On the Reynolds time-averaged equations and the long-time behavior of Leray-Hopf weak solutions, with applications to ensemble averages

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Abstract

We consider the three dimensional Navier-Stokes equations and we prove that for Leray-Hopf weak solutions it is possible to characterize (up to sub-sequences) their long-time averages, which satisfy the Reynolds averaged equations. Moreover, we show the validity of the Boussinesq hypothesis, without any additional assumption. Finally, in the last section we consider ensemble averages of solutions and we prove that the fluctuations continue to have a dissipative effect on the mean flow.

Keywords: Navier-Stokes equations, time-averaging, Reynolds equations, Boussinesq hypothesis.

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1 Introduction

We start by describing the problem we will analyze, with particular emphasis on the role of the external force and on the functional setting which turns out appropriate to describe the long-time behavior.

1.1 Framework and motivations

In this paper we study the effect of time-averaging on (Leray-Hopf) weak solutions to the 3D initial-boundary value problem for the Navier-Stokes equations (NSE for simplicity in the sequel)

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\,\mathbf{v} - \nu\Delta\mathbf{v} + \nabla p = \mathbf{f} & \text{in }]0,+\infty[\times\Omega, \\ \nabla\cdot\mathbf{v} = 0 & \text{in }]0,+\infty[\times\Omega, \\ \mathbf{v} = \mathbf{0} & \text{on }]0,+\infty[\times\Gamma, \\ \mathbf{v}(0,\mathbf{x}) = \mathbf{v}_0(\mathbf{x}) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\nu > 0$ is the kinematic viscosity while $\Omega \subset \mathbb{R}^3$ is an open Lipschitz and bounded set. We will often use the alternative expression $\nabla\cdot(\mathbf{v}\otimes\mathbf{v})$ for the convective term, which is equivalent due to the divergence-free constraint[1].

In order to properly set what we mean by time-averaging, let $\psi:\ \mathbb{R}^+\times\Omega \to \mathbb{R}^N$ be any tensor field related to a given turbulent flow (N being its order). The time-average over a time interval $[0,t]$ is defined by

$$M_t(\psi)(\mathbf{x}) := \frac{1}{t}\int_0^t \psi(s,\mathbf{x})\,ds \qquad \text{for } t > 0. \tag{1.2}$$

The main aim of this paper is to apply the averaging operator M_t to NSE (1.1) and, according to standard turbulence modeling process, to study the limit when $t \to +\infty$. We will adopt the following standard notation for the long-time average

$$\overline{\psi}(\mathbf{x}) := \lim_{t\to+\infty} M_t(\psi)(\mathbf{x}), \tag{1.3}$$

whenever the limit exists. We recall that time-averaging has been introduced by O. Reynolds [25], at least for large values of t, and the ideas have been widely developed by L. Prandtl [23] in the case of turbulent channel flows. The same ideas have been also later considered in the case of fully developed homogeneous and isotropic turbulence, such as grid-generated turbulence. In this case the velocity field is postulated as oscillating around a mean smoother steady state, see for instance G.-K. Batchelor [1].

The standard assumption about such turbulent flows is the ergodic one, so that –according to the Birkhoff theorem– the time-averaging coincide with statistical means (U. Frisch [12]), which is one of the basis for the derivation of RANS turbulent models such as the k–ε model (Mohammadi and Pironneau [22] and Chacon and Lewandowski [7]). However, it is worth noting that even for a given $A \subset \mathbb{R}$ and λ the Lebesgue measure, the mapping

$$A \mapsto \lim_{t\to+\infty} \frac{1}{t}\lambda(A\cap[0,t]) = \lim_{t\to+\infty} M_t(\mathbb{1}_A) = \mu(A),$$

[1] Being given two vector fields $\mathbf{u} = (u_1,u_2,u_3)$ and $\mathbf{w} = (w_1,w_2,w_3)$, the tensor product of $\mathbf{u}$ by $\mathbf{w}$ is defined by $(\mathbf{u}\otimes\mathbf{w})_{ij} = u_i w_j$, for $1 \le i,j \le 3$. It is easily checked that, at least formally, $\nabla\cdot\mathbf{v} = 0$ implies that $(\mathbf{v}\cdot\nabla)\,\mathbf{v} = \nabla\cdot(\mathbf{v}\otimes\mathbf{v})$.

is not –strictly speaking– a probability measure since it is not σ-additive[2]. Therefore, the quantity $\overline{\psi}$ is not rigorously a statistic, even if practitioners could be tempted to write it (in a suggestive and evocative meaningful way) as follows:

$$\overline{\psi}(\mathbf{x}) = \int_{\mathbb{R}_+} \psi(s, \mathbf{x})\, d\mu(s).$$

Concerning the application of time-averaging to turbulent flows one first main problem is due to the fact that the integration with respect to any variable is a linear operation, which is not well behaved on quadratic terms. In particular (but the same holds true even for finite-time averaging)

$$\overline{\mathbf{v} \otimes \mathbf{v}} \neq \overline{\mathbf{v}} \otimes \overline{\mathbf{v}}, \tag{1.4}$$

and the modeling of the difference between the left and right-hand side of (1.4) is one of the generic open questions in any large scale modeling. This is known as the *interior closure problem*, (cf. Berselli, Iliescu, and Layton [3], Chacon and Lewandowski [7], and Sagaut [26]).

By following the standard notation let $\mathbf{v}' := \mathbf{v} - \overline{\mathbf{v}}$ denote the velocity fluctuation. When we formally apply the "bar operator" (1.3) to the NSE (1.1), it appears the Reynolds stress tensor $\overline{\mathbf{v}' \otimes \mathbf{v}'}$ in the resulting equations through the term $-\text{div}(\overline{\mathbf{v}' \otimes \mathbf{v}'})$ in the right-hand side, due to this nonlinear phenomenon. One of the main themes in turbulence modeling is the Boussinesq hypothesis:

"on average the fluctuations are dissipative,"

which means that the term $-\text{div}(\overline{\mathbf{v}' \otimes \mathbf{v}'})$ can be considered as a viscous force exerted by the fluctuations (of the velocity) over the means (of the velocity itself). This naturally leads to the introduction of the concept of *eddy-viscosity* and it is at the basis of many eddy-viscosity models which have been proposed. It is still a great mathematical challenge today to obtain rigorous mathematical proofs of the validity of this process, which is of paramount importance in large scale modeling, hence in the mathematical analysis of turbulent flows, see refs. [3, 7, 26].

1.2 On the source term and an existence result.

Since we aim to consider long-time averages for the NSE, we must consider solutions which are global-in-time (defined for all positive times). Due to the well-known open problems related to the NSE, this forces to restrict to weak solutions. By using a most natural setting we take the initial datum $\mathbf{v}_0$ in

$$H := \{\mathbf{u} \in L^2(\Omega)^3 : \ \nabla \cdot \mathbf{u} = 0, \ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

where $\Gamma = \partial\Omega$ is the boundary of Ω, and $\mathbf{n}$ denotes the outward normal unit vector. The classical Leray-Hopf result of existence (but without uniqueness)

[2]The mapping μ satisfies $\mu(A \cup B) = \mu(A) + \mu(B)$ for $A \cap B = \emptyset$ but, on the other hand, we have $\sum_{n=0}^{\infty} \mu([n, n+1[) = 0 \neq 1 = \mu(\bigcup_{n=0}^{\infty} [n, n+1[)$.

of a global weak solution $\mathbf{v}$ to the NSE holds when $\mathbf{f} \in L^2(\mathbb{R}_+; V')$, and the velocity $\mathbf{v}$ satisfies

$$\mathbf{v} \in L^2(\mathbb{R}_+, V) \cap L^\infty(\mathbb{R}_+, H),$$

where

$$V := \{\mathbf{u} \in H_0^1(\Omega)^3 : \ \nabla \cdot \mathbf{u} = 0\},$$

and V' is its topological dual. We will also denote by $< \, , >$ the duality pairing[3] between V' and V.

Such source term $\mathbf{f}$ verifies $\int_t^\infty \|\mathbf{f}(s)\|_{V'}^2 \, ds \to 0$ when $t \to +\infty$. Therefore, we guess that it cannot maintain any turbulent motion for large t, which is not relevant for our purpose. This is why we must choose a broader class for the source terms. According to the usual folklore in mathematical analysis, we decided to consider the space $L^2_{uloc}(\mathbb{R}_+; V')$ made of all measurable vector fields $\mathbf{f} : \ \mathbb{R}_+ \to V'$ such that

$$\|\mathbf{f}\|_{L^2_{uloc}(\mathbb{R}_+;V')} := \left[\sup_{t \geq 0} \int_t^{t+1} \|\mathbf{f}(s)\|_{V'}^2 \, ds \right]^{1/2} < +\infty.$$

We will see in the following, that the above space, which strictly contains $L^2(\mathbb{R}_+; V')$, is well suited for our framework. We will prove the following existence result, in order to make the paper self-contained.

Theorem 1.1. *Let $\mathbf{v}_0 \in H$, and let $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$. Then, there exists a weak solution $\mathbf{v}$ to the NSE (1.1) global-in-time, obtained by approximations, such that*

$$\mathbf{v} \in L^2_{loc}(\mathbb{R}_+; V) \cap L^\infty(\mathbb{R}_+; H),$$

and which satisfies for all $t \geq 0$,

$$\|\mathbf{v}(t)\|^2 \leq \|\mathbf{v}_0\|^2 + \left(3 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu},$$

and

$$\nu \int_0^t \|\nabla \mathbf{v}(s)\|^2 ds \leq \|\mathbf{v}_0\|^2 + ([t] + 1) \frac{\mathcal{F}^2}{\nu},$$

where $\mathcal{F} := \|\mathbf{f}\|_{L^2_{uloc}(\mathbb{R}_+;V')}$.

Remark 1.2. *The weak solution $\mathbf{v}$ shares most of the properties of the Leray-Hopf weak solutions, with estimates valid for all positive times. Notice that we do not know whether or not this solution is unique. Anyway, it will not get "regular" as $t \to +\infty$, which is the feature of interest for our study. As usual by regular we mean that is does not necessarily have the L^2-norm of the gradient (locally) bounded, hence that it is not a strong solution.*

[3]Generally speaking and when no risk of confusion occurs, we always denote by $< \, , >$ the duality pairing between any Banach space X and its dual X', without mentioning explicitly which spaces are involved.

