IDEMPOTENT ULTRAFILTERS WITHOUT ZORN'S LEMMA

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ABSTRACT. We introduce the notion of *additive filter* and present a new proof of the existence of idempotent ultrafilters on $\mathbb{N}$ without using *Zorn's Lemma* in its entire power, and where one only assumes the *Ultrafilter Theorem* for the *continuum*.

INTRODUCTION

Idempotent ultrafilters are a central object in Ramsey theory of numbers. Over the last forty years they have been extensively studied in the literature, producing a great amount of interesting combinatorial properties (see the extensive monography [12]). As reported in [11], it all started in 1971 when F. Galvin realized that by assuming the existence of ultrafilters on $\mathbb{N}$ that are "almost translation invariant", one could produce a short proof of a conjecture of R. Graham and B. Rothschild, that was to become a cornerstone of Ramsey theory of numbers: "For every finite coloring of the natural numbers there exists an infinite set X such that all finite sums of distinct elements of X have the same color." However, at that time the problem was left open as whether such special ultrafilters could exist at all. In 1972, N. Hindman [9] showed that the *continuum hypothesis* suffices to construct those ultrafilters, but their existence in ZFC remained unresolved. Eventually, in 1974, N. Hindman [10] proved the Graham-Rothschild conjecture (now known as Hindman's Theorem) with a long and intricate combinatorial argument that avoided the use of ultrafilters. Shortly afterwards, in 1975, S. Glazer observed that "almost translation invariant" ultrafilters are precisely the idempotent elements of the semigroup $(\beta\mathbb{N}, \oplus)$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of the discrete space $\mathbb{N}$ (which can be identified with the space of ultrafilters on $\mathbb{N}$) and where $\oplus$ is a suitable pseudo-sum operation between ultrafilters. The existence of idempotent ultrafilters is then immediate, since any compact Hausdorff right-topological semigroup has idempotents, a well-known fact in semigroup theory known as *Ellis-Numakura's Lemma.* The proof of that lemma heavily relies on the axiom of choice, as it consists in a clever and elegant use of *Zorn's Lemma* jointly with the topological properties of a compact Hausdorff space. In 1989, T. Papazyan [16] introduced the notion of "almost translation invariant *filter*", and proved that the maximal filters in that class, obtained by applying *Zorn's Lemma*, are necessarily ultrafilters, and hence idempotent ultrafilters.

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Despite their central role in a whole area of combinatorics of numbers, no other proofs are known for the existence of idempotent ultrafilters. However, as it often happens with fundamental objects of mathematics, alternative proofs seem desirable because they may give a better insight and potentially lead to new applications. It is worth mentioning that generalizations of idempotent ultrafilters have been recently considered both in the usual set-theoretic context, and in the general framework of model theory: see [15] where P. Krautzberger thoroughly investigated the almost translation invariant filters (appropriately named "idempotent filters"), and see [1] where U. Andrews and I. Goldbring studied a model-theoretic notion of idempotent type and its relationship with Hindman's Theorem.

In this paper we introduce the notion of *additive filter*, which is weaker than the notion of idempotent filter. By suitably modifying the argument used in *Ellis-Numakura's Lemma*, we show that *Zorn's Lemma* is not needed to prove that every additive filter can be extended to a maximal additive filter, and that every maximal additive filter is indeed an idempotent ultrafilter. Precisely, we will only assume the following restricted form of the *Ultrafilter Theorem* (a strictly weaker form of the axiom of choice): "Every filter on $\mathbb{R}$ can be extended to an ultrafilter."

1. Preliminary facts

Although the notions below could also be considered on arbitrary sets, here we will focus only on the set of natural numbers $\mathbb{N}$. We agree that a natural number is a positive integer, so $0 \notin \mathbb{N}$.

Recall that a *filter* $\mathscr{F}$ is a nonempty family of nonempty sets that is closed under supersets and under (finite) intersections.[1] An *ultrafilter* is a filter that is maximal with respect to inclusion; equivalently, a filter $\mathscr{U}$ is an ultrafilter if whenever $A \notin \mathscr{U}$, the complement $A^c \in \mathscr{U}$. Trivial examples are given by the principal ultrafilters $\mathscr{U}_n = \{A \subseteq \mathbb{N} \mid n \in A\}$. Notice that an ultrafilter is non-principal if and only if it extends the *Fréchet filter* $\{A \subseteq \mathbb{N} \mid A^c \text{ finite}\}$ of cofinite sets. In the following, $\mathscr{F}, \mathscr{G}$ will denote filters on $\mathbb{N}$, and $\mathscr{U}, \mathscr{V}, \mathscr{W}$ will denote ultrafilters on $\mathbb{N}$.

Recall that the *Stone-Čech compactification* $\beta\mathbb{N}$ of the discrete space $\mathbb{N}$ can be identified with the space of all ultrafilters on $\mathbb{N}$ endowed with the Hausdorff topology that has the family $\{\{\mathscr{U} \in \beta\mathbb{N} \mid A \in \mathscr{U}\} \mid A \subseteq \mathbb{N}\}$ as a base of (cl)open sets. (We note here that the above identification is feasible in ZF, *i.e.*, in the Zermelo–Fraenkel set theory minus the axiom of choice AC; see [7, Theorems 14, 15].)

The existence of non-principal ultrafilters is established by the

- *Ultrafilter Theorem* UT: "For every set X, every proper filter on X can be extended to an ultrafilter."

The proof is a direct application of *Zorn's Lemma*.[2] It is a well-known fact that UT is a strictly weaker form of AC (see, *e.g.*, [13] and references therein). This means that one cannot prove UT in ZF alone, and that ZF+UT does not prove AC.

[1] More formally, $\mathscr{F} \subseteq \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\}$ is a filter if the following three properties are satified: (1) $\mathbb{N} \in \mathscr{F}$, (2) $B \supseteq A \in \mathscr{F} \Rightarrow B \in \mathscr{F}$, (3) $A, B \in \mathscr{F} \Rightarrow A \cap B \in \mathscr{F}$.

[2] In the study of weak forms of choice, one usually considers the equivalent formulation given by the *Boolean Prime Ideal Theorem* BPI: "Every nontrivial Boolean algebra has a prime ideal". (See [13] where BPI is Form 14 and UT is Form 14 A.)

Definition 1.1. The *pseudo-sum* of two filters $\mathscr{F}$ and $\mathscr{G}$ is defined by letting for every set $A \subseteq \mathbb{N}$:

$$A \in \mathscr{F} \oplus \mathscr{G} \iff \{n \mid A - n \in \mathscr{G}\} \in \mathscr{F}$$

where $A - n = \{m \in \mathbb{N} \mid m + n \in A\}$ is the rightward shift of A by n.

Notice that if $\mathscr{U}$ and $\mathscr{V}$ are ultrafilters, then also their pseudo-sum $\mathscr{U} \oplus \mathscr{V}$ is an ultrafilter. It is verified in a straightforward manner that the space of ultrafilters $\beta\mathbb{N}$ endowed with the pseudo-sum operation has the structure of a *right-topological semigroup*; that is, $\oplus$ is associative, and for every ultrafilter $\mathscr{V}$ the "product-on-the-right" $\mathscr{U} \mapsto \mathscr{U} \oplus \mathscr{V}$ is a continuous function on $\beta\mathbb{N}$ (see [12] for all details).

Definition 1.2. An *idempotent ultrafilter* is an ultrafilter $\mathscr{U}$ which is idempotent with respect to the pseudo-sum operation, *i.e.*, $\mathscr{U} = \mathscr{U} \oplus \mathscr{U}$.

We remark that the notion of idempotent ultrafilter is considered and studied in the general setting of semigroups (see [12]; see also the recent book [18]); however, for simplicity, here we will stick to idempotents in $(\beta\mathbb{N}, \oplus)$.

For sets $A \subseteq \mathbb{N}$ and for ultrafilters $\mathscr{V}$, let us denote by

$$A_{\mathscr{V}} \;=\; \{n \mid A - n \in \mathscr{V}\}.$$

So, by definition, $A \in \mathscr{F} \oplus \mathscr{V}$ if and only if $A_{\mathscr{V}} \in \mathscr{F}$. Notice that for every A, B one has $A_{\mathscr{V}} \cap B_{\mathscr{V}} = (A \cap B)_{\mathscr{V}}$, $A_{\mathscr{V}} \cup B_{\mathscr{V}} = (A \cup B)_{\mathscr{V}}$, and $(A_{\mathscr{V}})^c = (A^c)_{\mathscr{V}}$.

The following construction of filters will be useful in the sequel.

