

Global regularity for systems with p -structure depending on the symmetric gradient

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ABSTRACT. In this paper we study on smooth bounded domains the global regularity (up to the boundary) for weak solutions to systems having p -structure depending only on the symmetric part of the gradient.

Keywords. Regularity of weak solutions, symmetric gradient, boundary regularity, natural quantities.

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1. Introduction

In this paper we study regularity of weak solutions to the boundary value problem

$$\begin{aligned} -\operatorname{div} \mathbf{S}(\mathbf{Du}) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \quad (1.1)$$

where $\mathbf{Du} := \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$ denotes the symmetric part of the gradient $\nabla \mathbf{u}$ and where¹ $\Omega \subset \mathbb{R}^3$ is a bounded domain with a $C^{2,1}$ boundary $\partial\Omega$. Our interest in this system comes from the p -Stokes system

$$\begin{aligned} -\operatorname{div} \mathbf{S}(\mathbf{Du}) + \nabla \pi &= \mathbf{f} && \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega. \end{aligned} \quad (1.2)$$

In both problems the typical example for \mathbf{S} we have in mind is

$$\mathbf{S}(\mathbf{Du}) = \mu(\delta + |\mathbf{Du}|)^{p-2} \mathbf{Du},$$

where $p \in (1, 2]$, $\delta \geq 0$, and $\mu > 0$. In previous investigations of (1.2) only suboptimal results for the regularity up to the boundary have been proved. Here we mean suboptimal in the sense that the results are weaker than the results known for p -Laplacian systems, cf. [1, 13, 14]. Clearly, the system (1.1) is obtained from (1.2) by dropping the divergence constraint and the resulting pressure gradient. Thus the system (1.1) lies in between system (1.2) and p -Laplacian systems, which depend on the full gradient $\nabla \mathbf{u}$.

We would like to stress that the system (1.1) is of own independent interest, since it is studied within plasticity theory, when formulated in the framework of deformation theory (cf. [11, 24]). In this context the unknown is the displacement vector field $\mathbf{u} = (u^1, u^2, u^3)^\top$, while the external body force $\mathbf{f} = (f^1, f^2, f^3)^\top$ is given. The stress tensor \mathbf{S} , which is the tensor of small elasto-plastic deformations,

¹We restrict ourselves to the problem in three space dimensions, even if results can be easily transferred to the problem in \mathbb{R}^d for all $d \geq 2$.

depends only on \mathbf{Du} . Physical interpretation and discussion of both systems (1.1) and (1.2) and the underlying models can be found, e.g., in [5, 11, 15, 19, 20].

We study global regularity properties of weak solutions to (1.1) in sufficiently smooth and bounded domains Ω ; we obtain for all $p \in (1, 2]$ the optimal result, namely that $\mathbf{F}(\mathbf{Du})$ belongs to $W^{1,2}(\Omega)$, where the nonlinear tensor-valued function \mathbf{F} is defined in (2.8). This result has been proved near a flat boundary in [24] and is the same result as for p -Laplacian systems (cf. [1, 13, 14]). The situation is quite different for (1.2). There the optimal result, i.e. $\mathbf{F}(\mathbf{Du}) \in W^{1,2}(\Omega)$, is only known for (i) two-dimensional bounded domains (cf. [16] where even the p -Navier-Stokes system is treated), (ii) the space-periodic problem in \mathbb{R}^d , $d \geq 2$, which follows immediately from interior estimates, i.e. $\mathbf{F}(\mathbf{Du}) \in W_{\text{loc}}^{1,2}(\Omega)$, which are known in all dimensions and the periodicity of the solution, (iii) if the no-slip boundary condition is replaced by perfect slip boundary conditions (cf. [17]), and (iv) in the case of small \mathbf{f} (cf. [6]). We also observe that the above results for the p -Stokes system (apart those in the space periodic setting) require the stress tensor to be non-degenerate, that is $\delta > 0$. In the case of homogeneous Dirichlet boundary conditions and three- and higher-dimensional bounded, sufficiently smooth domains only suboptimal results are known. To our knowledge the state of the art for general data is that $\mathbf{F}(\mathbf{Du}) \in W_{\text{loc}}^{1,2}(\Omega)$, tangential derivatives of $\mathbf{F}(\mathbf{Du})$ near the boundary belong to L^2 , while the normal derivative of $\mathbf{F}(\mathbf{Du})$ near the boundary belongs to some L^q , where $q = q(p) < 2$ (cf. [2, 4] and the discussion therein). We would also like to mention a result for another system between (1.2) and p -Laplacian system, namely if (1.2) is considered with \mathbf{S} depending on the full velocity gradient $\nabla \mathbf{u}$. In this case it is proved in [7] that $\mathbf{u} \in W^{2,r}(\mathbb{R}^3) \cap W_0^{1,p}(\mathbb{R}^3)$ for some $r > 3$, provided $p < 2$ is very close to 2.

In the present paper we extend to the general case of bounded sufficiently smooth domains and to possibly degenerate stress tensors, that is the case $\delta = 0$, the optimal regularity result for (1.1) of Seregin and Shilkin [24] in the case of a flat boundary. The precise result we prove is the following:

THEOREM 1.3. *Let the tensor field \mathbf{S} in (1.1) have (p, δ) -structure for some $p \in (1, 2]$, and $\delta \in [0, \infty)$, and let \mathbf{F} be the associated tensor field to \mathbf{S} . Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,1}$ boundary and let $\mathbf{f} \in L^{p'}(\Omega)$. Then, the unique weak solution $\mathbf{u} \in W_0^{1,p}(\Omega)$ of the problem (1.1) satisfies*

$$\int_{\Omega} |\nabla \mathbf{F}(\mathbf{Du})|^2 dx \leq c,$$

where c denotes a positive function which is non-decreasing in $\|\mathbf{f}\|_{p'}$ and δ , and which depends on the domain through its measure $|\Omega|$ and the $C^{2,1}$ -norms of the local description of $\partial\Omega$. In particular, the above estimate implies that $\mathbf{u} \in W^{2, \frac{3p}{p+1}}(\Omega)$.

2. Preliminaries and main results

In this section we introduce the notation we will use, state the precise assumptions on the extra stress tensor \mathbf{S} , and formulate the main results of the paper.

2.1. Notation. We use c, C to denote generic constants, which may change from line to line, but are independent of the crucial quantities. Moreover, we write $f \sim g$ if and only if there exists constants $c, C > 0$ such that $cf \leq g \leq Cf$. In some cases we need to specify the dependence on certain parameters, and consequently we denote by $c(\cdot)$ a positive function which is non-decreasing with respect to all its arguments.

We use standard Lebesgue spaces $(L^p(\Omega), \|\cdot\|_p)$ and Sobolev spaces $(W^{k,p}(\Omega), \|\cdot\|_{k,p})$, where $\Omega \subset \mathbb{R}^3$, is a sufficiently smooth bounded domain. The space

$W_0^{1,p}(\Omega)$ is the closure of the compactly supported, smooth functions $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. Thanks to the Poincaré inequality we equip $W_0^{1,p}(\Omega)$ with the gradient norm $\|\nabla \cdot\|_p$. When dealing with functions defined only on some open subset $G \subset \Omega$, we denote the norm in $L^p(G)$ by $\|\cdot\|_{p,G}$. As usual we use the symbol \rightharpoonup to denote weak convergence, and \rightarrow to denote strong convergence. The symbol $\text{spt } f$ denotes the support of the function f . We do not distinguish between scalar, vector-valued or tensor-valued function spaces. However, we denote vectors by boldface lower-case letter as e.g. \mathbf{u} and tensors by boldface upper case letters as e.g. \mathbf{S} . For vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ we denote $\mathbf{u} \otimes \mathbf{v} := \frac{1}{2}(\mathbf{u} \otimes \mathbf{v} + (\mathbf{u} \otimes \mathbf{v})^\top)$, where the standard tensor product $\mathbf{u} \otimes \mathbf{v} \in \mathbb{R}^{3 \times 3}$ is defined as $(\mathbf{u} \otimes \mathbf{v})_{ij} := u_i v_j$. The scalar product of vectors is denoted by $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^3 u_i v_i$ and the scalar product of tensors is denoted by $\mathbf{A} \cdot \mathbf{B} := \sum_{i,j=1}^3 A_{ij} B_{ij}$.

Greek lower-case letters take only values 1, 2, while Latin lower-case ones take the values 1, 2, 3. We use the summation convention over repeated indices only for Greek lower-case letters, but not for Latin lower-case ones.

2.2. (p, δ) -structure. We now define what it means that a tensor field \mathbf{S} has (p, δ) -structure, see [8, 23]. For a tensor $\mathbf{P} \in \mathbb{R}^{3 \times 3}$ we denote its symmetric part by $\mathbf{P}^{\text{sym}} := \frac{1}{2}(\mathbf{P} + \mathbf{P}^\top) \in \mathbb{R}_{\text{sym}}^{3 \times 3} := \{\mathbf{P} \in \mathbb{R}^{3 \times 3} \mid \mathbf{P} = \mathbf{P}^\top\}$. We use the notation $|\mathbf{P}|^2 = \mathbf{P} \cdot \mathbf{P}$.

