YOU SEE (MOSTLY) WHAT YOU PREDICT: THE POWER OF GEOMETRIC PREDICTION

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We consider geometric prediction (GP), as a mental process through which a figure is manipulated, and its change imagined, while certain properties are maintained invariant. In this report on a recent study, we concentrate: 1) on capturing processes of GP before explorations are carried out in a dynamic geometry environment (DGE), to gain insight into possible characteristics of such processes; 2) on possible implications it can have in a subsequent process of dynamic exploration of a DGE figure, in particular in the solver's interpretation of feedback from the DGE.

INTRODUCTION AND THEORETICAL GROUNDING

As mathematics students, it has probably happened to many of us to listen to our professors quickly reason about a geometric configuration, reaching an "obvious" conclusion that could "clearly be seen" on the paper or the board in front of them. With uneasiness, and some embarrassment, we would nod and run to our room to try to see what was supposedly so clear, and maybe then try and prove it.

In this paper, we focus on *geometric prediction (GP)*, a mental process through which a figure is manipulated, and its change imagined, while certain properties are maintained invariant (Miragliotta, Baccaglini-Frank & Tomasi, 2017; Mariotti & Baccaglini-Frank, in press). In the vignette above, the expert geometry professors (for the sake of this paper let us think of Euclidean Geometry) are so skillful in carrying out GPs that they conceive new configurations or geometrical objects that others cannot even see. We are interested in gaining insight into processes through which GPs are accomplished, and how these predictions might condition subsequent explorations.

Indeed, on the one hand, mathematics educators have recognized the importance of helping students learn to think like mathematicians; for example, Cuoco, Goldenberg and Mark have proposed to organize curriculum around *mathematical habits of mind*. Among these, we find "visualizing" and "tinkering" (Cuoco, Goldenberg & Mark, 1996). Within the types of visualization discussed, the researchers include:

> reasoning about simple subsets of plane or three-dimensional space with or without the aid of drawings and pictures. […] Visualizing change. Seeing how a phenomenon varies continuously is one of the most useful habits of classical mathematics. Sometimes the phenomenon simply moves between states […]. Other times one thing blends into another […]. This habit cuts across many of the others […] (ibid, pp. 382-383).

These seem to be well aligned with Presmeg's description (2006), which is

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> taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics (ibid, p. 206).

Tinkering is described as being “at the heart of mathematical research” and it consists in “taking ideas apart and putting them back together”, and asking “what happens if…” (Cuoco et al., 1996, p. 379). Our notion of GP seems to have a lot in common both with the forms of visualization described and with the idea of tinkering, which justify its educational significance.

On the other hand, studying GP and its relationships with dynamic geometry environments (DGEs) seems especially important since this sort of technology can affect conceptualization and problem solving processes in Mathematics (e.g., Arcavi & Hadas, 2000). We believe GP to be key in geometrical problem solving, and we believe it can be trained, possibly using the support of a DGE (Mariotti & Baccaglini-Frank, in press). Moreover, we conjecture that once a GP is carried out, it can affect the subsequent process of dynamic exploration in a DGE, influencing what the solver can or cannot “see”. Exploring this conjecture has particular educational significance.

In the study we report on here, we concentrate firstly on capturing processes of GP before dynamic explorations are carried out, to gain insight into possible characteristics of such processes; and, secondly, on possible implications it can have in a subsequent process of dynamic exploration of a DGE figure, in particular on the solver’s interpretation of feedback from the DGE.

RESEARCH QUESTIONS AND METHODOLOGY

The data we present are part of a doctoral research project on geometric prediction for which 15 geometrical problems were designed and proposed to 18 Italian high school students (ages 14-18), undergraduates and graduate students majoring in mathematics (ages 19-33), during the months of November and December 2017. The problems were designed to elicit processes of GP and they were used within clinical interviews conducted by the first author of this paper. Although all data has not yet been thoroughly analyzed we wish to use some of it to present a preliminary report on the following questions: 1) When GP is used (either spontaneously or prompted by the interviewer), by what kinds of verbal or gestural descriptions is it accompanied? 2) Once a GP is advanced and the student is given the opportunity to interact with a DGE figure corresponding to the configuration reasoned upon, how does the student interpret the feedback from the DGE? Does such feedback lead him/her to change the GP?

All interviews were carried out in a quiet room and each student spent 60 minutes with the interviewer and worked through as many interview problems as they could.

The “locus of P” problem

The following task is a variation of a geometric problem described by De Finetti (1967). The task used in this study is composed of two parts. The first one is:

“Read and perform the following step-by-step construction: fix two points A and B; connect them with a segment AB; choose a point P on the plane; connect A and P with a segment AP; construct M as the midpoint of AP; construct the segment MB. Imagine moving the point P. If the length of the segment MB must always be constant, what can you say about the point P?”