1.3 Long-time averaging

We consider $\mathbf{v}_0 \in H$ and $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, and let $\mathbf{v}$ be any global-in-time weak solution to the NSE corresponding to these data. We will prove that there exists a second order stress tensor $\boldsymbol{\sigma}^{(\mathrm{R})} = (\sigma_{ij})$, for $1 \leq i, j \leq 3$, such that the time-average $M_t(\mathbf{v})$ converges as $t \to +\infty$ (up to a sub-sequence) to a vector field denoted by $\overline{\mathbf{v}}$, which is a solution of the steady-state Reynolds averaged equations:

$$\begin{cases} (\overline{\mathbf{v}} \cdot \nabla)\, \overline{\mathbf{v}} - \nu \Delta \overline{\mathbf{v}} + \nabla \overline{p} + \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{f}} & \text{in } \Omega, \\ \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = \mathbf{0} & \text{on } \Gamma. \end{cases} \tag{1.5}$$

By analogy with turbulence modeling, we identify $\boldsymbol{\sigma}^{(\mathrm{R})}$ as the Reynolds stress tensor[4]. In addition, we show that when the source term $\mathbf{f}(t)$ converges to a constant when $t \to +\infty$ –in some specific sense– then $\boldsymbol{\sigma}^{(\mathrm{R})}$ is dissipative "in average". This means that the averaged work over Ω done by the term $\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}$ is non negative, that is

$$0 \leq \frac{1}{|\Omega|} \int_\Omega (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \overline{\mathbf{v}} \, d\mathbf{x}. \tag{1.6}$$

The result holds without any extra-assumption of uniqueness or regularity of $\mathbf{v}$. Similar results were initially obtained in [19]. However, we substantially improve them since:

i) We do not need to assume that Γ (the boundary of Ω) to be of class $C^{9/4,1}$, as it was the case in [19]; here Ω bounded and with Lipschitz boundary is enough;

ii) We are not considering only constant source terms $\mathbf{f}$ as in [19]; our results are valid for the broad class $L^2_{uloc}(\mathbb{R}_+; V')$. However, we do not know whether this class is optimal or not, in the sense that it may exists a space $\mathbb{F}$, that strictly contains $L^2_{uloc}(\mathbb{R}_+; V')$, and such that the same result still holds for all $\mathbf{f} \in \mathbb{F}$.

Next, we consider the long-time averages of families of solutions corresponding to different external forces. We show by means of methods of convex analysis that it is possible to identify also in this case a limit, among the families of both Reynolds velocities and stress tensors. In some sense the structure of the result is not changed if we perform different experiments considering oscillatory external forces, or if the solution is affected by perturbations, due to the lack of precision in measurement. Observe that our results do not follows from the existence (and special properties) of statistical solutions, as for the problems analyzed in [11] for instance, but are valid for *all* weak solutions and generalize and improve the results in [7, 17, 19], relaxing many of the hypotheses.

[4]The convergence result is proved through a variational formulation. The pressure $\overline{p}$ is derived from De Rham Theorem, after passing to the limit.

To be more specific, our two main results are the following. The notation, the definitions of the function spaces, and the precise formulation are given in Section 2.1.

Theorem 1.3. *Let be given* $\mathbf{v}_0 \in H$, $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, *and let* $\mathbf{v}$ *a global-in-time weak solution to the NSE* (1.1). *Then, there exists*

a) a sequence $\{t_n\}_{n\in\mathbb{N}}$ *such that* $\lim_{n\to\infty} t_n = +\infty$;

b) a vector field $\overline{\mathbf{v}} \in V$;

c) vector field $\overline{\mathbf{f}} \in V'$;

d) a vector field $\mathbf{B} \in L^{3/2}(\Omega)^3$;

e) a second order tensor field $\boldsymbol{\sigma}^{(\mathrm{R})} \in L^3(\Omega)^9$;

such that it holds:

i) when $n \to \infty$,

$$\begin{aligned} &M_{t_n}(\mathbf{v}) \rightharpoonup \overline{\mathbf{v}} && \text{in } V, \\ &M_{t_n}(\mathbf{f}) \rightharpoonup \overline{\mathbf{f}} && \text{in } V', \\ &M_{t_n}\big((\mathbf{v}\cdot\nabla)\,\mathbf{v}\big) \rightharpoonup \mathbf{B} && \text{in } L^{3/2}(\Omega)^3, \\ &M_{t_n}(\mathbf{v}'\otimes\mathbf{v}') \rightharpoonup \boldsymbol{\sigma}^{(\mathrm{R})} && \text{in } L^3(\Omega)^9; \end{aligned}$$

ii) the Reynolds equation (1.5) *holds true in the weak sense;*

iii) the following equalities $\mathbf{F} = \mathbf{B} - (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}} = \nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})}$ *are valid in* $\mathcal{D}'(\Omega)$;

iv) the following energy balance holds true

$$\nu\|\nabla\overline{\mathbf{v}}\|^2 + \int_\Omega \mathbf{F}\cdot\overline{\mathbf{v}}\,d\mathbf{x} = <\overline{\mathbf{f}}, \overline{\mathbf{v}}>;$$

v) if in addition the source term verifies:

$$\exists\, \widetilde{\mathbf{f}} \in V', \quad \text{such that} \quad \lim_{t\to+\infty} \int_t^{t+1} \|\mathbf{f}(s) - \widetilde{\mathbf{f}}\|^2_{V'}\,ds = 0, \tag{1.7}$$

then $\overline{\mathbf{f}} = \widetilde{\mathbf{f}}$ *and the tensor* $\boldsymbol{\sigma}^{(\mathrm{R})}$ *is dissipative in average, that is (1.6) holds true.*

Our second result has to be compared with results in Layton *et al.* [16, 17], where the long-time averages are taken for an ensemble of solutions.

Theorem 1.4. *Let be given a sequence* $\{\mathbf{f}_k\}_{k\in\mathbb{N}} \subset L^q(\Omega)$ *converging weakly to some* $\langle \mathbf{f}\rangle$ *in* $L^q(\Omega)$*, with* $q > \frac{6}{5}$ *and let* $\{\overline{\mathbf{v}}^k\}_{k\in\mathbb{N}}$ *be the associated long-time average of velocities, whose existence has been proved in Theorem 1.3. Then, the sequence of arithmetic averages of the long-time limits* $\{\langle \mathbf{v}\rangle^n\}_{n\in\mathbb{N}}$*, defined as*

$$\langle \mathbf{v}\rangle^n := \frac{1}{n}\sum_{k=1}^{n} \overline{\mathbf{v}}^k$$

converges weakly, as $n \to +\infty$*, in* V *to some* $\langle \mathbf{v}\rangle$*, which satisfies the following system of Reynolds type*

$$\begin{cases} (\langle \mathbf{v}\rangle \cdot \nabla)\langle \mathbf{v}\rangle - \nu\Delta\langle \mathbf{v}\rangle + \nabla\langle p\rangle + \nabla\cdot\langle \boldsymbol{\sigma}^{(\mathrm{R})}\rangle = \langle \mathbf{f}\rangle & \textit{in } \Omega, \\ \nabla\cdot\langle \mathbf{v}\rangle = 0 & \textit{in } \Omega, \\ \langle \mathbf{v}\rangle = \mathbf{0} & \textit{on } \Gamma, \end{cases}$$

where $\langle \boldsymbol{\sigma}^{(\mathrm{R})}\rangle$ *is dissipative in average, that is more precisely*

$$0 \le \frac{1}{|\Omega|}\int_\Omega (\nabla\cdot\langle \boldsymbol{\sigma}^{(\mathrm{R})}\rangle)\cdot\langle \mathbf{v}\rangle\, d\mathbf{x}.$$

Plan of the paper In Section 2 we recall the main notation and we prove some results about the existence and the global estimates which are valid for the NSE, with uniformly-local source terms. In Section 3 we present a short overview of the long-time filtering, applied to the turbulent flows (especially we recall the Reynolds rules). In Section 4 we prove the first result (Thm. 1.3) about convergence to solutions of the Reynolds equations and the dissipative effect of the Reynolds tensor. Finally, in Section 5 we prove the result (Thm. 1.4) about the behavior of an ensemble of solutions.

2 Navier-Stokes equations with uniformly-local source terms

This section is devoted to sketch a proof of Theorem 1.1. Most of the arguments are quite standard and we will give appropriate references at each step, to focus on what seems (at least to us) non-standard when $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$; especially the proof of the uniform L^2-estimate (2.4), which is the building block for the results of the present paper. Before doing this, we introduce the function spaces we will use, and precisely define the notion of weak solutions we will deal with.

2.1 Function spaces

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary $\partial\Omega$. This is a sort of minimal assumption of regularity on the domain, in order to have the usual properties for Sobolev spaces and to characterize in a proper way divergence-free vector fields in the context of Sobolev spaces, see for instance Constantin and Foias [8], Galdi [13, 14], Girault and Raviart [15], Tartar [27].

We use the customary Lebesgue spaces $(L^p(\Omega), \|\,.\,\|_p)$ and Sobolev spaces $(W^{1,p}(\Omega), \|\,.\,\|_{1,p})$. For simplicity, we denote the L^2-norm simply by $\|\,.\,\|$ and we write $H^1(\Omega) := W^{1,2}(\Omega)$. For a given sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$, where $(X, \|\,.\,\|_X)$ is Banach space, we denote by $x_n \to x$ the strong convergence, while by $x_n \rightharpoonup x$ the weak one.

As usual in mathematical fluid dynamics, we use the following spaces

$$\begin{aligned}
&\mathcal{V} = \{\boldsymbol{\varphi} \in \mathcal{D}(\Omega)^3,\ \nabla \cdot \boldsymbol{\varphi} = 0\},\\
&H = \left\{\mathbf{v} \in L^2(\Omega)^3,\ \nabla \cdot \mathbf{v} = 0,\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\right\},\\
&V = \left\{\mathbf{v} \in H_0^1(\Omega)^3,\ \nabla \cdot \mathbf{v} = 0\right\},
\end{aligned}$$

and we recall that $\mathcal{V}$ is dense in H and V for their respective topologies [15, 27].

Let $(X, \|\,.\,\|_X)$ be a Banach space, we will use the Bochner spaces $L^p(I; X)$, for $I = [0, T]$ (for some fixed $T > 0$) or $I = \mathbb{R}_+$ equipped with the norm

$$\|u\|_{L^p(I;X)} := \left(\int_I \|u(s)\|_X^p ds\right)^{\frac{1}{p}}.$$

The existence of weak solutions for the NSE (1.1) is generally proved in the literature when $\mathbf{v}_0 \in H$ and the source term $\mathbf{f} \in L^2(I; V')$, or alternatively when the source term is a given constant element of V'. In order to study the long-time behavior of weak solutions of the NSE (1.1), we aim to enlarge the class of function spaces allowed for the source term $\mathbf{f}$, to catch a more complex behavior than that coming from constant external forces, as initially developed in [19]. To do so, we deal with "uniformly-local" spaces, as defined below in the most general setting.