It is convenient to define for $t \geq 0$ a special N-function² $\varphi(\cdot) = \varphi_{p,\delta}(\cdot)$, for $p \in (1, \infty)$, $\delta \geq 0$, by

$$\varphi(t) := \int_0^t (\delta + s)^{p-2} s \, ds. \quad (2.1)$$

The function φ satisfies, uniformly in t and independently of δ , the important equivalence

$$\varphi''(t)t \sim \varphi'(t), \quad (2.2)$$

$$\varphi'(t)t \sim \varphi(t), \quad (2.3)$$

$$t^p + \delta^p \sim \varphi(t) + \delta^p. \quad (2.4)$$

We use the convention that if $\varphi''(0)$ does not exist, the left-hand side in (2.2) is continuously extended by zero for $t = 0$. We define the shifted N-functions $\{\varphi_a\}_{a \geq 0}$, cf. [8, 9, 23], for $t \geq 0$ by

$$\varphi_a(t) := \int_0^t \frac{\varphi'(a+s)s}{a+s} \, ds$$

Note that the family $\{\varphi_a\}_{a \geq 0}$ satisfies the Δ_2 -condition uniformly with respect to $a \geq 0$, i.e. $\varphi_a(2t) \leq c(p)\varphi_a(t)$ holds for all $t \geq 0$.

DEFINITION 2.5 ((p, δ) -structure). *We say that a tensor field $\mathbf{S}: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}$ belonging to $C^0(\mathbb{R}^{3 \times 3}, \mathbb{R}_{\text{sym}}^{3 \times 3}) \cap C^1(\mathbb{R}^{3 \times 3} \setminus \{\mathbf{0}\}, \mathbb{R}_{\text{sym}}^{3 \times 3})$, satisfying $\mathbf{S}(\mathbf{P}) = \mathbf{S}(\mathbf{P}^{\text{sym}})$, and $\mathbf{S}(\mathbf{0}) = \mathbf{0}$ possesses (p, δ) -structure, if for some $p \in (1, \infty)$, $\delta \in [0, \infty)$, and the N-function $\varphi = \varphi_{p,\delta}$ (cf. (2.1)) there exist constants $\kappa_0, \kappa_1 > 0$ such that*

$$\sum_{i,j,k,l=1}^3 \partial_{kl} S_{ij}(\mathbf{P}) Q_{ij} Q_{kl} \geq \kappa_0 \varphi''(|\mathbf{P}^{\text{sym}}|) |\mathbf{Q}^{\text{sym}}|^2, \quad (2.6)$$

$$|\partial_{kl} S_{ij}(\mathbf{P})| \leq \kappa_1 \varphi''(|\mathbf{P}^{\text{sym}}|)$$

²For the general theory of N-functions and Orlicz spaces we refer to [21].

are satisfied for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3 \times 3}$ with $\mathbf{P}^{\text{sym}} \neq \mathbf{0}$ and all $i, j, k, l = 1, 2, 3$. The constants κ_0, κ_1 , and p are called the characteristics of \mathbf{S} .

REMARK 2.7. (i) Assume that \mathbf{S} has (p, δ) -structure for some $\delta \in [0, \delta_0]$. Then, if not otherwise stated, the constants in the estimates depend only on the characteristics of \mathbf{S} and on δ_0 , but are independent of δ .

(ii) An important example of a tensor field \mathbf{S} having (p, δ) -structure is given by $\mathbf{S}(\mathbf{P}) = \varphi'(|\mathbf{P}^{\text{sym}}|)|\mathbf{P}^{\text{sym}}|^{-1}\mathbf{P}^{\text{sym}}$. In this case the characteristics of \mathbf{S} , namely κ_0 and κ_1 , depend only on p and are independent of $\delta \geq 0$.

To a tensor field \mathbf{S} with (p, δ) -structure we associate the tensor field $\mathbf{F}: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}$ defined through

$$\mathbf{F}(\mathbf{P}) := (\delta + |\mathbf{P}^{\text{sym}}|)^{\frac{p-2}{2}} \mathbf{P}^{\text{sym}}. \quad (2.8)$$

The connection between \mathbf{S} , \mathbf{F} , and $\{\varphi_a\}_{a \geq 0}$ is best explained by the following proposition (cf. [8], [23]).

PROPOSITION 2.9. *Let \mathbf{S} have (p, δ) -structure, and let \mathbf{F} be defined in (2.8). Then*

$$(\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})) \cdot (\mathbf{P} - \mathbf{Q}) \sim |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2, \quad (2.10a)$$

$$\sim \varphi_{|\mathbf{P}^{\text{sym}}|}(|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|), \quad (2.10b)$$

$$\sim \varphi''(|\mathbf{P}^{\text{sym}}| + |\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|)|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|^2, \quad (2.10c)$$

$$|\mathbf{S}(\mathbf{P}) - \mathbf{S}(\mathbf{Q})| \sim \varphi'_{|\mathbf{P}^{\text{sym}}|}(|\mathbf{P}^{\text{sym}} - \mathbf{Q}^{\text{sym}}|), \quad (2.10d)$$

uniformly in $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{3 \times 3}$. Moreover, uniformly in $\mathbf{Q} \in \mathbb{R}^{3 \times 3}$,

$$\mathbf{S}(\mathbf{Q}) \cdot \mathbf{Q} \sim |\mathbf{F}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}^{\text{sym}}|). \quad (2.10e)$$

The constants depend only on the characteristics of \mathbf{S} .

For a detailed discussion of the properties of \mathbf{S} and \mathbf{F} and their relation to Orlicz spaces and N-functions we refer the reader to [23, 3]. Since in the following we shall insert into \mathbf{S} and \mathbf{F} only symmetric tensors, we can drop in the above formulas the superscript “sym” and restrict the admitted tensors to symmetric ones.

We recall that the following equivalence, which is proved in [3, Lemma 3.8],

$$|\partial_i \mathbf{F}(\mathbf{Q})|^2 \sim \varphi''(|\mathbf{Q}|)|\partial_i \mathbf{Q}|^2, \quad (2.11)$$

valid for all smooth enough symmetric tensor fields $\mathbf{Q} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$. The proof of this equivalence is based on Proposition 2.9. This Proposition and the theory of divided differences also imply (cf. [4, (2.26)]) that

$$|\partial_\tau \mathbf{F}(\mathbf{Q})|^2 \sim \varphi''(|\mathbf{Q}|)|\partial_\tau \mathbf{Q}|^2 \quad (2.12)$$

for all smooth enough symmetric tensor fields $\mathbf{Q} \in \mathbb{R}_{\text{sym}}^{3 \times 3}$.

A crucial observation in [24] is that the quantities in (2.11) are also equivalent to several further quantities. To formulate this precisely we introduce for $i = 1, 2, 3$ and for sufficiently smooth symmetric tensor fields \mathbf{Q} the quantity

$$\mathcal{P}_i(\mathbf{Q}) := \partial_i \mathbf{S}(\mathbf{Q}) \cdot \partial_i \mathbf{Q} = \sum_{k,l,m,n=1}^3 \partial_{kl} S_{mn}(\mathbf{Q}) \partial_i Q_{kl} \partial_i Q_{mn}. \quad (2.13)$$

Recall, that in the definition of $\mathcal{P}_i(\mathbf{Q})$ there is no summation convention over the repeated Latin lower-case index i in $\partial_i \mathbf{S}(\mathbf{Q}) \cdot \partial_i \mathbf{Q}$. Note that if \mathbf{S} has (p, δ) -structure, then $\mathcal{P}_i(\mathbf{v}) \geq 0$, for $i = 1, 2, 3$. There hold the following important equivalences, first proved in [24]:

PROPOSITION 2.14. *Assume that \mathbf{S} has (p, δ) -structure. Then the following equivalences are valid, for all smooth enough symmetric tensor fields \mathbf{Q} and all $i = 1, 2, 3$*

$$\mathcal{P}_i(\mathbf{Q}) \sim \varphi''(|\mathbf{Q}|)|\partial_i \mathbf{Q}|^2 \sim |\partial_i \mathbf{F}(\mathbf{Q})|^2, \quad (2.15)$$

$$\mathcal{P}_i(\mathbf{Q}) \sim \frac{|\partial_i \mathbf{S}(\mathbf{Q})|^2}{\varphi''(|\mathbf{Q}|)}, \quad (2.16)$$

with constants only depending on the characteristics of \mathbf{S} .

PROOF. The assertions are proved in [24] using a different notation. For the convenience of the reader we sketch the proof here. The equivalences in (2.15) follow from (2.11), (2.13) and the fact that \mathbf{S} has (p, δ) -structure. Furthermore, we have, using (2.15),

$$|\mathcal{P}_i(\mathbf{Q})|^2 \leq |\partial_i \mathbf{S}(\mathbf{Q})|^2 |\partial_i \mathbf{Q}|^2 \leq c |\partial_i \mathbf{S}(\mathbf{Q})|^2 \frac{\mathcal{P}_i(\mathbf{Q})}{\varphi''(|\mathbf{Q}|)},$$

which proves one inequality of (2.16). The other follows from

$$|\partial_i \mathbf{S}(\mathbf{Q})|^2 \leq c \sum_{k,l=1}^3 |\partial_{kl} \mathbf{S}(\mathbf{Q}) \partial_i Q_{kl}|^2 \leq c (\varphi''(|\mathbf{Q}|))^2 |\partial_i \mathbf{Q}|^2 \leq c \varphi''(|\mathbf{Q}|) \mathcal{P}_i(\mathbf{Q}),$$

where we used (2.6) and (2.15). \square

2.3. Existence of weak solutions. In this section we define weak solutions of (1.1), recall the main results of existence and uniqueness and discuss a perturbed problem, which is used to justify the computations that follow. From now on we restrict ourselves to the case $p \leq 2$.