The step-by-step construction could be accomplished with paper and pencil (obtaining a construction like that in Fig. 1a); the question about P one was proposed first mentally, then solvers were offered the possibility of drawing ideas on paper.

Once a construction was made in a DGE (*GeoGebra*) and the solver had proposed a solution or stated that s/he was not able to find one, part two of the task was given. The interviewer would ask the solver to move P in the DGE figure, consistently with her/his prediction, or else to explore the dynamic figure to help reach a solution.

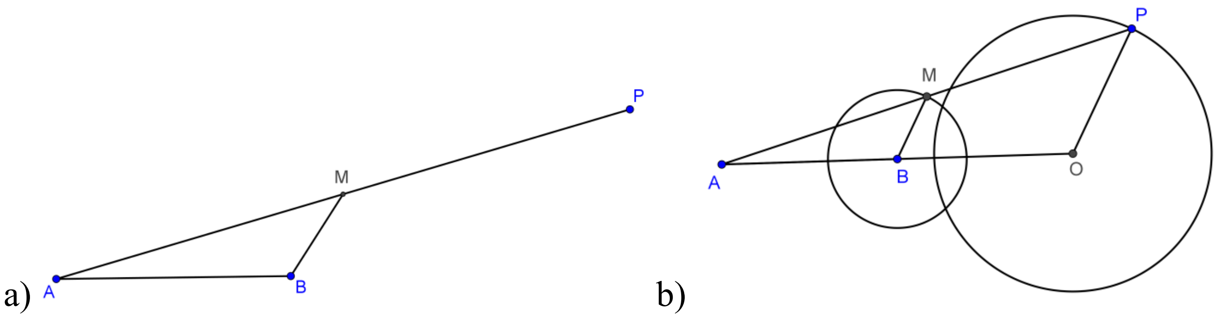


Figure 1: a) Configuration obtained from part 1; b) loci of M and P.

To solve the task, the following mathematical facts are important to note: 1) M is the midpoint of AP, so AM is always equal to 1/2 of AP; 2) MB must always have the same length, the locus of M is a circle with center in B and radius BM.

Solvers can reason in different ways; here, we describe a few possible steps leading to a solution. The discursive element “*MB must always be constant”* may foster recollection of the definition of circle, leading to immediate recognition of the locus of M. It is also possible to recognize such locus when looking for “good positions” of P, that is, positions for which the length of MB remains constant. The solver can imagine moving M along the circle, or draw it, and observe different positions of P, discovering that also P lies on a circle. Using the relationship AP = 2AM, s/he can view the locus of P as the circle corresponding to the locus of M through a homogenous dilation of factor 2. This theoretical consideration may also help the solver find the center and radius of the locus of P: the center is a point O on the line through A and B, satisfying the relationship AO = 2AB; the radius has length 2MB (Fig. 1b).

We conjectured that the task would foster various processes of GP, in particular for the loci of P and of M; indeed, recognizing the locus of M seemed a likely stepping stone. In the second part of the task, we expected the solver to use several dragging modalities (Arzarello, Olivero, Paola, & Robutti, 2002) and, in particular, maintaining dragging (MD) (Baccaglini-Frank & Mariotti, 2010) to maintain certain predicted properties.

STUDENTS' ANSWERS

In light of our research questions, we analyzed the videos and transcripts of all students' interviews in the following way. We searched for and coded all excerpts containing use of GP (spontaneous or in response to interviewer prompts), marking its being mathematically correct or not, and whether it was accompanied by explanations or not (whether they were mathematically correct or not). We also labelled all verbal or gestural descriptions by which these were accompanied. We then identified the excerpts in which the solvers explored the constructed DGE figure, and marked: whether the students spoke about new or contradictory properties (changing the product of their GP), whether the solvers seemed surprised by the feedback from the DGE, and whether they eventually reached a correct solution to the problem.

Here we report on the analyses of the interviews of the eight students who were assigned the "locus of P problem". Each of them uses different words and gestures. We will give more detail on two of the interviews in the next section. Table 1 summarizes results of the analyses of the outcomes of the most common of these students' GPs: P is fixed (predicted by 5 students); M on a circle (predicted by 2 students); P on a circle (predicted by 3 students).

