Chapter 16
Special Needs in Research and Instruction in Whole Number Arithmetic

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16.1 General Introduction

Lieven Vershaffel

Many children have difficulties or problems with learning mathematics. While these difficulties or problems may occur at any stage in learners' mathematical development, by far the most attention of researchers and practitioners goes to the domain of early and elementary mathematics and, more specifically, to the domain of whole number arithmetic (WNA). Even though the issues of diagnosis of and instruction

Electronic Supplementary Material The online version of this chapter (doi:10.1007/978-3-319-63555-2_16) contains supplementary material.

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M.G. Bartolini Bussi, X.H. Sun (eds.), *Building the Foundation: Whole Numbers in the Primary Grades*, New ICMI Study Series,
https://doi.org/10.1007/978-3-319-63555-2_16

for children with special mathematical learning needs are getting increasing research attention, research in this area is still lagging behind compared with other academic subjects such as reading. Hereafter, we list some major open questions for research and practice.

First, there is the terminological issue. Defining mathematical learning difficulties, problems or disabilities (hereafter abbreviated as MLD) is not an easy task (Berch and Mazzocco 2007). Despite the solid knowledge base that has been achieved in this field, more substantial progress in understanding and addressing MLD would be facilitated by establishing agreement on consistently used terminology and use of standardised criteria concerning the nature and seriousness of the disability. While certain definitions explicitly refer to a biologically based disorder, others emphasise the discrepancy between the child's mathematical achievement and his/her general intelligence as the main criterion, and others still focus on the response to intervention. But the field of MLD also lacks coherence and consensus about what constitutes 'mathematics' in MLD. Within MLD research, there is a history of predominance to focus on memorisation of arithmetic facts and automatisation of arithmetic procedures. A less (neuro)psychologically dominated and more interdisciplinary approach might bring a broader, more coherent and balanced perspective that takes into account both the views about mathematics learning as arithmetic and other equally important perspectives such as spatial and geometrical reasoning, mathematical relations and patterns and other forms of mathematical thinking with more potential towards abstraction and generalisation (Hord and Xin 2015; Mulligan 2011). Evidently, besides children with MLD, there are also other children requiring special mathematics educational support, but they are not diagnosed as MLD, such as children with intellectual disabilities; children with auditory, visual or motoric impairments; children with serious emotional and/or behavioural problems; or, finally, children with long-standing inappropriate instruction or environmental deprivation (De Smedt et al. 2013).

A second major concern of researchers in the field is to characterise the various cognitive mechanisms that are implicated in the development of MLD. Several cognitive explanations for the presence of MLD have been put forward. Most of the available research on MLD has dealt with domain-general cognitive factors, such as poor working memory and difficulties with the retrieval of phonological information of long-term memory. More recently (and against the background of findings from neuroimaging research), it has been proposed that MLD arises as a consequence of domain-specific impairments in number sense or the ability to represent and manipulate numerical magnitudes (Butterworth 2005; Landerl et al. 2004). For example, children with MLD have particular difficulties in comparing two numerical magnitudes and in putting numbers on a number line, both of which are thought to measure one's understanding of numerical magnitude. Although various cognitive candidates have been put forward to explain MLD, the existing body of data is still in its infancy. According to Karagiannakis et al. (2014), although the field has witnessed the development of many classifications, no single framework or model can be used for a comprehensive and fine interpretation of students' mathematical difficulties, not only for research purposes but also for informing mathematics educators. Starting from a multi-deficit neurocognitive approach and building on the available literature, these authors have recently proposed a classification model for MLD describing four cognitive domains within which specific deficits may reside.

Third, initial accounts of MLD in the 1970s suggested that MLD was due to brain abnormalities. With the advent of modern neuroimaging techniques, researchers have begun to address this issue. There is converging evidence for the existence of a frontoparietal network that is active during number processing and arithmetic (Ansari 2008). Studies that examine this network in children with MLD are currently slowly but steadily emerging. These few studies consistently indicate that children with MLD have both structural and functional alterations in the above-mentioned frontoparietal network, particularly in the intraparietal sulcus, which is the brain circuitry that supports the processing of numerical magnitudes, and (pre)frontal cortex, which is assumed to have an auxiliary role in the maintenance of intermediate mental operations in working memory. Furthermore, it has been suggested that these brain abnormalities in children with MLD are probably of a genetic origin, yet the genetic basis of MLD remains largely unknown and no genes responsible for mathematics (dis)abilities have been identified. Studies in the field of medical genetics have revealed that some disorders of a known genetic origin, such as Turner syndrome and 22q11 deletion syndrome, show a consistent pattern of MLD. Furthermore, there is some early evidence of links to autism spectrum disorders and Asperger's.

The fourth and final issue relates to the question: what are appropriate educational interventions for children with MLD? Originally, general perceptuo-motor training was the dominant way of remediating learning disorders, but the effects of this type of training have been discounted. Interventions that target those specific components of mathematics with which a child with MLD has difficulty appear to be the most effective (Dowker 2008). Such intervention involves the assessment of a child's strengths and weaknesses in mathematics, and this profile is taken as an input to remediate specific components of mathematical skill. However, several major questions remain: what is the appropriate moment to diagnose MLD and to start specific interventions? Do MLD children profit more from individualised interventions organised out of the regular mathematics class or do they profit more from being integral part of the regular mathematics class? Do these children need a special kind of intervention or do they profit most from the same kind of instruction as children without MLD? More specifically, is conceptually based and constructivist-oriented mathematics instruction also suitable for children with learning disabilities (Xin and Hord 2013; Xin et al. 2016)? Another issue is whether we do not have a blind spot when making assumptions about what children with MLD can do, rather than what they cannot do (Peltenburg et al. 2013). Finally, does the remedial instruction of children with MLD pay enough attention to other aspects of mathematics than whole number sense, such as to conceptual relationships that may develop from spatial reasoning? Clearly, it may not be productive to try to answer these major educational questions for all categories of children who have serious trouble with learning mathematics.