Definition 2.1. *Let be given $p \in [1, +\infty[$. We define $L^p_{uloc}(\mathbb{R}_+; X)$ as the space of strongly measurable functions $f : \mathbb{R}_+ \to X$ such that*

$$\|f\|_{L^p_{uloc}(X)} := \left[\sup_{t\geq 0} \int_t^{t+1} \|f(s)\|_X^p ds\right]^{1/p} < +\infty.$$

It is easily checked that the spaces $L^p_{uloc}(\mathbb{R}_+; X)$ are Banach spaces containing the constant X-valued functions, and strictly containing $L^p(\mathbb{R}_+; X)$, as illustrated by the following elementary lemma.

Lemma 2.2. *Let be given $f \in C(\mathbb{R}_+; X)$ converging a limit $\ell \in X$ when $t \to +\infty$. Then, for any $p \in [1, +\infty[$, we have that $f \in L^p_{uloc}(\mathbb{R}_+; X)$, and there exists $T > 0$ such that*

$$\|f\|_{L^p_{uloc}(X)} \leq \left[\sup_{t\in[0,T+1]} \|f(t)\|_X^p + (1 + \|\ell\|_X)^p\right]^{\frac{1}{p}}.$$

Proof. As $\ell = \lim_{t\to+\infty} f(t)$, there exists $T > 0$ such that: $\forall\, t > T$, $\|f(t)-\ell\|_X \le 1$. In particular, it holds

$$\int_t^{t+1} \|f(s)\|_X^p \, ds \le (1 + \|\ell\|_X)^p \qquad \text{for } t > T,$$

while for all $t \in [0, T]$,

$$\int_t^{t+1} \|f(s)\|_X^p \, ds \le \sup_{t\in[0,T+1]} \|f(t)\|_X^p,$$

hence the result. □

However, it easy to find examples of discontinuous functions in $L^p_{uloc}(\mathbb{R}_+; X)$ which are not converging when $t \to +\infty$, and which are not belonging to $L^p(\mathbb{R}_+; X)$.

2.2 Weak solutions

There are many ways of defining weak solutions to the NSE (see also P.-L. Lions [21]). Since we are considering the incompressible case, the pressure is treated as a Lagrange multiplier. Following the pioneering idea developed by Leray [18], the NSE are projected over spaces of divergence-free functions. This is why when we talk about solutions the NSE, only the velocity $\mathbf{v}$ is mentioned, not the pressure.

Following J.-L. Lions [20], we give the following definition of weak solution, see also Temam [28, Ch. III].

Definition 2.3 (Weak solution)**.** *Given* $\mathbf{v}_0 \in H$ *and* $\mathbf{f} \in L^2(I; V')$ *we say that* $\mathbf{v}$ *is a weak solution over the interval* $I = [0, T]$ *if the following items are fulfilled:*

i) the vector field $\mathbf{v}$ *has the following regularity properties*

$$\mathbf{v} \in L^2(I; V) \cap L^\infty(I; H),$$

and is weakly continuous from I *to* H*, while* $\lim_{t\to 0^+} \|\mathbf{v}(t) - \mathbf{v}_0\|_H = 0$*;*

ii) for all $\boldsymbol{\varphi} \in \mathcal{V}$*,*

$$\frac{d}{dt}\int_\Omega \mathbf{v}(t,\mathbf{x}) \cdot \boldsymbol{\varphi}(\mathbf{x})\, d\mathbf{x} - \int_\Omega \mathbf{v}(t,\mathbf{x}) \otimes \mathbf{v}(t,\mathbf{x}) : \nabla\boldsymbol{\varphi}(\mathbf{x})\, d\mathbf{x} + \nu \int_\Omega \nabla\mathbf{v}(t,\mathbf{x}) : \nabla\boldsymbol{\varphi}(\mathbf{x})\, d\mathbf{x} = < \mathbf{f}(t), \boldsymbol{\varphi} >,$$

holds true in $\mathcal{D}'(I)$*;*

iii) the energy inequality

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{v}(t)\|^2 + \nu\|\nabla\mathbf{v}(t)\|^2 \le < \mathbf{f}(t), \mathbf{v}(t) >, \tag{2.1}$$

holds in $\mathcal{D}'(I)$*, where we write* $\mathbf{v}(t)$ *instead of* $\mathbf{v}(t,\cdot)$ *for simplicity.*

When $\mathbf{f} \in L^2(0, T; V')$ *and* $\mathbf{v}$ *is a weak solution in* $I = [0, T]$*, and this holds true for all* $T > 0$*, we speak of "global-in-time solution", or simply a "global solution". In particular, ii) is satisfied in the sense of* $\mathcal{D}'(0, +\infty)$*.*

There are several ways to prove the existence of (at least) a weak solution to the NSE. Among them, in what follows, we will use the Faedo-Galerkin method. Roughly speaking, let $\{\boldsymbol{\varphi}_n\}_{n\in\mathbb{N}}$ denote a Hilbert basis of V, and let, for $n \in \mathbb{N}$, $V_n := \text{span}\{\boldsymbol{\varphi}_1, \cdots, \boldsymbol{\varphi}_n\}$. By assuming $\mathbf{f} \in L^2(I; V')$, it can be proved by the Cauchy-Lipschitz theorem (see [20]) the existence of a unique $\mathbf{v}_n \in C^1(I; V_n)$ such that for all $\boldsymbol{\varphi}_k$, with $k = 1, \ldots, n$ it holds

$$\begin{aligned} \frac{d}{dt}\int_\Omega \mathbf{v}_n(t,\mathbf{x})\cdot\boldsymbol{\varphi}_k(\mathbf{x})\,d\mathbf{x} - \int_\Omega \mathbf{v}_n(t,\mathbf{x})\otimes\mathbf{v}_n(t,\mathbf{x}) : \nabla\boldsymbol{\varphi}_k(\mathbf{x})\,d\mathbf{x} \\ + \nu\int_\Omega \nabla\mathbf{v}_n(t,\mathbf{x})\cdot\nabla\boldsymbol{\varphi}_k(\mathbf{x})\,d\mathbf{x} = <\mathbf{f}(t),\boldsymbol{\varphi}_k>, \end{aligned} \tag{2.2}$$

and which naturally satisfies the energy balance (equality)

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{v}_n(t)\|^2 + \nu\|\nabla\mathbf{v}_n(t)\|^2 = <\mathbf{f},\mathbf{v}_n> . \tag{2.3}$$

It can be also proved (always see again [20]) that from the sequence $\{\mathbf{v}_n\}_{n\in\mathbb{N}}$ one can extract a sub-sequence converging, in an appropriate sense, to a weak solution to the NSE. When $I = \mathbb{R}_+$ we get a global solution.

However, when we now assume $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, this does not work so straightforward. Of course, for any given $T > 0$, we have

$$L^2_{uloc}(\mathbb{R}_+; V')\big|_{[0,T]} \hookrightarrow L^2([0,T]; V'),$$

where $L^2_{uloc}(\mathbb{R}_+; V')_{|[0,T]}$ denotes the restriction of a function in $L^2_{uloc}(\mathbb{R}_+; V')$ to $[0, T]$. Therefore, no doubt that the construction above holds over any time-interval $[0, T]$. In such case letting T go to $+\infty$ to get a global solution is not obvious, and we do not know any reference explicitly dealing with this issue, which deserves to be investigated more carefully. This is the aim of the next subsection, where we prove the most relevant a-priori estimates

2.3 A priori estimates

Let be given $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, and let $\mathbf{v}_n = \mathbf{v}$ be the solution of the Galerkin projection of the NSE over the finite dimensional space V_n. The function $\mathbf{v}$ satisfies (2.2) and (2.3) (we do not write the subscript n for simplicity), is smooth, unique, and can be constructed by Cauchy-Lipschitz over any finite time interval $[0, T]$. Hence, we observe that by uniqueness it can be extended to $\mathbb{R}_+$. We then denote $\mathcal{F} := \|\mathbf{f}\|_{L^2_{uloc}(\mathbb{R}_+;V')}$ and then after a delicate manipulation of the energy balance combined with the Poincaré inequality, we get the following lemma.

Lemma 2.4. *For all $t \geq 0$ we have*

$$\|\mathbf{v}(t)\|^2 \leq \|\mathbf{v}_0\|^2 + \left(3 + \frac{C_\Omega}{\nu}\right)\frac{\mathcal{F}^2}{\nu}, \tag{2.4}$$

as well as

$$\nu \int_0^t \|\nabla \mathbf{v}(s)\|^2 \, ds \leq \|\mathbf{v}_0\|^2 + ([t] + 1)\frac{\mathcal{F}^2}{\nu}, \tag{2.5}$$

where C_Ω denotes the constant in the Poincaré inequality $\|\mathbf{u}\|^2 \leq C_\Omega \|\nabla \mathbf{u}\|^2$, valid for all $\mathbf{u} \in V$.

Proof. We focus on the proof of the estimate (2.4), the estimate (2.5) being a direct consequence of the energy balance.