GP	Student	Explained	New ideas	Surprise	Correct solution
P is fixed (GP1)	S2	Yes	No	No	No
	S3	Yes	Yes	Yes	No
	S4	Yes	No	No	No
	S5	Yes	No	Yes	No
	S6	No	Yes	No	No
	S8	Yes	Yes	No	Yes
M on a circle (GP2)	S1	No	Yes	No	Yes
	S2	Yes	No	No	Yes
P on a circle (GP3)	S1	No	Yes	No	Yes
	S2	No	Yes	No	No
	S7	Yes	Yes	Yes	Yes

Table 1: Analyses of the outcomes of 8 students' GPs on the "locus of P" problem

Only 2 of the 8 students (S1, S7) find that the locus of P is a circumference; S2 declares this possibility, but then suddenly discards it. No student succeeds in correctly predicting the center or radius of the circle: some predict the center to be B. The solution processes of S1 and S7 have in common that the locus of M is made explicit, either verbally or by a drawing. As predicted, a process of GP leading to the locus of M seems to be essential in reaching a GP of the locus of P. Other common aspects of these students' processes of GP are: the GP on the locus of M is supported by the discursive element "*constant/invariant length*" of MB; the GP on the locus of P is accompanied by visual-spatial considerations that do not seem to have strong theoretical grounding. Indeed, S1 does not explain why she imagines a circle; while S7 describes the motion of P very well, but he does not refer to any theoretical elements supporting his GP. This lack of theoretical grounding of the GPs could explain the difficulties of both students in defining the circle's center and radius.

Another interesting consideration is that in all cases in which the incorrect GP "P is fixed" (GP1) is made explicit, it becomes dominant in the problem solving process and it impedes to find other "good positions" for P. In particular, 4 out of the 5 students who make GP1 and never reach the correct solution of the problem try to explain why they see P as fixed. Although 2 of these get new ideas from the DGE exploration, leading them to partially change their mind, none are able to generate and interpret the feedback in a way that allows them to reach the correct locus of P. Moreover, only 2 of these students seem surprised by the DGE feedback. These data support the hypothesis that students who succeed in explaining (incorrectly) their GPs have more difficulty in grasping DGE feedback in contradiction with them.

On the other hand, S1, who fails to explain why the locus of P is a circle, and S7, who merely refers to spatial elements, seem more prepared to recognize and possibly modify some characteristics of the locus they had imagined. In particular, S1, in her dynamic exploration, looks for its center and radius; and S7 approximates the locus of P first with "a curve" then with "an ellipse" and finally with "a circle".

In general, it seems that all (correct and incorrect) GPs, produced before the DGE exploration – and more so when accompanied by explanations – strongly influence the subsequent exploration, conditioning how students interpret the DGE feedback. In particular, in the case of (mathematically) incorrect outcomes of processes of GP produced before a dynamic exploration, the DGE feedback in general does not help students to completely change their minds and recognizing a new contradictory geometrical property. In the case of (mathematically) correct outcomes of processes of GP produced before a dynamic exploration, the DGE feedback does help students refine the outcomes of their GPs. We now present to more detailed analyses of excerpts of two students' interviews.

S1: unexplained semi-correct GP and proper interpretation of DGE feedback

Shortly after reading the problem and having seen the configuration on the computer screen, but before dragging anything, the student S1 said:

1	S1:	The length of MB has to be constant…so I… so instinctively I would answer that P has to move on a circle […]
2	S1:	I'm not…I'm not that sure it is a circle. I mean, intuitively I imagine it, but I wouldn't actually know what center and what radius, I can't imagine it.

S1 immediately describes the product of a GP: the locus of P as a circle (GP3 in Table 1). GP3 is identified seemingly quite rapidly (line 1) without passing through the locus of M. However, GP3 is not completely clear to S1, who realizes she has not identified some of its characterizing features (line 2). The outcome of the GP is strong enough to allow S1 to trace the circle with her finger (Fig. 2a, 2b).

a) b) c)

Figure 2: a, b) S1 traces GP3 with her finger on the screen; c) S1 traces the locus of M (GP2) using her fingers as a compass.

Then S1 tries to predict the center and radius of GP3. To do this she predicts the locus of M (GP2 in Table 1), using her fingers as a compass (Fig. 2c):

3	S1:	For this [MB] to be constant…
4	S1:	[she places the thumb in B and index of her right hand on M and rotates her index around the thumb] Yes, P should…maybe moves along a circle…with center at B.

At this point S1 has produced a verbal and gestural description of GP3 and a gestural description of GP2. When she is asked to check her answers using the DGE figure.

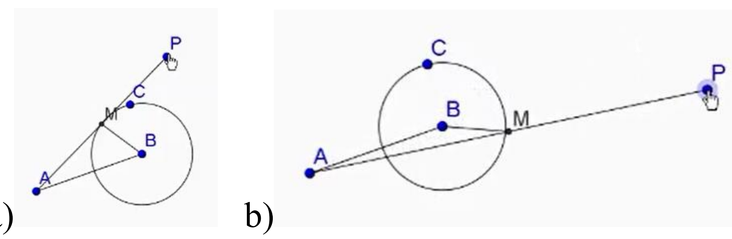
Initially S1 moves P continuously but only along a short arc of a circle, the one she had predicted. Doing this, she focuses on M and she verbally describes its locus:

5	S1:	So, yes, in order for MB to always have the same length, it is fundamental for M to move on a circle with center at B and…if M moves along a circle with center at B…also P will move on a circle.