So, although the last decades have witnessed a serious growth in research into the diagnosis, remediation and prevention of MLD, much work remains to be done. Longitudinal research is needed to identify developmental precursors and to delineate developmental trajectories of MLD. The neural basis of these difficulties and their association with classroom performance certainly need to be further explored. Understanding the different characteristics of MLD at different levels – the behavioural, the cognitive and the neurobiological – will inform appropriate educational interventions. The design and evaluation of these remedial interventions needs to be a priority on the agenda for future research. These interventions may not only treat the difficulties but also prevent them. And, finally, there is a great need to look beyond diagnoses and interventions that are merely focused on counting and arithmetic to those also involve other aspects of mathematical thinking which hold more promise for abstraction and generalisation.

The goal of the ICMI Study 23 panel on special needs was to explore and discuss the above issues and challenges, with a strong emphasis on the last issue, namely, instructional goals and interventions for children with MLD. The panel consisted of four scholars with complementary specialisations in the domain of children with MLD and other special needs in the curricular domain of whole number arithmetic, namely, Anna Baccaglini-Frank ('La Sapienza' University, Rome, Italy), Joanne Mulligan (Macquarie University, Sydney, NSW, Australia), Marja van den Heuvel-Panhuizen (Utrecht University, Utrecht, The Netherlands) and Yan Ping Xin (Purdue University, West Lafayette, IN, USA), complemented by one of the keynote speakers of the ICMI Study 23 Conference, Prof. Brian Butterworth (University College, London, UK), a world-leading scholar in the domain of the (neuro)cognitive roots of dyscalculia and its treatment, who acted as discussant.

16.2 Does 'Dyscalculia' Depend on Initial Primary School Instruction?[1]

Anna Baccaglini-Frank

In this contribution, I address the questions of whether (1) MLD children profit more from individualised interventions organised out of the regular mathematics class or from being an integral part of the regular mathematics class; (2) these children need a special kind of intervention or whether they profit most from the same kind of instruction as children without MLD; (3) the answers to the above questions are the same for all categories of children with MLD.

Let me start with the last question. Assuming that 'categories of children with MLD' is a well-defined construct (though I do not believe it yet is), in my opinion the answer is 'no'.

First of all, for the same child, the answers may vary at different stages of his/her life. For example, before any diagnosis is made (and some would argue, even after), many would probably claim that, at least initially (and perhaps always), the child should be in the 'regular' classroom and experience conceptually based,

[1] The study was possible thanks to the PerContare Project, coordinated by Fondazione ASPHI onlus, with the support of Compagnia di San Paolo and the operative support of Fondazione per la Scuola of Compagnia di San Paolo of Torino.

constructivist-oriented instruction (to use the same terms as in questions 2 and 3). But what if during a whole year, or even worse a whole school cycle (5–6 years), the child – for a variety of reasons – does not participate in the classroom discourse during the mathematics hours? This can be the case, for example, if the classroom culture is heavily based on written language and the child has not overcome difficulties related to the use of this medium, a frequent condition in cases of dyslexia. The child will have wasted years of his/her life, or even worse, he/she will have developed aversion for the discourse he/she failed to become a part of. Perhaps the child's environment would have been more aware of his/her difficulties with written language, if the child had spent time in a special education classroom, offering a context in which participation was fostered in a more appropriate way, leading to experiences of participation and success in mathematics.

However, after many years of (induced or voluntary) exclusion from mathematical discourse, throughout which the child – now adolescent – has never actually done mathematics, is it still appropriate to place him/her in a 'regular' classroom involving constructivist-oriented instruction that heavily builds on notions our student has never constructed? He/she will almost definitely fail mathematics for good.

On the other hand, it is possible that with an individualised remedial intervention that takes into account the student's (well-known) difficulties, he/she will rapidly regain confidence and start participating in a mathematical discourse that uses different means for acquiring and producing information and that can be appreciated by the teacher and by all the other participants in the mathematical discourse, even those within the 'regular' classroom. Throughout my experience in helping students learn mathematics in different settings, I have witnessed a number of cases similar to the prototypical one just described.

In the example, I mentioned difficulties in using written language; however, there can be many other cognitive conditions, such as a difficulty to remember procedures or facts, difficulties in encapsulating processes, difficulties in logical reasoning and many others that lead to experiencing failure and eventually to exclusion from the mathematical discourse produced in 'regular' classrooms. I believe it is fundamental (for the teacher, clinician or other educator) to identify these difficulties and 'work around' them, helping the student become aware of them while addressing and overcoming whichever ones are possible. Of course this is no trivial task and each student is quite different!

Returning to the last question, it also seems to be the case that, at a given point in time, different students can have different characteristics. For example, taking a cognitive perspective, it seems possible to regroup existing hypotheses on MLD into a fourfold model that can be used for describing students' mathematical (cognitive) learning profiles (Karagiannakis et al. 2014). Studies based on this assumption are showing that the profiles of students with similar (or identical) low scores on mathematical achievement tests (also those used for diagnosing MLD) are in fact different (Karagiannakis and Baccaglini-Frank 2014; Karagiannakis et al. 2018). In other words, the studies are suggesting that failure to overcome difficulties in mathematical learning, at a cognitive level, cannot always be associated with a single deficit in a particular domain of the model, nor can it be considered the consequence of *one* particular finalised combination (that is the same for all students) of deficits. This supports the claim that looking for a cognitive characterisation of all students with low achievement specifically in mathematics is not necessarily a fruitful direction of research.

What we can (and should) ask is, ‘why do some children fail to overcome difficulties in mathematical learning that others do overcome?’ Reasons may include students’ cognitive characteristics, as a result of ‘innate’ inclinations that are shaped by immersion in society (as mentioned above), students’ mathematical learning history, affective components of both the students and their teachers, teachers’ choices about what mathematical content to present and the means they choose (or do not choose) to introduce it, the way MLD is viewed within school policies and teachers’ perspectives, implicit or explicit assumptions on ‘what’ or ‘how much’ MLD students can learn, etc.

