On the Bardina's model in the whole space

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Abstract

We consider the Bardina's model for turbulent incompressible flows in the whole space with a cut-off frequency of order $\alpha^{-1} > 0$. We show that for any $\alpha > 0$ fixed, the model has a unique regular solution defined for all $t \in [0, \infty[$.

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1 Introduction

The purpose of this paper is the study of the Large Eddy Simulation (LES) Bardina's model in the whole space

$$
\begin{cases}
\partial_t \mathbf{u} + \operatorname{div}(\overline{\mathbf{u} \otimes \mathbf{u}}) - \nu \Delta \mathbf{u} + \nabla p = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
\operatorname{div} \mathbf{u} = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\
\mathbf{u}_{t=0} = \overline{\mathbf{u}_0} & \text{in } \mathbb{R}^3.
\end{cases}
\tag{1.1}
$$

In this system, $\mathbf{u} = \mathbf{u}(t, \mathbf{x}) = (u_1(t, \mathbf{x}), u_2(t, \mathbf{x}), u_3(t, \mathbf{x}))$ denotes the filtered velocity of a given turbulent flow, $p = p(t, \mathbf{x})$ the filtered pressure, $t \geq 0$, $\mathbf{x} \in \mathbb{R}^3$, and $\nu > 0$ is the kinematic viscosity. The filtering is obtained by bar-operator, which is given by solving, for a given $\alpha > 0$, the Helmholtz equation (1.2)

$$
-\alpha^2 \Delta \overline{\psi} + \overline{\psi} = \psi \quad \text{in } \mathbb{R}^3. \tag{1.2}
$$

This model is called the Navier-Stokes-Bardina-α model (NSEB-α in the following). Introduced by Bardina, Ferziger, and Reynolds [2] for weather forecasts, it is used in many practical applications (see for instance Adams and Stolz [1] and Chow, De Wekker, and Snyder [6]). It was studied mathematically speaking by Ilyin, Lunasin, and Titi [12] and by Layton with one of the present authors [13, 14] in the space-periodic case. In the periodic setting, for all $T > 0$ it is proved the existence of a unique weak solution $(\mathbf{u}, p)$, which satisfies

$$
\mathbf{u} \in L^\infty([0, T]; H^1_{per}) \cap L^2([0, T]; H^2_{per}).
$$

In this paper we consider the Cauchy problem in the whole space and new difficulties arise in the case of $\mathbb{R}^3$. To handle the problem, we revisit some extremely classical tools

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of potential theory and explicit representation formulas as developed by Leray [15] in his 1934 paper, where the path for modern research using Sobolev spaces for the Navier-Stokes equations has been paved. We also observe that contrary to Leray's approach, here the smoothing is not made with a mollification, but with the solution of the differential problem (1.2) and this implies weaker estimates on the smoothed fields. Moreover, here the whole quantity $\mathbf{u}\otimes\mathbf{u}$ is smoothed/filtered, while in Leray's model the filtering is applied only to the convective field. The special properties of the Helmholtz filter (1.2), which are very relevant in practical computations, are proved in Section 2.

Throughout the paper, the initial data $\mathbf{u}_0 \in L^2(\mathbb{R}^3)^3$ is given, and satisfies $\operatorname{div}\mathbf{u}_0 = 0$, so that $\overline{\mathbf{u}_0} \in H^2(\mathbb{R}^3)^3$ and $\operatorname{div}\overline{\mathbf{u}_0} = 0$. Our goal is to build a unique regular solution, global in time, to the NSEB-α model, for a given parameter $\alpha > 0$.

As it was firstly observed in [13], the key feature of the NSEB-α is the following energy balance (equality),

$$(1.3)\quad \begin{cases} \dfrac{1}{2}\left(\alpha^2\displaystyle\int_{\mathbf{R}^3}|\nabla\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbf{R}^3}|\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x}\right) + \nu\alpha^2\int_0^t\int_{\mathbf{R}^3}|\Delta\mathbf{u}(t',\mathbf{x})|^2\, d\mathbf{x}\, dt' + \\ \nu\displaystyle\int_0^t\int_{\mathbf{R}^3}|\nabla\mathbf{u}(t',\mathbf{x})|^2\, d\mathbf{x}dt' = \frac{1}{2}\left(\alpha^2\int_{\mathbf{R}^3}|\nabla\overline{\mathbf{u}_0}(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbf{R}^3}|\overline{\mathbf{u}_0}(\mathbf{x})|^2 d\mathbf{x}\right), \end{cases}$$

which is satisfied by any solution $(\mathbf{u}, p)$, belonging to $(L^2(\mathbb{R}^3))^4$ with its derivatives. This energy balance is the basic building block of our construction. Before stating our main result, let us precisely state what we mean by "regular solution". The following definition, which is largely inspired by that of regular solution in [15], will turn out to be well-suited to the NSEB-α model.

Definition 1.1. *We say that* $(\mathbf{u}, p)$ *is a regular solution of the NSEB-*α (1.1) *over the time interval* $[0, T[$ *(eventually* $T = +\infty$*) if*

i) $\mathbf{u}, \partial_t\mathbf{u}, \nabla\mathbf{u}, D^2\mathbf{u}, p, \nabla p$ *are well-defined and continuous for* $(t, \mathbf{x}) \in [0, T[\times\mathbb{R}^3$*, and they satisfy the relations (1.1.i) and (1.1.ii) in* $\mathbb{R}^3$ *for all* $t \in]0, T[$*;*

ii) $\forall\, \mathbf{x} \in \mathbb{R}^3, \quad \mathbf{u}(0, \mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x})$*;*

iii) $\forall\, \tau < T, \quad \mathbf{u} \in C([0, \tau]; H^2(\mathbb{R}^3)^3)$.

It is worth noting that when $(\mathbf{u}, p)$ is a regular solution, then the pressure p is solution of the Poisson equation

$$(1.4)\qquad \Delta p = -\operatorname{div}[\operatorname{div}(\overline{\mathbf{u}\otimes\mathbf{u}})],$$

at any given time $t \in]0, T[$. When $T < \infty$, we say that $(\mathbf{u}, p)$ becomes "singular when $t \to T$" if it is a regular solution over $[0, T[$ and

$$\lim_{\substack{t\to T\\ t<T}} \|\mathbf{u}(t,\cdot)\|_{2,2} = \infty,$$

where $\|\mathbf{u}(t,\cdot)\|_{2,2}$ denotes the $H^2(\mathbb{R}^3)$ norm of $\mathbf{x} \mapsto \mathbf{u}(t, \mathbf{x})$ at a given time t.

The main result of this paper is the following.

Theorem 1.1. *The NSEB-*α *model (1.1) has a unique regular solution* $(\mathbf{u}, p)$ *defined for any* $t \in [0, \infty[$*, which satisfies the energy balance (1.3), for all* $t > 0$*, such that*

$$\partial_t\mathbf{u} \in C([0, \infty[; L^2(\mathbb{R}^3)^3), \quad \textit{and} \quad p \in C([0, \infty[; H^4(\mathbb{R}^3)).$$

Moreover, $\forall\, \tau > 0$, $\forall\, m \geq 0$, $\quad (\mathbf{u}, p) \in C([\tau, \infty[; H^m(\mathbb{R}^3)^3 \times H^m(\mathbb{R}^3))$.

In particular, regular solutions of the NSEB-α model do not become singular in a finite time, and we found that the model exerts a strong regularizing effect on the pressure.

The convergence, as $\alpha \to 0^+$, of solutions to the NSEB-α model to weak solutions of the Navier-Stokes equations will studied in a forthcoming paper [3].

2 The Helmholtz filter

This section is devoted to the study of the Helmholtz equation (1.2) and its associated Green's kernel. We will:

i) set some notations;

ii) introduce the Green's kernel and deduce a few basic inequalities from usual results about convolutions;

iii) draw the link between the integral representation and the variational solutions of (1.2).