Definition 1.3. For filters $\mathscr{F}, \mathscr{G}$ and ultrafilter $\mathscr{V}$, let

$$\mathscr{F}(\mathscr{V}, \mathscr{G}) \;=\; \{B \subseteq \mathbb{N} \mid B \supseteq F \cap A_{\mathscr{V}} \text{ for some } F \in \mathscr{F} \text{ and some } A \in \mathscr{G}\}.$$

Notice that, whenever it satisfies the *finite intersection property*, the family $\mathscr{F}(\mathscr{V}, \mathscr{G})$ is the smallest filter that contains both $\mathscr{F}$ and $\{A_{\mathscr{V}} \mid A \in \mathscr{G}\}$. Families $\mathscr{F}(\mathscr{V}, \mathscr{G})$ satisfy the following properties that will be relevant to our purposes.

Proposition 1.4 (ZF). *Let $\mathscr{F}$ be a filter, and let $\mathscr{V}$ be an ultrafilter. Then for every filter $\mathscr{G} \supseteq \mathscr{F} \oplus \mathscr{V}$, the family $\mathscr{F}(\mathscr{V}, \mathscr{G})$ is a filter such that $\mathscr{F} \subseteq \mathscr{F}(\mathscr{V}, \mathscr{G})$ and $\mathscr{G} \subseteq \mathscr{F}(\mathscr{V}, \mathscr{G}) \oplus \mathscr{V}$.*

Proof. The inclusion $\mathscr{F}(\mathscr{V}, \mathscr{G}) \supseteq \mathscr{F}$ follows from the trivial observation that $\mathbb{N}_{\mathscr{V}} = \mathbb{N}$, and hence $F = F \cap \mathbb{N} = F \cap \mathbb{N}_{\mathscr{V}} \in \mathscr{F}(\mathscr{V}, \mathscr{G})$ for every $F \in \mathscr{F}$. All sets in $\mathscr{F}(\mathscr{V}, \mathscr{G})$ are nonempty; indeed if $F \cap A_{\mathscr{V}} = \emptyset$ for some $F \in \mathscr{F}$ and some $A \in \mathscr{G}$, then $F \subseteq (A_{\mathscr{V}})^c = (A^c)_{\mathscr{V}} \Rightarrow (A^c)_{\mathscr{V}} \in \mathscr{F} \Leftrightarrow A^c \in \mathscr{F} \oplus \mathscr{V} \subseteq \mathscr{G} \Rightarrow A \notin \mathscr{G}$. Since $\mathscr{F}(\mathscr{V}, \mathscr{G})$ is closed under supersets and under finite intersections, it is a filter. Finally, $A \in \mathscr{G} \Rightarrow A_{\mathscr{V}} = \mathbb{N} \cap A_{\mathscr{V}} \in \mathscr{F}(\mathscr{V}, \mathscr{G}) \Leftrightarrow A \in \mathscr{F}(\mathscr{V}, \mathscr{G}) \oplus \mathscr{V}$. □

Corollary 1.5 (ZF). *Let $\mathscr{F}$ be a filter, and let $\mathscr{V}, \mathscr{W}$ be ultrafilters where $\mathscr{W} \supseteq \mathscr{F} \oplus \mathscr{V}$. Then for every ultrafilter $\mathscr{U}$ we have:*

(1) *If $\mathscr{U} \supseteq \mathscr{F}(\mathscr{V}, \mathscr{W})$ then $\mathscr{U} \oplus \mathscr{V} = \mathscr{W}$;*
(2) *If $\mathscr{U} \oplus \mathscr{V} = \mathscr{W}$ and $\mathscr{F} \subseteq \mathscr{U}$ then $\mathscr{F}(\mathscr{V}, \mathscr{W}) \subseteq \mathscr{U}$.*

Proof. (1) If $\mathscr{U} \supseteq \mathscr{F}(\mathscr{V}, \mathscr{W})$ then $\mathscr{U} \supseteq \mathscr{F}$ and $\mathscr{W} \subseteq \mathscr{F}(\mathscr{V}, \mathscr{W}) \oplus \mathscr{V} \subseteq \mathscr{U} \oplus \mathscr{V}$, and hence $\mathscr{W} = \mathscr{U} \oplus \mathscr{V}$, since no inclusion between ultrafilters can be proper, by their maximality.

(2) By definition, $\mathscr{U} \oplus \mathscr{V} = \mathscr{W}$ if and only if $A_{\mathscr{V}} \in \mathscr{U}$ for all $A \in \mathscr{W}$. Since $F \in \mathscr{U}$ for all $F \in \mathscr{F}$, it follows that $\mathscr{F}(\mathscr{V}, \mathscr{W}) \subseteq \mathscr{U}$. □

Let us now denote by $\mathsf{UT}(X)$ the restriction of UT to the set X, namely the property that every filter on X is extended to an ultrafilter. In particular, in the sequel we will consider $\mathsf{UT}(\mathbb{N})$ and $\mathsf{UT}(\mathbb{R})$.[3]

Corollary 1.6 (ZF+$\mathsf{UT}(\mathbb{N})$)**.** *Let $\mathscr{F}$ be a filter and let $\mathscr{V},\mathscr{W}$ be ultrafilters. Then $\mathscr{W}\supseteq\mathscr{F}\oplus\mathscr{V}$ if and only if $\mathscr{W}=\mathscr{U}\oplus\mathscr{V}$ for some ultrafilter $\mathscr{U}\supseteq\mathscr{F}$.*

Proof. One direction is trivial, because $\mathscr{U}\supseteq\mathscr{F}$ directly implies $\mathscr{U}\oplus\mathscr{V}\supseteq\mathscr{F}\oplus\mathscr{V}$ for every $\mathscr{V}$. Conversely, given an ultrafilter $\mathscr{W}\supseteq\mathscr{F}\oplus\mathscr{V}$, by $\mathsf{UT}(\mathbb{N})$ we can pick an ultrafilter $\mathscr{U}\supseteq\mathscr{F}(\mathscr{V},\mathscr{W})\supseteq\mathscr{F}$, and the equality $\mathscr{U}\oplus\mathscr{V}=\mathscr{W}$ is satisfied by (1) of the previous corollary. ☐

2. Additive filters

The central notion in this paper is the following.

Definition 2.1. A filter $\mathscr{F}$ is *additive* if for every ultrafilter $\mathscr{V}\supseteq\mathscr{F}$, the pseudo-sum $\mathscr{F}\oplus\mathscr{V}\supseteq\mathscr{F}$; that is, $A_{\mathscr{V}}\in\mathscr{F}$ for every $A\in\mathscr{F}$.

Since Definition 2.1 is given in the setting of ZF, we should note here that if a filter $\mathscr{F}$ on $\mathbb{N}$ cannot be extended to an ultrafilter, then $\mathscr{F}$ is vacuously additive. In consequence, if there are no non-principal ultrafilters on $\mathbb{N}$, then any non-principal filter on $\mathbb{N}$ is additive. Recall that if ZF is consistent, then also ZF+ "every ultrafilter on $\mathbb{N}$ is principal," and hence $\mathsf{ZF}+\neg\mathsf{UT}(\mathbb{N})$, are consistent.[4]

Remark 2.2. In 1989, T. Papazyan [16] considered the *almost translation invariant filters* $\mathscr{F}$ such that $\mathscr{F}\subseteq\mathscr{F}\oplus\mathscr{F}$, and showed that every maximal filter in that class is necessarily an ultrafilter, and hence an idempotent ultrafilter. We remark that almost translation invariance is a stronger notion with respect to addivity; indeed, it is straightforwardly seen that any almost translation invariant filter is additive, *but not conversely.* (For the latter assertion, see Example 3.9 in the next section.) That same class of filters, named *idempotent filters*, has been thoroughly investigated by P. Krautzberger in [15].

A first trivial example of an additive filter is given by $\mathscr{F}=\{\mathbb{N}\}$; another easy example is given by the Fréchet filter $\{A\subseteq\mathbb{N}\mid A^c \text{ finite}\}$ of cofinite sets. More interesting examples are obtained by considering "additively large sets". For any $X\subseteq\mathbb{N}$, the set of all (finite) sums of distinct elements of X is denoted by

$$\mathrm{FS}(X)=\left\{\sum_{x\in F}x\;\middle|\;F\subseteq X \text{ is finite nonempty}\right\}.$$

Recall that a set $A\subseteq\mathbb{N}$ is called *additively large* if it contains a set $\mathrm{FS}(X)$ for some infinite X. A stronger version of *Hindman's Theorem* states that the family of additively large sets is *partition regular*, *i.e.*, if an additively large set is partitioned into finitely many pieces, then one of the pieces is still additively large. By using a model-theoretic argument, it was shown that this property is a ZF-theorem, although no explicit proof is known where the use of the axiom of choice is avoided (see §4.2 of [4]).

[3] $\mathsf{UT}(\mathbb{N})$ is Form 225 in [13].

[4] See models $\mathcal{M}2$, $\mathcal{M}5(\aleph)$ and $\mathcal{M}15$ in [13] by Feferman, Solovay and Blass, respectively.