DEFINITION 2.17. *We say that $\mathbf{u} \in W_0^{1,p}(\Omega)$ is a weak solution to (1.1) if for all $\mathbf{v} \in W_0^{1,p}(\Omega)$*

$$\int_{\Omega} \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}.$$

We have the following very standard result:

PROPOSITION 2.18. *Let the tensor field \mathbf{S} in (1.1) have (p, δ) -structure for some $p \in (1, 2]$, and $\delta \in [0, \infty)$. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,1}$ boundary and let $\mathbf{f} \in L^{p'}(\Omega)$. Then, there exists a unique weak solution \mathbf{u} to (1.1) such that*

$$\int_{\Omega} \varphi(|\mathbf{D}\mathbf{u}|) \, d\mathbf{x} \leq c(\|\mathbf{f}\|_{p'}, \delta).$$

PROOF. The assertions follow directly from the assumptions, by using the theory of monotone operators. \square

In order to justify some of the following computations we find it convenient to consider a perturbed problem, where we add to the tensor field \mathbf{S} with (p, δ) -structure a linear perturbation. Using again the theory of monotone operators one can easily prove:

PROPOSITION 2.19. *Let the tensor field \mathbf{S} in (1.1) have (p, δ) -structure for some $p \in (1, 2]$, and $\delta \in [0, \infty)$ and let $\mathbf{f} \in L^{p'}(\Omega)$ be given. Then, there exists a unique weak solution $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega)$ of the problem*

$$\begin{aligned} -\operatorname{div} \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u}_\varepsilon &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned} \quad (2.20)$$

where

$$\mathbf{S}^\varepsilon(\mathbf{Q}) := \varepsilon \mathbf{Q} + \mathbf{S}(\mathbf{Q}), \quad \text{with } \varepsilon > 0,$$

i.e. \mathbf{u}_ε satisfies for all $\mathbf{v} \in W_0^{1,2}(\Omega)$

$$\int_{\Omega} \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot \mathbf{D}\mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}.$$

The solution \mathbf{u}_ε satisfies the estimate

$$\varepsilon \int_{\Omega} |\nabla \mathbf{u}_\varepsilon|^2 + \varphi(|\mathbf{D}\mathbf{u}_\varepsilon|) \, d\mathbf{x} \leq c(\|\mathbf{f}\|_{p'}, \delta). \quad (2.21)$$

REMARK 2.22. *In fact, one could already prove more at this point. Namely, that for $\varepsilon \rightarrow 0$, the unique solution \mathbf{u}_ε converges to the unique weak solution \mathbf{u} of the unperturbed problem (1.1). Let us sketch the argument only, since later we get the same result with different easier arguments. From (2.21) and the properties of \mathbf{S} follows that*

$$\begin{aligned} \mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{in } W_0^{1,p}(\Omega), \\ \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon) &\rightharpoonup \boldsymbol{\chi} && \text{in } L^{p'}(\Omega). \end{aligned}$$

Passing to the limit in the weak formulation of the perturbed problem, we get

$$\int_{\Omega} \boldsymbol{\chi} \cdot \mathbf{D}\mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v} \in W_0^{1,p}(\Omega).$$

One can not show directly that $\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \varepsilon \mathbf{D}\mathbf{u}_\varepsilon \cdot (\mathbf{D}\mathbf{u}_\varepsilon - \mathbf{D}\mathbf{u}) \, d\mathbf{x} = 0$, since $\mathbf{D}\mathbf{u}$ belongs to $L^p(\Omega)$ only. Instead one uses the Lipschitz truncation method (cf. [10, 22]). Denoting by $\mathbf{v}^{\varepsilon,j}$ the Lipschitz truncation of $\xi(\mathbf{u}_\varepsilon - \mathbf{u})$, where $\xi \in C_0^\infty(\Omega)$ is a localization, one can show, using the ideas from [10, 22], that

$$\limsup_{\varepsilon \rightarrow 0} \left| \int_{\Omega} (\mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon) - \mathbf{S}(\mathbf{D}\mathbf{u})) \cdot \mathbf{D}\mathbf{v}^{\varepsilon,j} \, d\mathbf{x} \right| = 0, \quad (2.23)$$

which implies $\mathbf{D}\mathbf{u}_\varepsilon \rightarrow \mathbf{D}\mathbf{u}$ almost everywhere in Ω . Consequently, we have $\boldsymbol{\chi} = \mathbf{S}(\mathbf{D}\mathbf{u})$, since weak and a.e. limits coincide.

2.4. Description and properties of the boundary. We assume that the boundary $\partial\Omega$ is of class $C^{2,1}$, that is for each point $P \in \partial\Omega$ there are local coordinates such that in these coordinates we have $P = 0$ and $\partial\Omega$ is locally described by a $C^{2,1}$ -function, i.e., there exist $R_P, R'_P \in (0, \infty)$, $r_P \in (0, 1)$ and a $C^{2,1}$ -function $a_P : B_{R_P}^2(0) \rightarrow B_{R'_P}^1(0)$ such that

- (b1) $\mathbf{x} \in \partial\Omega \cap (B_{R_P}^2(0) \times B_{R'_P}^1(0)) \iff x_3 = a_P(x_1, x_2)$,
- (b2) $\Omega_P := \{(x, x_3) \mid x = (x_1, x_2)^\top \in B_{R_P}^2(0), a_P(x) < x_3 < a_P(x) + R'_P\} \subset \Omega$,
- (b3) $\nabla a_P(0) = \mathbf{0}$, and $\forall x = (x_1, x_2)^\top \in B_{R_P}^2(0) \quad |\nabla a_P(x)| < r_P$,

where $B_r^k(0)$ denotes the k -dimensional open ball with center 0 and radius $r > 0$. Note that r_P can be made arbitrarily small if we make R_P small enough. In the sequel we will also use, for $0 < \lambda < 1$, the following scaled open sets, $\lambda\Omega_P \subset \Omega_P$ defined as follows

$$\lambda\Omega_P := \{(x, x_3) \mid x = (x_1, x_2)^\top \in B_{\lambda R_P}^2(0), a_P(x) < x_3 < a_P(x) + \lambda R'_P\}.$$

To localize near to $\partial\Omega \cap \partial\Omega_P$, for $P \in \partial\Omega$, we fix smooth functions $\xi_P : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that

$$(\ell 1) \quad \chi_{\frac{1}{2}\Omega_P}(\mathbf{x}) \leq \xi_P(\mathbf{x}) \leq \chi_{\frac{3}{4}\Omega_P}(\mathbf{x}),$$

where $\chi_A(\mathbf{x})$ is the indicator function of the measurable set A . For the remaining interior estimate we localize by a smooth function $0 \leq \xi_{00} \leq 1$ with $\text{spt } \xi_{00} \subset \Omega_{00}$, where $\Omega_{00} \subset \Omega$ is an open set such that $\text{dist}(\partial\Omega_{00}, \partial\Omega) > 0$. Since the boundary $\partial\Omega$ is compact, we can use an appropriate finite sub-covering which, together with the interior estimate, yields the global estimate.

Let us introduce the tangential derivatives near the boundary. To simplify the notation we fix $P \in \partial\Omega$, $h \in (0, \frac{R_P}{16})$, and simply write $\xi := \xi_P$, $a := a_P$. We use the standard notation $\mathbf{x} = (x', x_3)^\top$ and denote by $\mathbf{e}^i, i = 1, 2, 3$ the canonical orthonormal basis in \mathbb{R}^3 . In the following lower-case Greek letters take values 1, 2. For a function g with $\text{spt } g \subset \text{spt } \xi$ we define for $\alpha = 1, 2$

$$g_\tau(x', x_3) = g_{\tau_\alpha}(x', x_3) := g(x' + h \mathbf{e}^\alpha, x_3 + a(x' + h \mathbf{e}^\alpha) - a(x')),$$

and if $\Delta^+ g := g_\tau - g$, we define tangential divided differences by $d^+ g := h^{-1} \Delta^+ g$. It holds that, if $g \in W^{1,1}(\Omega)$, then we have for $\alpha = 1, 2$

$$d^+ g \rightarrow \partial_\tau g = \partial_{\tau_\alpha} g := \partial_\alpha g + \partial_\alpha a \partial_3 g \quad \text{as } h \rightarrow 0, \quad (2.24)$$

almost everywhere in $\text{spt } \xi$, (cf. [18, Sec. 3]). Conversely uniform L^q -bounds for $d^+ g$ imply that $\partial_\tau g$ belongs to $L^q(\text{spt } \xi)$.

For simplicity we denote $\nabla a := (\partial_1 a, \partial_2 a, 0)^\top$. The following variant of integration per parts will be often used.

LEMMA 2.25. *Let $\text{spt } g \cup \text{spt } f \subset \text{spt } \xi$ and h small enough. Then*

$$\int_{\Omega} f g_{-\tau} d\mathbf{x} = \int_{\Omega} f_\tau g d\mathbf{x}.$$

Consequently, $\int_{\Omega} f d^+ g d\mathbf{x} = \int_{\Omega} (d^- f) g d\mathbf{x}$. Moreover, if in addition f and g are smooth enough and at least one vanishes on $\partial\Omega$, then

$$\int_{\Omega} f \partial_\tau g d\mathbf{x} = - \int_{\Omega} (\partial_\tau f) g d\mathbf{x}.$$

3. Proof of the main result

In the proof of the main result we use finite differences to show estimates in the interior and in tangential directions near the boundary and calculations involving directly derivatives in "normal" directions near the boundary. In order to justify that all occurring quantities are well posed, we perform the estimate for the approximate system (2.20).