2.1 General setting

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$ be a multi-index and let $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$, then we denote as usual

$$D^\alpha \mathbf{u} = (D^\alpha u_1, D^\alpha u_2, D^\alpha u_3), \quad \text{with} \quad D^\alpha u_i = \frac{\partial^{|\alpha|} u_i}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}}.$$

For any given $m \in \mathbb{N}$, when we write $D^m \mathbf{u}$ we assume that $D^\alpha \mathbf{u}$ is well defined whatever α is a multi-index such that $|\alpha| = m$, and in practical calculations we will use the following notation

$$|D^m \mathbf{u}| = \sup_{|\alpha|=m} |D^\alpha \mathbf{u}|.$$

space $W^{m,p}(\mathbb{R}^3)$ is equipped with the norm

$$\|w\|_{m,p} = \sum_{j=0}^{m} \|D^j w\|_{L^p(\mathbb{R}^3)},$$

as usual $H^m(\mathbb{R}^3) = W^{m,2}(\mathbb{R}^3)$. Throughout the paper, we will use the following consequence of the Green's formula in $\mathbb{R}^3$ (see [15, Eq. (1.11)]):

$$\forall\, u \in W^{1,p}(\mathbb{R}^3), \forall\, v \in W^{1,p'}(\mathbb{R}^3), \qquad \int_{\mathbb{R}^3} \left[u(\mathbf{x}) \frac{\partial v}{\partial x_i}(\mathbf{x}) + \frac{\partial u}{\partial x_i}(\mathbf{x}) v(\mathbf{x}) \right] d\mathbf{x} = 0, \tag{2.1}$$

for $1 < p < \infty$, where as usual $1/p + 1/p' = 1$.

2.2 The Helmholtz kernel

In order to prove the main estimates it turns useful to use the integral representation formula for solutions of (1.2). Let H_α denotes, for $\alpha > 0$ the kernel given by

$$H_\alpha(\mathbf{x}) = \frac{1}{4\pi\alpha^2} \frac{e^{-\frac{|\mathbf{x}|}{\alpha}}}{|\mathbf{x}|} \qquad \text{defined for } \mathbf{x} \neq \mathbf{0}. \tag{2.2}$$

We notice that $\|H_\alpha\|_{0,1} = 1$ and

$$\forall\, \mathbf{x} \neq 0, \qquad -\alpha^2 \Delta H_\alpha(\mathbf{x}) + H_\alpha(\mathbf{x}) = 0,$$

which leads to the following result

Lemma 2.1. *Let $\varphi \in C_c^\infty(\mathbb{R}^3)$. Then*

$$\forall \mathbf{x} \in \mathbb{R}^3, \qquad \int_{\mathbb{R}^3} (-\alpha^2 \Delta\varphi(\mathbf{y}) + \varphi(\mathbf{y})) H(\mathbf{x}-\mathbf{y})\, d\mathbf{y} = \varphi(\mathbf{x}). \tag{2.3}$$

The classical proof is very close to that of the Green's representation formula in Gilbarg and Trudinger [11, Ch. 2], so we skip the details. Thereby, we get

$$\forall\, \alpha > 0 \qquad -\alpha^2 \Delta H_\alpha + H_\alpha = \delta_0,$$

in the sense of distributions over $\mathbb{R}^3$, where δ_0 denotes the Dirac (delta) measure centered at the origin.
Let ψ be any measurable function and let $\overline{\psi}$ denote the filtered function:

$$\overline{\psi}(\mathbf{x}) = H_\alpha \star \psi(\mathbf{x}) = \int_{\mathbb{R}^3} H_\alpha(\mathbf{x}-\mathbf{y})\psi(\mathbf{y})\, d\mathbf{y}, \tag{2.4}$$

as long as the integral converges. We deduce from the Cauchy-Schwarz and Young inequalities the series of formal inequalities:

$$\|\overline{\psi}\|_{0,p} \le \|\psi\|_{0,p}, \quad \text{for all } 1 \le p \le \infty, \tag{2.5}$$

$$\|\nabla\overline{\psi}\|_{0,2} \le \frac{2}{\alpha}\|\psi\|_{0,2}, \tag{2.6}$$

$$\|\overline{\psi}\|_{0,\infty} \le \frac{1}{\sqrt{8\pi}\alpha^{\frac{3}{2}}}\|\psi\|_{0,2}, \tag{2.7}$$

$$\|\overline{\psi}\|_{0,2} \le \frac{1}{\sqrt{8\pi}\alpha^{\frac{3}{2}}}\|\psi\|_{0,1}. \tag{2.8}$$

Remark 2.1. *We notice that $H_\alpha \in L^q(\mathbb{R}^3)$, for all $q < 3$, and it is continuous for $\mathbf{x} \neq 0$. Then, if $\psi \in L^2(\mathbb{R}^3)$, we have that $\overline{\psi} \in C^0(\mathbb{R}^3)$, but it does not necessary belong to $C^1(\mathbb{R}^3)$. On the other hand, when $\psi \in L^p(\mathbb{R}^3)$ for some $p > 3$, then as $\nabla H_\alpha \in L^r(\mathbb{R}^3)$, $r < 3/2$ and is continuous for $\mathbf{x} \neq 0$, we obtain that $\overline{\psi} \in C^1(\mathbb{R}^3)$.*
In any case, assuming just that $\psi \in L^q(\mathbb{R}^3)$ it is not possible to prove that $\overline{\psi} \in C^2(\mathbb{R}^3)$ since $D^2 H_\alpha \notin L^p(\mathbb{R}^3)$, whatever the value of $p > 1$ is taken. However, when $\psi \in H^1(\mathbb{R}^3)$, then we have $\overline{\psi} \in C_b^1(\mathbb{R}^3) \cap H^2(\mathbb{R}^3)$.

2.3 Variational formulation and related properties

With the estimates (2.5), the relation (2.3) combined with the rotational symmetry of the kernel H_α and the Green's formula (2.1), we obtain:

Lemma 2.2. *Let $\psi \in L^2(\mathbb{R}^3)$. Then $\overline{\psi} \in H^1(\mathbb{R}^{3)}$ is the unique weak solution to the Helmholtz equation (1.2), in the following sense:*

$$\forall\, \varphi \in H^1(\mathbb{R}^3), \quad \alpha^2 \int_{\mathbb{R}^3} \nabla\overline{\psi}(\mathbf{x}) \cdot \nabla\varphi(\mathbf{x})\, d\mathbf{x} + \int_{\mathbb{R}^3} \overline{\psi}(\mathbf{x})\, \varphi(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^3} \psi(\mathbf{x})\, \varphi(\mathbf{x})\, d\mathbf{x}. \tag{2.9}$$

Proof. The existence and uniqueness of a solution to the variational problem (2.9) is a consequence of the Lax-Milgram Theorem in $H^1(\mathbb{R}^3)$. We must check that this coincides with the convolution formula (2.4). Let $\varphi \in C_c^\infty(\mathbb{R}^3)$; by Fubini's Theorem and the

Green's formula, we have

$$
\begin{aligned}
&\alpha^2 \int_{\mathbf{R}^3} \nabla\overline{\psi}(\mathbf{x}) \cdot \nabla\varphi(\mathbf{x})\, d\mathbf{x} + \int_{\mathbf{R}^3} \overline{\psi}(\mathbf{x})\, \varphi(\mathbf{x})\, d\mathbf{x} \\
&= \int_{\mathbf{R}^3} \int_{\mathbf{R}^3} [\alpha^2 \nabla_{\mathbf{x}} H_\alpha(\mathbf{x}-\mathbf{y}) \cdot \nabla\varphi(\mathbf{x}) + H_\alpha(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})]\psi(\mathbf{y})\, d\mathbf{y} d\mathbf{x} \\
&= \int_{\mathbf{R}^3} \psi(\mathbf{y}) d\mathbf{y} \left(\int_{\mathbf{R}^3} [\alpha^2 \nabla_{\mathbf{x}} H_\alpha(\mathbf{x}-\mathbf{y}) \cdot \nabla\varphi(\mathbf{x}) + H_\alpha(\mathbf{x}-\mathbf{y})\varphi(\mathbf{x})]\, d\mathbf{x} \right) \\
&= \int_{\mathbf{R}^3} \psi(\mathbf{y}) d\mathbf{y} \left(\int_{\mathbf{R}^3} [-\alpha^2 \Delta\varphi(\mathbf{x}) + \varphi(\mathbf{x})] H_\alpha(\mathbf{y}-\mathbf{x})\, d\mathbf{x} \right) = \int_{\mathbf{R}^3} \psi(\mathbf{y})\varphi(\mathbf{y})\, d\mathbf{y},
\end{aligned}
$$

where we have used (2.3) and the fact that $H_\alpha(\mathbf{x}) = H_\alpha(-\mathbf{x})$. We conclude by a density argument[1]. $\square$

The following corollary is straightforward.

Corollary 2.1. *The bar operator is self-adjoint with respect to the* L^2*-scalar product* $(.\, , .)$, *which means:*

$$\forall\, \psi, \varphi \in L^2(\mathbb{R}^3), \qquad (\overline{\psi}, \varphi) = (\psi, \overline{\varphi}).$$

The convergence of $\overline{\psi}$ to ψ as $\alpha \to 0^+$ is the aim of the next lemma.