I believe that research conducted by mathematics educators should address how to minimise failure in mathematics due to children’s individual specific learning characteristics, as early as possible – starting at least at the beginning of formal instruction (kindergarten or first grade in most countries). This is what we attempted to do in a 3-year project recently carried out in Italy (2011–2014). For this project, a team of mathematics educators and psychologists designed curricular material for mathematics, framed within the theories of semiotic mediation (Bartolini Bussi and Mariotti 2008) and embodied cognition (Gallese and Lakoff 2005), with the aim of providing all students (in first and second grade) with ‘hands-on’ (kinaesthetic-tactile) experiences that involve manipulation of physical artefacts to develop mathematical meanings (including procedures) from these and from consequent mathematical discussions.

For example, to help children learn what are known in English as the ‘multiplication tables’, children were introduced to the manipulation of rectangles cut out of squared paper (see Sect. 7.4.2 of Chap. 7). Children learned to cut and paste these rectangles together to figure out unknown products. The physical procedures were then carried out simply by drawing (in notebooks or on the blackboard), and eventually children started to use them with no further external support, as strategies of mental calculation (see the example in Sect. 7.4.2 of Chap. 7). A fundamental aspect of the mathematical activity stemming from activities such as the ones described is the sharing and discussing of strategies, during which all students were invited to (and did!) contribute.

In the episodes shown in part 4 of the video (Electronic Supplementary Material: Baccaglini-Frank 2017b), the teacher has asked the children to share strategies they used to figure out 8×6, showing their procedure on the blackboard. One student has decided to break the segment 8 into three parts (5, 2 and 1), which for him ‘make it easy’ because they are numbers he knows how to count by. He then counts up by 5s to obtain the first piece, mentally rotates the second piece and remembers that $6 + 6 = 12$ and recognises the last piece as 1×6. So he finally adds $30 + 12 + 6$. The student in general performs at an average-low level, but he was able to keep up with the class using the proposed activities and occasional extra practice at home.

Another student had decided to decompose 8 into 10 − 2 and describes her reasoning through 'ghost rectangles',[2] a terminology that very quickly catches on in the classroom. These are rectangles that appear to make the calculation easier, but then they need to be taken away. She uses ghost rectangles to think of 8 as a part of 10, to reach the product 10 × 6 (= 60), and then subtract off 2 × 6 (= 12). In the final mental calculation (60 – 12), she makes a mistake: at first, she forgets to take a second 10 from 60 and ends up with 58 instead of 48. Then, prompted by the teacher, she quickly corrects the mistake. Both students seem to be very much at ease when implicitly using the distributive property (it was not presented formally).

An important finding of the project was that working with the experimental materials through first and second grade significantly reduced the number of children who could be classified as MLD by third grade (Baccaglini-Frank and Scorza 2013; Baccaglini-Frank and Bartolini Bussi 2015; Baccaglini-Frank 2015). Moreover, the children exposed to the PerContare teaching-learning experience developed a variety of strategies for addressing different mathematical situations. In particular, with respect to calculation, for these children, the acquisition of numerical facts occurred with greater accuracy, variety of strategies and eventually speed. The 'cost' was a 3-month lag in fact and automatisation compared with the higher performing children in the control classes.

Insisting on the finding that persistent use of particular curricular materials can significantly reduce the number of children who tested positively for dyscalculia in third grade, we find an apparent contradiction with the literature claiming that dyscalculia is an innate deficit. Indeed, our sample of students seems to show that testing positively for dyscalculia can depend on initial primary school instruction, an extremely 'cultural' experience. Of course, one can solve the dilemma in a number of ways, for example by attacking the effectiveness of diagnostic test batteries (at least those used in Italy) or the diagnostic criteria more in general, or by speaking more loosely of MLD without giving a clear definition, which indeed, unsurprisingly, has not yet been agreed upon across groups of research (e.g. Mazzocco and Räsänen 2013).

This brings me back to my earlier plea: as educators, we should continue studying why fewer students fail in mathematics when they participate in particular types of early mathematical experiences. Let us call these good practices. I believe that particular effort should be put in developing good practices and studying their effect with different samples of children. At this point, when a set of good practices has been identified, we can ask whether there are students who still fail in mathematics and set up studies to explore why this is the case, then possibly use such knowledge to further ameliorate the practices or, in parallel, develop ad hoc remedial interventions. My personal belief is that it is unlikely that many students now classified as MLD will benefit more from individualised interventions than from whole-class learning situations where they make use of good practices which afford multiple

[2] This may also be seen as a *pivot sign* according to the Theory of Semiotic Mediation (Bartolini Bussi and Mariotti 2008) and was exploited as such by the teacher.

means of participation in the mathematical discourse and which are considerate of the learning inclinations of all students. But of course this is yet an open question.

16.3 Are MLD Linked to a Lack of Underlying Awareness of Mathematical Patterns and Relationships that Are More Linked to Spatial Ability than Development of Number?

Joanne Mulligan

To address some of the issues articulated in the introduction of this chapter, I will adopt an integrated perspective, in order to provide a more coherent view of the underlying cognitive bases of mathematical development and MLD based on an awareness of mathematical 'pattern and structure'. Rather than focusing on the domain of WNA, my research has focused on supporting the development of mathematical generalisation – through early identification of patterns and relationships and the development of interrelated spatial processes.

In recent years, mathematics education research has turned increasing attention to other research domains and interdisciplinary studies to explain and describe the wide variation in mathematical competence in the early years.

Studies of early mathematical competencies have largely emphasised children's numerical competencies, e.g. counting, subitising, representing number, numerical magnitudes and positioning on an empty number line (De Smedt et al. 2013; Fias and Fischer 2005). Another approach has focused on children's spontaneous focusing on number and quantitative relations (Hannula and Lehtinen 2005) found to be predictive of later achievement. Related studies highlight the critical role of perceptual subitising (McDonald 2015) and the spatial structuring of groups in arrays (Starkey and McCandliss 2014). Neurocognitive studies (Butterworth et al. 2011) also provide complementary evidence of the connection between the development of number and arithmetic and spatial processes. Number concepts depend on processes such as subitising (the rapid and accurate perception of small numerosities), comparison of numerical magnitudes, location on a number line, axis differentiation and symmetry (e.g. Dehaene 2009). Some interventions have incorporated some of these aspects for students with MLD and those performing below specified benchmarks, but with a focus on counting and arithmetic rather than underlying mathematical attributes.