The first intermediate step is the following result for the approximate problem.

PROPOSITION 3.1. *Let the tensor field \mathbf{S} in (1.1) have (p, δ) -structure for some $p \in (1, 2]$, and $\delta \in (0, \infty)$, and let \mathbf{F} be the associated tensor field to \mathbf{S} . Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with $C^{2,1}$ boundary and let $\mathbf{f} \in L^{p'}(\Omega)$. Then, the unique weak solution $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega)$ of the approximate problem (2.20) satisfies*

$$\begin{aligned} \varepsilon \int_{\Omega} \xi_0^2 |\nabla^2 \mathbf{u}_\varepsilon|^2 + \xi_0^2 |\nabla \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 d\mathbf{x} &\leq c(\|\mathbf{f}\|_{p'}, \|\xi_0\|_{2,\infty}, \delta), \\ \varepsilon \int_{\Omega} \xi_P^2 |\partial_\tau \mathbf{D}\mathbf{u}_\varepsilon|^2 + \xi_P^2 |\partial_\tau \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 d\mathbf{x} &\leq c(\|\mathbf{f}\|_{p'}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}, \delta). \end{aligned} \quad (3.2)$$

Here ξ_0 is a cut-off function with support in the interior of Ω , while for arbitrary $P \in \partial\Omega$ the function ξ_P is a cut-off function with support near to the boundary

$\partial\Omega$, as defined in Sec. 2.4. The tangential derivative ∂_τ is defined locally in Ω_P by (2.24). Moreover, there exists a constant $C_1 > 0$ such that

$$\varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 + \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \leq c(\|\mathbf{f}\|_{p'}, \|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}, \delta^{-1}, \varepsilon^{-1}, C_1) \quad (3.3)$$

provided that in the local description of the boundary there holds $r_P < C_1$ in (b3).

In particular, these estimates imply that $\mathbf{u}_\varepsilon \in W^{2,2}(\Omega)$ and that (2.20) holds pointwise a.e. in Ω .

The two estimates (3.2) are uniform with respect to ε and could be also proved directly for the problem (1.1). However, the third estimate (3.3) depends on ε but is needed to justify all subsequent steps, which will give the proof of an estimate uniformly in ε , by using a different technique.

PROOF OF PROPOSITION 3.1. The proof of this result is very similar, being in fact a simplification (due to the fact that there is no pressure term involved), to the proof of the results in [4, Theorems 2.27, 2.28]. For the convenience of the reader we recall the main steps here.

Fix $P \in \partial\Omega$ and use in Ω_P

$$\mathbf{v} = d^-(\xi^2 d^+(\mathbf{u}_\varepsilon|_{\frac{1}{2}\Omega_P})),$$

where $\xi := \xi_P$, $a := a_P$, and $h \in (0, \frac{R_P}{16})$, as a test function in the weak formulation of (2.20). This yields

$$\begin{aligned} & \int_{\Omega} \xi^2 d^+ \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot d^+ \mathbf{D}\mathbf{u}_\varepsilon dx \\ &= - \int_{\Omega} \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot (\xi^2 d^+ \partial_3 \mathbf{u}_\varepsilon - (\xi_{-\tau} d^- \xi + \xi d^- \xi) \partial_3 \mathbf{u}_\varepsilon) \overset{\circ}{\otimes} d^- \nabla a dx \\ & \quad - \int_{\Omega} \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot \xi^2 (\partial_3 \mathbf{u}_\varepsilon)_\tau \overset{\circ}{\otimes} d^- d^+ \nabla a - \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot d^- (2\xi \nabla \xi \overset{\circ}{\otimes} d^+ \mathbf{u}_\varepsilon) dx \\ & \quad + \int_{\Omega} \mathbf{S}^\varepsilon((\mathbf{D}\mathbf{u}_\varepsilon)_\tau) \cdot (2\xi \partial_3 \xi d^+ \mathbf{u}_\varepsilon + \xi^2 d^+ \partial_3 \mathbf{u}_\varepsilon) \overset{\circ}{\otimes} d^+ \nabla a dx \\ & \quad + \int_{\Omega} \mathbf{f} \cdot d^-(\xi^2 d^+ \mathbf{u}_\varepsilon) dx =: \sum_{j=1}^8 I_j. \end{aligned}$$

From the assumption on \mathbf{S} , Proposition 2.9, and [4, Lemma 3.11] we have the following estimate

$$\begin{aligned} & \varepsilon \int_{\Omega} \xi^2 |d^+ \nabla \mathbf{u}_\varepsilon|^2 + \xi^2 |\nabla d^+ \mathbf{u}_\varepsilon|^2 + |d^+ \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 + \varphi(\xi |\nabla d^+ \mathbf{u}|) + \varphi(\xi |d^+ \nabla \mathbf{u}|) dx \\ & \leq c \int_{\Omega} \xi^2 d^+ \mathbf{S}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \cdot d^+ \mathbf{D}\mathbf{u}_\varepsilon dx + c(\|\xi\|_{1,\infty}, \|a\|_{C^{1,1}}) \int_{\Omega \cap \text{spt } \xi} \varphi(|\nabla \mathbf{u}_\varepsilon|) dx. \end{aligned}$$

The terms I_1 – I_7 are estimated exactly as in [4, (3.17)–(3.22)], while I_8 is estimated as the term I_{15} in [4, (4.20)]. Thus, we get

$$\begin{aligned} & \int_{\Omega} \varepsilon \xi^2 |d^+ \nabla \mathbf{u}_\varepsilon|^2 + \varepsilon \xi^2 |\nabla d^+ \mathbf{u}_\varepsilon|^2 + \xi^2 |d^+ \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 + \varphi(\xi |d^+ \nabla \mathbf{u}_\varepsilon|) + \varphi(\xi |\nabla d^+ \mathbf{u}_\varepsilon|) dx \\ & \leq c(\|\mathbf{f}\|_{p'}, \|\xi\|_{2,\infty}, \|a\|_{C^{2,1}}, \delta). \end{aligned}$$

This proves the second estimate in (3.2) by standard arguments. The first estimate in (3.2) is proved in the same way with many simplifications, since we work in the interior where the method works for all directions. This estimate implies that $\mathbf{u}_\varepsilon \in W_{\text{loc}}^{2,2}(\Omega)$ and that the system (2.20) is well-defined point-wise a.e. in Ω .

To estimate the derivatives in the x_3 direction we use equation (2.20). Denoting, for $\alpha, \gamma = 1, 2$, $A_{\alpha\gamma} := \partial_{\gamma 3} S_{\alpha 3}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon)$, $\mathbf{b}_\gamma := \partial_3 D_{\gamma 3} \mathbf{u}_\varepsilon$, and³ $\mathbf{f}_\alpha := f_\alpha + \partial_{33} S_{\alpha 3}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{33} \mathbf{u}_\varepsilon + \partial_{\gamma\sigma} S_{\alpha 3}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 D_{\gamma\sigma} \mathbf{u}_\varepsilon + \sum_{k,l=1}^3 \partial_{kl} S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_\beta D_{kl} \mathbf{u}_\varepsilon$, we can re-write the first two equations in (2.20) as follows

$$-2A_{\alpha\gamma} \mathbf{b}_\gamma = \mathbf{f}_\alpha \quad \text{a.e. in } \Omega.$$

By straightforward manipulations (cf. [4, Sections 3.2 and 4.2]) we get

$$\begin{aligned} & (\varepsilon + \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)) |\mathbf{b}| \\ & \leq c (|\mathbf{f}| + |\mathbf{f}| \|\nabla a\|_\infty + (\varepsilon + \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)) (|\partial_\tau \nabla \mathbf{u}_\varepsilon| + \|\nabla a\|_\infty |\nabla^2 \mathbf{u}_\varepsilon|)). \end{aligned}$$

Note that we can deduce from this inequality information about $\tilde{\mathbf{b}}_\gamma := \partial_{33}^2 u_\varepsilon^\gamma$, because $|\mathbf{b}| \geq 2|\tilde{\mathbf{b}}| - |\partial_\tau \nabla \mathbf{u}_\varepsilon| - \|\nabla a\|_\infty |\nabla^2 \mathbf{u}_\varepsilon|$. Adding on both sides, for $\alpha = 1, 2$ and $i, k = 1, 2, 3$ the term

$$(\varepsilon + \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)) (|\partial_\alpha \partial_i u_\varepsilon^k| + |\partial_{33}^2 u_\varepsilon^3|),$$

we finally arrive at

$$\begin{aligned} & (\varepsilon + \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)) |\nabla^2 \mathbf{u}_\varepsilon| \\ & \leq c (|\mathbf{f}| + |\mathbf{f}| \|\nabla a\|_\infty + (\varepsilon + \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)) (|\partial_\tau \nabla \mathbf{u}_\varepsilon| + \|\nabla a\|_\infty |\nabla^2 \mathbf{u}_\varepsilon|)), \end{aligned}$$

where the constant c only depends on the characteristics of \mathbf{S} . Next, we can choose the open sets Ω_P in such a way that $\|\nabla a_P(x)\|_{\infty, \Omega_P}$ is small enough, so that we can absorb the last term from the right hand side, which yields

$$(\varepsilon + \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)) |\nabla^2 \mathbf{u}_\varepsilon| \leq c (|\mathbf{f}| + (\varepsilon + \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)) |\partial_\tau \nabla \mathbf{u}_\varepsilon|) \quad \text{a.e. in } \Omega_P,$$

where again the constant c only depends on the characteristics of \mathbf{S} . By neglecting the second term on the left-hand side (which is non-negative), raising the remaining inequality to the power 2, and using that \mathbf{S} has (p, δ) -structure for $p < 2$ we obtain

$$\varepsilon \int_{\Omega} \xi_P^2 |\nabla^2 \mathbf{u}_\varepsilon|^2 dx \leq c \int_{\Omega} |\mathbf{f}|^2 dx + \frac{(\varepsilon + \delta^{2(p-2)})}{\varepsilon} \left(\varepsilon \int_{\Omega} \xi_P^2 |\partial_\tau \nabla \mathbf{u}_\varepsilon|^2 dx \right)$$