Lemma 2.3. *Let* $1 \le p \le \infty$ *and let* $\psi \in W^{2,p}(\mathbb{R}^3)$*. Then*

$$\|\overline{\psi} - \psi\|_{0,p} \le \alpha^2 \|\Delta\psi\|_{0,p}. \tag{2.10}$$

Proof. It is enough to prove the estimate (2.10) when $\psi \in C_c^\infty(\mathbb{R}^3)$. In this case, we deduce from (2.2) and (2.4) that $\overline{\psi}, \nabla\overline{\psi} = \mathcal{O}(e^{-\frac{|\mathbf{x}|}{\alpha}})$, which allows the following integrations by parts. Let $\delta\psi = \overline{\psi} - \psi$, that satisfies the equation

$$-\alpha^2 \Delta\delta\psi + \delta\psi = \Delta\psi. \tag{2.11}$$

Let $1 \le p < \infty$ be given. We take $\delta\psi|\delta\psi|^{p-2}$ as test function in the equation (2.11), which yields[2]

$$(p-1)\, \alpha^2 \int_{\mathbf{R}^3} |\nabla\delta\psi|^2 |\delta\psi|^{p-2} + \int_{\mathbf{R}^3} |\delta\psi|^p \le \alpha^2 \int_{\mathbf{R}^3} |\Delta\psi|\, |\delta\psi|^{p-1}.$$

The estimate (2.10) follows From the Hölder inequality we get

$$\|\delta\psi\|_{0,p}^p \le \alpha^2 \|\Delta\psi\|_{0,p} \|\delta\psi\|_{0,p}^{p-1} \qquad 1 \le p < +\infty,$$

from which we get the thesis when $1 \le p < \infty$. As $\psi \in C_c^\infty$, then $\|\psi\|_{0,\infty} = \lim_{p\to+\infty} \|\psi\|_{0,p}$ and in view of the decay of $\overline{\psi}$ at infinity, we get, by passing to the limit when $p \to \infty$ in the right-hand side of (2.10),

$$\limsup_{p\to+\infty} \|\overline{\psi} - \psi\|_{0,p} \le \alpha^2 \|\Delta\psi\|_{0,\infty},$$

which proves the estimate also in the limit case. $\square$

From Lemma 2.3 and a straightforward density argument, we get:

[1] Applying Lemma 1.1 in Galdi and Simader [10], we deduce that any $\varphi \in H^1(\mathbb{R}^3)$ goes to zero at infinity. So density of $C_c^\infty(\mathbf{R}^3)$ can be obtained by mollifying and truncating.

[2] Strictly speaking we should first take $\delta\psi(\varepsilon + \delta\psi^2)^{(p-2)/2}$ as test function, and then pass to the limit when $\varepsilon \to 0^+$ once the estimate is established. This is standard, so that we skip the details. The reader can see this for instance in Di Perna-Lions [7], where similar calculations are carried out.

Lemma 2.4. *Let* $1 \le p < \infty$, $\psi \in L^p(\mathbb{R}^3)$. *Then* $\overline{\psi} \to \psi$ *in* $L^p(\mathbb{R}^3)$ *when* $\alpha \to 0^+$.

We also will need the following Leibniz like formula:

Lemma 2.5. *Let* $\varphi, \psi \in H^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. *Then*

$$D\,\overline{\varphi\psi} = \overline{\psi D\varphi} + \overline{\varphi D\psi}, \tag{2.12}$$

where D *denotes any partial derivative* $\frac{\partial}{\partial x_i}$, $i = 1, 2, 3$.

Proof. According to the assumptions about φ and ψ, the products $\varphi\,\psi$, $\psi\,D\varphi$ and $\varphi\,D\psi$ are all in $L^2(\mathbb{R}^3)$. Therefore, we deduce from Lemma 2.2 that (at least in a weak sense),

$$-\alpha^2\Delta\overline{\varphi\psi} + \overline{\varphi\psi} = \varphi\psi,$$

which yields by differentiating (at least in the sense of the distributions),

$$-\alpha^2\Delta D\overline{\varphi\psi} + D\overline{\varphi\psi} = \varphi D\psi + \psi D\varphi.$$

Moreover, again by Lemma 2.2, we have

$$-\alpha^2\Delta(\overline{\psi D\varphi}) + \overline{\psi D\varphi} = \psi D\varphi \quad \text{and} \quad -\alpha^2\Delta(\overline{\varphi D\psi}) + \overline{\varphi D\psi} = \varphi D\psi,$$

hence (2.12) follows, due to the uniqueness of the solution. $\square$

Finally, by applying the basic elliptic regularity to the Helmholtz equation (1.2), we also have the estimate

$$\|\overline{\psi}\|_{2,2} \le \frac{C}{\alpha}\|\psi\|_{0,2}, \tag{2.13}$$

and more generally, according to the standard elliptic theory (see, e.g, Brézis [4]), we always have

$$\forall\, s \ge 0, \qquad \|\overline{\psi}\|_{s+2,2} \le \frac{C}{\alpha}\|\psi\|_{s,2}. \tag{2.14}$$

As a consequence we have the following result.

Lemma 2.6. *Let* $(\psi_n)_{n\in\mathbb{N}}$ *be a sequence that converges to* ψ *in* $H^s(\mathbb{R}^3)$. *Then* $(\overline{\psi}_n)_{n\in\mathbb{N}}$ *converges to* $\overline{\psi}$ *in* $H^{s+2}(\mathbb{R}^3)$.

3 A priori estimates and energy balance

To avoid repetition, we will assume throughout the rest of the paper that $\mathbf{u}_0 \in L^2(\mathbb{R}^3)^3$ and $\operatorname{div}\mathbf{u}_0 = 0$. Regular solutions to the NSEB-α model are defined in Definition 1.1. For any fixed time t, then $\|\mathbf{u}(t,\cdot)\|_{s,p}$ denotes the norm in $W^{s,p}(\mathbb{R}^3)$ of the field $\mathbf{x} \mapsto \mathbf{u}(t,\mathbf{x})$ for a given fixed t.

Throughout this section, $(\mathbf{u}, p)$ denotes an *à priori* regular solution to the NSEB-α model. We aim to figure out the optimal regularity of this solution and to show that it satisfies the energy balance (1.3). To do so, we will:

i) precise few notations and practical functions linked to norms of $\mathbf{u}$;

ii) give the Oseen's integral representation for the calculation of the velocity $\mathbf{u}$;

iii) deduce from this representation additional regularity for $\partial_t\mathbf{u}$ and p. in order to get the energy balance;

iv) find H^m estimates for $(\mathbf{u}, p)$ on the interval $[\tau, T[$ for any $0 < \tau < T$.

Finally, as the pressure p is linked to the velocity $\mathbf{u}$ by the equation (1.4), we sometime will refer to $\mathbf{u}$ as the solution of the NSEB-α model, without talking about p.

3.1 A few notations

Let $W(t)$ and $J(t)$ denote the following functions:

$$W(t) := \|\mathbf{u}(t,\cdot)\|_{0,2}^2 \qquad \text{and} \qquad J(t) =: \|\nabla\mathbf{u}(t,\cdot)\|_{0,2}.$$

At this stage, they could be not finite for some positive t. The energy balance (1.3) is related to the function $E_\alpha(t)$ defined as follows

$$E_\alpha(t) := \alpha^2 \int_{\mathbf{R}^3} |\nabla\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbf{R}^3} |\mathbf{u}(t,\mathbf{x})|^2 d\mathbf{x} = \alpha^2 J^2(t) + W(t),$$

and we set

$$E_{\alpha,0} := \alpha^2 \int_{\mathbf{R}^3} |\nabla\overline{\mathbf{u}_0}(\mathbf{x})|^2 d\mathbf{x} + \int_{\mathbf{R}^3} |\ \overline{\mathbf{u}_0}(\mathbf{x})|^2 d\mathbf{x}.$$

The following inequalities hold true:

$$W(t) \leq E_\alpha(t) \quad \text{and} \quad J(t) \leq \alpha^{-1}\sqrt{E_\alpha(t)},$$

and the energy balance (1.3) can be rewritten as follows

$$\frac{1}{2}E_\alpha(t) + \int_0^t \left(\alpha^2\|\Delta\mathbf{u}(t',\cdot)\|_{0,2}^2 + \nu J(t')^2\right)\, dt' = \frac{1}{2}E_{\alpha,0}. \tag{3.1}$$

In particular, each solution $\mathbf{u}$ of the NSEB-α model for which (3.1) holds is such that the function $t \mapsto E_\alpha(t)$ is non-increasing and it satisfies

$$\forall t \in [0,T], \quad E_\alpha(t) \leq E_{\alpha,0} \leq 5\|\mathbf{u}_0\|_{0,2}^2,$$

where the latter inequality is deduced from the estimates (2.5) and (2.6). In the sequel we will use $E_{\alpha,0}$ as control parameter for the NSEB-α rather than $\|\mathbf{u}_0\|_{0,2}^2$.