First at all, note that if $t \in [0, 1]$, then (2.4) is automatically satisfied for such a time t. Let us now integrate the energy balance (2.3) between two arbitrary times $0 \leq \xi \leq \tau$, and use the Young inequality. We get:

$$\|\mathbf{v}(\tau)\|^2 + \nu \int_\xi^\tau \|\nabla \mathbf{v}(s)\|^2 \, ds \leq \|\mathbf{v}(\xi)\|^2 + \frac{1}{\nu}\int_\xi^\tau \|\mathbf{f}(s)\|_{V'}^2 \, ds. \tag{2.6}$$

From there, we distinguish three different cases.

a) The time $t \geq 1$ is such that

$$\|\mathbf{v}(t)\| \leq \|\mathbf{v}(t+1)\|. \tag{2.7}$$

b) The time $t \geq 1$ is such that

$$\|\mathbf{v}(t+1)\| < \|\mathbf{v}(t)\|, \tag{2.8}$$

and there exists $n \in \mathbb{N}$ for $t - n - 1 > 1$ satisfying

$$\forall\, k = 0, \cdots, n-1, \quad \|\mathbf{v}(t-k)\| < \|\mathbf{v}(t-k-1)\|, \tag{2.9}$$

and

$$\|\mathbf{v}(t-n)\| \leq \|\mathbf{v}(t-n-1)\|. \tag{2.10}$$

c) The time t is such that $t \geq 1$, the inequality (2.8) holds true, and

$$\forall\, k = 1, \cdots, [t], \quad \|\mathbf{v}(t-k)\| < \|\mathbf{v}(t-k-1)\|. \tag{2.11}$$

Case a) We consider (2.6) with $\xi = t$ and $\tau = t + 1$ and by using (2.7), we get

$$\begin{aligned}\nu \int_t^{t+1} \|\nabla \mathbf{v}(s)\|^2 \, ds &\leq \|\mathbf{v}(t+1)\|^2 - \|\mathbf{v}(t)\|^2 + \nu \int_t^{t+1} \|\nabla \mathbf{v}(s)\|^2 \, ds \\ &\leq \frac{1}{\nu}\int_t^{t+1} \|\mathbf{f}(s)\|_{V'}^2 \, ds \leq \frac{\mathcal{F}^2}{\nu}.\end{aligned}$$

Hence, by the Poincaré inequality we have

$$\nu \int_t^{t+1} \|\mathbf{v}(s)\|^2 \, ds \leq C_\Omega \nu \int_t^{t+1} \|\nabla \mathbf{v}(s)\|^2 \, ds \leq \frac{C_\Omega \mathcal{F}^2}{\nu}. \tag{2.12}$$

Let be given now $\epsilon > 0$ and let $\xi \in [t, t+1]$ be such that

$$\|\mathbf{v}(\xi)\|^2 < \inf_{s \in [t,t+1]} \|\mathbf{v}(s)\|^2 + \epsilon \leq \|\mathbf{v}(s)\|^2 + \epsilon \qquad \forall \, s \in [t, t+1]. \tag{2.13}$$

Combining (2.7), (2.13), and with some manipulations implies

$$\begin{aligned}
\|\mathbf{v}(t)\|^2 \leq \|\mathbf{v}(t+1)\|^2 &\leq \left| \|\mathbf{v}(t+1)\|^2 - \|\mathbf{v}(\xi)\|^2 \right| + \|\mathbf{v}(\xi)\|^2 \\
&\leq \left| \|\mathbf{v}(t+1)\|^2 - \|\mathbf{v}(\xi)\|^2 \right| + \int_t^{t+1} \|\mathbf{v}(\xi)\|^2 \, ds \\
&\leq \left| \|\mathbf{v}(t+1)\|^2 - \|\mathbf{v}(\xi)\|^2 \right| + \int_t^{t+1} (\|\mathbf{v}(s)\|^2 + \epsilon) \, ds.
\end{aligned}$$

Hence, by using (2.6) with $\tau = t + 1$ and the inequality (2.12) we get

$$\|\mathbf{v}(t)\|^2 \leq \left(1 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu} + \epsilon.$$

In conclusion if we fix ϵ small enough (such that $\epsilon \leq \mathcal{F}^2/\nu$), then

$$\|\mathbf{v}(t)\|^2 \leq \left(2 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu}.$$

Therefore, the estimate (2.4) is satisfied for times t such that (2.7) holds. In particular, for such times the estimate is better, being the initial data not involved.

Case b) We argue by induction. If $n = 0$, therefore (2.10) becomes for such time t:

$$\|\mathbf{v}(t-1)\| \leq \|\mathbf{v}(t)\|.$$

Then, we deduce from the previous case, replacing t by $t - 1 \geq 0$, that

$$\|\mathbf{v}(t-1)\|^2 \leq \left(2 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu}. \tag{2.14}$$

Combining this last inequality (2.14) with the energy balance (2.6), taking $\tau = t$ and $\xi = t - 1$, yields

$$\|\mathbf{v}(t)\|^2 \leq \left(3 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu}. \tag{2.15}$$

For a given n, we get from (2.10) and the previous calculations, that

$$\|\mathbf{v}(t-n-1)\|^2 \leq \left(3 + \frac{C_\Omega}{\nu}\right) \frac{\mathcal{F}^2}{\nu},$$

hence (2.15) follows by using (2.9). Therefore, (2.4) is satisfied for such t.

Case c) Inequalities (2.11) read

$$\|\mathbf{v}(t)\|^2 < \|\mathbf{v}(t-1)\|^2 < \|\mathbf{v}(t-2)\|^2 < \cdots < \|\mathbf{v}(t-[t])\|^2,$$

that is

$$\|\mathbf{v}(t)\|^2 \leq \sup_{s\in[0,1]} \|\mathbf{v}(s)\|^2,$$

on which integration of the standard energy inequality (2.1) gives

$$\|\mathbf{v}(t)\|^2 \leq \sup_{s\in[0,1]} \|\mathbf{v}(s)\|^2 \leq \|\mathbf{v}_0\|^2 + \frac{1}{\nu}\int_0^1 \|\mathbf{f}(s)\|_{V'}^2\, ds \leq \|\mathbf{v}_0\|^2 + \frac{\mathcal{F}^2}{\nu},$$

which leads to (2.4).

We have covered all possible cases, which completes the proof of (2.4). Estimate (2.5) is obtained by (2.6) taking $\xi = 0$ and $\tau = t$:

$$\nu \int_0^t \|\nabla \mathbf{v}(s)\|^2\, ds \leq \|\mathbf{v}_0\|^2 + \frac{1}{\nu}\int_0^{[t]+1} \|\mathbf{f}(s)\|_{V'}^2\, ds \leq \|\mathbf{v}_0\|^2 + \frac{[t]+1}{\nu}\mathcal{F}^2,$$

which concludes the proof. □

Once we have proved that the uniform (independently of $n \in \mathbb{N}$) L^2-estimate is satisfied by the Galerkin approximate functions, it is rather classical to prove that we can extract a sub-sequence that converges weakly to a weak solution to the NSE. We refer to the references already mentioned for this point.

3 Properties of the time-averaging filter

We sketch the standard routine, concerning time-averaging, when used in turbulence modeling practice. In particular, we recall the Reynolds decomposition and the Reynolds rules. Then, we give a few technical properties of the time-averaging operator M_t, for a given fixed time $t > 0$, as defined by (1.2).

3.1 General setup of turbulence modeling

We recall that M_t is a linear filtering operator which commutes with differentiation with respect to the space variables (the so called Reynolds rules). In particular, one has the following result (its proof is straightforward), which is essential for our modeling process.

Lemma 3.1. *Let be given $\psi \in L^1([0,T], W^{1,p}(\Omega))$, then*

$$DM_t(\psi) = M_t(D\psi) \qquad \forall\, t > 0,$$

for any first order differential operator D acting on the space variables $\mathbf{x} \in \Omega$.

By denoting the long-time average of any field ψ by $\overline{\psi}$ as in (1.3), we consider the fluctuations ψ' around the mean value, given by the Reynolds decomposition

$$\psi := \overline{\psi} + \psi'.$$

Observe that long-time averaging has many convenient *formal* mathematical properties.

Lemma 3.2. *The following formal properties holds true, provided the long-time averages do exist.*

1. *The "bar operator" preserves the no-slip boundary condition. In other words, if* $\psi|_\Gamma = 0$*, then* $\overline{\psi}|_\Gamma = 0$*;*
2. *Fluctuation are in the kernel of the bar operator, that is* $\overline{\psi'} = 0$*;*
3. *The bar operator is idempotent, that is* $\overline{\overline{\psi}} = \overline{\psi}$*, which also yields* $\overline{\overline{\psi}\,\varphi} = \overline{\psi}\,\overline{\varphi}$*.*

Accordingly, the velocity components can be decomposed following the Reynolds decomposition as follows.

$$\mathbf{v}(t, \mathbf{x}) = \overline{\mathbf{v}}(\mathbf{x}) + \mathbf{v}'(t, \mathbf{x}).$$

Let us determine (at least formally) the equation satisfied by $\overline{\mathbf{v}}$. To do so, we use the above Reynolds rules to expand the nonlinear quadratic term into

$$\overline{\mathbf{v} \otimes \mathbf{v}} = \overline{\mathbf{v}} \otimes \overline{\mathbf{v}} + \overline{\mathbf{v}' \otimes \mathbf{v}'},$$

which follows by observing that $\overline{\mathbf{v}' \otimes \overline{\mathbf{v}}} = \overline{\overline{\mathbf{v}} \otimes \mathbf{v}'} = \mathbf{0}$. Long-time averaging applied to the Navier-Stokes equations (in a strong formulation) gives the following "equilibrium problem" for the long-time average $\overline{\mathbf{v}}(\mathbf{x})$,

$$\begin{cases} -\nu\Delta\overline{\mathbf{v}} + \nabla \cdot (\overline{\mathbf{v} \otimes \mathbf{v}}) + \nabla\overline{p} = \overline{\mathbf{f}} & \text{in } \Omega, \\ \nabla \cdot \overline{\mathbf{v}} = 0 & \text{in } \Omega, \\ \overline{\mathbf{v}} = \mathbf{0} & \text{on } \Gamma, \end{cases}$$

where the first equation can be rewritten also as follows (by using the decomposition into averages and fluctuations)

$$-\nu\Delta\overline{\mathbf{v}} + \nabla \cdot (\overline{\mathbf{v}} \otimes \overline{\mathbf{v}}) + \nabla\overline{p} = -\nabla \cdot (\overline{\mathbf{v}' \otimes \mathbf{v}'}) + \overline{\mathbf{f}},$$

Beside convergence issues, a relevant point is to characterize the average of product of fluctuations from the right-hand side, which is the divergence of the so called Reynolds stress tensor, defined as follows

$$\boldsymbol{\sigma}^{(\mathrm{R})} = \overline{\mathbf{v}' \otimes \mathbf{v}'}. \tag{3.1}$$

The Boussinesq hypothesis, formalized in [5] (see also [7, Ch. 3 & 4], for a comprehensive and modern presentation) corresponds then to a closure hypothesis with the following linear constitutive equation:

$$\boldsymbol{\sigma}^{(\mathrm{R})} = -\nu_t \frac{\nabla\overline{\mathbf{v}} + \nabla\overline{\mathbf{v}}^T}{2} + \frac{2}{3} k \,\mathrm{Id}, \tag{3.2}$$

where $\nu_t \geq 0$ is a scalar coefficient, called turbulent viscosity or eddy-viscosity (sometimes called "effective viscosity"), and

$$k = \frac{1}{2}\overline{|\mathbf{v}'|^2},$$

is the turbulent kinetic energy. Formula (3.2) is a linear relation between stress and strain tensors, and shares common formal points with the linear constitutive equation valid for Newtonian fluids. In particular, this assumptions motivates the fact that $\boldsymbol{\sigma}^{(\mathrm{R})}$ must be dissipative[5]. Some recent results in the numerical verification of the hypothesis can be found in the special issue [4] dedicated to Boussinesq. Here we show that, beside the validity of the modeling assumption (3.2), the Reynolds stress tensor $\boldsymbol{\sigma}^{(\mathrm{R})}$ is dissipative, under minimal assumptions on the regularity of the data of the problem.