The already proven results on tangential derivatives and Korn's inequality imply that the last integral from right-hand side is finite. Thus, the properties of the covering imply the last estimate in (3.2). \square

3.1. Improved estimates for normal derivatives. The proof of (3.3) used the system (2.20) and resulted in an estimate that is not uniform with respect to ε . In this section, by following the ideas in [24], we proceed differently and estimate \mathcal{P}_3 in terms of quantities occurring in (3.2). The main technical step of the paper is the proof of the following result:

PROPOSITION 3.4. *Let the same hypotheses as in Theorem 1.3 be satisfied with $\delta > 0$ and let the local description a_P of the boundary and the localization function ξ_P satisfy (b1)–(b3) and (l1) (cf. Section 2.4). Then, there exist a constant $C_2 > 0$*

³Recall that we use the summation convention over repeated Greek lower-case letters from 1 to 2.

such that the weak solution $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega)$ of the approximate problem (2.20) satisfies for every $P \in \partial\Omega$

$$\varepsilon \int_{\Omega} \xi_P^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + \int_{\Omega} \xi_P^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \leq C(\|\mathbf{f}\|_{p'}, \|\xi_P\|_{2,\infty}, \|a_P\|_{C^{2,1}}, \delta, C_2),$$

provided $r_P < C_2$ in (b3).

PROOF. Let us fix an arbitrary point $P \in \partial\Omega$ and a local description $a = a_P$ of the boundary and the localization function $\xi = \xi_P$ satisfying (b1)–(b3) and (l1). In the following we denote by C constants that depend only on the characteristics of \mathbf{S} . First we observe that, by the results of Proposition 2.14 there exists a constant C_0 , depending only on the characteristics of \mathbf{S} , such that

$$\frac{1}{C_0} |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 \leq \mathcal{P}_3(\mathbf{D}\mathbf{u}_\varepsilon) \quad \text{a.e. in } \Omega.$$

Thus, we get, using also the symmetry of $\mathbf{D}\mathbf{u}_\varepsilon$ and \mathbf{S} ,

$$\begin{aligned} & \varepsilon \sum_{j=1}^3 \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + \frac{1}{C_0} \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\ & \leq \int_{\Omega} \xi^2 (\varepsilon \partial_3 \mathbf{D}\mathbf{u}_\varepsilon + \partial_3 \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)) \cdot \partial_3 \mathbf{D}\mathbf{u}_\varepsilon dx \\ & = \int_{\Omega} \sum_{i,j=1}^3 \xi^2 (\varepsilon \partial_3 D_{ij} \mathbf{u}_\varepsilon + \partial_3 S_{ij}(\mathbf{D}\mathbf{u}_\varepsilon)) \partial_3 \partial_j u^i dx \\ & = \int_{\Omega} \xi^2 (\varepsilon \partial_3 D_{\alpha\beta} \mathbf{u}_\varepsilon + \partial_3 S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon)) \partial_3 D_{\alpha\beta} \mathbf{u}_\varepsilon dx \\ & \quad + \int_{\Omega} \xi^2 (\varepsilon \partial_3 D_{3\alpha} \mathbf{u}_\varepsilon + \partial_3 S_{3\alpha}(\mathbf{D}\mathbf{u}_\varepsilon)) \partial_\alpha D_{33} \mathbf{u}_\varepsilon dx \\ & \quad + \int_{\Omega} \sum_{j=1}^3 \xi^2 \partial_3 (\varepsilon D_{j3} \mathbf{u}_\varepsilon + S_{j3}(\mathbf{D}\mathbf{u}_\varepsilon)) \partial_3^2 u_\varepsilon^j dx \\ & =: I_1 + I_2 + I_3. \end{aligned}$$

To estimate I_2 we multiply and divide by the quantity $\sqrt{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)} \neq 0$, use Young's inequality and Proposition 2.14. This yields that for all $\lambda > 0$ there exists $c_\lambda > 0$ such that

$$\begin{aligned} |I_2| & \leq \sum_{\alpha=1}^2 \int_{\Omega} \xi^2 |\partial_3 \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)| |\partial_\alpha \mathbf{D}\mathbf{u}_\varepsilon| \frac{\sqrt{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)}}{\sqrt{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)}} dx \\ & \quad + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda \varepsilon \sum_{\alpha=1}^2 \int_{\Omega} \xi^2 |\partial_\alpha \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\ & \leq \lambda \int_{\Omega} \xi^2 \frac{|\partial_3 \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)|^2}{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)} dx + c_\lambda \sum_{\alpha=1}^2 \int_{\Omega} \xi^2 \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|) |\partial_\alpha \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\ & \quad + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda \varepsilon \sum_{\alpha=1}^2 \int_{\Omega} \xi^2 |\partial_\alpha \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \end{aligned}$$

$$\begin{aligned} &\leq C\lambda \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \sum_{a=1}^2 \int_{\Omega} \xi^2 |\partial_a \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\ &\quad + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda \varepsilon \sum_{\alpha=1}^2 \int_{\Omega} \xi^2 |\partial_\alpha \mathbf{D}\mathbf{u}_\varepsilon|^2 dx. \end{aligned}$$

Here and in the following we denote by c_λ constants that may depend on the characteristics of \mathbf{S} and on λ^{-1} , while C denotes constants that may depend on the characteristics of \mathbf{S} only.

To treat the third integral I_3 we proceed as follows: We use the well-known algebraic identity, valid for smooth enough vectors \mathbf{v} and $i, j, k = 1, 2, 3$,

$$\partial_j \partial_k v^i = \partial_j D_{ik} \mathbf{v} + \partial_k D_{ij} \mathbf{v} - \partial_i D_{jk} \mathbf{v}, \quad (3.5)$$

and the equations (2.20) point-wise, which can be written for $j = 1, 2, 3$ as,

$$\partial_3 (\varepsilon D_{j3} \mathbf{u}_\varepsilon + S_{j3}(\mathbf{D}\mathbf{u}_\varepsilon)) = -f^j - \partial_\beta (\varepsilon D_{j\beta} \mathbf{u}_\varepsilon + S_{j\beta}(\mathbf{D}\mathbf{u}_\varepsilon)) \quad \text{a.e. in } \Omega.$$

This is possible due to Proposition 3.1. Hence, we obtain

$$|I_3| \leq \sum_{j=1}^3 \left| \int_{\Omega} \xi^2 (-f^j - \partial_\beta S_{j\beta}(\mathbf{D}\mathbf{u}_\varepsilon) - \varepsilon \partial_\beta D_{j\beta} \mathbf{u}_\varepsilon) (2\partial_3 D_{j3} \mathbf{u}_\varepsilon - \partial_j D_{33} \mathbf{u}_\varepsilon) dx \right|.$$

The right-hand side can be estimated similarly as I_2 . This yields that for all $\lambda > 0$ there exists $c_\lambda > 0$ such that estimated by

$$\begin{aligned} |I_3| &\leq \int_{\Omega} \xi^2 (|\mathbf{f}| + \sum_{\beta=1}^2 |\partial_\beta \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)|) (2|\partial_3 \mathbf{D}\mathbf{u}_\varepsilon| + \sum_{\alpha=1}^2 |\partial_\alpha \mathbf{D}\mathbf{u}_\varepsilon|) \frac{\sqrt{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)}}{\sqrt{\varphi'(|\mathbf{D}\mathbf{u}_\varepsilon|)}} dx \\ &\quad + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\ &\leq \lambda C \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\ &\quad + c_\lambda \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda \int_{\Omega} \frac{\xi^2 |\mathbf{f}|^2}{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)} dx \\ &\leq \lambda C \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\ &\quad + c_\lambda \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda (\|\mathbf{f}\|_{p'}^{p'} + \|\mathbf{D}\mathbf{u}_\varepsilon\|_p^p + \delta^p). \end{aligned}$$

Observe that we used $p \leq 2$ to estimate the term involving \mathbf{f} .