3.2 Oseen representation

Let be given any Navier-Stokes-like system in $\mathbb{R}^3$ of the form

$$\begin{cases} \partial_t\mathbf{u} + B(\mathbf{u},\mathbf{u}) - \nu\Delta\mathbf{u} + \nabla p = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathbf{u}_{t=0} = F(\mathbf{u}_0) & \text{in } \mathbb{R}^3. \end{cases}$$

In the case of the NSEB-α model,

$$F(\mathbf{u}_0) = \overline{\mathbf{u}_0} \quad \text{and} \quad B(\mathbf{u},\mathbf{u}) = \overline{\operatorname{div}(\mathbf{u}\otimes\mathbf{u})} = \overline{(\mathbf{u}\cdot\nabla)\mathbf{u}}.$$

Modern analysis often describes a regular solution to this system by the abstract differential equation

$$\mathbf{u}(t) = e^{-\nu t\Delta}F(\mathbf{u}_0) + \int_0^t e^{-\nu(t-t')\Delta}PB(\mathbf{u}(t'),\mathbf{u}(t'))\, dt', \tag{3.2}$$

P being the Leray's projector on L^2 divergence-free vector fields, where, given any vector field $V = (V_1, V_2, V_3)$, then

$$PV = (PV_1, PV_2, PV_3), \quad \text{where} \quad PV_i = V_i - \Delta^{-1}\partial_i\partial_j V_j.$$

This abstract formulation has been extensively exploited in the case of the Navier-Stokes equations, for which $B(\mathbf{u},\mathbf{u}) = (\mathbf{u}\cdot\nabla)\,\mathbf{u}$, $F(\mathbf{u}_0) = \mathbf{u}_0$, probably starting from the famous paper by Fujita and Kato [8]. However, according to Tao [21], it seems that this approach has reached its limits. Unfortunately, we have not found and alternate formulation that will revolutionize the field, yet.
On the contrary, we will be very old-fashioned in using the Oseen representation formula which gives an explicit expression by convolutions in space of the integral relation (3.2) through a "semi singular" kernel. This kernel was first determined by Oseen [17] for the evolutionary Stokes problem, and developed by Leray [15] to study the Navier-Stokes equations. We introduce in what follows the Oseen's kernel.
Let us consider the evolutionary Stokes problem with a continuous source term $\mathbf{f}$ and a continuous initial data $\mathbf{v}_0$:

$$\left\{\begin{array}{ll} \partial_t\mathbf{v} - \nu\Delta\mathbf{v} + \nabla q = \mathbf{f} & \text{in } \mathbb{R}^+\times\mathbb{R}^3, \\ \operatorname{div}\mathbf{v} = 0 & \text{in } \mathbb{R}^+\times\mathbb{R}^3, \\ \mathbf{v}_{t=0} = \mathbf{v}_0 & \text{in } \mathbb{R}^3. \end{array}\right. \tag{3.3}$$

It is well-known (see Oseen [17, 18]) that there exists a tensor $\mathbf{T} = (T_{ij})_{1\le i,j\le 3}$ such that, being $(\mathbf{v}, q)$ a regular solution of (3.3), then

$$\mathbf{v}(t,\mathbf{x}) = (Q\star\mathbf{v}_0)(t,\mathbf{x}) + \int_0^t\int_{\mathbf{R}^3}\mathbf{T}(t-t',\mathbf{x}-\mathbf{y})\cdot\mathbf{f}(t',\mathbf{y})\,d\mathbf{y},$$

where

$$Q(t,\mathbf{x}) := \frac{1}{(4\pi\nu t)^{3/2}}e^{-\frac{|\mathbf{x}|^2}{4\nu t}},$$

is the heat kernel and

$$(Q\star\mathbf{v}_0)(t,\mathbf{x}) := \int_{\mathbf{R}^3} Q(t,\mathbf{x}-\mathbf{y})\mathbf{v}_0(\mathbf{y})d\mathbf{y}.$$

The components of $\mathbf{T}$ are detailed for instance in [16], where the following estimates are proved:

$$\forall\, m\ge 0,\ \exists C_m > 0: \quad |D^m\mathbf{T}(t,\mathbf{x})| \le \frac{C_m}{(|\mathbf{x}|^2+\nu t)^{\frac{m+3}{2}}} \qquad \forall\,(t,\mathbf{x}) \ne (0,\mathbf{0}),$$

C_m being some constant depending only on $m\in\mathbb{N}$. As a consequence we have (see for instance [16]) the following result.

Lemma 3.1. *Let $t>0$. Then, $\forall\, t'\in[0,t[$, the tensor field $\mathbf{x}'\mapsto\mathbf{T}(t-t',\mathbf{x}')$ belongs to $L^1_{\mathbf{x}'}(\mathbb{R}^3)$. Moreover, $t'\mapsto\|\nabla\,\mathbf{T}(t-t',\cdot)\|_{0,1}\in L^1([0,t])$ and*

$$\|\nabla\,\mathbf{T}(t-t',\cdot)\|_{0,1} \le \frac{C}{\sqrt{\nu(t-t')}}. \tag{3.4}$$

Lemma 8 in [15] applies also to the case of the NSEB-α model (we skip the details), and we have also the following result

Lemma 3.2. *Let $(\mathbf{u},p)$ be a regular solution of the NSEB-α model (1.1) over the time interval $[0,T[$, then for all $t\in[0,T[$,*

$$\mathbf{u}(t,\mathbf{x}) = (Q\,\star\,\overline{\mathbf{u}_0})(t,\mathbf{x}) + \int_0^t\int_{\mathbf{R}^3}\nabla\mathbf{T}(t-t',\mathbf{x}-\mathbf{y}) : \overline{\mathbf{u}\otimes\mathbf{u}}(t',\mathbf{y})\,d\mathbf{y}dt', \tag{3.5}$$

$$p(t,\mathbf{x}) = \frac{1}{4\pi}\int_{\mathbf{R}^3}\nabla^2\left(\frac{1}{r}\right)\overline{\mathbf{u}\otimes\mathbf{u}}(t,\mathbf{y})\,d\mathbf{y}, \tag{3.6}$$

where $r := |\mathbf{x}-\mathbf{y}|$.

In formula (3.6), $\nabla^2\left(\frac{1}{r}\right)$ is a Dirac tensor so that the integral (3.6) must be understood as a singular operator, which satisfies the assumptions of the Calderón-Zygmund Theorem (see Stein [20] for a general setting and Galdi [9] for the implementation within the Navier-Stokes equations framework).

3.3 Regularity and energy balance

Recall that the notion of regular solution is given in Definition 1.1. The goal is to prove that any regular solution satisfies the energy balance. We know that a regular solution has H^2-regularity, which is *à priori* not enough to get this energy balance. We also need extra integrability conditions for $\partial_t \mathbf{u}$ and ∇p, which is the main goal of this section. We will observe that the model exerts a strong regularizing effect of the pressure, even near $t = 0$.

Lemma 3.3. *For all* $0 < \tau < T$,

$$p \in C([0,\tau]; H^4(\mathbb{R}^3)) \tag{3.7}$$

Proof. Here $\tau \in]0, T[$ and $t \in [0, \tau]$, and we split the proof into 3 steps:

i) H^4 regularity of $\overline{\mathbf{u}\otimes\mathbf{u}}$ from the Helmholtz equation;

ii) L^2 regularity of p by using the integral representation (3.6) and the Calderón-Zygmund Theorem;

iii) H^4 regularity of p by using the equation (1.4) and the elliptic theory.

Step i): Since $H^2(\mathbb{R}^3)$ is an algebra and $\mathbf{u} \in C([0,\tau]; H^2(\mathbb{R}^3)^3)$, then $\mathbf{u}\otimes\mathbf{u} \in C([0,\tau]; H^2(\mathbb{R}^3)^9)$. Therefore, by (2.14), it follows that $\overline{\mathbf{u}\otimes\mathbf{u}} \in C([0,\tau]; H^4(\mathbb{R}^3)^9)$ and we have

$$\|\overline{\mathbf{u}\otimes\mathbf{u}}(t,\cdot)\|_{4,2} \leq \frac{C}{\alpha^2}\|\mathbf{u}(t,\cdot)\|_{2,2}^2. \tag{3.8}$$

We also deduce $\mathrm{div}[\mathrm{div}(\overline{\mathbf{u}\otimes\mathbf{u}})] \in C([0,\tau]; H^2(\mathbb{R}^3)^9)$, which will be useful in step iii).