3.2 time-averaging of uniformly-local fields

We list in this section some technical properties of the operator M_t acting on uniform-local fields, and the corresponding global weak solutions to the NSE. We start with the following corollary of Bochner theorem (see Yosida [29, p. 132]).

Lemma 3.3. *Assume that, for some $t > 0$ we have $\psi \in L^p([0,t];X)$ (namely ψ is a Bochner p-summable function over $[0,t]$, with values in the Banach space X). Then, it holds*

$$\|M_t(\psi)\|_X \leq \frac{1}{t^{\frac{1}{p}}}\|\psi\|_{L^p([0,t];X)}. \tag{3.3}$$

A simple consequence is then the following

Lemma 3.4. *Let $1 < p < \infty$ and let be given $f \in L^p_{uloc}(\mathbb{R}_+;X)$. Then*

$$\forall\, t \geq 1, \qquad \|M_t(f)\|_X \leq 2\|f\|_{L^p_{uloc}(\mathbb{R}_+;X)}.$$

Proof. Applying some straightforward inequalities yields

$$\|M_t(f)\|_X \leq \frac{1}{t}\int_0^t \|f\|_X\,ds \leq \frac{1}{t}\int_0^{[t]+1} \|f\|_X\,ds \leq \frac{1}{t}\sum_{k=0}^{[t]}\int_k^{k+1} \|f\|_X\,ds.$$

Therefore by the Hölder inequality we get:

$$\begin{aligned}\|M_t(f)\|_X &\leq \frac{1}{t}\sum_{k=0}^{[t]}\left(\int_k^{k+1} \|f\|_X^p\,ds\right)^{1/p}\left(\int_k^{k+1} 1\,ds\right)^{1/p'} \\ &\leq \frac{[t]+1}{t}\|f\|^p_{L^p_{uloc}(X)} \leq 2\|f\|^p_{L^p_{uloc}(\mathbb{R}_+;X)},\end{aligned}$$

the last inequality being satisfied for $t \geq 1$. □

[5]The sign adopted in (3.1) is a convention consistent with our mathematical approach. However, according to the analogy of the Reynolds stress with viscous forces, it is common to define it as $\boldsymbol{\sigma}^{(\mathrm{R})} = -\overline{\mathbf{v}' \otimes \mathbf{v}'}$, which does not change anything.

We finish this section with a last technical result, that we will need to prove the item v) of Theorem 1.3.

Lemma 3.5. *Let* $1 < p < \infty$ *and let be given* $f \in L^p_{uloc}(\mathbb{R}_+; X)$*, which satisfies in addition*

$$\exists\ \widetilde{f} \in X, \quad \textit{such that} \quad \lim_{t\to+\infty} \int_t^{t+1} \|f(s) - \widetilde{f}\|_X^p \, ds = 0. \tag{3.4}$$

Then, we have

$$\lim_{t\to+\infty} \frac{1}{t} \int_0^t \|f(s) - \widetilde{f}\|_X^p \, ds = 0. \tag{3.5}$$

Moreover, $M_t(f)$ *weakly converges to* $\widetilde{f}$ *in* X *when* $t \to +\infty$*.*

Proof. By the hypothesis (3.4), we have that

$$\forall\, \varepsilon > 0 \quad \exists\, M \in \mathbb{N} : \quad \int_t^{t+1} \|f(s) - \widetilde{f}\|_X^p \, ds < \frac{\varepsilon}{2} \qquad \forall\, t > M.$$

Hence, for $t \geq M$, then

$$\begin{aligned} \frac{1}{t} \int_0^t \|f(s) - \widetilde{f}\|_X^p \, ds &= \frac{1}{t} \int_0^M \|f(s) - \widetilde{f}\|_X^p \, ds + \frac{1}{t} \int_M^t \|f(s) - \widetilde{f}\|_X^p \, ds \\ &\leq \frac{M}{t}\big(\|f\|^p_{L^p_{uloc}(X)} + \|\widetilde{f}\|_X^p\big) + \frac{[t] + 1 - M}{t} \frac{\varepsilon}{2}. \end{aligned}$$

It follows that one can choose M large enough such that

$$\frac{1}{t} \int_0^t \|f(s) - \widetilde{f}\|_X^p \, ds < \varepsilon \qquad \forall\, t > M,$$

hence, being this valid for arbitrary $\varepsilon > 0$, it follows (3.5).

It remains to prove the weak convergence of $M_t(f)$ to $\widetilde{f} \in X$ when $t \to +\infty$. To this end, let be given $\varphi \in X'$. Then, we have

$$< \varphi, M_t(f) > - < \varphi, \widetilde{f} > = \frac{1}{t} \int_0^t < \varphi, f(s) - \widetilde{f} > \, ds,$$

which leads to

$$| < \varphi, M_t(f) > - < \varphi, \widetilde{f} > | \leq \frac{1}{t} \int_0^t \|\varphi\|_{X'} \|f(s) - \widetilde{f}\|_X \, ds,$$

and by Hölder inequality,

$$\Big| < \varphi, M_t(f) > - < \varphi, \widetilde{f} > \Big| \leq \|\varphi\|_{X'} \left(\frac{1}{t} \int_0^t \|f(s) - \widetilde{f}\|_X^p \, ds \right)^{\frac{1}{p}},$$

yielding, by (3.5), to $\lim_{t\to+\infty} < \varphi, M_t(f) > = < \varphi, \widetilde{f} >$, hence concluding the proof. ∎

In the next section, we will focus on the case $p = 2$ and $X = V'$. In order to complete our toolbox, we conclude this section by the following result, which is a consequence of Lemma 2.4.

Lemma 3.6. *Let be given* $\mathbf{v}_0 \in H$, $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, *and let* $\mathbf{v}$ *be a global weak solution to the NSE corresponding to the above data. Then, we have*

$$M_t(\|\nabla \mathbf{v}\|^2) \leq \frac{\|\mathbf{v}_0\|^2}{\nu t} + 2\frac{\mathcal{F}^2}{\nu^2}, \qquad \forall\, t \geq 1, \tag{3.6}$$

where $\mathcal{F} = \|\mathbf{f}\|_{L^2_{uloc}(\mathbb{R}_+;V')}$.

Proof. It suffices to divide estimate (2.5) by νt and use the inequality $\frac{[x]+1}{x} \leq 2$ which is valid for all $x \geq 1$. Therefore, it follows that

$$\frac{1}{t}\int_0^t \|\nabla \mathbf{v}(s)\|^2 \, ds \leq \frac{\|\mathbf{v}_0\|^2}{\nu t} + 2\frac{\mathcal{F}^2}{\nu^2},$$

ending the proof. □

4 Proof of Theorem 1.3

In all this section we have as before $\mathbf{v}_0 \in H$, $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$, and $\mathbf{v}$ is a global weak solution to the NSE (1.1) corresponding to the above data. We split the proof of Theorem 1.3 into two steps. We first apply the operator M_t to the NSE, then we extract sub-sequences and take the limit in the equations. In the second step we make the identification with the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$ and show that it is dissipative in average, at least when $\mathbf{f}$ satisfies in addition (1.7).

4.1 Extracting sub-sequences

We set:

$$\mathbf{V}_t(\mathbf{x}) := M_t(\mathbf{v})(\mathbf{x}).$$

Applying the operator M_t on the NSE we see that for almost all $t \geq 0$ and for all $\boldsymbol{\phi} \in V$, the field $\mathbf{V}_t$ is a weak solution of the following steady Stokes problem (where $t > 0$ is simply a parameter)

$$\begin{aligned}\nu \int_\Omega \nabla \mathbf{V}_t : \nabla \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega M_t((\mathbf{v}\cdot\nabla)\,\mathbf{v})\cdot\boldsymbol{\phi}\, d\mathbf{x} = &< M_t(\mathbf{f}), \boldsymbol{\phi} > \\ &+ \int_\Omega \frac{\mathbf{v}_0 - \mathbf{v}(t)}{t}\cdot\boldsymbol{\phi}\, d\mathbf{x}.\end{aligned} \tag{4.1}$$

The full justification of the equality (4.1) starting from the definition of global weak solutions can be obtained by following a very well-known path used for instance to show that Leray-Hopf weak solutions can be re-defined on a set of zero Lebesgue measure in $[0, t]$ in such a way that $\mathbf{v}(s) \in H$ for *all* $s \in [0, t]$, see for instance Galdi [13, Lemma 2.1]. In fact, by following ideas developed among the others by Prodi [24], one can take $\chi_{[a,b]}$ the characteristic function of an interval $[a, b] \subset \mathbb{R}$, and use as test function its regularization multiplied by $\boldsymbol{\phi} \in V$. Passing to the limit as the regularization parameter vanishes one gets (4.1).

The process of extracting sub-sequences, which is the core of the main result, is reported in the following proposition.

Proposition 4.1. *Let be given a global solution* $\mathbf{v}$ *to the NSE, corresponding to the data* $\mathbf{v}_0 \in H$ *and* $\mathbf{f} \in L^2_{uloc}(\mathbb{R}_+; V')$*. Then, there exists*

a) a sequence $\{t_n\}_{n\in\mathbb{N}}$ *that goes to* $+\infty$ *when* n *goes to* $+\infty$*;*

b) a vector field $\overline{\mathbf{f}} \in V'$*;*

c) a vector field $\overline{\mathbf{v}} \in V$*;*

d) a vector field $\mathbf{B} \in L^{3/2}(\Omega)^3$*;*

such that

$$\begin{aligned} &M_{t_n}(\mathbf{f}) \rightharpoonup \overline{\mathbf{f}} \qquad \text{in } V', \\ &M_{t_n}(\mathbf{v}) \rightharpoonup \overline{\mathbf{v}} \qquad \text{in } V, \\ &M_{t_n}\big((\mathbf{v}\cdot\nabla)\,\mathbf{v}\big) \rightharpoonup \mathbf{B} \qquad \text{in } L^{3/2}(\Omega)^3 \subset V', \end{aligned}$$

in such a way that for all $\boldsymbol{\phi} \in V$

$$\nu \int_\Omega \nabla\overline{\mathbf{v}} : \nabla\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \mathbf{B}\cdot\boldsymbol{\phi}\, d\mathbf{x} = <\overline{\mathbf{f}}, \boldsymbol{\phi}> . \tag{4.2}$$

Moreover, by defining

$$\mathbf{F} := \mathbf{B} - (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}} \in L^{3/2}(\Omega)^3, \tag{4.3}$$

we can also rewrite (4.2) *as follows*

$$\nu \int_\Omega \nabla\overline{\mathbf{v}} : \nabla\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega (\overline{\mathbf{v}}\cdot\nabla)\,\overline{\mathbf{v}}\cdot\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \mathbf{F}\cdot\boldsymbol{\phi}\, d\mathbf{x} = <\overline{\mathbf{f}}, \boldsymbol{\phi}> . \tag{4.4}$$

Proof of Proposition 4.1. As $\mathbf{f} \in L^2(\mathbb{R}_+; V')$, we deduce from Lemma 3.4 that $\{M_t(\mathbf{f})\}_{t>0}$ is bounded in V'. Hence, we can use weak pre-compactness of bounded sets in the Hilbert space V' to infer the existence of t_n and $\overline{\mathbf{f}} \in V'$ such that $M_{t_n}(\mathbf{f}) \rightharpoonup \overline{\mathbf{f}}$ in V'. Next, estimate (3.6) from Lemma 3.6, combined with estimate (3.3) from Lemma 3.3, leads to the bound

$$\exists\, c > 0 : \qquad \|\nabla M_t(\mathbf{v})\| = \|M_t(\nabla\mathbf{v})\| \le c \qquad \forall\, t > 0,$$

proving (up to a the extraction of a further sub-sequence from $\{t_n\}$, which we call with the same name) that $M_{t_n}(\mathbf{v}) \rightharpoonup \overline{\mathbf{v}}$ in V'.