As mentioned in the introduction, idempotent ultrafilters can be used to give a short and elegant proof of Hindman's Theorem; indeed, in ZF, all sets in an idempotent ultrafilter are additively large, whereas, in ZFC, for every additively large set A there exists an idempotent ultrafilter $\mathscr{U}$ such that $A \in \mathscr{U}$.[5] For completeness, let us recall here a proof of the former combinatorial property, whose simplicity and elegance was the main motivation for the interest in that special class of ultrafilters.

Notice first that if $\mathscr{V} \oplus \mathscr{V} = \mathscr{V}$ then $\mathscr{V}$ is non-principal.[6] If $A \in \mathscr{V}$, then $A^\star = A \cap A_\mathscr{V} \in \mathscr{V}$. It is readily verified that $A^\star - a \in \mathscr{V}$ for every $a \in A^\star$. Pick any $x_1 \in A^\star$. Then $A_1 = A^\star \cap (A^\star - x_1) \in \mathscr{V}$, and we can pick $x_2 \in A_1$ where $x_2 > x_1$. Since $x_1, x_2, x_1+x_2 \in A^\star$, the set $A_2 = A^\star \cap (A^\star - x_1) \cap (A^\star - x_2) \cap (A^\star - x_1 - x_2) \in \mathscr{V}$, and we can pick $x_3 \in A_2$ where $x_3 > x_2$. By iterating the process, one obtains an infinite sequence $X = \{x_1 < x_2 < x_3 < \ldots\}$ such that $\mathrm{FS}(X) \subseteq A^\star \subseteq A$, as desired.[7]

Every additively large set determines an additive filter, as the next ZF-example clarifies.

Example 2.3. Given an infinite set $X = \{x_1 < x_2 < \ldots\}$, denote by

$$\mathcal{FS}_X \;=\; \{A \subseteq \mathbb{N} \mid A \supseteq \mathrm{FS}(X \setminus F) \text{ for some finite } F \subset X\}\,.$$

Clearly, $\mathcal{FS}_X$ is a filter that contains $\mathrm{FS}(X)$. It just takes a quick check to verify that $\mathcal{FS}_X \subseteq \mathcal{FS}_X \oplus \mathcal{FS}_X$, and hence $\mathcal{FS}_X$ is additive.

Proposition 2.4 below provides a necessary and sufficient condition for a filter to be additive, and shows that additive filters directly correspond to the closed sub-semigroups of $(\beta\mathbb{N}, \oplus)$.

Proposition 2.4 (ZF+$\mathsf{UT}(\mathbb{N})$). *A filter $\mathscr{F}$ is additive if and only if $\mathscr{F} \subseteq \mathscr{U} \oplus \mathscr{V}$ for every pair of ultrafilters $\mathscr{U}, \mathscr{V} \supseteq \mathscr{F}$.*

Proof. For the "only if" implication, notice first that ZF+$\mathsf{UT}(\mathbb{N})$ implies that $\mathscr{F}$ equals the intersection of all ultrafilters $\mathscr{U} \supseteq \mathscr{F}$. By the hypothesis, for every $A \in \mathscr{F}$ one has that $A_\mathscr{V} \in \mathscr{U}$ for all ultrafilters $\mathscr{U} \supseteq \mathscr{F}$, and hence $A_\mathscr{V} \in \mathscr{F}$. Conversely, assume there exists an ultrafilter $\mathscr{V} \supseteq \mathscr{F}$ with $\mathscr{F} \nsubseteq \mathscr{F} \oplus \mathscr{V}$, and pick a set $A \in \mathscr{F}$ with $A \notin \mathscr{F} \oplus \mathscr{V}$, that is, $A_\mathscr{V} \notin \mathscr{F}$. Then there exists an ultrafilter $\mathscr{U} \supseteq \mathscr{F}$ with $A_\mathscr{V} \notin \mathscr{U}$ (note that the family $\{F \cap (A_\mathscr{V})^c \mid F \in \mathscr{F}\}$ has the finite intersection property, thus it can be extended to an ultrafilter by $\mathsf{UT}(\mathbb{N})$). Then $A \notin \mathscr{U} \oplus \mathscr{V}$ and the set A is a witness of $\mathscr{F} \nsubseteq \mathscr{U} \oplus \mathscr{V}$. ☐

Remark 2.5. If C is a nonempty closed sub-semigroup of $(\beta\mathbb{N}, \oplus)$ then

$$\mathrm{Fil}(C) \;=\; \bigcap_{\mathscr{U} \in C} \mathscr{U}$$

is an additive filter. To show this, notice first that if $\mathscr{V} \supseteq \mathrm{Fil}(C)$ is an ultrafilter, then $\mathscr{V} \in \overline{C} = C$. So, for all ultrafilters $\mathscr{V}, \mathscr{V}' \supseteq \mathrm{Fil}(C)$ one has that $\mathscr{V} \oplus \mathscr{V}' \in C$ by the property of sub-semigroup, and hence $\mathscr{V} \oplus \mathscr{V}' \supseteq \mathrm{Fil}(C)$. Conversely, if $\mathscr{F}$ is an additive filter, then $\mathsf{UT}(\mathbb{N})$ implies that

$$\mathrm{Cl}(\mathscr{F}) \;=\; \{\mathscr{U} \in \beta\mathbb{N} \mid \mathscr{U} \supseteq \mathscr{F}\}$$

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[5] See Theorem 5.12 and Lemma 5.11 of [12], respectively.

[6] The only possible principal idempotent ultrafilter would be generated by an element m such that $m + m = m$, whereas we agreed that $0 \notin \mathbb{N}$.

[7] For detailed proofs of other basic properties of idempotent ultrafilters the reader is referred to [12].

is a nonempty closed sub-semigroup. Moreover, the two operations are one the inverse of the other, since $\mathrm{Cl}(\mathrm{Fil}(C)) = C$ and $\mathrm{Fil}(\mathrm{Cl}(\mathscr{F})) = \mathscr{F}$ for every nonempty closed sub-semigroup C and for every additive filter $\mathscr{F}$.

Next, we show two different ways of extending additive filters that preserve the additivity property.

Proposition 2.6 (ZF+UT($\mathbb{N}$)). *Let $\mathscr{F}$ be an additive filter. Then for every ultrafilter $\mathscr{V} \supseteq \mathscr{F}$, the filter $\mathscr{F} \oplus \mathscr{V}$ is additive.*

Proof. Take any ultrafilter $\mathscr{W} \supseteq \mathscr{F} \oplus \mathscr{V}$. Then, by Corollary 1.6, there exists an ultrafilter $\mathscr{U} \supseteq \mathscr{F}$ such that $\mathscr{W} = \mathscr{U} \oplus \mathscr{V}$. By additivity of $\mathscr{F}$, we have that $\mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{U} \subseteq \mathscr{V} \oplus \mathscr{U} \Rightarrow \mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{V} \oplus \mathscr{U} \Rightarrow \mathscr{F} \oplus \mathscr{V} \subseteq \mathscr{F} \oplus \mathscr{V} \oplus \mathscr{U} \oplus \mathscr{V} = (\mathscr{F} \oplus \mathscr{V}) \oplus \mathscr{W}$. □

Proposition 2.7 (ZF). *Let $\mathscr{F}$ be an additive filter. Then for every ultrafilter $\mathscr{V}$ where $\mathscr{V} \supseteq \mathscr{F} \oplus \mathscr{V}$, $\mathscr{F}(\mathscr{V}, \mathscr{V})$ is an additive filter.*

Proof. Let $\mathscr{U}_1, \mathscr{U}_2 \supseteq \mathscr{F}(\mathscr{V}, \mathscr{V})$ be ultrafilters. We want to show that $\mathscr{F}(\mathscr{V}, \mathscr{V}) \subseteq \mathscr{U}_1 \oplus \mathscr{U}_2$. Since $\mathscr{F}$ is additive and $\mathscr{U}_1, \mathscr{U}_2 \supseteq \mathscr{F}(\mathscr{V}, \mathscr{V}) \supseteq \mathscr{F}$ by Proposition 1.4, we have that $\mathscr{F} \subseteq \mathscr{U}_1 \oplus \mathscr{U}_2$. By Corollary 1.5, we have $\mathscr{U}_1 \oplus \mathscr{V} = \mathscr{U}_2 \oplus \mathscr{V} = \mathscr{V}$, and so $\mathscr{V} = \mathscr{U}_1 \oplus \mathscr{U}_2 \oplus \mathscr{V}$. But then for every $A \in \mathscr{V}$ the set $A_{\mathscr{V}} \in \mathscr{U}_1 \oplus \mathscr{U}_2$, and the proof is complete. □

Theorem 2.8 (ZF+UT($\mathbb{N}$)). *If a filter is maximal among the additive filters then it is an idempotent ultrafilter.*