Step ii): The L^2-regularity of p is a consequence of the integral representation (3.6), Calderón-Zygmund Theorem, and (3.8). The time continuity with values in $L^2(\mathbb{R}^3)$ is straightforward, and we have in particular

$$\|p(t,\cdot)\|_{0,2} \leq \frac{C}{\alpha^2}\|\mathbf{u}(t,\cdot)\|_{2,2}^2.$$

Step iii): We use the equation (1.4) for the pressure, that we write under the form

$$-\Delta p + p = F, \tag{3.9}$$

where

$$F = \mathrm{div}[\mathrm{div}(\overline{\mathbf{u}\otimes\mathbf{u}})] + p \in C([0,\tau]; L^2(\mathbb{R}^3)), \tag{3.10}$$

by the results of the two previous steps. Then (3.10) is a consequence of the standard elliptic theory (see Brézis [4]) and yields $p \in C([0,\tau]; H^2(\mathbb{R}^3))$. Finally as $\mathrm{div}[\mathrm{div}(\overline{\mathbf{u}\otimes\mathbf{u}})] \in C([0,\tau]; H^2(\mathbb{R}^3)^9)$ by step i), then F given by (3.10) is in $C([0,\tau]; H^2(\mathbb{R}^3)^9)$, hence $p \in C([0,\tau]; H^4(\mathbb{R}^3)^9)$ from (3.9) and we have in particular

$$\|p(t,\cdot)\|_{4,2} \leq \frac{C}{\alpha^2}\|\mathbf{u}(t,\cdot)\|_{2,2}^2,$$

which is optimal. $\square$

We have also the following result.

Lemma 3.4. *For all* $0 < \tau_1 < \tau_2 < \infty$, $\mathbf{u} \in C([\tau_1, \tau_2]; H^4(\mathbb{R}^3)^3)$.

Proof. Let $m = 3, 4$, $0 < \tau_1 < \tau_2 < T$. An argument similar to that of Lemma 3.2 in [16] shows that we can differentiate under the integral sign in the Oseen's representation and therefore we skip the details. We get

$$D^m \mathbf{u}(t, \mathbf{x}) = D^{m-2}(Q \star D^2 \overline{\mathbf{u}_0})(t, \mathbf{x}) + \int_0^t \int_{\mathbb{R}^3} \nabla \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) : D^m(\overline{\mathbf{u} \otimes \mathbf{u}})(t', \mathbf{y})\, d\mathbf{y} dt',$$

hence by standard results about the heat kernel, Cauchy-Schwarz and Young inequalities, inequalities (3.4) and (3.8), and the fact that $H^2(\mathbb{R}^3)$ is an algebra, we obtain

$$\|D^m \mathbf{u}(t, \mathbf{x})\|_{0,2} \le C \left(\frac{1}{(\nu t)^{\frac{m-2}{2}}} \|(Q \star D^2 \overline{\mathbf{u}_0})(t, \cdot)\|_{0,2} + \frac{1}{\alpha^2} \int_0^t \frac{\|\mathbf{u}(t', \cdot)\|_{2,2}^2}{\sqrt{\nu(t - t')}\, dt'} \right),$$

which gives

$$\|D^m \mathbf{u}(t, \mathbf{x})\|_{0,2} \le C \left(\frac{1}{\alpha(\nu \tau_1)^{\frac{m-2}{2}}} \|\mathbf{u}_0\|_{0,2} + \frac{1}{\alpha^2} \sup_{t' \in [0, \tau_2]} \|\mathbf{u}(t', \cdot)\|_{2,2}^2 \sqrt{\frac{\nu t}{\nu}} \right),$$

hence the result, the continuity being a consequence of standard results in analysis. □

We have some relevant corollaries of the previous results

Corollary 3.1. *For all* $0 < \tau < T$, $\partial_t \mathbf{u} \in C([0, \tau]; L^2(\mathbb{R}^3)^3)$, *and for all* $0 < \tau_1 < \tau_2 < T$, *it follows that* $\partial_t \mathbf{u} \in C([\tau_1, \tau_2]; H^2(\mathbb{R}^3)^3)$.

Proof. It is enough to write

$$\partial_t \mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} - \operatorname{div}(\overline{\mathbf{u} \otimes \mathbf{u}}),$$

and to apply the previous results to the right-hand side. □

Corollary 3.2. *Let* $\mathbf{u}$ *be a regular solution to the NSEB-*α *model. Then, the velocity* $\mathbf{u}$ *satisfies the energy balance (3.1).*

Proof. Since for all $t \in [0, T[$ we have that $\mathbf{u}(t, \cdot) \in H^2(\mathbb{R}^3)^3$, we can take $-\alpha^2 \Delta \mathbf{u} + \mathbf{u}$ as test vector field in (1.1) and integrate over $\mathbb{R}^3$ by using the Stokes formula. In particular, we have (as $H^2(\mathbb{R}^3)$ is an algebra) that $(\mathbf{u} \otimes \mathbf{u})(t, \cdot) \in H^2(\mathbb{R}^3)^9$, hence $(\overline{\mathbf{u} \otimes \mathbf{u}})(t, \cdot) \in H^4(\mathbb{R}^3)^9$. Therefore, since the bar operator is self-adjoint, the following equalities hold true

$$(\operatorname{div}(\overline{\mathbf{u} \otimes \mathbf{u}}), -\alpha^2 \Delta \mathbf{u} + \mathbf{u}) = (\operatorname{div}(\mathbf{u} \otimes \mathbf{u}), -\alpha^2 \Delta \overline{\mathbf{u}} + \overline{\mathbf{u}}) = (\operatorname{div}(\mathbf{u} \otimes \mathbf{u}), \mathbf{u}) = 0.$$

Moreover, let $t \in]0, T[$. As $\mathbf{u}, \partial_t \mathbf{u} \in C([0, t + \varepsilon]; L^2(\mathbb{R}^3)^3)$, where $\varepsilon > 0$ is such that $t + \varepsilon < T$, we obtain by applying a standard results (see Temam [22] for instance),

$$(\partial_t \mathbf{u}(t, \cdot), \mathbf{u}(t, \cdot) = \frac{d}{2dt} \int_{\mathbb{R}^3} |\mathbf{u}(t, \mathbf{x})|^2 d\mathbf{x}.$$

By the previous results, $\partial_t \mathbf{u} \in C([\varepsilon, t + \varepsilon]; H^1(\mathbb{R}^3)^3)$, $\partial_t \nabla \mathbf{u} \in C([\varepsilon, t + \varepsilon]; L^2(\mathbb{R}^3)^9)$, $\nabla \mathbf{u} \in C([\varepsilon, t + \varepsilon]; L^2(\mathbb{R}^3)^9)$. Therefore, we also have

$$\forall\, t \in]0, T[\qquad -(\partial_t \mathbf{u}(t, \cdot), \Delta \mathbf{u}(t, \cdot)) = \frac{d}{2dt} \int_{\mathbb{R}^3} |\nabla \mathbf{u}(t, \mathbf{x})|^2 d\mathbf{x}.$$

Finally, since $\operatorname{div} \mathbf{u} = \operatorname{div}(\Delta \mathbf{u}) = 0$, and due to the integrability results of Lemma 3.3 and Lemma 3.4 about $(\mathbf{u}, p)$ we also have $(\nabla p, -\alpha^2 \Delta \mathbf{u} + \mathbf{u}) = 0$. The rest of the proof is now straightforward. □

Remark 3.1. *Once the regularity of any regular solution is established and the energy balance is checked, uniqueness follows from the same process, based on energy balances by the regularity results, combined with an application of Gronwall's lemma. One has only to reproduce what is written in Section 2.2 of [14], with minor modifications. To avoid repetitions, we skip the details.*

4 Construction of a regular solution

In this section, we start with a continuation principle, that states that a regular solution on a finite time interval $[0, T[$ (with $T < \infty$) cannot develop any singularity when $t \to T^-$ and can be extended by continuity up to time T. Then, we set up the standard iteration process based on the integral Oseen representation formula. Finally, we will construct a regular solution, obtained as the limit of the iterations previously analyzed.

4.1 Continuation principle

The regular solution that will be constructed by the Picard (iteration) theorem is local-in-time. The results in this subsection are essential to understand the transition from a local-in-time regular solution, to a regular solution defined for all time $t \in [0, \infty[$, that we call a global time solution.