Then, we observe that, if $\mathbf{v} \in L^\infty(0, T; H) \cap L^2(0, T; V)$ by classical interpolation

$$(\mathbf{v}\cdot\nabla)\,\mathbf{v} \in L^r(0, T; L^s(\Omega)) \qquad \text{with} \quad \frac{2}{r} + \frac{3}{s} = 4, \quad r \in [1, 2].$$

In particular, we get

$$\|(\mathbf{v}\cdot\nabla)\,\mathbf{v}\|_{L^{3/2}(\Omega)} \le \|\mathbf{v}\|_{L^6}\|\nabla\mathbf{v}\|_{L^2} \le C_S\|\nabla\mathbf{v}\|^2,$$

where C_S is the constant of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$. Hence, by using the bound on $\mathbf{v}\in V$ we obtain that

$$\exists\, c: \qquad \|M_t\big((\mathbf{v}\cdot\nabla)\,\mathbf{v}\big)\|_{L^{3/2}(\Omega)} \le c, \qquad \forall\, t>0,$$

proving that, up to a further sub-sequence relabelled again as $\{t_n\}$,

$$M_{t_n}\big((\mathbf{v}\cdot\nabla)\,\mathbf{v}\big) \rightharpoonup \mathbf{B} \qquad \text{in } L^{3/2}(\Omega)^3,$$

for some vector field $\mathbf{B}\in L^{3/2}(\Omega)^3$.

Next, we use (2.4) which shows that

$$\int_\Omega \frac{\mathbf{v}_0-\mathbf{v}(t)}{t}\cdot\boldsymbol{\phi}\,d\mathbf{x} \to 0 \qquad \text{as } t\to+\infty.$$

Then, writing the weak formulation and by using the results of weak convergence previously proved, we get (4.2). Then, the identity (4.4) comes simply from the definition (4.3) of $\mathbf{F}$. □

4.2 Reynolds stress, energy balance and dissipation

In the first step we have identified a limit $(\overline{\mathbf{v}},\overline{\mathbf{f}})$ for the time-averages of both velocity and external force $(\mathbf{v},\mathbf{f})$. We need now to recast this in the setting of the Reynolds equations, in order to address the proof of the Boussinesq assumption.

Proof of Theorem 1.3. Beside the results in Proposition 4.1, in order to complete the proof of Theorem 1.3, we have to prove the following facts:

1) the proper identification of the limits with the Reynolds stress $\boldsymbol{\sigma}^{(\mathrm{R})}$;

2) the energy balance for $\overline{\mathbf{v}}$;

3) to prove that $\boldsymbol{\sigma}^{(\mathrm{R})}$ is dissipative in average.

We proceed in the same order.

Item 1. Since $\mathbf{v}\in L^2(0,T;V)\subset L^2(0,T;L^6(\Omega)^3)$, it follows that $\mathbf{v}\otimes\mathbf{v}\in L^1(0,T;L^3(\Omega))$. Hence, the same argument as in the previous subsection shows that (possibly up to the extraction of a further sub-sequence) there exists a second order tensor $\boldsymbol{\theta}\in L^3(\Omega)^9$ such that

$$M_{t_n}(\mathbf{v}\otimes\mathbf{v}) \rightharpoonup \boldsymbol{\theta} \qquad \text{in } L^3(\Omega)^9.$$

Let us set

$$\boldsymbol{\sigma}^{(\mathrm{R})} := \boldsymbol{\theta} - \overline{\mathbf{v}}\otimes\overline{\mathbf{v}}.$$

Since the operator M_t commutes with the divergence operator, the equation (4.1) becomes

$$\nu \int_\Omega \nabla \mathbf{V}_t : \nabla \boldsymbol{\phi} \, d\mathbf{x} - \int_\Omega M_t(\mathbf{v} \otimes \mathbf{v}) : \nabla \boldsymbol{\phi} \, d\mathbf{x} = < M_t(\mathbf{f}), \boldsymbol{\phi} > + \int_\Omega \frac{\mathbf{v}_0 - \mathbf{v}(t)}{t} \cdot \boldsymbol{\phi} \, d\mathbf{x}. \tag{4.5}$$

Then, by taking the limit along the sequence $t_n \to +\infty$ in (4.5), we get the equality

$$\mathbf{F} = \nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}.$$

Item 2. We use $\overline{\mathbf{v}} \in V$ in (1.5) as test function and we obtain the equality

$$\nu \|\nabla \overline{\mathbf{v}}\|^2 + \int_\Omega (\nabla \cdot \boldsymbol{\sigma}^{(\mathrm{R})}) \cdot \overline{\mathbf{v}} \, d\mathbf{x} = < \overline{\mathbf{f}}, \overline{\mathbf{v}} > . \tag{4.6}$$

We observe that due to the absence of the time-variable the following identity concerning the integral over the space variables is valid

$$\int_\Omega (\overline{\mathbf{v}} \cdot \nabla) \, \overline{\mathbf{v}} \cdot \overline{\mathbf{v}} \, d\mathbf{x} = \int_\Omega (\overline{\mathbf{v}} \cdot \nabla) \, \frac{|\overline{\mathbf{v}}|^2}{2} \, d\mathbf{x} = 0 \qquad \forall \, \overline{\mathbf{v}} \in V.$$

This is one of the main technical facts which are typical of the mathematical analysis of the steady Navier-Stokes equations and which allow to give precise results for the averaged Reynolds equations. On the other hand, we recall that if $\mathbf{v}(t, \mathbf{x})$ is a non-steady (Leray-Hopf) weak solution, then the space-time integral

$$\int_0^T \int_\Omega (\mathbf{v} \cdot \nabla) \, \mathbf{v} \cdot \mathbf{v} \, d\mathbf{x} \, dt,$$

is not well defined and consequently the above integral vanishes only formally.

Item 3. From now, we assume that the assumption (1.7) in the statement of Theorem 1.3 holds true. We integrate the energy inequality (2.1) between 0 and t_n and we divide the result by $t_n > 0$, which leads to

$$\frac{\|\mathbf{v}(t)\|^2}{2t_n} + \frac{1}{t_n} \int_0^{t_n} \|\nabla \mathbf{v}(s)\|^2 \, ds \leq \frac{\|\mathbf{v}_0\|^2}{2t_n} + \frac{1}{t_n} \int_0^{t_n} < \mathbf{f}, \mathbf{v} > \, ds. \tag{4.7}$$

Recall that by Lemma 2.4

$$\frac{\|\mathbf{v}(t)\|^2}{2t} \to 0 \quad \text{and} \quad \frac{\|\mathbf{v}_0\|^2}{2t} \to 0 \quad \text{as } t \to +\infty.$$

From Proposition 4.1 it holds that

$$\frac{1}{t_n} \int_0^{t_n} \nabla \mathbf{v}(s) \, ds \rightharpoonup \nabla \overline{\mathbf{v}} \quad \text{in } L^2(\Omega)^9,$$

hence, we have the lower semi-continuity estimate

$$
\begin{aligned}
\|\nabla\overline{\mathbf{v}}\|^2 &\leq \liminf_{t_n\to+\infty}\left\|\frac{1}{t_n}\int_0^{t_n}\nabla\mathbf{v}(s)\,ds\right\|^2 \leq \liminf_{t_n\to\infty}\frac{1}{t_n^2}\int_0^{t_n}\|\nabla\mathbf{v}\|^2\,ds\cdot\int_0^{t_n}1\,ds \\
&\leq \liminf_{t_n\to\infty}\frac{1}{t_n}\int_0^{t_n}\|\nabla\mathbf{v}(s)\|^2\,ds.
\end{aligned}
$$

Then, passing to the limit as $n\to+\infty$ in (4.7) leads to the following inequality

$$
\nu\|\nabla\overline{\mathbf{v}}\|^2 \leq \lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n} < \mathbf{f},\mathbf{v} > \,ds, \tag{4.8}
$$

provided that the limit exists. It remains to identify the right-hand side of (4.8). We observe that, by combining (1.7) with Lemma 3.5, leads to $\tilde{\mathbf{f}}=\overline{\mathbf{f}}$. Let us write the following decomposition:

$$
\frac{1}{t_n}\int_0^{t_n} < \mathbf{f},\mathbf{v} > \,ds = \frac{1}{t_n}\int_0^{t_n} < \mathbf{f}-\overline{\mathbf{f}},\mathbf{v} > \,ds + \frac{1}{t_n}\int_0^{t_n} < \overline{\mathbf{f}},\mathbf{v} > \,ds.
$$

On one hand since $\overline{\mathbf{f}}\in V$ is independent of t, we obviously have

$$
\frac{1}{t_n}\int_0^{t_n} < \overline{\mathbf{f}},\mathbf{v} > \,ds \to < \overline{\mathbf{f}},\overline{\mathbf{v}} > . \tag{4.9}
$$

On the other hand, we have also

$$
\begin{aligned}
\left|\frac{1}{t_n}\int_0^{t_n} < \mathbf{f}-\overline{\mathbf{f}},\mathbf{v} > \,ds\right| &\leq \frac{1}{t_n}\int_0^{t_n}\|\mathbf{f}-\overline{\mathbf{f}}\|_{V'}\|\nabla\mathbf{v}\|\,ds \\
&\leq \left(\frac{1}{t_n}\int_0^{t_n}\|\mathbf{f}-\overline{\mathbf{f}}\|_{V'}^2\,ds\right)^{1/2}\left(\frac{1}{t_n}\int_0^{t_n}\|\nabla\mathbf{v}\|^2\,ds\right)^{1/2}.
\end{aligned} \tag{4.10}
$$

Combining (3.5) with (3.6) shows that the right-hand side in (4.10) vanishes as $n\to\infty$. Therefore, we deduce by (4.9) that

$$
\nu\|\nabla\overline{\mathbf{v}}\|^2 \leq < \overline{\mathbf{f}},\overline{\mathbf{v}} >,
$$

which, combined with (4.6), shows by comparison that

$$
0 \leq \frac{1}{|\Omega|}\int_\Omega(\nabla\cdot\boldsymbol{\sigma}^{(\mathrm{R})})\cdot\overline{\mathbf{v}}\,d\mathbf{x},
$$

ending the proof of Theorem 1.3. ☐

Remark 4.1. *It is important to observe that*

$$
M_t\big(< \mathbf{f},\mathbf{v} > \big) \to < \overline{\mathbf{f}},\overline{\mathbf{v}} >,
$$

is –in some sense– an assumption on the (long-time) behavior of the "covariance" between the external force and the solution itself. Cf. Layton [17, Thm. 5.2] for a related result in the case of ensemble averages.