To estimate I_1 we employ the algebraic identity (3.5) to split the integral as follows

$$\begin{aligned} I_1 &= \int_{\Omega} \xi^2 (\varepsilon \partial_3 D_{\alpha\beta} \mathbf{u}_\varepsilon + \partial_3 S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon)) (\partial_\alpha D_{3\beta} \mathbf{u}_\varepsilon + \partial_\beta D_{3\alpha} \mathbf{u}_\varepsilon) dx \\ &\quad - \int_{\Omega} \xi^2 (\varepsilon \partial_3 D_{\alpha\beta} \mathbf{u}_\varepsilon + \partial_3 S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon)) \partial_\beta \partial_\alpha u_\varepsilon^3 dx \\ &=: A + B. \end{aligned}$$

The first term is estimated similarly as I_2 , yielding for all $\lambda > 0$

$$\begin{aligned} |A| &\leq C\lambda \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\ &\quad + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 dx. \end{aligned}$$

To estimate B we observe that by the definition of the tangential derivative we have

$$\partial_\alpha \partial_\beta u_\varepsilon^3 = \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 - (\partial_\alpha \partial_\beta a) D_{33} \mathbf{u}_\varepsilon - (\partial_\beta a) \partial_\alpha D_{33} \mathbf{u}_\varepsilon,$$

and consequently the term B can be split into the following three terms:

$$\begin{aligned} & - \int_{\Omega} \xi^2 (\varepsilon \partial_3 D_{\alpha\beta} \mathbf{u}_\varepsilon + \partial_3 S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon)) (\partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 - (\partial_\alpha \partial_\beta a) D_{33} \mathbf{u}_\varepsilon - (\partial_\beta a) \partial_\alpha D_{33} \mathbf{u}_\varepsilon) dx \\ & =: B_1 + B_2 + B_3. \end{aligned}$$

We estimate B_2 as follows

$$\begin{aligned} |B_2| &\leq \int_{\Omega} \xi^2 |\partial_3 \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)| |\nabla^2 a| |\mathbf{D}\mathbf{u}_\varepsilon| \frac{\sqrt{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)}}{\sqrt{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)}} + \varepsilon \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon| |\nabla^2 a| |\mathbf{D}\mathbf{u}_\varepsilon| dx \\ &\leq \lambda \int_{\Omega} \xi^2 \frac{|\partial_3 \mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)|^2}{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)} dx + c_\lambda \|\nabla^2 a\|_\infty^2 \int_{\Omega} \xi^2 |\mathbf{D}\mathbf{u}_\varepsilon|^2 \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|) dx \\ &\quad + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda \varepsilon \|\nabla^2 a\|_\infty^2 \int_{\Omega} \xi^2 |\mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\ &\leq \lambda C \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \|\nabla^2 a\|_\infty^2 \rho_\varphi(|\mathbf{D}\mathbf{u}_\varepsilon|) \\ &\quad + \frac{\varepsilon}{8} \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + 2\varepsilon \|\nabla^2 a\|_\infty^2 \|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2. \end{aligned}$$

The term B_3 is estimated similarly as I_2 , yielding for all $\lambda > 0$

$$\begin{aligned} |B_3| &\leq \lambda C \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \|\nabla a\|_\infty^2 \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\ &\quad + \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda \varepsilon \|\nabla a\|_\infty^2 \sum_{\beta=1}^2 \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 dx. \end{aligned}$$

Concerning the term B_1 , we would like to perform some integration by parts, which is one of the crucial observations we are adapting from [24]. Neglecting the localization ξ in B_1 we would like to use that

$$\int_{\Omega} \partial_3 S_{\alpha\beta}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 dx = \int_{\Omega} \partial_\alpha S_{\alpha\beta}^\varepsilon(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 dx. \quad (3.6)$$

This formula can be justified by using an appropriate approximation, that exists for $\mathbf{u}_\varepsilon \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ since $\partial_\tau \mathbf{u}_\varepsilon = \mathbf{0}$ on $\partial\Omega$. More precisely, to treat the term B_1 we use that the solution \mathbf{u}_ε of (2.20) belongs to $W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$. Thus, $\partial_\tau(\mathbf{u}_\varepsilon|_{\Omega_P}) = \mathbf{0}$ on $\partial\Omega_P \cap \partial\Omega$, hence $\xi_P \partial_\tau(\mathbf{u}_\varepsilon^3) = \mathbf{0}$ on $\partial\Omega$. This implies that we can find a sequence $(\mathcal{S}_n, \mathcal{U}_n) \in C^\infty(\Omega) \times C_0^\infty(\Omega)$ such that $(\mathcal{S}_n, \mathcal{U}_n) \rightarrow (\mathcal{S}^\varepsilon, \partial_\tau \mathbf{u}_\varepsilon)$ in $W^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ and perform calculations with $(\mathcal{S}_n, \mathcal{U}_n)$, showing then that all formulas of integration by parts are valid. Passage to the limit as $n \rightarrow +\infty$ is

done only in the last step. For simplicity we drop the details of this well-known argument (sketched also in [24]) and we write directly formulas without this smooth approximation. Thus, performing several integrations by parts, we get

$$\begin{aligned} & \int_{\Omega} \xi^2 \partial_3 S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} \\ &= \int_{\Omega} (\partial_\alpha \xi^2) S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} - \int_{\Omega} (\partial_3 \xi^2) S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} \\ & \quad + \int_{\Omega} \xi^2 \partial_\alpha S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} \end{aligned}$$

and

$$\begin{aligned} & \varepsilon \int_{\Omega} \xi^2 \partial_3 D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} \\ &= \varepsilon \int_{\Omega} (\partial_\alpha \xi^2) D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} - \varepsilon \int_{\Omega} (\partial_3 \xi^2) D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} \\ & \quad + \varepsilon \int_{\Omega} \xi^2 \partial_\alpha D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x}. \end{aligned}$$

This shows that

$$\begin{aligned} B_1 &= \int_{\Omega} 2\xi \partial_\alpha \xi S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} - \int_{\Omega} 2\xi \partial_3 \xi S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} \\ & \quad + \int_{\Omega} \xi^2 \partial_\alpha S_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon) \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} + \varepsilon \int_{\Omega} 2\xi \partial_\alpha \xi D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} \\ & \quad - \varepsilon \int_{\Omega} 2\xi \partial_3 \xi D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} + \varepsilon \int_{\Omega} \xi^2 \partial_\alpha D_{\alpha\beta} \mathbf{u}_\varepsilon \partial_3 \partial_{\tau_\beta} u_\varepsilon^3 \, d\mathbf{x} \\ & =: B_{1,1} + B_{1,2} + B_{1,3} + B_{1,4} + B_{1,5} + B_{1,6}. \end{aligned}$$

To estimate $B_{1,1}, B_{1,3}, B_{1,4}, B_{1,6}$ we observe that

$$\partial_3 \partial_{\tau_\beta} u_\varepsilon^3 = \partial_{\tau_\beta} \partial_3 u_\varepsilon^3 = \partial_{\tau_\beta} D_{33} \mathbf{u}_\varepsilon.$$

By using Young inequality, the growth properties of \mathbf{S} in (2.10d) and (2.12) we get

$$\begin{aligned} |B_{1,1}| &\leq \|\nabla \xi\|_\infty^2 \int_{\Omega} \frac{|\mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)|^2}{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)} \, d\mathbf{x} + C \sum_{\beta=1}^2 \int_{\Omega} \xi^2 \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|) |\partial_{\tau_\beta} \mathbf{D}\mathbf{u}_\varepsilon|^2 \, d\mathbf{x} \\ &\leq \|\nabla \xi\|_\infty^2 \rho_\varphi(|\mathbf{D}\mathbf{u}_\varepsilon|) + C \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_{\tau_\beta} \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 \, d\mathbf{x} \end{aligned}$$

and

$$\begin{aligned} |B_{1,3}| &\leq \sum_{\beta=1}^2 \int_{\Omega} \xi^2 \frac{|\partial_\beta \mathbf{S}_{\alpha\beta}(\mathbf{D}\mathbf{u}_\varepsilon)|^2}{\varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|)} \, d\mathbf{x} + \sum_{\beta=1}^2 \int_{\Omega} \xi^2 \varphi''(|\mathbf{D}\mathbf{u}_\varepsilon|) |\partial_{\tau_\beta} \mathbf{D}\mathbf{u}_\varepsilon|^2 \, d\mathbf{x} \\ &\leq C \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 + \xi^2 |\partial_{\tau_\beta} \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 \, d\mathbf{x}. \end{aligned}$$

Similarly we get

$$|B_{1,4}| \leq C \varepsilon \|\nabla \xi\|_\infty^2 \|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2 + C \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_{\tau_\beta} \mathbf{D}\mathbf{u}_\varepsilon|^2 dx$$

and

$$|B_{1,6}| \leq C \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 + \xi^2 |\partial_{\tau_\beta} \mathbf{D}\mathbf{u}_\varepsilon|^2 dx.$$

To estimate $B_{1,2}$ and B_{15} we observe that, using the algebraic identity (3.5) and the definition of the tangential derivative,

$$\begin{aligned} \partial_\alpha \partial_{\tau_\beta} u_\varepsilon^3 &= \partial_\alpha (\partial_\beta u_\varepsilon^3 + \partial_\beta a \partial_3 u_\varepsilon^3) \\ &= \partial_\alpha \partial_\beta u_\varepsilon^3 + \partial_\alpha \partial_\beta a D_{33} \mathbf{u}_\varepsilon + \partial_\beta a \partial_\alpha D_{33} \mathbf{u}_\varepsilon \\ &= \partial_\alpha D_{\beta 3} \mathbf{u}_\varepsilon + \partial_\beta D_{\alpha 3} \mathbf{u}_\varepsilon - \partial_3 D_{\alpha \beta} \mathbf{u}_\varepsilon + \partial_\alpha \partial_\beta a D_{33} \mathbf{u}_\varepsilon + \partial_\beta a \partial_\alpha D_{33} \mathbf{u}_\varepsilon. \end{aligned}$$