Lemma 4.1. *Assume $T < \infty$. Then a regular solution to (1.1) on $[0, T[$ do not develop a singularity when $t \to T^-$.*

Proof. We must prove that $\|\mathbf{u}(t, \cdot)\|_{2,2}$ remains bounded on the time interval $[0, T[$. We deduce from the integral representation formula and from the inequality (3.4) that

$$\|D^m \mathbf{u}(t, \cdot)\|_{0,2} \le \frac{C}{\alpha}\|\mathbf{u}_0\|_{0,2} + \int_0^t \frac{\|D^m(\overline{\mathbf{u} \otimes \mathbf{u}})(t', \cdot)\|_{0,2}}{\sqrt{\nu(t - t')}}\, dt' \quad \text{for } m = 0, 1, 2, \tag{4.1}$$

and, as soon as $m \le 2$, we get by (2.13), with the Cauchy-Schwarz and Sobolev inequalities, that

$$\begin{aligned} \|D^m(\overline{\mathbf{u} \otimes \mathbf{u}})(t', \cdot)\|_{0,2} &\le C\alpha^{-1}\|(\mathbf{u} \otimes \mathbf{u})(t', \cdot)\|_{0,2} \\ &\le C\alpha^{-1}\|\mathbf{u}(t', \cdot)\|_{0,4}^2 \\ &\le C\alpha^{-1}\|\mathbf{u}(t', \cdot)\|_{1,2}^2 \\ &\le C\alpha^{-3}E_\alpha(t), \end{aligned}$$

which leads to, by the energy balance,

$$\|D^m(\overline{\mathbf{u} \otimes \mathbf{u}})(t', \cdot)\|_{0,2} \le C\alpha^{-3}E_{\alpha,0},$$

that we combine with (4.1) to get,

$$\forall\, m \le 2, \qquad \|D^m \mathbf{u}(t, \cdot)\|_{0,2} \le C\left(\frac{1}{\alpha}\|\mathbf{u}_0\|_{0,2} + \frac{1}{\alpha^3}E_{\alpha,0}\sqrt{\frac{t}{\nu}}\right),$$

concluding the proof. □

We note that the Duhamel principle applies, by Lemma 8 in [15]. Therefore, we have for all $0 < \tau \le t < T$,

$$\mathbf{u}(t, \mathbf{x}) = (Q \star \mathbf{u}(\tau, \cdot))(t, \mathbf{x}) + \int_\tau^t \int_{\mathbf{R}^3} \nabla T(t - t', \mathbf{x} - \mathbf{y}) : (\overline{\mathbf{u} \otimes \mathbf{u}})(t', \mathbf{y})\, d\mathbf{y}dt'.$$

Therefore, from this principle, by induction and following the same proofs of Lemma 3.3, Lemma 3.4, and Lemma 4.1, and Lemma 2.12, we can easily show the following result.

Lemma 4.2. *For all $0 < \tau_1 < \tau_2 < T$ and for $m > 2$, we have that $(\mathbf{u}, p) \in C([\tau_1, \tau_2]; H^m(\mathbb{R}^3)^3 \times H^m(\mathbb{R}^3))$ and there exists $C = C(m, \tau_1, T, \|\mathbf{u}_0\|_{0,2}, \nu, \alpha)$ such that*

$$\forall\, t \in [\tau_1, \tau_2], \qquad \|\mathbf{u}(t, \cdot)\|_{m,2} \leq C(m, \tau_1, T, \|\mathbf{u}_0\|_{0,2}, \nu, \alpha). \tag{4.2}$$

We stress that the constant in (4.2) depends on T as $\mathcal{O}(\sqrt{T})$, and remains finite, whatever the value of $T < \infty$ is considered. We also notice that we cannot let τ_1 to go to zero when $m > 2$, which is due to the Helmholtz filter that only regularizes the initial data up to $H^2(\mathbb{R}^3)$ and not better, since $\mathbf{u}_0 \in L^2(\mathbb{R}^3)^3$, which is the appropriate regularity assumption about the initial value in the framework of the Navier-Stokes equations. This is one of the main differences with respect to the filtering made by convolution with smooth functions.

Lemma 4.3. *Let us assume $T > 0$. Any regular solution on $[0, T[$ can be extended up to $t = T$.*

Proof. Let $0 < t_1 < t_2 < T$. We write

$$\mathbf{u}(t_2, \mathbf{x}) = (Q \star \mathbf{u}(t_1, \cdot))(t_2, \mathbf{x}) + \int_{t_1}^{t_2} \int_{\mathbf{R}^3} \nabla T(t_2 - t', \mathbf{x} - \mathbf{y}) : (\overline{\mathbf{u} \otimes \mathbf{u}})(t', \mathbf{y})\, d\mathbf{y} dt'.$$

Therefore, by the same calculation as above, using the again the energy balance, we get

$$\|\mathbf{u}(t_1, \cdot) - \mathbf{u}(t_2, \cdot)\|_{2,2} \leq \|(Q \star \mathbf{u}(t_1, \cdot))(t_2, \cdot) - \mathbf{u}(t_1, \cdot)\|_{2,2} + \frac{C}{\alpha^3} E_{\alpha,0} \sqrt{\frac{t_2 - t_1}{\nu}}. \tag{4.3}$$

Standard results about the heat equation yield the estimate

$$\|(Q \star \mathbf{u}(t_1, \cdot))(t_2, \cdot) - \mathbf{u}(t_1, \cdot)\|_{2,2} \leq \frac{\nu(t_2 - t_1)}{2} \|\mathbf{u}(t_1, \cdot)\|_{3,2}.$$

Therefore, by Lemma 4.2, which provides a uniform bound in t_1 about $\|\mathbf{u}(t_1, \cdot)\|_{3,2}$ for $t_1 \geq \tau > 0$ (the time $\tau < T$ being fixed), combined with (4.3), we see that $\mathbf{u}(t, \cdot)$ satisfies a uniform Cauchy criterion in the Banach space $H^2(\mathbb{R}^3)^3$ over $[\tau, T[$, in particular when both $t_1, t_2 \to T$. Therefore, $\mathbf{u}(t, \cdot)$ admits a limit when $t \to T$, concluding the proof. $\square$

4.2 Iterative procedure

Let $\tau > 0$ be a given time, which will be fixed later. We equip the Banach space $C([0, \tau]; H^2(\mathbb{R}^3)^3)$ with its natural uniform norm

$$\|\mathbf{v}(t, \cdot)\|_{\tau;2,2} := \sup_{t \in [0,\tau]} \|\mathbf{v}(t, \cdot)\|_{2,2}.$$

We consider in $C([0, \tau]; H^2(\mathbb{R}^3)^3)$ the sequence $(\mathbf{u}^{(n)})_{n \in \mathbf{N}}$ defined by

$$\begin{cases} \mathbf{u}^{(0)}(t, \mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}), \\ \mathbf{u}^{(n)}(t, \mathbf{x}) = (Q \star \overline{\mathbf{u}_0})(t, \mathbf{x}) + \\ \qquad \displaystyle\int_0^t \int_{\mathbf{R}^3} \nabla \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) : \overline{\mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)}}(t', \mathbf{y})\, d\mathbf{y} dt'. \end{cases} \tag{4.4}$$

By the same reasoning of the previous section, it is easily checked by induction that each $\mathbf{u}^{(n)}$ lies indeed in $C([0, \tau]; H^2(\mathbb{R}^3)^3)$ (but in fact and much better space). In the following,

C denotes a constant that only depends on Sobolev constants and the Oseen tensor. Let $t \mapsto E_\alpha^{(n)}(t)$ be the function defined by

$$E_\alpha^{(n)}(t) := \|\mathbf{u}^{(n)}(t,\cdot)\|_{0,2}^2 + \alpha^2 \|\nabla \mathbf{u}^{(n)}(t,\cdot)\|_{0,2}^2.$$

For technical conveniences, we will assume in the following that $\alpha \leq 1$[3].
The goal is to prove that the sequence $(\mathbf{u}^{(n)})_{n\in\mathbb{N}}$ satisfies a contraction property on a time interval $[0, \tau_{Lip}]$. We will therefore conclude to the convergence of the sequence $(\mathbf{u}^{(n)})_{n\in\mathbb{N}}$ by the Picard theorem. To do so, we need to estimate $E_\alpha^{(n)}(t)$, at least over a small time interval $[0, \tau[$ to begin. This is the purpose of the next lemma.