The control of the (average/expectation of) kinetic energy in terms of the energy input is one of the remarkable features of classes of statistical solutions, making the stochastic Navier-Stokes equations very appealing in this context. See the review, with applications to the determination of the Lilly constant, in Ref. [2]. See also [10].

5 On ensemble averages

In this section we show how to use the results of Theorem 1.3 to give new insight to the analysis of ensemble averages of solutions. In this case we study suitable averages of the long-time behavior and not the long-time behavior of statistics, as in Layton [17].

Since we first take long-time limits and then we average the Reynolds equations, the initial datum is not so relevant. In fact due to the fact that it holds

$$\frac{\|\mathbf{v}_0\|^2}{t} \to 0 \qquad \text{as } t \to +\infty,$$

then the mean $\overline{\mathbf{v}}$ is not affected by the initial datum.

As claimed in the introduction, we consider now the problem of having several external forces, say a whole family $\{\mathbf{f}^k\}_{k\in\mathbb{N}} \subset V'$, all independent of time. We can think as different experiments with slightly different external forces, whose difference can be due to errors in measurement or in the uncertainty intrinsic in any measurement method. In particular, one can consider for a given force $\mathbf{f}$ and that $\{\mathbf{f}^k\}$ will represent small oscillations around it, hence we can freely assume that we have an uniform bound

$$\exists\, C > 0: \qquad \|\mathbf{f}^k\|_{V'} \leq C \qquad \forall\, k \in \mathbb{N}. \tag{5.1}$$

Having in mind this physical setting, we denote by $\overline{\mathbf{v}^k} \in V$ the long-time average of the solution corresponding to the external force $\mathbf{f}^k \in V'$ and, as explained before (without loss of generality) to the initial datum $\mathbf{v}_0 = \mathbf{0}$. The vector $\overline{\mathbf{v}^k} \in V$ satisfies for all $\boldsymbol{\phi} \in V$ the following equivalent equalities for all $k \in \mathbb{N}$

$$\nu \int_\Omega \nabla\overline{\mathbf{v}^k} : \nabla\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \mathbf{B}^k \cdot \boldsymbol{\phi}\, d\mathbf{x} = < \mathbf{f}^k, \boldsymbol{\phi} >,$$
$$\nu \int_\Omega \nabla\overline{\mathbf{v}^k} : \nabla\boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega (\overline{\mathbf{v}^k} \cdot \nabla)\, \overline{\mathbf{v}^k} \cdot \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \mathbf{F}^k \cdot \boldsymbol{\phi}\, d\mathbf{x} = < \mathbf{f}^k, \boldsymbol{\phi} >,$$

for appropriate $\mathbf{B}^k, \mathbf{F}^k \in L^3(\Omega)^{3/2}$. Since both V and V' are Hilbert spaces, by using (5.1) it follows that there exists $\langle \mathbf{f} \rangle \in V'$ and a sub-sequence (still denoted by $\{\mathbf{f}^k\}$) such that

$$\mathbf{f}^k \rightharpoonup \langle \mathbf{f} \rangle \qquad \text{in } V'.$$

Our intention is to characterize, if possible, the limit of $\{\overline{\mathbf{v}}^k\}_{k\in\mathbb{N}}$. If the forces are fluctuations around a mean value, then the field $\overline{\mathbf{v}^k}$ will remain bounded in V, but possibly without converging to some limit. From an heuristic point of view one can expect that averaging the sequence of velocities (which corresponds to averaging the result over different realizations) one can identify a proper limit, which retains the "average" effect of the flow.

Again, it comes into the system, the main idea at the basis of Large Scale methods: the average behavior of solutions seems the only quantity which can be measured or simulated. It is well-known that one of the most used *summability technique* is that of Cesàro and consists in taking the mean values, hence we focus on the arithmetic mean of velocities

$$\mathbf{S}^n := \frac{1}{n}\sum_{k=1}^{n}\overline{\mathbf{v}^k}.$$

It is a basic calculus result that if a real sequence $\{x_j\}_{j\in\mathbb{N}}$ converges to $x \in \mathbb{R}$, then also its Cesàro mean $S_n = \frac{1}{n}\sum_{j=1}^{n} x_j$ will converge to the same value x. On the other hand, the converse is false; sufficient conditions on the sequence $\{x_j\}_{j\in\mathbb{N}}$ implying that if the Cesàro mean converges, then the original sequence converges, are known in literature as Tauberian theorems. This is a classical topic in the study of divergent sequences/series. In the case of X-valued sequences $\{\mathbf{u}^k\}_{k\in\mathbb{N}}$ (the space X being an infinite dimensional Banach space) one has again that if a sequence converges strongly or weakly, then its Cesàro mean will converge to the same value, strongly or weakly in X, respectively.

The fact that averaging generally improves the properties of a sequence, is reflected also in the setting of Banach spaces even if with additional features coming into the theory. Two main results we will consider are two theorems known as Banach-Saks and Banach-Mazur.

Banach and Saks originally in 1930 formulated the result in $L^p(0, 1)$, but it is valid in more general Banach spaces.

Theorem 5.1 (Banach-Saks)**.** *Let be given a bounded sequence $\{x_j\}_{j\in\mathbb{N}}$ in a reflexive Banach space X. Then, there exists a sub-sequence $\{x_{j_k}\}_{k\in\mathbb{N}}$ such that the sequence $\{S_m\}_{m\in\mathbb{N}}$ defined by*

$$S_m := \frac{1}{m}\sum_{k=1}^{m} x_{j_k},$$

converges strongly in X.

The reader can observe that in some cases it is not needed to extract a subsequence (think of any orthonormal set in an Hilbert space, which is weakly converging to zero, and the Cesàro averages converge to zero strongly), but in general one cannot infer that the averages of the full sequence converge strongly. One sufficient condition is that of *uniform weak convergence*. We recall that $\{x_j\} \subset X$ *uniformly weakly* converges to zero if for any $\epsilon > 0$ there exists $j \in \mathbb{N}$, such that for all $\phi \in X'$, with $\|\phi\|_{X'} \leq 1$, it holds true that

$$\#\big\{j \in N : \ |\phi(x_j)| \geq \varepsilon\big\} \leq j.$$

See also Brezis [6, p. 168].

Another way of improving the weak convergence to the strong one is by the by the convex-combination theorem (cf. Yosida [29, p.120]).

Theorem 5.2 (Banach-Mazur)**.** *Let $\{x_j\} \subset X$ with X a Banach space be a sequence such that $x_j \rightharpoonup x$ as $j \to +\infty$.*

Then, one can find for each $n \in \mathbb{N}$, real coefficients $\{\alpha_j^n\}$, for $j = 1, \dots, n$ such that

$$\alpha_j^n \geq 0 \qquad \text{and} \qquad \sum_{j=1}^{n} \alpha_j^n = 1,$$

such that

$$\sum_{j=1}^{n} \alpha_j^n x_j \to x \qquad \text{in } X, \quad \text{as} \quad n \to +\infty,$$

that is we can find a "convex combination" of $\{x_j\}$, which strongly converges to $x \in X$.

One basic point will be that of considering averages of the external forces, which we will denote by $\langle \mathbf{f} \rangle^n$ and considering the same averages of the solution of the Reynolds equations $\langle \overline{\mathbf{v}} \rangle^n$. They are both bounded and hence, weakly converging (up to a sub-sequence) to $\langle \mathbf{f} \rangle \in V'$ and $\langle \overline{\mathbf{v}} \rangle \in V$, respectively. Then, in order to prove that the dissipativity is preserved one has to handle the following limit of the products

$$\lim_{n \to +\infty} < \langle \mathbf{f} \rangle^n, \langle \overline{\mathbf{v}} \rangle^n >,$$

which cannot be characterized, unless (at least) one of the two terms converges strongly. This is why we have to use special means instead of the simple Cesàro averages

The first result of this section is then the following:

Proposition 5.1. *Let be given $\{\mathbf{f}^k\}_{k \in \mathbb{N}}$ uniformly bounded in V'. Then one can find either a Banach-Saks sub-sequence or a convex combination of $\{\mathbf{v}^k\}_{k \in \mathbb{N}}$, which are converging weakly to some $\langle \mathbf{v} \rangle \in V$, which satisfies a Reynolds system (5.4), with an additional dissipative term.*

Proof of Theorem 5.1. We define $\langle \mathbf{f} \rangle^n$ and $\langle \mathbf{v} \rangle^n$ to be either

$$\langle \mathbf{f} \rangle^n := \frac{1}{n} \sum_{k=1}^{n} \mathbf{f}^{j_k} \qquad \text{and} \qquad \langle \mathbf{v} \rangle^n := \frac{1}{n} \sum_{k=1}^{n} \overline{\mathbf{v}}^{j_k}$$

or

$$\langle \mathbf{f} \rangle^n := \sum_{j=1}^{n} \alpha_j^n \mathbf{f}^j \qquad \text{and} \qquad \langle \mathbf{v} \rangle^n := \sum_{j=1}^{n} \alpha_j^n \overline{\mathbf{v}^j}$$

where the sub-sequence $\{\mathbf{f}^{j_k}\}_{k \in \mathbb{N}}$ or the coefficients $\{\alpha_j^n\}_{j,\, n \in \mathbb{N}}$ are chosen accordingly to the Banach-Saks of Banach-Mazur theorems in such a way that

$$\langle \mathbf{f} \rangle^n \to \langle \mathbf{f} \rangle \qquad \text{in } V'.$$