Hence by substituting and again by the same inequalities as before we arrive to the following estimates

$$\begin{aligned} |B_{1,2}| &\leq \lambda C \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx + C(1 + \|\nabla a\|_\infty^2) \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\ &\quad + c_\lambda (1 + \|\nabla^2 a\|_\infty) \|\nabla \xi\|_\infty^2 \rho_\varphi(|\mathbf{D}\mathbf{u}_\varepsilon|), \\ |B_{1,5}| &\leq \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + c_\lambda (1 + \|\nabla a\|_\infty^2) \sum_{\beta=1}^2 \varepsilon \int_{\Omega} \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\ &\quad + c_\lambda (1 + \|\nabla^2 a\|_\infty) \|\nabla \xi\|_\infty^2 \varepsilon \|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2. \end{aligned}$$

Collecting all estimates and using that $\|\nabla a\|_\infty \leq r_P \leq 1$, we finally obtain

$$\begin{aligned} &\varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + \frac{1}{C_0} \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\ &\leq \lambda \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + \lambda C \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\ &\quad + c_\lambda \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 + \xi^2 |\partial_{\tau_\beta} \mathbf{F}(\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_\beta \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\ &\quad + c_\lambda (1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2) (\|\mathbf{f}\|_{p'}^{p'} + \rho_\varphi(|\mathbf{D}\mathbf{u}_\varepsilon|) + \rho_\varphi(\delta)) \\ &\quad + c_\lambda (1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2) \|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2. \end{aligned}$$

The quantities that are bounded uniformly in $L^2(\Omega_P)$ are the tangential derivatives of $\varepsilon \mathbf{D}\mathbf{u}_\varepsilon$ and of $\mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)$. By definition we have

$$\begin{aligned} \partial_\beta \mathbf{D}\mathbf{u}_\varepsilon &= \partial_{\tau_\beta} \mathbf{D}\mathbf{u}_\varepsilon - \partial_\beta a \partial_3 \mathbf{D}\mathbf{u}_\varepsilon, \\ \partial_\beta \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon) &= \partial_{\tau_\beta} \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon) - \partial_\beta a \partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon), \end{aligned}$$

and if we substitute we obtain

$$\begin{aligned}
& \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + \frac{1}{C_0} \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\
& \leq \varepsilon (\lambda + 4 \|\nabla a\|_\infty^2) \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + (\lambda C + c_\lambda \|\nabla a\|_\infty^2) \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\
& \quad + c_\lambda \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_{\tau_\beta} \mathbf{F}(\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_{\tau_\beta} \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\
& \quad + c_\lambda (1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2) (\|\mathbf{f}\|_{p'}^{p'} + \rho_\varphi(|\mathbf{D}\mathbf{u}_\varepsilon|) + \rho_\varphi(\delta)) \\
& \quad + c_\lambda (1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2) \|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2.
\end{aligned}$$

By choosing first $\lambda > 0$ small enough such that $\lambda C < 4^{-1}C_0$ and then choosing in the local description of the boundary $R = R_P$ small enough such that $c_\lambda \|\nabla a\|_\infty < 4^{-1}C_0$, we can absorb the first two terms from the right-hand side into the left-hand side to obtain

$$\begin{aligned}
& \varepsilon \int_{\Omega} \xi^2 |\partial_3 \mathbf{D}\mathbf{u}_\varepsilon|^2 dx + \frac{1}{C_0} \int_{\Omega} \xi^2 |\partial_3 \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)|^2 dx \\
& \leq c_\lambda \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_{\tau_\beta} \mathbf{F}(\mathbf{u}_\varepsilon)|^2 dx + c_\lambda \varepsilon \sum_{\beta=1}^2 \int_{\Omega} \xi^2 |\partial_{\tau_\beta} \mathbf{D}\mathbf{u}_\varepsilon|^2 dx \\
& \quad + c_\lambda (1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2) (\|\mathbf{f}\|_{p'}^{p'} + \rho_\varphi(|\mathbf{D}\mathbf{u}_\varepsilon|) + \rho_\varphi(\delta)) \\
& \quad + c_\lambda (1 + \|\nabla^2 a\|_\infty^2 + (1 + \|\nabla^2 a\|_\infty^2) \|\nabla \xi\|_\infty^2) \|\mathbf{D}\mathbf{u}_\varepsilon\|_2^2,
\end{aligned}$$

where now c_λ depends on the fixed parameter λ , the characteristics of \mathbf{S} and on C_2 . The right-hand side is bounded uniformly with respect to $\varepsilon > 0$, due to Proposition 3.1, proving the assertion of the proposition. \square

Choosing now an appropriate finite covering of the boundary (for the details see also [4]), Propositions 3.1-3.4 yield the following result:

THEOREM 3.7. *Let the same hypotheses as in Theorem 1.3 with $\delta > 0$ be satisfied. Then, it holds*

$$\varepsilon \|\nabla \mathbf{D}\mathbf{u}_\varepsilon\|_2^2 + \|\nabla \mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)\|_2^2 \leq C(\|\mathbf{f}\|_{p'}, \delta, \partial\Omega).$$

3.2. Passage to the limit. Once this has been proved, by means of appropriate limiting process we can show that the estimate is inherited by $\mathbf{u} = \lim_{\varepsilon \rightarrow 0} \mathbf{u}_\varepsilon$, since \mathbf{u} is the unique solution to the boundary value problem (1.1). We can now give the proof of the main result

PROOF (OF THEOREM 1.3). Let us firstly assume that $\delta > 0$. From Proposition 2.19, Proposition 2.9 and Theorem 3.7 we know that $\mathbf{F}(\mathbf{D}\mathbf{u}_\varepsilon)$ is uniformly bounded with respect to ε in $W^{1,2}(\Omega)$. This also implies (cf. [3, Lemma 4.4]) that \mathbf{u}_ε is uniformly bounded with respect to ε in $W^{2,p}(\Omega)$. The properties of \mathbf{S} and Proposition 2.19 also yield that $\mathbf{S}(\mathbf{D}\mathbf{u}_\varepsilon)$ is uniformly bounded with respect to ε in $L^{p'}(\Omega)$. Thus, there exists a subsequence $\{\varepsilon_n\}$ (which converges to 0 as $n \rightarrow +\infty$),

$\mathbf{u} \in W^{2,p}(\Omega)$, $\tilde{\mathbf{F}} \in W^{1,2}(\Omega)$, and $\chi \in L^{p'}(\Omega)$ such that

$$\begin{aligned} \mathbf{u}_{\varepsilon_n} &\rightharpoonup \mathbf{u} && \text{in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ \mathbf{D}\mathbf{u}_{\varepsilon_n} &\rightarrow \mathbf{D}\mathbf{u} && \text{a.e. in } \Omega, \\ \mathbf{F}(\mathbf{D}\mathbf{u}_{\varepsilon_n}) &\rightharpoonup \tilde{\mathbf{F}} && \text{in } W^{1,2}(\Omega), \\ \mathbf{S}(\mathbf{D}\mathbf{u}_{\varepsilon_n}) &\rightharpoonup \chi && \text{in } L^{p'}(\Omega). \end{aligned}$$

The continuity of \mathbf{S} and \mathbf{F} and the classical result stating that the weak limit and the a.e. limit in Lebesgue spaces coincide (cf. [12]) imply that

$$\tilde{\mathbf{F}} = \mathbf{F}(\mathbf{D}\mathbf{u}) \quad \text{and} \quad \chi = \mathbf{S}(\mathbf{D}\mathbf{u}).$$

These results enable us to pass to the limit in the weak formulation of the perturbed problem (2.20), which yields

$$\int_{\Omega} \mathbf{S}(\mathbf{D}\mathbf{u}) \cdot \mathbf{D}\mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} \quad \forall \mathbf{v} \in C_0^\infty(\Omega),$$

where we also used that $\lim_{\varepsilon_n \rightarrow 0} \int_{\Omega} \varepsilon_n \mathbf{D}\mathbf{u}_{\varepsilon_n} \cdot \mathbf{D}\mathbf{v} \, d\mathbf{x} = 0$. By density we thus know that \mathbf{u} is the unique weak solution of problem (1.1). Finally the lower semi-continuity of the norm implies that

$$\int_{\Omega} |\nabla \mathbf{F}(\mathbf{D}\mathbf{u})|^2 \, d\mathbf{x} \leq \liminf_{\varepsilon_n \rightarrow 0} \int_{\Omega} |\nabla \mathbf{F}(\mathbf{D}\mathbf{u}_{\varepsilon_n})|^2 \, d\mathbf{x} \leq c,$$

ending the proof in the case $\delta > 0$.