Lemma 4.4. *Let us define*

$$\tau_{max}(\sigma) := \frac{\nu \alpha^6}{4C^2 \sigma} \quad \textit{for } \sigma > 0. \tag{4.5}$$

Then

$$\forall\, n \in \mathbb{N} \textit{ and } \forall\, t \in [0, \tau_{max}(E_{\alpha,0})], \qquad E_\alpha^{(n)}(t) \leq 8E_{\alpha,0}. \tag{4.6}$$

Proof. We argue by induction. We have for all $t \in [0, \infty[$, $E_\alpha^{(0)}(t) = E_{\alpha,0} \leq 8E_{\alpha,0}$. Let $n \geq 0$ be given. On one hand we have

$$\|\mathbf{u}^{(n)}(t,\cdot)\|_{0,2} \leq \|\overline{\mathbf{u}_0}\|_{0,2} + \int_0^t \frac{\|(\overline{\mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)}})(t',\cdot)\|_{0,2}}{\sqrt{\nu(t-t')}}\, dt', \tag{4.7}$$

and, by (2.8), we also have for $t' \in [0, t]$

$$\begin{aligned} \|(\overline{\mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)}})(t',\cdot)\|_{0,2} &\leq C\alpha^{-\frac{3}{2}} \|(\mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)})(t',\cdot)\|_{0,1} \\ &\leq C\alpha^{-\frac{3}{2}} \|\mathbf{u}^{(n-1)}(t',\cdot)\|_{0,2}^2. \end{aligned} \tag{4.8}$$

On the other hand, we have

$$\|\nabla \mathbf{u}^{(n)}(t,\cdot)\|_{0,2} \leq \|\nabla \overline{\mathbf{u}_0}\|_{0,2} + \int_0^t \frac{\|\nabla(\overline{\mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)}})(t',\cdot)\|_{0,2}}{\sqrt{\nu(t-t')}}\, dt', \tag{4.9}$$

and consequently, by using Sobolev inequality,

$$\begin{aligned} \|\nabla(\overline{\mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)}})(t',\cdot)\|_{0,2} &\leq C\alpha^{-1} \|(\mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)})(t',\cdot)\|_{0,2} \\ &\leq C\alpha^{-1} \|\mathbf{u}^{(n-1)}\|_{0,4}^2 \\ &\leq C\alpha^{-1} \left[\|\mathbf{u}^{(n-1)}\|_{0,2}^2 + \|\nabla \mathbf{u}^{(n-1)}\|_{0,2}^2 \right]. \end{aligned} \tag{4.10}$$

We then combine (4.7), (4.8), (4.9), and (4.10) with elementary algebraic inequalities, and we get

$$\sqrt{E_\alpha^{(n)}(t)} \leq \sqrt{2E_{\alpha,0}} + \frac{C}{\alpha^3} \int_0^t \frac{E_\alpha^{(n-1)}(t')}{\sqrt{\nu(t-t')}}\, dt' = F(E_\alpha^{(n-1)})(t). \tag{4.11}$$

Then, we deduce from (4.11) by induction, that the inequality "$E_\alpha^{(n-1)}(t) \leq 8E_{\alpha,0}$" yields "$E_\alpha^{(n)}(t) \leq 8E_{\alpha,0}$" for all $t \in [0, \tau]$, and for any time $\tau > 0$ such that

$$F(8E_{\alpha,0})(\tau) \leq 2\sqrt{2}\sqrt{E_{\alpha,0}}.$$

[3]We can take any other upper bound for α, which only yields technical complications

After easy calculations (where we use $\alpha \leq 1$ to balance the terms of the form α^{-s}) we can explicitly prove the upper bound

$$\tau \leq \frac{\nu \alpha^6}{4C^2 E_{\alpha,0}} = \tau_{max}(E_{\alpha,0}),$$

hence the result. □

It is very important to notice that the function $\sigma \mapsto \tau_{max}(\sigma)$ is non-increasing. From this local time estimate of $E_\alpha^{(n)}(t)$, we can now prove the following result.

Lemma 4.5. *There exists $\tau_{Lip} = \tau_{Lip}(E_{\alpha,0}) \in]0, \tau_{max}(E_{\alpha,0})]$ such that*

$$\forall\, n \geq 1, \qquad \|\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}\|_{\tau_{Lip};2,2} \leq \frac{1}{2} \|\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)}\|_{\tau_{Lip};2,2}.$$

Moreover, the function $\sigma \mapsto \tau_{Lip}(\sigma)$ is a non-increasing function.

Proof. Let $t \leq \tau_{max}(E_{\alpha,0})$, in order to apply (4.6). We deduce from (3.4) and (4.4) that

$$\|(\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)})(t,\cdot)\|_{2,2} \leq \int_0^t \frac{\|\overline{(\mathbf{u}^{(n)} \otimes \mathbf{u}^{(n)} - \mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)})(t',\cdot)}\|_{2,2}}{\sqrt{\nu(t-t')}}\, dt'. \tag{4.12}$$

Repeating the same reasoning as above yields, for $t' \in [0,t]$,

$$\begin{aligned} &\|\overline{(\mathbf{u}^{(n)} \otimes \mathbf{u}^{(n)} - \mathbf{u}^{(n-1)} \otimes \mathbf{u}^{(n-1)})(t',\cdot)}\|_{2,2} \\ &\qquad \leq \frac{C}{\alpha} \left[\|\mathbf{u}^{(n)}(t',\cdot)\|_{1,2} + \|\mathbf{u}^{(n-1)}(t',\cdot)\|_{1,2} \right] \|(\mathbf{u}^{(n)} - \mathbf{u}^{(n-1)})(t',\cdot)\|_{1,2}. \end{aligned} \tag{4.13}$$

By (4.6), we have $\|\mathbf{u}^{(p)}(t',\cdot)\|_{1,2} \leq \sqrt{2}\alpha^{-1}\sqrt{E_\alpha(t')} \leq 4\alpha^{-1}\sqrt{E_{\alpha,0}}$, in particular for $p = n, n-1$. Therefore, combining (4.12) and (4.13) we obtain that, for all $\tau \in]0, \tau_{max}(E_{\alpha,0})]$, and for all $t \in [0,\tau]$,

$$\|(\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)})(t,\cdot)\|_{2,2} \leq \frac{4C}{\alpha^2}\sqrt{\frac{\tau E_{\alpha,0}}{\nu}} \|\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}\|_{\tau;2,2},$$

leading to

$$\|\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}\|_{\tau;2,2} \leq \frac{4C}{\alpha^2}\sqrt{\frac{\tau E_{\alpha,0}}{\nu}} \|\mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}\|_{\tau;2,2}.$$

The conclusion follows by taking

$$\tau_{Lip}(E_{\alpha,0}) = \inf\left\{ \frac{\nu\alpha^4}{16C^2 E_{\alpha,0}}, \tau_{max}(E_{\alpha,0}) \right\}. \tag{4.14}$$

By (4.5) and (4.14), we see that the function $\sigma \mapsto \tau_{Lip}(\sigma)$ is a non-increasing function. □

We deduce from Lemma 4.5 and Picard's Theorem, that the sequence $(\mathbf{u}^{(n)})_{n\in\mathbb{N}}$ is convergent in the Banach space $C([0,\tau_{Lip}]; H^2(\mathbb{R}^3)^3)$ to some $\mathbf{u}$. To conclude the proof of Theorem 1.1 it remains to prove that this field is indeed a solution to the NSEB-α model. This is the aim of the next section.

4.3 Existence of a regular solution

In this subsection we prove the final results leading to the existence of a regular solution. This subsection is divided into three steps:

i) we prove that $\mathbf{u}$ satisfies the relation (3.6);

ii) we show that there exists $p \in C([0, \tau_{Lip}]; H^2(\mathbb{R}^3)^3)$, such that $(\mathbf{u}, p)$ is a regular solution of the NSEB-α model on the time interval $[0, \tau_{Lip}[$;

iii) we show that $(\mathbf{u}, p)$ can be extended for all $t \in [0, \infty[$.