The relative influence of the various components and how they interrelate in mathematical development, especially for students with MLD, remains unclear. Moreover the influence of one or more of these components on the individual's mathematical development may vary widely.

Recent developmental studies have indicated the positive impact of the early development of spatial skills on mathematical development (Verdine et al. 2014). Other studies in mathematics education highlight the sustained development of spatial reasoning skills from an early age – these are malleable and can be augmented over time but can weaken if not supported (Davis 2015). Spatial ability has also been linked to development of patterning and early algebraic skills (Clements and Sarama 2011; Papic et al. 2011) and the relationship with other concepts such as number and measurement (Mulligan et al. 2013; Mulligan et al. 2015). This raises critical questions about the need for differentiated teaching, assessment and intervention programmes for learners with poor spatial skills, not exclusive to those identified with MLD.

The Australian Pattern and Structure Mathematics Awareness Project (see also Sect. 7.3.3 of Chap. 7) is a suite of related studies with 4–8-year-olds focused on the assessment of mathematical structures for children representing a wide range of abilities including children with MLD (Mulligan et al. 2013). These studies have taken into account the complexity of various components of mathematical competency by adopting a more integrated view: what are common salient features of early mathematical development? Does the ability to recognise patterns and structures reflect innate ability or can it be developed? Why do some children with MLD lack this ability? What is the role of spatial reasoning?

The project, spanning over a decade, involved the development and validation of an interview-based assessment instrument the *Pattern and Structure Assessment – Early Mathematics* (*PASA*) (Mulligan et al. 2015) and the evaluation of the *Pattern and Structure Mathematics Awareness Program* (*PASMAP*) (Xin et al. in press). On the basis of students' PASA responses drawn from a range of studies, five levels of structural development were identified and described: prestructural, emergent, partial, structural and advanced structural (see Mulligan et al. 2013). Students with low AMPS operated generally at the prestructural or emergent level: for example, they had difficulty subitising larger sets, recognising a unit of repeat in simple patterns or utilising the structural features of arrays. They were most likely to represent idiosyncratic or superficial features in their models, drawings and explanations.

Based on early studies on patterning, counting, the numeration system and multiplicative thinking, the research focused on identifying and describing common characteristics, later coined as the construct *Awareness of Mathematical Pattern and Structure* (*AMPS*). AMPS has two interdependent components: one cognitive (knowledge of structure) and one meta-cognitive (a tendency to seek and analyse patterns). The AMPS construct involves the following structural components:

- *Sequences*: recognising a (linear) series of objects or symbols arranged in a definite order or using repetitions, i.e. repeating and growing patterns and number sequences.
- *Structured counting and grouping*: subitising, counting in groups, such as counting by 2s or 5s or on a numeral track with the equal grouping structure recognised as multiplicative.
- *Shape and alignment*: recognising structural features of two- and three-dimensional (2D and 3D) shapes and graphical representations, constructing units of measure, such as colinearity (horizontal and vertical coordination), similarity and congruence and such properties as equal sides, opposite and adjacent sides, right angles, horizontal and vertical parallel and perpendicular lines.

- *Equal spacing*: partitioning of lengths, other 2D or 3D spaces and objects into equal parts, such as constructing units of measure. It is fundamental to representing fractions, scales and intervals.
- *Partitioning*: division of lengths, other 2D or 3D spaces, objects and quantities, into unequal or equal parts, including fractions and units of measure.

Remedial or intervention initiatives in early numeracy for students with MLD typically focus on number and arithmetic without paying attention to patterning and spatial processes. Yet increasing evidence from a number of disciplines points to other components contributing to numerical competence. Our studies have traced the early development of number and other mathematical concepts to the development of AMPS. The PASMAP intervention studies, which examine the development of spatial aspects of patterns and spatial structures across mathematics concepts, indicated that such features as differentiation of foreground/background, alignment (collinear or axis), unitising and equal grouping, transformation and recognition of shape and equal areas are critical to mathematical development. It was found that these aspects can be improved through intervention for some children with MLD (Mulligan et al. 2013).

The design of the PASMAP intervention takes account of the assessment (PASA) which measures the child's level of AMPS; however, the programme can be utilised in conjunction with other assessments and intervention strategies. The PASMAP intervention programme was designed and trialled with students with wide-ranging abilities including those with MLD. PASMAP focused on five structures described above and is flexible in its implementation because the teacher can target specific mathematical structures with which a child with MLD has most difficulty. The pedagogy is designed to move students towards identifying similarities and differences, with a view to representing and abstracting core structural elements. The use of visual memory to record spatial representations is emphasised.

I propose that development of the various components of mathematical competence described in the literature earlier must have interrelated influences on mathematical development, but there is a common underlying thread. I am not suggesting that components are simply amalgamated into the construct that we call AMPS. Our empirical evidence supports the promotion of structural features rather than emphasis on counting and arithmetic. Fine-grained analysis of children's development over time suggests a complex network of these PASMAP components: the common denominator is the ability to see patterns and structural features that are essentially or initially spatial in nature. Hence the importance of focusing on children's development of structures such as grouping and partitioning, unitising, subitising, colinearity and benchmarking numerical magnitudes.

Conceptual relationships in mathematics depend on AMPS: spatial structuring and recognising patterns may provide the inextricable link between spatial and number development. Our recent studies have linked a measure of AMPS to standardised measures of early numeracy (Mulligan et al. 2015). Further analysis utilising network analysis (Woolcott et al. 2015) provides visual links between AMPS structures as a map of connectivity. However, the role of spatial reasoning in the development and use of AMPS is not fully understood; what our studies have described are domain-specific aspects of AMPS such as spatial structuring, partitioning and structuring linear, two-dimensional and three-dimensional space and relations between pattern and number.