Proof. Let $\mathscr{F}$ be maximal among the additive filters. By UT($\mathbb{N}$) we can pick an ultrafilter $\mathscr{V} \supseteq \mathscr{F}$. We will show that $\mathscr{V} = \mathscr{F}$ and $\mathscr{V} \oplus \mathscr{V} = \mathscr{V}$. By additivity $\mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{V}$, and since $\mathscr{F} \oplus \mathscr{V}$ is additive, by maximality $\mathscr{F} = \mathscr{F} \oplus \mathscr{V}$. Since $\mathscr{F}$ is additive and the ultrafilter $\mathscr{V} \supseteq \mathscr{F} \oplus \mathscr{V}$, also the filter $\mathscr{F}(\mathscr{V}, \mathscr{V})$ is additive by the previous proposition and so, again by maximality, $\mathscr{F}(\mathscr{V}, \mathscr{V}) = \mathscr{F}$. In particular, for every $A \in \mathscr{V}$ one has that $A_{\mathscr{V}} \in \mathscr{F}(\mathscr{V}, \mathscr{V}) = \mathscr{F}$, that is $A \in \mathscr{F} \oplus \mathscr{V} = \mathscr{F}$. This shows that $\mathscr{V} \subseteq \mathscr{F}$, and hence $\mathscr{V} = \mathscr{F}$. Finally, since $\mathscr{V} \supseteq \mathscr{F}(\mathscr{V}, \mathscr{V})$ and $\mathscr{V} \supseteq \mathscr{F} \oplus \mathscr{V}$, we have $\mathscr{V} \oplus \mathscr{V} = \mathscr{V}$ by Corollary 1.5. □

Thanks to the above properties of additive filters, one proves the existence of idempotent ultrafilters with a straight application of *Zorn's Lemma*.

Theorem 2.9 (ZFC). *Every additive filter can be extended to an idempotent ultrafilter.*

Proof. Given an additive filter $\mathscr{F}$, consider the following family

$$\mathbb{F} \;=\; \{\mathscr{G} \supseteq \mathscr{F} \mid \mathscr{G} \text{ is an additive filter}\}.$$

It is easily verified that if $\langle \mathscr{G}_i \mid i \in I \rangle$ is an increasing sequence of filters in $\mathbb{F}$, then the union $\bigcup_{i \in I} \mathscr{F}_i$ is an additive filter. So, *Zorn's Lemma* applies, and one gets a maximal element $\mathscr{G} \in \mathbb{F}$. By the previous theorem, $\mathscr{G} \supseteq \mathscr{F}$ is an idempotent ultrafilter. □

Remark 2.10. As already pointed out in the introduction, with the only exception of [16], the only known proof of existence of idempotent ultrafilters is grounded on *Ellis-Numakura's Lemma*, a general result in semigroup theory that establishes the existence of idempotent elements in every compact Hausdorff right-topological semigroup. An alternate argument to prove the above Theorem 2.9 can be obtained by same pattern. Indeed, given an additive filter $\mathscr{F}$, by Remark 2.5 we know that $C = \mathrm{Cl}(\mathscr{F})$ is a closed nonempty sub-semigroup of the compact right-topological semigroup $(\beta\mathbb{N}, \oplus)$. In consequence, $(C, \oplus)$ is itself a compact right-topological semigroup, so *Ellis-Numakura's Lemma* applies, and one gets the existence of an idempotent element $\mathscr{U} \in C$; clearly, $\mathscr{U} \supseteq \mathscr{F}$.

As *Zorn's Lemma* was never used in this section except for the last theorem above, we are naturally led to the following question:

- Can one prove Theorem 2.9 without using *Zorn's Lemma*?

Clearly, at least some weakened form of the *Ultrafilter Theorem* must be assumed, as otherwise there may be no non-principal ultrafilters at all (see [13]). We will address the above question in the next section.

3. Avoiding Zorn's Lemma

Proposition 3.1 (ZF)**.** *Assume there exists a choice function Φ that associates to every additive filter $\mathscr{F}$ an ultrafilter $\Phi(\mathscr{F}) \supseteq \mathscr{F}$. Then there exists a choice function Ψ that associates to every additive filter $\mathscr{F}$ an ultrafilter $\Psi(\mathscr{F}) \supseteq \mathscr{F}$ such that $\Psi(\mathscr{F}) \supseteq \mathscr{F} \oplus \Psi(\mathscr{F})$.*

Proof. Given an additive filter $\mathscr{F}$, let us define a sequence of filters by transfinite recursion as follows. At the base step, let $\mathscr{F}_0 = \mathscr{F}$. At successor steps, let $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$ if $\Phi(\mathscr{F}_\alpha) \supseteq \mathscr{F}_\alpha \oplus \Phi(\mathscr{F}_\alpha)$, and let $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha \oplus \Phi(\mathscr{F}_\alpha)$ otherwise. Finally, at limit steps λ, let $\mathscr{F}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{F}_\alpha$. It is readily seen by induction that all $\mathscr{F}_\alpha$ are additive filters, and that $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ for $\alpha \leq \beta$. If it was $\mathscr{F}_{\alpha+1} \neq \mathscr{F}_\alpha$ for all α, then the sequence $\langle \mathscr{F}_\alpha \mid \alpha \in \mathbf{ON} \rangle$ would be strictly increasing.[8] This is not possible, even without assuming AC. Indeed, if $\langle \mathscr{F}_\alpha \mid \alpha \in \mathbf{ON} \rangle$ were strictly increasing, then one could consider the function f defined on $\mathcal{P}(\mathcal{P}(\mathbb{N}))$ by setting $f(X) = \alpha$ if $X = \mathscr{F}_\alpha$ for some ordinal α and $f(X) = 0$ otherwise; by the replacement axiom, we would have that $\mathrm{range}(f) = \mathbf{ON}$ is a set, which is absurd.

Then define $\Psi(\mathscr{F}) = \Phi(\mathscr{F}_\alpha)$ where α is the least ordinal such that $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. Such an ultrafilter $\Psi(\mathscr{F})$ satisfies the desired properties. Indeed, $\Phi(\mathscr{F}_\alpha) \supseteq \mathscr{F}_\alpha \supseteq \mathscr{F}_0 = \mathscr{F}$. Moreover, if it was $\Phi(\mathscr{F}_\alpha) \not\supseteq \mathscr{F} \oplus \Phi(\mathscr{F}_\alpha)$, then also $\Phi(\mathscr{F}_\alpha) \not\supseteq \mathscr{F}_\alpha \oplus \Phi(\mathscr{F}_\alpha)$, and so $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha \oplus \Phi(\mathscr{F}_\alpha)$. But then, since $\Phi(\mathscr{F}_\alpha) \supseteq \mathscr{F}_\alpha$ and $\Phi(\mathscr{F}_\alpha) \not\supseteq \mathscr{F}_{\alpha+1}$, it would follow that $\mathscr{F}_{\alpha+1} \neq \mathscr{F}_\alpha$, against the hypothesis. □

Theorem 3.2 (ZF)**.** *Assume there exists a choice function Φ that associates to every additive filter $\mathscr{F}$ an ultrafilter $\Phi(\mathscr{F}) \supseteq \mathscr{F}$. Then there exists a choice function Θ that associates to every additive filter $\mathscr{F}$ an idempotent ultrafilter $\Theta(\mathscr{F}) \supseteq \mathscr{F}$.*

Proof. Fix a function Ψ as given by the previous proposition. Given an additive filter $\mathscr{F}$, by transfinite recursion let us define the sequence $\langle \mathscr{F}_\alpha \mid \alpha \in \mathbf{ON} \rangle$ as follows. At the base step, let $\mathscr{F}_0 = \mathscr{F}$. At successor steps $\alpha + 1$, consider the ultrafilter $\mathscr{V}_\alpha = \Psi(\mathscr{F}_\alpha)$, and let $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$ if $\mathscr{V}_\alpha \supseteq \mathscr{F}_\alpha(\mathscr{V}_\alpha, \mathscr{V}_\alpha)$, and let $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha(\mathscr{V}_\alpha, \mathscr{V}_\alpha)$ otherwise. At limit steps λ, let $\mathscr{F}_\lambda = \bigcup_{\alpha<\lambda} \mathscr{F}_\alpha$. It is shown by induction that all $\mathscr{F}_\alpha$ are additive filters, and that $\mathscr{F}_\alpha \subseteq \mathscr{F}_\beta$ for $\alpha \leq \beta$. Indeed, notice that at successor steps $\mathscr{F}_\alpha(\mathscr{V}_\alpha, \mathscr{V}_\alpha) \supseteq \mathscr{F}_\alpha$ is additive by Proposition 2.7, since $\mathscr{V}_\alpha = \Psi(\mathscr{F}_\alpha) \supseteq \mathscr{F}_\alpha \oplus \mathscr{V}_\alpha$. By the same argument as used in the proof of the previous

[8] By **ON** we denote the proper class of all ordinals.