i) We first pass to the limit in the recursive relation (4.4), that can be written in an abstract way as $\mathbf{u}^{(n+1)} = F(\mathbf{u}^{(n)})$, and our aim is to prove $\mathbf{u} = F(\mathbf{u})$. This will also prove the continuity of the function $\mathbf{v} \mapsto F(\mathbf{v})$ in the Banach space $C([0, \tau_{Lip}]; H^2(\mathbb{R}^3)^3)$.
Let $(t, \mathbf{x}) \in [0, \tau_{Lip}] \times \mathbb{R}^3$ be fixed, and for $(t', \mathbf{y}) \in [0, t[\times \mathbb{R}^3$, we set:

$$\begin{cases} \psi_n(t, \mathbf{x}, t', \mathbf{y}) &= \nabla T(t - t', \mathbf{x} - \mathbf{y}) : \overline{\mathbf{u}^{(n)} \otimes \mathbf{u}^{(n)}}(t', \mathbf{y}), \\ \psi(t, \mathbf{x}, t', \mathbf{y}) &= \nabla T(t - t', \mathbf{x} - \mathbf{y}) : \overline{\mathbf{u} \otimes \mathbf{u}}(t', \mathbf{y}), \\ \varphi_n(t, t', \mathbf{x}) &= \displaystyle\int_{\mathbb{R}^3} \psi_n(t, \mathbf{x}; t', \mathbf{y})\, d\mathbf{y}, \\ \varphi(t, t', \mathbf{x}) &= \displaystyle\int_{\mathbb{R}^3} \psi(t, \mathbf{x}; t', \mathbf{y})\, d\mathbf{y}. \end{cases}$$

We must check that

$$\lim_{n \to \infty} \int_0^t \varphi_n(t, t', \mathbf{x})\, dt' = \int_0^t \varphi(t, t', \mathbf{x})\, dt'. \tag{4.15}$$

To do so we will apply Lebesgue dominated convergence Theorem twice. From the previous results we deduce that

$$\lim_{n \to \infty} \psi_n(t, \mathbf{x}; t', \mathbf{y}) = \psi(t, \mathbf{x}; t', \mathbf{y}) = \nabla T(t - t', \mathbf{x} - \mathbf{y}) : \overline{\mathbf{u} \otimes \mathbf{u}}(t', \mathbf{y}),$$

uniformly in $\mathbf{y} \in \mathbb{R}^3$, for any given $t' \in [0, t[$. Moreover, as $(\mathbf{u}^{(n)})_{n \in \mathbb{N}}$ is a sequence converging in $C([0, \tau_{Lip}]; H^2(\mathbb{R}^3)^3)$, it is bounded in the same space. Consequently, by the results of Section 2, it is easily checked that

$$|\psi_n(t, \mathbf{x}; t', \mathbf{y})| \le C \sup_{n \in \mathbb{N}} \|\mathbf{u}^{(n)}\|^2_{\tau_{Lip};2,2} |\nabla T(t - t', \mathbf{x} - \mathbf{y})| \in L^1_{\mathbf{y}}(\mathbb{R}^3),$$

by (3.4), since $t' < t$. Therefore, we have by Lebesgue's theorem,

$$\lim_{t \to \infty} \varphi_n(t, t', \mathbf{x}) = \varphi(t, t', \mathbf{x}).$$

Similarly,

$$|\varphi_n(t, t', \mathbf{x})| \le \frac{C}{\sqrt{\nu(t - t')}} \in L^1([0, t]),$$

hence (4.15) holds by Lebesgue's theorem once again. In conclusion, $\mathbf{u}$ satisfies the integral relation (3.6) as claimed.

ii) Let us now consider the unsteady (linear) Stokes problem:

$$
\left\{
\begin{array}{lll}
\partial_t \mathbf{v} - \nu \Delta \mathbf{v} + \nabla p = -\operatorname{div}(\overline{\mathbf{u} \otimes \mathbf{u}}) & \text{in} & [0, \tau_{Lip}] \times \mathbb{R}^3, \\
\operatorname{div} \mathbf{v} = 0 & \text{in} & [0, \tau_{Lip}] \times \mathbb{R}^3, \\
\mathbf{v}_{t=0} = \overline{\mathbf{u}_0} & \text{in} & \mathbb{R}^3.
\end{array}
\right. \tag{4.16}
$$

This Stokes problem has a source term in $C([0, \tau_{Lip}]; H^3(\mathbb{R}^3)^3)$ and an initial data in $H^2(\mathbb{R}^3)^3$. By standard results (see for instance Caffarelli, Kohn, and Nirenberg [5], Solonnikov [19], and Temam [22]) we already know the existence of a unique variational solution $(\mathbf{v}, p)$ to the problem (4.16), obtained by the Galerkin method, and with (at least) the regularity

$$
\begin{aligned}
&\mathbf{v} \in C([0, \tau_{Lip}]; H^2(\mathbb{R}^3)^3) \cap L^2([0, \tau_{Lip}]; H^4(\mathbb{R}^3)^3), \quad \partial_t \mathbf{v} \in L^2([0, \tau_{Lip}]; H^2(\mathbb{R}^3)^3), \\
&p \in L^2([0, \tau_{Lip}]; H^1(\mathbb{R}^3)^3).
\end{aligned}
$$

From this, it is easily checked that $(\mathbf{v}, p)$ is a strong solution to (4.16), therefore a regular solution in the sense of Definition 1.1, which satisfies by Lemma 8 in [15] the integral formulation

$$
\mathbf{v}(t, \mathbf{x}) = (Q \star \overline{\mathbf{u}_0})(t, \mathbf{x}) + \int_0^t \int_{\mathbb{R}^3} \nabla \mathbf{T}(t - t', \mathbf{x} - \mathbf{y}) : \overline{\mathbf{u} \otimes \mathbf{u}}(t', \mathbf{y}) \, d\mathbf{y} dt'.
$$

Hence, $\mathbf{u} = \mathbf{v}$, and $(\mathbf{u}, p)$ is indeed a regular solution to the NSEB-α model over $[0, \tau_{Lip}]$.

iii) It remains to check that the solution $\mathbf{u}$ can be extended up to $[0, \infty[$. We already know by Lemma 4.3 that $\mathbf{u}$ can be extended to $t = \tau_{Lip}$. We also know by Corollary 3.2 that this solution satisfies the energy balance. Therefore, the function $t \mapsto E_\alpha(t)$ is non increasing over $[0, \tau_{Lip}]$, and we have in particular

$$
E_{\alpha,1} = E_\alpha(\tau_{Lip}(E_{\alpha,0})) \leq E_{\alpha,0}. \tag{4.17}
$$

The construction carried out in Subsection 4.2 and step i) can be reproduced starting from the time $t = \tau_{Lip}(E_{\alpha,0})$ instead of $t = 0$, and with initial data $\mathbf{u}(\tau_{Lip}(E_{\alpha,0}), \cdot)$ instead of $\overline{\mathbf{u}_0}$. We then get a regular solution to the NSEB-α model over the time interval $[\tau_{Lip}(E_{\alpha,0}), \tau_{Lip}(E_{\alpha,0}) + \tau_{Lip}(E_{\alpha,1})]$. As the solution is left continuous at $t = \tau_{Lip}(E_{\alpha,0})$ in $H^2(\mathbb{R}^3)^3$, and also right continuous at the same time, it is continuous, and therefore we constructed

$$
\mathbf{u} \in C([0, \tau_{Lip}(E_{\alpha,0}) + \tau_{Lip}(E_{\alpha,1})]; H^3(\mathbb{R}^3)^3).
$$

We first observe that this $\mathbf{u}$ is a weak solution to the NSEB-α, but then we can easily check from the equation that the gluing at $t = \tau_{Lip}(E_{\alpha,0})$ is of class C^1 in time, so that we get a regular solution over $[0, \tau_{Lip}(E_{\alpha,0}) + \tau_{Lip}(E_{\alpha,1})]$. The main point that allows to iterate this process, is that the functions $\sigma \mapsto \tau_{Lip}(\sigma)$ and $t \mapsto E_\alpha(t)$ are both non-increasing. In particular we get by (4.17),

$$
\tau_{Lip}(E_{\alpha,0}) \leq \tau_{Lip}(E_{\alpha,1}).
$$

This suggests to build the sequence $(T_n)_{n \in \mathbb{N}}$, by setting

$$
T_0 = \tau_{Lip}(E_{\alpha,0}),
$$

and assuming that we have constructed a regular solution of the NSEB-α model over $[0, T_n]$. Then we extend the solution as above, starting from $t = T_n$ and $\mathbf{u}(T_n, \cdot)$ over the time interval $[T_n, T_n + \tau_{Lip}(E_{\alpha,n})]$, where $E_{\alpha,n} = E_\alpha(T_n)$. Therefore,

$$
T_{n+1} = T_n + \tau_{Lip}(E_{\alpha,n}).
$$

Since

$$\tau_{Lip}(E_{\alpha,n}) \geq \tau_{Lip}(E_{\alpha,n-1}) \geq \cdots \geq \tau_{Lip}(E_{\alpha,0}),$$

we have

$$T_n \geq n\tau_{Lip}(E_{\alpha,0}),$$

hence

$$\lim_{n\to\infty} T_n = \infty,$$

and the solution is indeed constructed for all $t \in [0, \infty[$.

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