Our future studies focus on evaluating the impact of various structures within the PASMAP intervention with children with MLD, moreover identifying critical differences for individuals in terms of AMPS over time. This may require a considered review of what constitutes critical components in early mathematical development and improved cross-disciplinary collaboration to inform research agendas and more effective pedagogical innovations.

16.4 It Is Time to Reveal What MLD Students Know, Rather than What They Do Not Know

Marja van den Heuvel-Panhuizen

Good teaching starts with getting to know what students know. Although this applies to all students, it is particularly true for students who have mathematical learning difficulties (MLD). Unfortunately, the problem with these students is that they usually have low scores on mathematics tests, which may automatically lead to the conclusion that they are ignorant, that they are unable to solve demanding mathematical problems and that it cannot be expected that they can come up with their own solution methods. Unmasking these and other prejudiced ideas is of vital importance for MLD students, because it may open new opportunities for teaching them mathematics. However, the burning question is how we can reveal what MLD students *do* know. In this contribution, I will discuss some research findings that give rise to reconsidering the presumed limitations of MLD students.

My research activities in this field started at the beginning of the 1980s when I got acquainted with an approach to mathematics education that proposes to start from students' informal and context-related mathematical knowledge, to offer students models to eventually reach more general and formal levels of understanding, to go beyond the sole focus on whole number operations, but also includes other mathematical domains, to give students an active role in the learning process, to elicit reflection, to stimulate classroom interaction about different solution strategies and to aim not only at learning facts and skills, but also at gaining insight.

As a former special education teacher, I was surprised that special educationalists rejected this kind of teaching for students in special education. According to these educationalists, it would be better to teach MLD students only a fixed solution strategy; otherwise, they would get confused. Also, it would be better to teach MLD students bare number problems, because problems situated in contexts would make problems too complex for them. Furthermore, building on students' own informal solution methods would be an illusion, because MLD students can hardly come up with solutions by themselves (see more about these assumptions of special educationalists in van den Heuvel-Panhuizen 1986, 1996).

To challenge these, in my view, incorrect assumptions, in 1990, I set up a study (van den Heuvel-Panhuizen 1996) in two special education schools for mildly mentally retarded students. The age of the students was between 10.5 and 13 years. The students' mathematical levels were far behind their peers and lay between grades 2 and 4 of regular primary school. The topic I chose for this study was ratio, which is generally considered beyond the reach of students in schools for mildly mentally retarded students and which accordingly was not taught to the students who participated in the study. In order to provide evidence that this was an underestimation of their mathematical ability, I administered a test on ratio including 16 ratio problems all referring to contextual situations familiar to the students and not including formal notations of ratio. Instead, as Freudenthal suggested to me when I designed the test, I made use of the visual roots of ratio. The results revealed that the MLD students, without having had instruction on ratio, were quite able to solve the problems. The percentage of correct answers for the problems ranged from 13% to 64%. Both the teachers of these students and the experts (two special education school inspectors and two special educationists) who were asked to predict the students' scores in many cases underestimated them. Moreover, the scrap papers added to the test sheets showed clear traces of self-invented strategies and notations.

The often-heard claim that students who are weak at mathematics can be better taught only one fixed standard strategy for every operation (e.g. see Gelderblom 2008, p. 36: 'Letting students who are weak in mathematics discover strategies by themselves is fatal. Lead them by the hand, tell them which strategies they have to use'; translation into English by author) induced me and my PhD student Marjolijn Peltenburg in 2010 to set up a study in which we investigated how special education students solve subtraction problems up to 100. The standard strategy that MLD students are taught for this type of problems is the take-away strategy. On purpose, we also included in the subtraction test for this study problems that might elicit an adding-on strategy (e.g. bare number problems such as 62 − 58 and context problems with an adding-on context). What we found was that the MLD students, without being taught, made spontaneous use of the adding-on strategy. Moreover, they were rather flexible in what strategy they applied, and they were quite successful when applying the adding-on strategy (Peltenburg et al. 2012).

Besides offering MLD students assessment problems in which they could show their competence on topics that belong to the regular mathematics curriculum, we also did further research on a topic that is far beyond what is taught in special primary school education and even is lacking in regular primary school. In this research we investigated what happened when MLD students were asked to solve a number of combinatorics problems. Here we found that the MLD students in our study were equally successful in solving the combinatorics problems as their comparable peers in regular education, who were younger but at the same level of understanding number and operations. Moreover, on average the MLD students equally often applied a systematic strategy to find all possible combinations as the students in regular education, and in both school types, a significant increase in the use of systematic strategies could be observed (Peltenburg et al. 2013; Peltenburg 2012, Chap. 6).