proposition, it cannot be $\mathscr{F}_{\alpha+1} \neq \mathscr{F}_\alpha$ for all ordinals. So, we can define $\Theta(\mathscr{F}) = \Psi(\mathscr{F}_\alpha)$ where α is the least ordinal such that $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha$. Let us verify that the ultrafilter $\Theta(\mathscr{F})$ satisfies the desired properties. First of all, $\Theta(\mathscr{F}) = \Psi(\mathscr{F}_\alpha) \supseteq \mathscr{F}$. Now notice that $\mathscr{V}_\alpha \supseteq \mathscr{F}_\alpha(\mathscr{V}_\alpha, \mathscr{V}_\alpha)$, as otherwise $\mathscr{F}_{\alpha+1} = \mathscr{F}_\alpha(\mathscr{V}_\alpha, \mathscr{V}_\alpha)$ and we would have $\mathscr{F}_\alpha \neq \mathscr{F}_{\alpha+1}$, since $\mathscr{V}_\alpha \supseteq \mathscr{F}_\alpha$ but $\mathscr{V}_\alpha \not\supseteq \mathscr{F}_{\alpha+1}$. So, $\Theta(\mathscr{F}) = \Psi(\mathscr{F}_\alpha) = \mathscr{V}_\alpha$, and by Corollary 1.5, we finally obtain that $\mathscr{V}_\alpha \oplus \mathscr{V}_\alpha = \mathscr{V}_\alpha$. □

Corollary 3.3 (ZF)**.** *Assume there exists a choice function Φ that associates to every additive filter $\mathscr{F}$ an ultrafilter $\Phi(\mathscr{F}) \supseteq \mathscr{F}$. Then for every additively large set A there exists an idempotent ultrafilter $\mathscr{U}$ where $A \in \mathscr{U}$.*

Proof. Let X be an infinite set with $\mathrm{FS}(X) \subseteq A$, and consider the additive filter $\mathcal{FS}_X$ of Example 2.3. Then $A \in \mathcal{FS}_X$ and, by the previous theorem, $\mathcal{FS}_X$ is included in an idempotent ultrafilter. □

In order to prove that every additive filter extends to an idempotent ultrafilter, one does not need the full axiom of choice, and indeed we will see that a weakened version of the *Ultrafilter Theorem* suffices.

The result below was proved in [8, Lemma 4(ii)] as the outcome of a chain of results about the relative strength of $\mathsf{UT}(\mathbb{R})$ with respect to properties of the Tychonoff products $2^{\mathcal{P}(\mathbb{R})}$ and $2^{\mathbb{R}}$, where $2 = \{0, 1\}$ has the discrete topology.[9] In order to keep our paper self-contained, we give here an alternative direct proof where explicit topological notions are avoided.

Proposition 3.4 (ZF+UT($\mathbb{R}$))**.** *There exists a choice function Φ that associates to every filter $\mathscr{F}$ on $\mathbb{N}$ an ultrafilter $\Phi(\mathscr{F}) \supseteq \mathscr{F}$.*

Proof. Every filter is an element of $\mathcal{P}(\mathcal{P}(\mathbb{N}))$, which is in bijection with $\mathcal{P}(\mathbb{R})$. So, in ZF, one has a 1-1 enumeration of all filters $\{\mathscr{F}_Y \mid Y \in \mathfrak{F}\}$ for a suitable family $\mathfrak{F} \subseteq \mathcal{P}(\mathbb{R})$. Fix a bijection $\psi : \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$, let $I = \mathrm{Fin}(\mathbb{R}) \times \mathrm{Fin}(\mathrm{Fin}(\mathbb{R}))$ (where for a set X, $\mathrm{Fin}(X)$ denotes the set of finite subsets of X), and for every $(A, Y) \in \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{R})$, let

$$X(A, Y) \;=\; \{(F, S) \in I \mid S \subseteq \mathcal{P}(F);\ \psi(A, Y) \cap F \in S\}.$$

Notice that for every $\mathfrak{B} \subseteq \mathcal{P}(\mathbb{N}) \times \mathcal{P}(\mathbb{R})$, the family

$$\langle \mathfrak{B} \rangle \;=\; \{X(A, Y) \mid (A, Y) \in \mathfrak{B}\} \cup \{X(A, Y)^c \mid (A, Y) \notin \mathfrak{B}\}$$

has the finite intersection property. Indeed, given pairwise distinct $(A_1, Y_1), \ldots, (A_k, Y_k) \in \mathfrak{B}$ and pairwise distinct $(B_1, Z_1), \ldots, (B_h, Z_h) \notin \mathfrak{B}$, for every i, j pick an element $u_{i,j} \in \psi(A_i, Y_i) \triangle \psi(B_j, Z_j)$ (where $\triangle$ denotes the symmetric difference of sets). If we let

$$F = \{u_{i,j} \mid i = 1, \ldots, k;\ j = 1, \ldots, h\} \quad \text{and} \quad S = \{\psi(A_i, Y_i) \cap F \mid i = 1, \ldots, k\},$$

then it is readily seen that $(F, S) \in \bigcap_{i=1}^{k} X(A_i, Y_i) \cap \bigcap_{j=1}^{h} X(B_j, Z_j)^c$.

For every $Y \in \mathfrak{F}$, let us now consider the following family of subsets of I:

$$\mathscr{G}_Y \;=\; \{X(A, Y) \mid A \in \mathscr{F}_Y\} \cup \{\Lambda(A, B, Y) \mid A, B \subseteq \mathbb{N}\} \cup \{\Gamma(A, Y) \mid A \subseteq \mathbb{N}\}$$

---

[9] Precisely, a proof of Proposition 3.4 is obtained by combining the following ZF-results: (a) $\mathsf{UT}(\mathbb{R})$ if and only $2^{\mathcal{P}(\mathbb{R})}$ is compact; (b) $\beta\mathbb{N}$ embeds as a closed subspace of $2^{\mathbb{R}}$; (c) $\mathsf{UT}(\mathbb{R})$ implies that $2^{\mathbb{R}}$, and hence $\beta\mathbb{N}$, is both compact and Loeb. Recall that a topological space is *Loeb* if there exists a choice function on the family of its nonempty closed subspaces. Recall also that nonempty closed subspaces of $\beta\mathbb{N}$ correspond to filters (see Remark 2.5).

$$\cup \{\Delta(A,B,Y) \mid A \subseteq B \subseteq \mathbb{N}\}$$

where

- $\Lambda(A,B,Y) = X(A,Y)^c \cup X(B,Y)^c \cup X(A \cap B, Y)$;
- $\Gamma(A,Y) = X(A,Y) \cup X(A^c,Y)$;
- $\Delta(A,B,Y) = X(A,Y)^c \cup X(B,Y)$.

We want to show that every finite union $\bigcup_{i=1}^{k} \mathscr{G}_{Y_i}$ where $Y_i \in \mathfrak{F}$ has the finite intersection property, and hence also $\mathscr{G} = \bigcup_{Y \in \mathfrak{F}} \mathscr{G}_Y$ has the finite intersection property. By $\mathsf{UT}(\mathbb{N})$, which follows from $\mathsf{UT}(\mathbb{R})$, we can pick ultrafilters $\mathscr{V}_i \supseteq \mathscr{F}_{Y_i}$ for $i = 1, \ldots, k$. Then

$$\mathscr{H} \;=\; \bigcup_{i=1}^{k} \left(\{X(A,Y_i) \mid A \in \mathscr{V}_i\} \cup \{X(A,Y_i)^c \mid A \notin \mathscr{V}_i\}\right)$$

has the finite intersection property, because $\mathscr{H} \subset \langle \mathfrak{B} \rangle$ where $\mathfrak{B} = \{(A,Y_i) \mid 1 \leq i \leq k; A \in \mathscr{V}_i\}$. Now let $G_1, \ldots, G_h \in \bigcup_{i=1}^{k} \mathscr{G}_{Y_i}$. For every G_j pick $H_j \in \mathscr{H}$ such that $H_j \subseteq G_j$ as follows. If $G_j = X(A,Y_i)$ for some $A \in \mathscr{F}_{Y_i}$ then let $H_j = G_j$; if $G_j = \Lambda(A,B,Y_i)$ then let $H_j = X(A,Y_i)^c$ if $A \notin \mathscr{V}_i$, let $H_j = X(B,Y_i)^c$ if $A \in \mathscr{V}_i$ and $B \notin \mathscr{V}_i$, and let $H_j = X(A \cap B, Y_i)$ if $A, B \in \mathscr{V}_i$; if $G_j = \Gamma(A,Y_i)$ then let $H_j = X(A,Y_i)$ if $A \in \mathscr{V}_i$, and let $H_j = X(A^c,Y_i)$ if $A \notin \mathscr{V}_i$; and if $G_j = \Delta(A,B,Y_i)$ (where $A \subseteq B$) then let $H_j = X(B,Y_i)$ if $A \in \mathscr{V}_i$ or $B \in \mathscr{V}_i$, and let $H_j = X(A,Y_i)^c$ if $A \notin \mathscr{V}_i$ and $B \notin \mathscr{V}_i$. But then $\bigcap_{j=1}^{h} G_j$ is nonempty because it includes $\bigcap_{j=1}^{h} H_j$ and the family $\mathscr{H}$ has the finite intersection property.