We define accordingly to the same rules $\langle \mathbf{B} \rangle^n$ and we observe that, by linearity, we have $\forall\, n \in \mathbb{N}$

$$\nu \int_\Omega \nabla \langle \mathbf{v} \rangle^n : \nabla \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \langle \mathbf{B} \rangle^n \cdot \boldsymbol{\phi}\, d\mathbf{x} = < \langle \mathbf{f} \rangle^n, \boldsymbol{\phi} > \qquad \forall \boldsymbol{\phi} \in V. \tag{5.2}$$

Then, we can define $\langle \mathbf{F} \rangle^n := \langle \mathbf{B} \rangle^n - (\langle \mathbf{v} \rangle^n \cdot \nabla)\, \langle \mathbf{v} \rangle^n$, to rewrite (5.2) also as follows

$$\nu \int_\Omega \nabla \langle \mathbf{v} \rangle^n : \nabla \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega (\langle \mathbf{v} \rangle^n \cdot \nabla)\, \langle \mathbf{v} \rangle^n \cdot \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \langle \mathbf{F} \rangle^n \cdot \boldsymbol{\phi}\, d\mathbf{x} = < \langle \mathbf{f} \rangle^n, \boldsymbol{\phi} > . \tag{5.3}$$

By the uniform bound on $\|\mathbf{f}^k\|_{V'}$ and by results of Section 4.2 on the Reynolds equations it follows that there exists C such that $\|\overline{\mathbf{v}}^k\|_V \leq C$, hence

$$\|\langle \mathbf{v} \rangle^n\|_V \leq C \qquad \forall\, n \in \mathbb{N},$$

and we can suppose that (up to sub-sequences) we have weak convergence of the convex combinations

$$\begin{aligned} \langle \mathbf{v} \rangle^n &\rightharpoonup \langle \mathbf{v} \rangle && \text{in } V, \\ \langle \mathbf{B} \rangle^n &\rightharpoonup \langle \mathbf{B} \rangle && \text{in } L^{3/2}(\Omega)^3, \\ \langle \mathbf{F} \rangle^n &\rightharpoonup \langle \mathbf{F} \rangle && \text{in } L^{3/2}(\Omega)^3, \end{aligned}$$

Hence, passing to the limit in (5.2), we obtain

$$\nu \int_\Omega \nabla \langle \mathbf{v} \rangle : \nabla \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \langle \mathbf{B} \rangle \cdot \boldsymbol{\phi}\, d\mathbf{x} = < \langle \mathbf{f} \rangle, \boldsymbol{\phi} > \qquad \forall \boldsymbol{\phi} \in V.$$

By the same reasoning used before we have, for $\langle \mathbf{F} \rangle := \langle \mathbf{B} \rangle - (\langle \mathbf{v} \rangle \cdot \nabla)\, \langle \mathbf{v} \rangle$,

$$\nu \int_\Omega \nabla \langle \mathbf{v} \rangle : \nabla \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega (\langle \mathbf{v} \rangle \cdot \nabla)\, \langle \mathbf{v} \rangle \cdot \boldsymbol{\phi}\, d\mathbf{x} + \int_\Omega \langle \mathbf{F} \rangle \cdot \boldsymbol{\phi}\, d\mathbf{x} = < \langle \mathbf{f} \rangle, \boldsymbol{\phi} > . \tag{5.4}$$

Then, if we take $\boldsymbol{\phi} = \langle \mathbf{v} \rangle$ in (5.4) we obtain

$$\nu \|\nabla \langle \mathbf{v} \rangle\|^2 + \int_\Omega \langle \mathbf{F} \rangle \cdot \langle \mathbf{v} \rangle\, d\mathbf{x} = < \langle \mathbf{f} \rangle, \langle \mathbf{v} \rangle > . \tag{5.5}$$

On the other hand, if we take $\boldsymbol{\phi} = \langle \mathbf{v} \rangle^n$ in (5.3) and by the result of the previous section, we have

$$\nu \|\nabla \langle \mathbf{v} \rangle^n\|^2 \leq < \langle \mathbf{f} \rangle^n, \langle \mathbf{v} \rangle^n >,$$

hence passing to the limit, by using the strong convergence of $\langle \mathbf{f} \rangle^n$ in V' and the weak convergence of $\langle \mathbf{v} \rangle^n$ in V we have

$$\nu \|\nabla \langle \mathbf{v} \rangle\|^2 \leq \liminf_{n \to +\infty} \nu \|\nabla \langle \mathbf{v} \rangle^n\|^2 \leq < \langle \mathbf{f} \rangle, \langle \mathbf{v} \rangle > .$$

If we compare with (5.5) we have finally the dissipativity

$$\frac{1}{|\Omega|} \int_\Omega (\nabla \cdot \langle \boldsymbol{\sigma}^{(\mathrm{R})} \rangle) \cdot \langle \mathbf{v} \rangle\, d\mathbf{x} = \frac{1}{|\Omega|} \int_\Omega \langle \mathbf{F} \rangle \cdot \langle \mathbf{v} \rangle\, d\mathbf{x} \geq 0,$$

that is a sort of ensemble/long-time Boussinesq hypothesis, cf. with the results from Ref. [17, 16]. □

In the previous theorem, we have a result which does not concern directly with the ensemble averages, but a selection of special coefficients is required. This is not completely satisfactory from the point of view of the numerical computations, where the full arithmetic mean should be considered. The main result can be obtained at the price of a slight refinement on the hypotheses on the external forces

To this end we recall a lemma, which is a sort of Rellich theorem in negative spaces (see also Galdi [14, Thm. II.5.3] and Feireisl [9, Thm. 2.8]).

Lemma 5.3. *Let $\Omega \subset \mathbb{R}^n$ be bounded and let be given $1 < p < n$. Let $\{f_k\}_{k\in\mathbb{N}}$ be a sequence uniformly bounded in $L^q(\Omega)$ with $q > (p^*)'$, where $p^* = \frac{np}{n-p}$ is the exponent in the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$. Then, there exists a sub-sequence $\{f_{k_m}\}_{m\in\mathbb{N}}$ and $f \in L^q(\Omega)$ such that*

$$\begin{aligned} f_{k_m} &\rightharpoonup f \qquad \text{in } L^q(\Omega), \\ f_{k_m} &\to f \qquad \text{in } W^{-1,p'}(\Omega), \end{aligned}$$

or, in other words, the embedding $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,p'}(\Omega)$ is compact.

We present the proof for the reader's convenience.

Proof of Lemma 5.3. Since by hypothesis $L^q(\Omega)$ is reflexive, by the Banach-Alaouglu-Bourbaki theorem we can find a sub-sequence f_{k_m} such that

$$f_{k_m} \rightharpoonup f \qquad \text{in } L^q(\Omega),$$

and by considering the sequence $\{f_{k_m} - f\}_{m\in\mathbb{N}}$ we can suppose that $f = 0$. We then observe that by the Sobolev embedding we have the continuous embeddings

$$W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \sim L^{(p^*)'}(\Omega) \hookrightarrow (W_0^{1,p}(\Omega))' \simeq W^{-1,p'}(\Omega),$$

where $L^{p^*}(\Omega) \sim L^{(p^*)'}(\Omega)$ is the duality identification, while the second one $(W_0^{1,p}(\Omega))' \simeq W^{-1,p'}(\Omega)$ is the Lax isomorphism. This shows $L^{(p^*)'}(\Omega) \subseteq W^{-1,p'}(\Omega)$.

Next, let be given a sequence $\{f_{k_m}\} \subset W^{-1,p'}(\Omega)$, then by reflexivity (since $1 < p < \infty$) there exists $\{\phi_{k_m}\} \subset W_0^{1,p}(\Omega)$ such that

$$\|f_{k_m}\|_{W^{-1,p'}(\Omega)} = f_{k_m}(\phi_{k_m}) = < f_{k_m}, \phi_{k_m} >,$$

with $\|\phi_{k_m}\|_{W_0^{1,p}(\Omega)} = \|\nabla \phi_{k_m}\|_{L^p(\Omega)} = 1$.

Hence, by using the classical Rellich theorem, we can find a sub-sequence $\{\phi_{k_j}\}_{j\in\mathbb{N}}$ such that

$$\phi_{k_j} \to \phi \qquad \text{in } L^r(\Omega) \qquad \forall\, r < p^*.$$

In particular, we fix $r = q'$ (observe that $q > (p^*)'$ implies $q' < p^*$) and we have

$$\|f_{k_m}\|_{W^{-1,p'}(\Omega)} = < f_{k_m}, \phi_{k_m} - \phi > + < f_{k_m}, \phi > .$$

The last term converges to zero, by the definition of weak convergence $f_{k_m} \rightharpoonup 0$, while the first one satisfies

$$|< f_{k_m}, \phi_{k_m} - \phi >| \leq \|f_{k_m}\|_{W^{-1,p'}} \|\phi_{k_m} - \phi\|_{W_0^{1,p}},$$

and since $\|f_{k_m}\|_{W^{-1,p'}}$ is uniformly bounded and $\|\phi_{k_m} - \phi\|_{W_0^{1,p}}$ goes to zero, then also this one vanishes as $j \to +\infty$. □

Proof of Theorem 1.4. The proof of this theorem can be obtained by following the same ideas of the Proposition 5.1. In fact, the main improvement is that the weak convergence $\mathbf{f}^j \rightharpoonup \langle \mathbf{f} \rangle$ in $L^q(\Omega)$ implies (without extracting sub-sequences) that

$$\mathbf{f}^k \to \mathbf{f} \qquad \text{in } V'.$$

This follows since from any sub-sequence we can find a further sub-sequence which is converging strongly, by Lemma 5.3. Then, by the weak convergence of the original sequence, the limit is always the same and this implies that the whole sequence $\{\mathbf{f}^k\}$ strongly converges to its weak limit.

Hence, we have

$$\frac{1}{n} \sum_{k=1}^{n} \mathbf{f}^k \to \mathbf{f} \qquad \text{in } V',$$

and then, since $\langle \mathbf{v} \rangle^n \rightharpoonup \langle \mathbf{v} \rangle$ in V, we can infer that

$$< \langle \mathbf{f} \rangle^n, \langle \mathbf{v} \rangle^n > = < \frac{1}{n} \sum_{k=1}^{n} \mathbf{f}^k, \frac{1}{n} \sum_{k=1}^{n} \overline{\mathbf{v}}^k > \to < \langle \mathbf{f} \rangle, \langle \mathbf{v} \rangle >,$$

and the rest follows as in Proposition 5.1. □

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