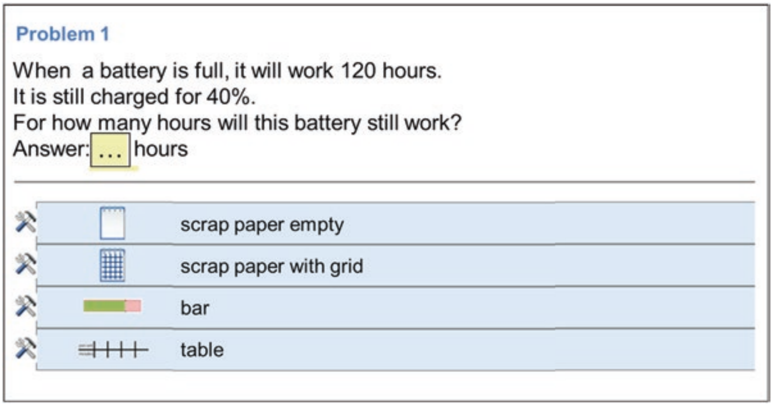

Fig. 16.1 Percentage problem in the DAE with optional auxiliary tools

Another avenue in our search for possibilities to make the hidden mathematical potential of MLD students visible is the use of a technology-enhanced assessment. For this we put a series of subtraction problems from a standardised test into an ICT environment and extended them with optional auxiliary tools. In one study, we used a digital interactive 100-board on which the students could represent the problems to be solved by dragging counters. In the other study, the optional auxiliary tool consisted of a digital interactive number line. Both studies showed that the proportions of correct answers were higher in the ICT-based test than on the standardised test (Peltenburg et al. 2010). This result appears rather obvious, but for teachers a test that not only tells them which students got which problems correct, but also which students made use of the auxiliary tools and how they used them, contains very valuable indications for further instruction. In fact, in this way the zone of proximal development of the students is opened. Moreover, we found that the MLD students were quite aware of whether they needed the help of the auxiliary tools. Students who made the most mistakes in the later administered standardised test more often chose to use an auxiliary tool in the earlier administered ICT-based test.

As a result of these positive experiences with optional auxiliary tools, this approach to assessment is now being further explored in the EU-funded FaSMEd project, which aims to research the use of technology in formative assessment classroom practices in ways that allow teachers to respond to the emerging needs of low-achieving students. The Dutch team of this project developed the Digital Assessment Environment (DAE) for mathematics education in the upper grades of primary school. Figure 16.1 shows an item on percentages with the optional auxiliary tools that can be chosen to solve this problem.

16.5 Conceptual Model-Based Problem-Solving: An Integration of Constructivist Mathematics Pedagogy and Explicit Strategy Instruction

Yan Ping Xin

The question whether students with learning disabilities should be educated in the inclusive classroom or in a segregated instructional environment has always been a hot topic. Here, I use the term 'students with learning disabilities or difficulties in mathematics' (LDM) to include all students whose mathematics performance is ranked below the 35th percentile (Bryant et al. 2011), so not necessarily only students with a biologically based disorder. With this broad definition in mind, a more pertinent question would be: 'Do these children need a special kind of intervention or do they profit most from the same kind of instruction as children without LDM?' In particular, 'is conceptually-based and constructivist-oriented mathematics instruction also suitable for children with learning disabilities?' For the most part, it depends on (a) how we support these students with instructional strategies that address their needs and (b) how much support or scaffolding we provide for these students so that they are able to make sense of the mathematical concept or relations or, from the instructional point of view, whether we can make the mathematical discourse or reasoning process explicit to the students so they can grasp the concept or knowledge. To this end, regardless of the placement, it is more important to consider whether the instructional strategies we employ will provide the needed support or scaffolding that will allow these students to have meaningful access to mathematics.

As the outcome of a collaborative piece of work that integrates research-based practices from math education and special education, Xin, Tzur and Si (2008), with the project team, have developed an intelligent tutor, PGBM-COMPS © (Xin, Tzur and Si 2017), to support the learning of multiplicative problem solving for students with LDM. The intelligent tutor draws on three research-based frameworks: a constructivist view of learning from mathematics education, data (or statistical) learning from computer science, and conceptual model-based problem solving (COMPS) (Xin 2012), from special education, that generalises word problem underlying structures. Rooted in a constructivist perspective on learning, we focused on how a student-adaptive teaching approach (Steffe 1990), which tailors goals and activities for students' learning to their available conceptions, can foster advances in multiplicative reasoning. This approach is not based on a deficit view of students with learning disabilities; rather, it focuses on and begins from what they do know and uses task-based activities to foster transformation into advanced, more powerful ways of knowing.

The PGBM-COMPS tutor is made of two parts: (a) 'Please Go Bring Me…' (PGBM) turn-taking games designed to nurture a learner's *construction* of fundamental ideas in multiplicative reasoning (Tzur et al. 2013) and (b) COMPS (Xin 2012) that emphasises understanding and representation of word problem structures in mathematical model equations. In particular, the PGBM turn-taking games were designed to nurture a learner's *construction* of fundamental ideas such as 'number as a composite unit'. A basic version of the PGBM platform game involves sending a student to a box with Unifix Cubes to produce and bring back a tower made of a few cubes. After taking two-to-nine ‘trips’ for bringing same-sized towers, students are asked how many towers (i.e. composite units, CU) they brought, how many cubes each tower has (i.e. unit rate, UR) and how many cubes (1s) there are in all. The PGBM game was devised to promote learners’ anticipated creation of and differentiation among 1s and CUs (Tzur et al. 2013). These two anticipations are crucial if the learner is to construct the mental operation of multiplicative double counting, which is fundamental to multiplicative reasoning. Multiplicative double counting integrates two counting sequences in a multiplicative problem situation (e.g. ‘Please bring me a tower with 6 cubes in each…. If you brought me 5 such towers; how many cubes in all?’): one sequence that quantifies how many CUs (i.e. towers) were produced and one sequence that monitors the corresponding accumulation of 1s (i.e. total # of cubes) contained within those CUs (i.e. towers). Double counting is considered to be ‘an advance over the more basic direct representation because it requires more abstract processing’ (Kouba 1989, p. 152).

A variety of activities following a PGBM format were designed to promote students’ construction of basic multiplicative concepts on the basis of continuous assessment of their existing knowledge and experiences. The learner will progress from a low to high level of tasks along the dimensions of (a) numerical numbers (e.g. 2, 5 or 10 - level I; 3 or 4 - level II and 6, 7, 8 or 9 - level III) involved in the problem and (b) cognitive demands of the task (i.e. operating with visible objects or invisible/covered objects with mental system).

On the other hand, COMPS generalises the understanding of multiplicative reasoning to the level of mathematical models. At this stage, students no longer rely on concrete models (such as cubes and towers) or drawing pictures or tally marks; the mathematical models directly drive the solution plan. The COMPS programme emphasises (a) the connection between the PGBM games (in the contexts of cubes and towers for instance) and the symbolic mathematical model equations, (b) students’ representation of various multiplicative problem situations in the mathematical model equations and (c) development of the solution plan that is directly driven by the model equations. Figure 16.2 presents two sample screenshots from the PGBM-COMPS programme. The upper panel shows how the programme engages students in making the connection between the concrete modelling (cubes and towers) and the mathematical expression; the lower panel shows how the problem should be represented in the COMPS model to find a solution.