Since $I = \mathrm{Fin}(\mathbb{R}) \times \mathrm{Fin}(\mathrm{Fin}(\mathbb{R}))$ is in bijection with $\mathbb{R}$, by $\mathsf{UT}(\mathbb{R})$ there exists an ultrafilter $\mathfrak{U} \supseteq \mathscr{G}$. Finally, for every $Y \in \mathfrak{F}$, the family

$$\mathscr{U}_Y \;=\; \{A \subseteq \mathbb{N} \mid X(A,Y) \in \mathfrak{U}\}$$

is an ultrafilter that extends $\mathscr{F}_Y$. Indeed, if $A \in \mathscr{F}_Y$ then $X(A,Y) \in \mathscr{G}_Y \subseteq \mathfrak{U}$, and so $A \in \mathscr{U}_Y$. Now assume $A, B \in \mathscr{U}_Y$, *i.e.* $X(A,Y), X(B,Y) \in \mathfrak{U}$. Since $\Lambda(A,B,Y) \in \mathscr{G}_Y \subseteq \mathfrak{U}$, we have $X(A \cap B, Y) = \Lambda(A,B,Y) \cap X(A,Y) \cap X(B,Y) \in \mathfrak{U}$, and so $A \cap B \in \mathscr{U}_Y$. Now let $A \in \mathscr{U}_Y$ and also let $B \supseteq A$. Since $\Delta(A,B,Y) \in \mathscr{G}_Y \subseteq \mathfrak{U}$, we have $X(A,Y) \cap \Delta(A,B,Y) \in \mathfrak{U}$. Furthermore, $X(B,Y) \supseteq X(A,Y) \cap X(B,Y) = X(A,Y) \cap \Delta(A,B,Y)$, thus $X(B,Y) \in \mathfrak{U}$, and consequently $B \in \mathscr{U}_Y$. Now let $A \subseteq \mathbb{N}$. If $A \notin \mathscr{U}_Y$, *i.e.* if $X(A,Y) \notin \mathfrak{U}$, then $X(A,Y)^c \in \mathfrak{U}$. But $\Gamma(A,Y) \in \mathscr{G}_Y \subseteq \mathfrak{U}$, so $X(A^c,Y) \supseteq \Gamma(A,Y) \cap X(A,Y)^c \in \mathfrak{U}$, and hence $A^c \in \mathscr{U}_Y$. Clearly, the correspondence $\mathscr{F}_Y \mapsto \mathscr{U}_Y$ yields the desired choice function. □

Remark 3.5. In ZF, the property that "there exists a choice function Φ that associates to every filter $\mathscr{F}$ on $\mathbb{N}$ an ultrafilter $\Phi(\mathscr{F}) \supseteq \mathscr{F}$" is equivalent to the property that "$\beta\mathbb{N}$ is compact and Loeb" (see [8, Proposition 1(ii)]). We remark that the latter statement is strictly weaker than $\mathsf{UT}(\mathbb{R})$ in ZF (see [14, Theorem 10]).

By putting together Proposition 3.4 with Theorem 3.2, one obtains:

Theorem 3.6 (ZF+$\mathsf{UT}(\mathbb{R})$)**.** *Every additive filter can be extended to an idempotent ultrafilter.*

Remark 3.7. Since every idempotent filter $\mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{F}$ is readily seen to be additive, as a straight corollary we obtain Papazyan's result [16] that every maximal idempotent filter is an idempotent ultrafilter.

Remark 3.8. The above Theorem 3.6 cannot be proved by ZF alone. Indeed, since the Fréchet filter $\{A \subseteq \mathbb{N} \mid \mathbb{N}\setminus A \text{ is finite}\}$ is additive, one would obtain the existence of a non-principal ultrafilter on $\mathbb{N}$ in ZF, against the well-known fact that there exist models of ZF with no non-principal ultrafilters on $\mathbb{N}$ (see [13]).

We conclude this section by showing an example of an additive filter $\mathscr{F}$ which is not idempotent, *i.e.*, $\mathscr{F} \not\subseteq \mathscr{F} \oplus \mathscr{F}$.

Recall that a nonempty family $\mathscr{P} \subseteq \mathcal{P}(\mathbb{N})$ is *partition regular* if in every finite partition $A = C_1 \cup \ldots \cup C_n$ where $A \in \mathscr{P}$, one of the pieces $C_i \in \mathscr{P}$; it also assumed that $\mathscr{P}$ is closed under supersets, *i.e.*, $A' \supseteq A \in \mathscr{P} \Rightarrow A' \in \mathscr{P}$. In this case, the dual family

$$\mathscr{P}^* \;=\; \{A \subseteq \mathbb{N} \mid A^c \notin \mathscr{P}\} \;=\; \{A \subseteq \mathbb{N} \mid A \cap B \neq \emptyset \ \text{ for every } B \in \mathscr{P}\}$$

is a filter; moreover, by assuming $\mathsf{UT}(\mathbb{N})$, one has $\mathscr{P}^* = \bigcap\{\mathscr{U} \in \beta\mathbb{N} \mid \mathscr{U} \supseteq \mathscr{P}\}$. All these facts follow from the definitions in a straightforward manner (see, *e.g.*, [12, Theorem 3.11] or [2]).

Call *finitely additively large* (*FAL* for short) a set $A \subseteq \mathbb{N}$ such that for every $n \in \mathbb{N}$ there exist $x_1 < \ldots < x_n$ with $\mathrm{FS}(\{x_i\}_{i=1}^n) \subseteq A$. Clearly every additively large set is FAL, but not conversely; *e.g.*, the set $A = \bigcup_{k\in\mathbb{N}} \mathrm{FS}(\{2^i \mid 2^{k-1} \leq i < 2^k\})$ is FAL but not additively large.[10]

Example 3.9. (ZF). The following family is an additive filter which is *not* idempotent:

$$\mathscr{F} \;=\; \{A \subseteq \mathbb{N} \mid A^c \text{ is not FAL}\}.$$

First of all, the dual family $\mathscr{F}$ is a filter because the family of FAL sets is partition regular. Recall that the latter property is a consequence of *Folkman's Theorem* in its finite version: "For every n and for every r there exists N such that for every r-coloring $\{1, \ldots, N\} = C_1 \cup \ldots \cup C_r$ there exists a set S of cardinality n with $\mathrm{FS}(S)$ monochromatic." (For a ZF-proof of Folkman's Theorem, see [6, Theorem 11, Lemma 12], pp. 81–82.)

We now turn to the proof that $\mathscr{F}$ is additive. If there is no ultrafilter $\mathscr{V} \supseteq \mathscr{F}$, then $\mathscr{F}$ is vacuously additive. Otherwise, fix any ultrafilter $\mathscr{V} \supseteq \mathscr{F}$; we want to show that $\mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{V}$. Notice that every $B \in \mathscr{V}$ is FAL, as otherwise $B^c \in \mathscr{F} \subseteq \mathscr{V}$ and we would have $\emptyset = B \cap B^c \in \mathscr{V}$. By the definitions, if $A \notin \mathscr{F} \oplus \mathscr{V}$ then $A_{\mathscr{V}} = \{n \mid A - n \in \mathscr{V}\} \notin \mathscr{F}$, *i.e.*, $(A_{\mathscr{V}})^c = \{n \mid A^c - n \in \mathscr{V}\}$ is FAL. Then for every n there exist $x_1 < \ldots < x_n$ such that $A^c - s \in \mathscr{V}$ for every $s \in \mathrm{FS}(\{x_i\}_{i=1}^n)$. Since the finite intersection $B = \bigcap\{A^c - s \mid s \in \mathrm{FS}(\{x_i\}_{i=1}^n)\}$ also belongs to $\mathscr{V}$, we can pick $y_1 < \ldots < y_n$ where $y_1 > x_1 + \ldots + x_n$ such that $\mathrm{FS}(\{y_i\}_{i=1}^n) \subseteq B$. It is readily verified that $x_1 + y_1 < \ldots < x_n + y_n$ and that $\mathrm{FS}(\{x_i + y_i\}_{i=1}^n) \subseteq A^c$. This shows that A^c is FAL, and hence $A \notin \mathscr{F}$.