S1 uses MD expressing her intention of having MB "always have the same length". This seems to lead her to perceive a relationship between the two predicted circles, expressed in the form "if …, also…" which seems logically close to the conditional form "if…, then…". The interviewer asks for additional information on the locus of P:

6	Int.:	Can you say anything else about this circle?
7	S1:	Eh, I am asking myself if its radius is…if its center is B, but I don't think so.

As S1 answers (line 7) she keeps on using MD, making bigger movements along a predicted circle, which leads her to reject her initial conjecture for the center of the locus of P.



a) b)

Figure 3: a) initial configuration from which S1 starts using MD to maintain the length of MB constant; b) P describes a circle as it is dragged using MD.

Finally, in light of her new GP supported by the DGE feedback (lines 5, 7), S1 decides to use MD on P trying to maintain M (Fig. 3b) on a constructed circle centered in B with a given radius (in Fig. 3 she chose point C randomly on the plane).

This convinces S1 that P moves along a circle, but definitely not with center at B. So S1 seems to have a very constructive interaction with the DGE, using the feedback to support and refine her GPs.

S2: explained incorrect GP and improper interpretation of DGE feedback

Student S2 starts working in a paper and pencil environment, and quickly predicts that P has to be fixed:

1 S2: If I move this [her finger runs along BM] it [M] moves but this distance [length of BM] increases…I would say that P is univocally determined, once A, B and m are fixed.

This GP (GP1 in Table 1) seems to arise because of the possible movement imagined for M and P: whenever S2 speaks of moving P she moves her index or the tip of a pen in the direction of PA, as it is drawn, without ever changing its inclination. This GP seems closely related to the kind of movement imagined for M: it is only possible to stretch or shrink MB, but not to move M maintaining constant m, the length of BM. Moreover, at times, there seems to be ambiguity between "m" (the length of BM) and point "M", leading to the idea that M must be fixed. Therefore, S2 predicts that, at most, A and P can move "coming closer and farther, to M in a proportional manner". With this in mind, once she realizes that A is fixed, S2 is sure that P must also be fixed.

The strength of her GP seems to inhibit S2's ability to constructively interpret the feedback obtained from the DGE. Indeed, when asked to explore the dynamic figure, at first S2 tries to move P maintaining m = 2:

2 S2: Ok, let's put 2, even though I will never be able to make it 2. I don't know.

Although at least to positions in which m = 2 appear on the screen, S2 does not seem to notice them, and she puts P back in the original position, an instance in which m = 3.

3 S2: Ehm, let's put this one. I can move P, let's activate the trace, maybe.

As she moves P, S2 mentions the possibility of P moving on a circle, but she seems unsure and rapidly discards that possibility, even though a few good positions for M had appeared on the screen (Fig. 4). Instead, S2 continues to speak about the "fixed-ness" of m that necessarily determines a single good position for P.

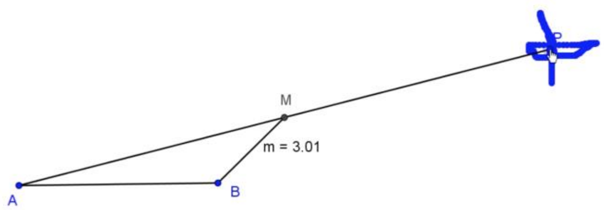


Figure 4: S2's attempts to drag P maintaing m = 3.

We see this as a case in which an incorrect GP with an explanation that seems to be very convincing for the student does not allow the student to generate and interpret constructively the feedback from the DGE. Indeed, although the DGE exploration makes use of MD, the student appears to be "blinded" by her original GP, to the extent that she cannot see any "good positions" for P other than the original configuration.

CONCLUSION

The analyses of students' videos and transcripts revealed that processes of GP (or at least the description of their outcomes) tend to be accompanied by verbal or gestural explanations: for the GPs considered in this study 7 out of 11 times these explanations were present. The verbal and gestural forms varied from student to student. Moreover, our data suggests that once a GP is advanced and the student is given the opportunity to interact with a DGE figure corresponding to the configuration reasoned upon, the GP has influence on the DGE exploration. In particular, if the students are quite convinced by their (correct or incorrect) GP, the exploration only serves to refine or confirm the predictions, and there seem to rarely be instances of surprise in the exploration. This phenomenon, that we refer to as "power of the GP", is particularly striking in cases in which the GP is incorrect and still drives the solver to see on the screen only what s/he has predicted and is, therefore, prepared to see.

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