To evaluate the effect of the PGBM-COMPS © intelligent tutor, Xin et al. (2017) compared the effectiveness of the PGBM-COMPS programme with school teacher-delivered instruction (TDI) on enhancing the multiplicative reasoning and problem-solving skills of students with LDM. Results indicated that the improvement rate of the PGBM-COMPS group was much greater than that of the TDI group (effect size [ES] = 1.99 on researcher-developed multiplicative reasoning tasks; ES = 2.26 on a range of multiplication and division word problem-solving tasks involving large numbers). In addition, the group difference was shown on a commercial/published standardised test, the Stanford Achievement Test (SAT,

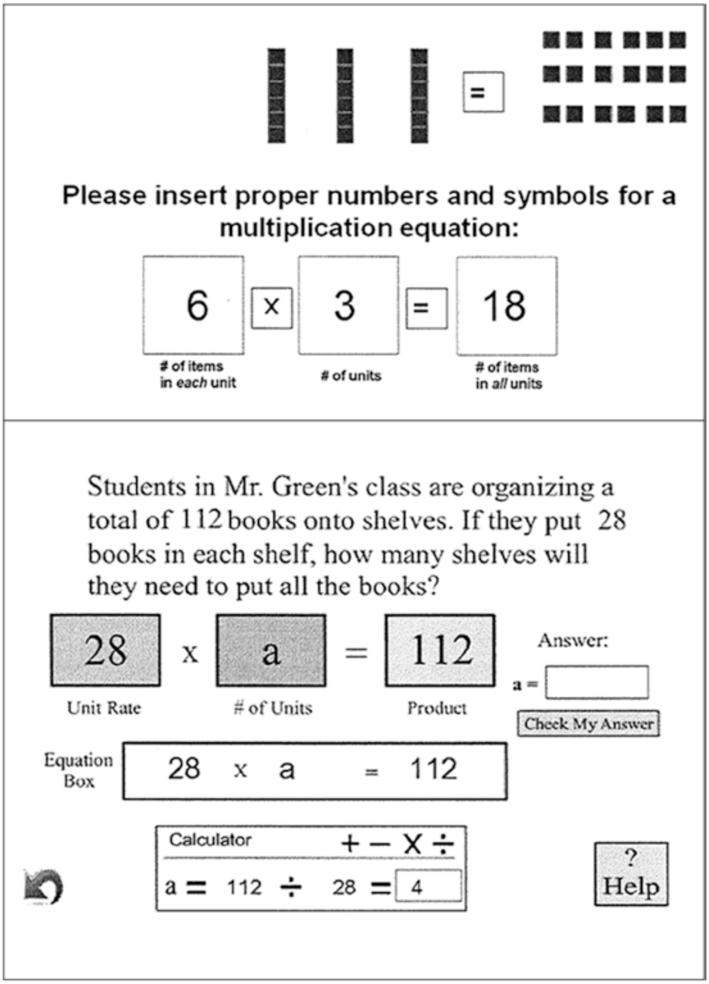

Fig. 16.2 Sample screenshots of the PGBM-COMPS intelligent tutor system (Xin, Tzur and Si 2017)

Harcourt Assessment Inc. 2004): *Mathematics Problem Solving* subtest, favouring the COMPS group (ES = 1.23).

Given that the Common Core State Standards for Mathematics (National Council of Teachers of Mathematics 2012) demand much deeper content knowledge from teachers of mathematics, the preliminary findings of the above study are encouraging. The PGBM-COMPS intelligent tutor, which integrates the best practices from general mathematics education and special education, seems to yield better outcomes in enhancing participating students' multiplicative problem solving. Through the integration of heuristic instruction (that facilitates concept construction) and the explicit model-based problem-solving instruction, it seems that the PGBM-COMPS programmes have promoted generalised problem-solving skills of students with LDM.

From the foregoing, here comes my answer to the question "whether concetually based and constructivist-oriented mathematics instruction also suitable for children with learning disabilities? With appropriate scaffolding and support, students with LDM are able to engage in conceptually based and constructivist-oriented mathematics instruction. The promising outcome of the PGBM-COMPS intervention programme (Xin, Tzur and Si 2017) is just one example.

16.6 Discussion

Brian Butterworth

If you want to get ahead, get a theory.

Verschaffel raised two fundamental issues in his introductory remarks to the panel. First, he asks what constitutes the 'mathematics' that MLD should address. Here I would like to start with a very simple approach. What constitutes 'a billable ICD-10-CM code that can be used to indicate a diagnosis for reimbursement purposes'? That is, what diagnosis will ensure that a child is entitled to special help for his or her mathematical difficulties? I take ICD 10 (The World Health Authorities list of diseases) because it is the clearest and most specific of the widely used classifications. In Sect. F81.2, the term used is a 'specific disorder of arithmetical skills'. This 'involves a specific impairment in arithmetical skills that is not solely explicable on the basis of general mental retardation or of inadequate schooling. The deficit concerns mastery of basic computational skills of addition, subtraction, multiplication, and division rather than of the more abstract mathematical skills involved in algebra, trigonometry, geometry, or calculus'. So, in this context, the answer to Verschaffel's question is simple: *arithmetic*. However, there are problems.

Notice that the ICD definition excludes an impairment in arithmetical skills that is solely explicable on the basis of general mental retardation. That is, the child cannot be both stupid and have MLD. Moreover, it excludes, in a later paragraph, 'arithmetical difficulties associated with a reading or spelling disorder'. Thus, the child cannot be both dyscalculic and dyslexic.