Let us now check that the filter $\mathscr{F}$ is *not* idempotent. To this end, we need some preliminary work. Denote by $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and let $\psi : \mathrm{Fin}(\mathbb{N}_0) \to \mathbb{N}_0$ be the bijection where $\psi(F) = \sum_{i\in F} 2^i$ for $F \neq \emptyset$ and $\psi(\emptyset) = 0$. Notice that $\psi(F) + \psi(G) = \psi(F \triangle G) + 2 \cdot \psi(F \cap G)$, and also observe that for every H one has $2 \cdot \psi(H) = \psi(1 + H)$ where $1 + H = \{1 + h \mid h \in H\}$. In consequence, if $\mathbb{E}$ is the set of even natural numbers, the following property is easily checked:

[10] This example is mentioned in [3], p. 4499; see also [2, Theorem 1.12], where FAL sets are called $\mathrm{IP}_{<\omega}$ sets.

($\star$) *Let* $F, G \in \mathrm{Fin}(\mathbb{E})$*. Then* $\psi(F) + \psi(G) = \psi(H)$ *for some* $H \in \mathrm{Fin}(\mathbb{E})$ *if and only if* $F \cap G = \emptyset$ *and* $F \cup G = H$*.*

Now fix a partition $\mathbb{E} = \bigcup_{n\in\mathbb{N}_0} A_n$ of the even natural numbers into infinitely many infinite sets, and define

$$X \;=\; \left\{\psi(F \cup G) \mid \emptyset \neq F \in \mathrm{Fin}(A_0) \;\&\; \emptyset \neq G \in \mathrm{Fin}(A_{\psi(F)})\right\}.$$

We will see that $X^c \in \mathscr{F}$ and $X^c \notin \mathscr{F} \oplus \mathscr{F}$, thus showing that $\mathscr{F} \not\subseteq \mathscr{F} \oplus \mathscr{F}$.

The first property follows from the fact that there are no triples $a, b, a + b \in X$, and hence X is not FAL. To see this, assume by contradiction that $\psi(F_1 \cup G_1) + \psi(F_2 \cup G_2) = \psi(F_3 \cup G_3)$ for suitable nonempty $F_1, F_2, F_3 \in \mathrm{Fin}(A_0)$ and $G_i \in \mathrm{Fin}(A_{\psi(F_i)})$. By the above property ($\star$), it follows that $(F_1 \cup G_1) \cap (F_2 \cup G_2) = \emptyset$, and hence $F_1 \cap F_2 = \emptyset$; moreover, $(F_1 \cup G_1) \cup (F_2 \cup G_2) = F_3 \cup G_3$, and hence $F_1 \cup F_2 = F_3$ and $G_1 \cup G_2 = G_3$. This is not possible because $F_1 \cap F_2 = \emptyset$ implies that $F_1, F_2 \neq F_3$, and so $(G_1 \cup G_2) \cap G_3 = \emptyset$.

By the definitions, $X^c \notin \mathscr{F} \oplus \mathscr{F}$ if and only if $\Xi = \{n \mid X^c - n \in \mathscr{F}\} \notin \mathscr{F}$ if and only if $\Xi^c = \{n \mid X - n \text{ is FAL}\}$ is FAL, and this last property is true. Indeed, for every nonempty $F \in \mathrm{Fin}(A_0)$ and for every nonempty $G \in \mathrm{Fin}(A_{\psi(F)})$, we have that $F \cap G = \emptyset$ and so $\psi(F \cup G) = \psi(F) + \psi(G)$. In consequence, the set $X - \psi(F) \supseteq \{\psi(G) \mid \emptyset \neq G \in \mathrm{Fin}(A_{\psi(F)})\} = \mathrm{FS}(A_{\psi(F)})$ is additively large, and hence FAL. But then also $\Xi^c \supseteq \{\psi(F) \mid \emptyset \neq F \in \mathrm{Fin}(A_0)\} = \mathrm{FS}(A_0)$ is FAL because it is additively large, as desired.

Remark 3.10. The above example is fairly related to Example 2.8 found in P. Krautzberger's thesis [15]; however there are relevant differences. Most notably, besides the fact that different semigroups are considered, our example is carried within ZF, whereas the proof in [15] requires certain weak forms of the axiom of choice. Let us see in more detail.

In [15] one first considers a *partial* semigroup $(\mathbb{F}, \cdot)$ on the family $\mathbb{F}$ of finite subsets of $\mathbb{N}$ where the partial operation is defined by means of disjoint unions, and then the corresponding semigroup of ultrafilters $(\delta\mathbb{F}, \cdot)$ where $\delta\mathbb{F}$ is a suitable closed subspace of $\beta\mathbb{F}$. (See [15, Definition 1.4] for details.) Recall that a *finite union set* is a set of the form $\mathrm{FU}(X) = \{\bigcup_{F\in\mathscr{X}} F \mid \emptyset \neq \mathscr{X} \in \mathrm{Fin}(X)\}$. By *Graham-Rothschild parameter-sets Theorem* [5], the family of sets that contain arbitrarily large finite union sets is partition regular and so the following closed set is nonempty:

$$H \;=\; \{\mathscr{U} \in \delta\mathbb{F} \mid (\forall A \in \mathscr{U})(\forall n \in \mathbb{N})(\exists x_1 < \ldots < x_n)\ \mathrm{FU}(\{x_i\}_{i=1}^n) \subseteq A\}.$$

(Notice that $\mathsf{UT}(\mathbb{N})$ suffices to prove $H \neq \emptyset$; indeed, any ultrafilter on $\mathbb{F}$ extending the filter $\{A \subseteq \mathbb{F} \mid A^c \text{ does not contain arbitrarily large union sets}\}$ is in H.) It is then shown that H is a sub-semigroup and that the filter

$$\mathscr{H} \;=\; \mathrm{Fil}(H) = \bigcap\{\mathscr{U} \mid \mathscr{U} \in H\}$$

is not idempotent. This last property is proved by showing the existence of an injective sequence of ultrafilters $\langle \mathscr{U}_n \mid n \in \mathbb{N}\rangle$ in H whose limit $\mathscr{U}\text{-}\lim_n(n+\mathscr{U}_n) \notin H$ for a suitable $\mathscr{U} \in H$; notice that here countably many choices are made.[11]

It is well-known that partition regularity results about finite unions can be (almost) directly translated into partition regularity results about finite sums, and

[11] More precisely, for every $n \in \mathbb{N}$ one picks an ultrafilter $\mathscr{U}_n \in H$ that contains a suitable set $A_n \in \mathscr{H}$. (See [15] for details.) In view of Proposition 3.4, instead of countable choice one could assume $\mathsf{UT}(\mathbb{R})$ to get such a sequence.

conversely (see, *e.g.*, [6, Theorem 13] and [12, pp.113-114]). Along these lines, our Example 3.9 can be seen as a translation of the above example to $(\beta\mathbb{N}, \oplus)$. We remark that, besides some non-trivial adjustments, we paid attention not to use any form of choice; to this end, we directly considered the dual filter

$$\mathscr{F} \;=\; \{A \subseteq \mathbb{N} \mid A^c \text{ is not FAL}\}.$$

instead of the corresponding closed sub-semigroup $\{\mathscr{U} \in \beta\mathbb{N} \mid (\forall A \in \mathscr{U})(A \text{ is FAL})\}$.

4. Final remarks and open questions

By only assuming a weaker property for a filter $\mathscr{F}$ than additivity, one can prove that every set $A \in \mathscr{F}$ is finitely additively large.

Proposition 4.1 (ZF). *Let $\mathscr{F}$ be a filter, and assume that there exists an ultrafilter $\mathscr{V} \supseteq \mathscr{F}$ such that $\mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{V}$. Then for every $A \in \mathscr{F}$ and for every k there exist k-many elements $x_1 < \ldots < x_k$ such that* $\mathrm{FS}(\{x_i\}_{i=1}^n) \subseteq A$.

Proof. Let $\mathscr{V}$ be an ultrafilter as given by the hypothesis. If $\mathscr{V} \supseteq \mathscr{F}$ is principal, say generated by $m \in \mathbb{N}$, then $A \in \mathscr{F} \Rightarrow m \in A$. Moreover, since $\mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{V}$, we also have that $A \in \mathscr{F} \Rightarrow A_{\mathscr{V}} = A - m \in \mathscr{F}$. But then every $A \in \mathscr{F}$ contains all multiples hm for $h \in \mathbb{N}$, and the thesis trivially follows. So, let us assume that $\mathscr{V}$ is non-principal.

For the sake of simplicity, here we will only consider the case $k = 4$; for arbitrary k, the proof is obtained by the same argument. Notice first that, since $\mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{V}$, we have that $A \in \mathscr{F} \Rightarrow A_{\mathscr{V}} \in \mathscr{F}$, and hence also $A_{\mathscr{V}^2}, A_{\mathscr{V}^3} \in \mathscr{F}$, where we denoted $\mathscr{V}^2 = \mathscr{V} \oplus \mathscr{V}$ and $\mathscr{V}^3 = \mathscr{V} \oplus \mathscr{V} \oplus \mathscr{V}$.

- Pick $x_1 \in A \cap A_{\mathscr{V}} \cap A_{\mathscr{V}^2} \cap A_{\mathscr{V}^3} \in \mathscr{F}$.