Let us now assume that $\delta = 0$. Proposition 3.1 and Proposition 3.4 are valid only for $\delta > 0$ and thus cannot be used directly for the case that \mathbf{S} has (p, δ) -structure with $\delta = 0$. However, it is proved in [3, Section 3.1] that for any stress tensor with $(p, 0)$ -structure \mathbf{S} , there exist stress tensors \mathbf{S}^κ , having (p, κ) -structure with $\kappa > 0$, and approximating \mathbf{S} in an appropriate way. Thus we approximate (2.20) by the system

$$\begin{aligned} -\operatorname{div} \mathbf{S}^{\varepsilon, \kappa}(\mathbf{D}\mathbf{u}_{\varepsilon, \kappa}) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned}$$

where

$$\mathbf{S}^{\varepsilon, \kappa}(\mathbf{Q}) := \varepsilon \mathbf{Q} + \mathbf{S}^\kappa(\mathbf{Q}), \quad \text{with } \varepsilon > 0, \kappa \in (0, 1).$$

For fixed $\kappa > 0$ we can use the above theory and use that fact that the estimates are uniformly in κ to pass to the limit as $\varepsilon \rightarrow 0$. Thus, we obtain that for all $\kappa \in (0, 1)$ there exists a unique $\mathbf{u}_\kappa \in W_0^{1,p}(\Omega)$ satisfying for all $\mathbf{v} \in W_0^{1,p}(\Omega)$

$$\int_{\Omega} \mathbf{S}^\kappa(\mathbf{D}\mathbf{u}_\kappa) \cdot \mathbf{D}\mathbf{v} \, d\mathbf{x} = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}$$

and

$$\int_{\Omega} |\mathbf{F}^\kappa(\mathbf{D}\mathbf{u}_\kappa)|^2 + |\nabla \mathbf{F}^\kappa(\mathbf{D}\mathbf{u}_\kappa)|^2 \, d\mathbf{x} \leq c(\|\mathbf{f}\|_{p'}, \partial\Omega), \quad (3.8)$$

where the constant is independent of $\kappa \in (0, 1)$ and $\mathbf{F}^\kappa: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_{\text{sym}}^{3 \times 3}$ is defined through

$$\mathbf{F}^\kappa(\mathbf{P}) := \left(\kappa + |\mathbf{P}^{\text{sym}}| \right)^{\frac{p-2}{2}} \mathbf{P}^{\text{sym}}.$$

Now we can proceed as in [3]. Indeed, from (3.8) and the properties of $\varphi_{p, \kappa}$ (in particular (2.4)) it follows that $\mathbf{F}^\kappa(\mathbf{D}\mathbf{u}_\kappa)$ is uniformly bounded in $W^{1,2}(\Omega)$, that \mathbf{u}_κ is uniformly bounded in $W_0^{1,p}(\Omega)$ and that $\mathbf{S}^\kappa(\mathbf{D}\mathbf{u}_\kappa)$ is uniformly bounded in $L^{p'}(\Omega)$.

Thus, there exist $\mathbf{A} \in W^{1,2}(\Omega)$, $\mathbf{u} \in W_0^{1,p}(\Omega)$, $\boldsymbol{\chi} \in L^{p'}(\Omega)$, and a subsequence $\{\kappa_n\}$, with $\kappa_n \rightarrow 0$, such that

$$\begin{aligned} \mathbf{F}(\mathbf{D}\mathbf{u}_{\kappa_n}) &\rightharpoonup \mathbf{A} && \text{in } W^{1,2}(\Omega), \\ \mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}_{\kappa_n}) &\rightarrow \mathbf{A} && \text{in } L^2(\Omega) \text{ and a.e. in } \Omega, \\ \mathbf{u}_{\kappa_n} &\rightharpoonup \mathbf{u} && \text{in } W_0^{1,p}(\Omega), \\ \mathbf{S}^{\kappa}(\mathbf{D}\mathbf{u}_{\kappa}) &\rightharpoonup \boldsymbol{\chi} && \text{in } L^{p'}(\Omega). \end{aligned}$$

Setting $\mathbf{B} := (\mathbf{F}^0)^{-1}(\mathbf{A})$, it follows from [3, Lemma 3.23] that

$$\mathbf{D}\mathbf{u}_{\kappa_n} = (\mathbf{F}^{\kappa_n})^{-1}(\mathbf{F}^{\kappa_n}(\mathbf{D}\mathbf{u}_{\kappa_n})) \rightarrow (\mathbf{F}^0)^{-1}(\mathbf{A}) = \mathbf{B} \quad \text{a.e. in } \Omega.$$

Since weak and a.e. limit coincide we obtain that

$$\mathbf{D}\mathbf{u}_{\kappa_n} \rightarrow \mathbf{D}\mathbf{u} = \mathbf{B} \quad \text{a.e. in } \Omega.$$

From [3, Lemma 3.16] and [3, Corollary 3.22] it now follows that

$$\begin{aligned} \mathbf{F}(\mathbf{D}\mathbf{u}_{\kappa_n}) &\rightharpoonup \mathbf{F}^0(\mathbf{D}\mathbf{u}) && \text{in } W^{1,2}(\Omega), \\ \mathbf{S}^{\kappa_n}(\mathbf{D}\mathbf{u}_{\kappa_n}) &\rightarrow \mathbf{S}(\mathbf{D}\mathbf{u}) && \text{a.e. in } \Omega. \end{aligned}$$

Since weak and a.e. limit coincide we obtain that

$$\mathbf{D}\mathbf{u} = \boldsymbol{\chi} \quad \text{a.e. in } \Omega.$$

Now we can finish the proof in the same way as in the case $\delta > 0$. \square

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References

- [1] E. Acerbi and N. Fusco. Regularity for minimizers of nonquadratic functionals: the case $1 < p < 2$. *J. Math. Anal. Appl.*, 140(1):115–135, 1989.
- [2] H. Beirão da Veiga, P. Kaplický, and M. Růžička. Boundary regularity of shear-thickening flows. *J. Math. Fluid Mech.*, 13:387–404, 2011.
- [3] L. C. Berselli, L. Diening, and M. Růžička. Existence of strong solutions for incompressible fluids with shear dependent viscosities. *J. Math. Fluid Mech.*, 12(1):101–132, 2010.
- [4] L. C. Berselli and M. Růžička. Global regularity properties of steady shear thinning flows. *J. Math. Anal. Appl.*, 45062:839–871, 2017.
- [5] R.B. Bird, R.C. Armstrong, and O. Hassager. *Dynamic of Polymer Liquids*. John Wiley, 1987. 2nd edition.
- [6] F. Crispo and C. R. Grisanti, On the existence, uniqueness and $C^{1,\gamma}(\overline{\Omega}) \cap W^{2,2}(\Omega)$ regularity for a class of shear-thinning fluids, *J. Math. Fluid Mech.* 10(4):455–487, 2008.
- [7] F. Crispo and P. Maremonti. A high regularity result of solutions to modified p -Stokes equations. *Nonlinear Anal.*, 118:97–129, 2015.
- [8] L. Diening and F. Ettwein. Fractional estimates for non-differentiable elliptic systems with general growth. *Forum Math.*, 20(3):523–556, 2008.
- [9] L. Diening and C. Kreuzer. Linear convergence of an adaptive finite element method for the p -Laplacian equation. *SINUM*, 46:614–638, 2008.
- [10] L. Diening, J. Málek, and M. Steinhauer. On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. *ESAIM: Control, Opt. Calc. Var.*, 14(2):211–232, 2008.
- [11] M. Fuchs and G. Seregin. *Variational methods for problems from plasticity theory and for generalized Newtonian fluids*. Springer-Verlag, Berlin, 2000.
- [12] H. Gajewski, K. Gröger, and K. Zacharias. *Nichtlineare Operatorgleichungen und Operatordifferentialgleichungen*. Akademie-Verlag, Berlin, 1974.
- [13] M. Giaquinta and G. Modica. Remarks on the regularity of the minimizers of certain degenerate functionals. *Manuscripta Math.*, 57(1):55–99, 1986.
- [14] E. Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

- [15] L. M. Kachanov. *Foundations of the theory of plasticity*. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., New York, revised edition, 1971. North-Holland Series in Applied Mathematics and Mechanics, Vol. 12.
- [16] P. Kaplický, J. Málek, and J. Stará. $C^{1,\alpha}$ -regularity of weak solutions to a class of nonlinear fluids in two dimensions - stationary Dirichlet problem. *Zap. Nauchn. Sem. Pt. Odel. Mat. Inst.*, 259:89–121, 1999.
- [17] P. Kaplický and J. Tichý. Boundary regularity of flows under perfect slip boundary conditions. *Cent. Eur. J. Math.*, 11(7):1243–1263, 2013.
- [18] J. Málek, J. Nečas, and M. Růžička. On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains. the case $p \geq 2$. *Adv. Differential Equations*, 6(3), 257–302, 2001.
- [19] J. Málek and K. R. Rajagopal. Mathematical issues concerning the Navier-Stokes equations and some of its generalizations. In *Evolutionary equations. Vol. II, Handb. Differ. Equ.*, pages 371–459. Elsevier/North-Holland, Amsterdam, 2005.
- [20] J. Nečas and I. Hlaváček. *Mathematical theory of elastic and elasto-plastic bodies: an introduction*, volume 3 of *Studies in Applied Mechanics*. Elsevier Scientific Publishing Co., Amsterdam-New York, 1980.
- [21] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1991.
- [22] M. Růžička. Analysis of generalized Newtonian fluids. In *Topics in mathematical fluid mechanics*, volume 2073 of *Lecture Notes in Math.*, pages 199–238. Springer, Heidelberg, 2013.
- [23] M. Růžička and L. Diening. Non-Newtonian fluids and function spaces. In *Nonlinear Analysis, Function Spaces and Applications, Proceedings of NAFSA 2006 Prague*, volume 8, pages 95–144, 2007.
- [24] G. A. Seregin and T. N. Shilkin. Regularity of minimizers of some variational problems in plasticity theory. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 243(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funktsii. 28):270–298, 342–343, 1997.

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