Spatial Abilities

ICD 10 does not mention spatial abilities, though it is known that, especially in the early years, there is a close link between them and arithmetical development (Rourke 1989). However, how this link operates is far from clear. Mulligan focuses on a specific set of spatial competences. In particular, she argues that a set of these underpins makes the conceptual relationships critical to arithmetical understanding. Specially designed interventions for weaknesses in this set of competences can make a big difference to the development of arithmetic.

'Mental Retardation'

Mental retardation does not prevent high levels of mathematical skill. We know from the study of savants, with very low measured IQ or with other indicators of limited cognitive ability, that they can be superb calculators (Butterworth 2006). We also know that IQ measures are poor predictors of mathematical competence, such that even individuals with very high measured IQ can be dyscalculic (Butterworth et al. 2011). In an original approach to this issue, van den Heuvel-Panhuizen reported studies she had carried out on the mathematical abilities in schools for children with special educational needs. Now these children will have low scores on standard tests and would be classified as MLD but would be excluded from a 'billable code' because of their measured IQ. Now it may well be that these children can be drilled to perform moderately well on arithmetical problems, but the question addressed is much more interesting: do they have the conceptual basis and cognitive ability to develop their own valid strategies for calculation?

van den Heuvel-Panhuizen has a clear answer to this question. 'What we found was that the MLD students, without being taught, made spontaneous use of the adding-on strategy. Moreover, they were rather flexible in what strategy they applied, and they were quite successful when applying the adding-on strategy.' They were also 'equally successful in solving the combinatorics problems'.

Verschaffel's second issue is what is the appropriate intervention for children with special needs, and this raises the ICD 10 exclusion criterion – 'inadequate schooling'. Now ICD 10 does not define this term, so it is not possible to determine whether the child is classified as MLD because of poor teaching. Baccaglini-Frank notes that this raises an important problem for the definition of MLD. She writes, 'Insisting on the finding that persistent use of particular curricular materials can significantly reduce the number of children who are positive to diagnostic tests for *dyscalculia* in third grade, we find an apparent contradiction with literature claiming that dyscalculia is an *innate deficit'*. I will return to this point below.

She also notes, quite properly, that 'if the classroom culture is heavily based on written language and the child has not overcome difficulties related to the use of this medium', then this could cause the child to fall behind in maths. The child, she says, would be better served in a 'special education classroom'. This would be another example of inadequate schooling.

Xin has developed an intelligent tutoring system designed to help all students with 'learning disabilities or difficulties in mathematics', by which she means students below the 35th percentile. For her, the nature of those difficulties, or their causes, appears not to be relevant. She argues that a conceptually based and constructivist-oriented mathematics instruction, developed for more typical students, is also suitable for children with learning disabilities. Her findings suggest that the intelligent tutoring system is more effective than teacher-delivered instruction.

A Theory-based Approach

At the root of the confusions about the criteria for MLD and about the appropriate intervention is the lack of theoretical perspective, and this is critical for understanding why a child fails to reach an expected level in maths.

Of course, what the expected level is will depend on social, economic and, importantly, political factors. One example is whether the educational authority – usually a government agency – recognises MLD as a ‘billable’ category. It may fail to do so out of ignorance, since mathematical competence is a proxy for intelligence or out of indolence, for example, if there is no parent pressure group to prod it into action. In the UK, and in many other countries, dyslexia is recognised precisely because there exist organisations that insist on its recognition. The authorities may not recognise MLD because it could entail a commitment to provide support for those assessed as MLD.

Without a theory, one is left with a criterion that could be set for economic or political reasons or could simply be arbitrary: 35th percentile, for example, or 1, 1.5, 2, 2.5 or 3 SDs below the population mean on a standardised test of arithmetic. None of these criteria tells you what the learner needs. The problem is compounded when one considers different populations. Consider an international comparison, for example, the PISA 2012 study. The proportion of children below level 2 in top ten countries was around 10%, but in the worst performing countries, it was between 60% and 75%. So what would count as MLD in Macao will be very different from what would count in Indonesia in terms of what the learner can and cannot do.

Here is the question that should be addressed: *why* is this child failing to understand what his or her classmates can understand? This is a theoretical question. For perhaps 5% of learners, the answer is that there is a deficit in very basic numerical concepts. That is, they will do poorly on tests that depend very little on the appropriateness of schooling or on social and economic status and even home background. These learners do poorly on tests of the enumeration of small sets of objects, typically displays of dots. They will be slower and less accurate than their peers, and this is a stable measure of individual difference and is a reliable predictor of the ease or difficulty of acquiring arithmetical competence (Reeve et al. 2012). Other very simple tests that rely little on education arrive at similar conclusions (Piazza et al. 2010). These are tests of a crucial component in the learner’s ‘starter kit’ for acquiring basic numerical competence that I have called ‘numerosity processing’ and means the ability to estimate the number of objects in a set. Poor performance on tests of this ability points to a congenital *core deficit* in numerosity processing. Not only is performance on these tests independent of schooling, it is independent of intelligence, of working memory and of literacy (Landerl et al. 2004). We call this special need ‘dyscalculia’.

The identification of a deficit in this core capacity has implications for intervention: more of the same, more slowly and with more repetition, does not work. As with dyslexia, specially designed interventions are needed, preferably using concrete materials, and adaptive digital games with virtual concrete materials, for much longer than would be needed with typically developing learners (Butterworth et al. 2011).

This approach also sheds light on the relationship between dyscalculia and other neurodevelopmental disorders. Dyslexia on this account cannot be a cause of dyscalculia, because it is due to a quite distinct core deficit, in most cases a phonological deficit (Butterworth and Kovas 2013; Landerl et al. 2009). This means that we must reject the ICD 10 exclusion criterion of reading disability and test for both core deficits.

Our approach also means that it is possible to have a core numerical deficit despite being highly intelligent, or indeed having low cognitive abilities. This is not to say that MLD may not have other causes, including inadequate schooling (an international problem), prematurity, poor diet and a difficult home environment (Benavides-Varela et al. 2016). In these cases, a different approach to intervention will be needed. In mathematical education, as in so many other things, one size does not fit all. Measure the customer first and then find the garment that fits best.

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