Then $x_1 \in A$, and $A - x_1, A_{\mathscr{V}} - x_1, A_{\mathscr{V}^2} - x_1 \in \mathscr{V}$.

- Pick $x_2 \in A \cap (A - x_1) \cap A_{\mathscr{V}} \cap (A_{\mathscr{V}} - x_1) \cap A_{\mathscr{V}^2} \cap (A_{\mathscr{V}^2} - x_1) \in \mathscr{V}$. As $\mathscr{V}$ is non-principal, we can take $x_2 > x_1$.

Then $x_2, x_2 + x_1 \in A$, and $A - x_2, A - x_1 - x_2, A_{\mathscr{V}} - x_2, A_{\mathscr{V}} - x_1 - x_2 \in \mathscr{V}$.

- Pick $x_3 \in A \cap (A - x_1) \cap (A - x_2) \cap (A - x_1 - x_2) \cap A_{\mathscr{V}} \cap (A_{\mathscr{V}} - x_1) \cap (A_{\mathscr{V}} - x_2) \cap (A_{\mathscr{V}} - x_1 - x_2) \in \mathscr{V}$. We can take $x_3 > x_2$.

Then $x_3, x_3 + x_1, x_3 + x_2, x_3 + x_2 + x_1 \in A$ and $A - x_3, A - x_1 - x_3, A - x_2 - x_3, A - x_1 - x_2 - x_3 \in \mathscr{V}$.

- Pick $x_4 \in A \cap (A - x_1) \cap (A - x_2) \cap (A - x_3) \cap (A - x_1 - x_2) \cap (A - x_1 - x_3) \cap (A - x_2 - x_3) \cap (A - x_1 - x_2 - x_3) \in \mathscr{V}$. We can take $x_4 > x_3$.

We finally obtain that $\mathrm{FS}(\{x_1 < x_2 < x_3 < x_4\}) \subseteq A$. □

Corollary 4.2 (ZF). *Assume that there are no non-principal ultrafilters. If $\mathscr{F}$ is an additive filter, then every $A \in \mathscr{F}$ is infinite.*

Proof. By contradiction, assume that the filter $\mathscr{F}$ contains a finite set, and pick a minimal finite set C in $\mathscr{F}$. It is easily seen that $\mathscr{F} = \{A \subseteq \mathbb{N} \mid A \supseteq C\}$. If c is any element of C, then the corresponding principal ultrafilter $\mathscr{U}_c \supseteq \mathscr{F}$; and since $\mathscr{F}$ is additive, one has $\mathscr{F} \subseteq \mathscr{F} \oplus \mathscr{U}_c$. By the same argument used at the beginning of the proof of Proposition 4.1, it is shown that all multiples $hc \in C$ for $h \in \mathbb{N}$, contradicting the finiteness of C. □

Notice that, by combining Theorem 3.6 with the fact that every set in an idempotent filter is additively large, one obtains the following stronger property.

Proposition 4.3 ($\mathsf{ZF} + \mathsf{UT}(\mathbb{R})$). *If $\mathscr{F}$ is an additive filter then every $A \in \mathscr{F}$ is additively large.*

The assumption of $\mathsf{UT}(\mathbb{R})$ is necessary. Indeed, the following holds:

Proposition 4.4 (ZF). *Assume that for every additive filter $\mathscr{F}$, every $A \in \mathscr{F}$ is additively large. Then there exists a non-principal ultrafilter on $\mathbb{N}$. In consequence, Proposition 4.3 cannot be proved in ZF alone.*

Proof. Let $\mathscr{F}_0 = \{A \subseteq \mathbb{N} \mid A^c \text{ finite}\}$ be the Fréchet filter of cofinite sets, and let $A = \{2^n \mid n \in \mathbb{N}\}$. Notice that A has no triples of the form $x, y, x+y$ with $x \neq y$ (such triples are called a *Schur triples*), and hence A is not additively large. Let $\mathscr{F}$ be the filter generated by $\mathscr{F}_0 \cup \{A\}$; clearly $\mathscr{F}$ is non-principal. If there are no non-principal ultrafilters on $\mathbb{N}$, then $\mathscr{F}$ is vacuously additive, but the set $A \in \mathscr{F}$ is not additively large. □

The result of Proposition 4.4 is in *striking contrast* with the ZF-result that every element of an idempotent ultrafilter is additively large. With regard to this, let us recall here that Hindman's Theorem is a theorem of ZF, although this fact was established only indirectly by a model-theoretic argument (see §4.2 of [4]), and as yet, no explicit ZF-proof of Hindman's Theorem is available.

Another corollary of Theorem 3.6 is the following:

Proposition 4.5 ($\mathsf{ZF} + \mathsf{UT}(\mathbb{R})$). *For every additive filter $\mathscr{F}$ and for every $B \subseteq \mathbb{N}$ there exists an additive filter $\mathscr{G} \supseteq \mathscr{F}$ such that either $B \in \mathscr{G}$ or $B^c \in \mathscr{G}$.*

As the referee pointed out to us, it is interesting to note that also the statement "every ultrafilter on $\mathbb{N}$ is principal" (which contradicts $\mathsf{UT}(\mathbb{R})$) implies the result of Proposition 4.5.

Proposition 4.6 (ZF). *Assume that there are no non-principal ultrafilters on $\mathbb{N}$. Then for every additive filter $\mathscr{F}$ and for every $B \subseteq \mathbb{N}$ there exists an additive filter $\mathscr{G} \supseteq \mathscr{F}$ such that either $B \in \mathscr{G}$ or $B^c \in \mathscr{G}$.*

Proof. Notice first that, by our assumption, there is no ultrafilter extending the Fréchet filter $\mathscr{F}_0$ of cofinite sets. By Corollary 4.2, every element of $\mathscr{F}$ is infinite. In consequence, one can show that at least one of the families $\mathscr{F} \cup \mathscr{F}_0 \cup \{B\}$ or $\mathscr{F} \cup \mathscr{F}_0 \cup \{B^c\}$ has the finite intersection property. In both cases, the generated filter $\mathscr{G}$ is vacuously additive, it extends $\mathscr{F}$, and it contains either B or B^c.[12] □

Next, we formulate a few open problems that arise naturally from the material presented in this paper.

(1) Is Proposition 4.5 provable in ZF?

Let us now consider the following statements:

(a) "Every additive filter can be extended to an idempotent ultrafilter."

(b) "Every idempotent filter can be extended to an idempotent ultrafilter."

(c) "There exists an idempotent ultrafilter on $\mathbb{N}$."

(d) "There exists a non-principal ultrafilter on $\mathbb{N}$."

[12] We remark that from the above discussion, it follows that the result of Proposition 4.5 is true in the models $\mathcal{M}2$, $\mathcal{M}5(\aleph)$ and $\mathcal{M}15$ of [13]. Furthermore, since UT is true in the Basic Cohen Model $\mathcal{M}1$ of [13], the result is also true in $\mathcal{M}1$.

In the previous section, we showed in ZF that $\mathsf{UT}(\mathbb{R}) \Rightarrow (a)$ and noticed that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. We also recalled that (d) cannot be proved by ZF alone. These facts suggest to investigate whether any of the above implications can be reversed.

(2) Does ZF prove that $(a) \Rightarrow \mathsf{UT}(\mathbb{R})$?
(3) Does ZF prove that $(b) \Rightarrow (a)$?
(4) Does ZF prove that $(c) \Rightarrow (b)$?
(5) Does ZF prove that $(d) \Rightarrow (c)$?

Remark 4.7. A detailed investigation of the strength of *Ellis-Numakura's Lemma* in the hierarchy of weak choice principles is found in [17]. In particular, in that paper it is shown that either one of the *Axiom of Multiple Choice* MC or the *Ultrafilter Theorem* UT (in its equivalent formulation given by the *Boolean Prime Ideal Theorem* BPI) suffices to prove Ellis-Numakura's Lemma.[13] (The key point of the proof is the fact that both MC and UT imply the existence of a choice function for the family of nonempty closed sub-semigroups of any compact Hausdorff right topological semigroup.) Recall that, as pointed out in Remark 2.5, under the assumption of $\mathsf{UT}(\mathbb{N})$ (or of MC, since $\mathsf{MC} \Rightarrow \mathsf{UT}(\mathbb{N})$), nonempty closed sub-semigroups of $(\beta\mathbb{N}, \oplus)$ exactly correspond to additive filters, and so one obtains that either one of MC or UT implies that every additive filter on $\mathbb{N}$ is extended to an idempotent ultrafilter.

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[13] MC postulates the existence of a "multiple choice" function for every family $\mathcal{A}$ of nonempty sets, *i.e.*, a function F such that $F(x)$ is a nonempty finite subset of x for every $x \in \mathcal{A}$. Recall that MC is equivalent to AC in ZF, but it is strictly weaker than AC in Zermelo–Fraenkel set theory with atoms ZFA (see [13]).

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