, \dots, m+p$

Proof. The proof if this result may be obtained as in Galdi [14, Lemma III.3.4], by replacing C_0^∞ with C_D in any occurrence involving f, f_i or g_i, by assuming Ω as their domain, and by extending ψ_i and χ_i by zero outside their supports. $\square$

Remark 18. *We remark explicitly that from i) and iii) it follows immediately the crucial property*

$$\int_{\Omega_i} F_i(x)\, dx = 0,$$

which, together with the properties of the covering, allows to apply the regularity result in Theorem 16 to the divergence problem "localized" at Ω_k.

The final result, which extends Theorem 16 to a considerably wider class of domains, will be now obtained by a localization argument.

Theorem 19. *Let Ω be a bounded subset of $\mathbb{R}^n$ with a locally Lipschitz boundary. Then, for any $F \in C_D(\Omega)$ with $\int_\Omega F(x)\,dx = 0$ there exists a solution $v \in C^1(\Omega) \cap C^0(\overline{\Omega})$ of the problem*

$$\begin{cases} \operatorname{div} v(x) = F(x) & \text{in } \Omega, \\ v \equiv 0 & \text{on } \partial\Omega. \end{cases}$$

Proof. Let Ω_k and F_k be defined as in the previous lemma, and let v_k be the solution in $C^1(\Omega_k) \cap C^0(\mathbb{R}^n)$ of the problem

$$\begin{cases} \operatorname{div} v_k(x) = F_k(x) & \text{in } \Omega_k, \\ v_k \equiv 0 & \text{on } \complement\Omega_k, \end{cases}$$

whose existence is ensured by Theorem 16, Lemma 17, and the last remark. Thus, by setting

$$v(x) = \sum_{1}^{m+p} v_k(x),$$

one obtains $v \in C^0(\mathbb{R}^n)$. Moreover, since v_k vanishes on $\complement\Omega_k$ then $v \equiv 0$ on $\partial\Omega$ and

$$v(x) = \sum_{k\colon \Omega_k \ni x} v_k(x),$$

and hence $v \in C^1(\Omega)$.

Finally, for any $x \in \Omega$

$$\operatorname{div} v(x) = \sum_{k\colon \Omega_k \ni x} \operatorname{div} v_k(x) = \sum_{k\colon \Omega_k \ni x} F_k(x),$$

and since by Lemma 17 *i)* and *ii)* the last term is equal to $F(x)$, and then v is the aimed solution. □

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