SUITABLE WEAK SOLUTIONS OF THE NAVIER-STOKES EQUATIONS CONSTRUCTED BY A SPACE-TIME NUMERICAL DISCRETIZATION

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Abstract. We prove that weak solutions obtained as limits of certain numerical space-time discretizations are suitable in the sense of Scheffer and Caffarelli-Kohn-Nirenberg. More precisely, in the space-periodic setting, we consider a full discretization in which the θ-method is used to discretize the time variable, while in the space variables we consider appropriate families of finite elements. The main result is the validity of the so-called local energy inequality.

1. Introduction

We consider the homogeneous incompressible 3D Navier-Stokes equations

$$\begin{aligned} \partial_t u - \Delta u + (u\cdot\nabla)\,u + \nabla p &= 0 \qquad \text{in } (0,T)\times\mathbb{T}^3,\\ \operatorname{div} u &= 0 \qquad \text{in } (0,T)\times\mathbb{T}^3, \end{aligned} \tag{1.1}$$

in the space periodic setting, with divergence-free initial datum

$$u|_{t=0} = u_0 \qquad \text{in } \mathbb{T}^3, \tag{1.2}$$

where $T>0$ is arbitrary and $\mathbb{T}^3$ is the three dimensional flat torus. Here, the unknowns are the vector field u and the scalar p, which are both with zero mean value. The aim of this paper is to consider a space-time discretization of the initial value problem (1.1)-(1.2) and to prove the convergence (as the parameters of the discretization vanish) to a Leray-Hopf weak solution, satisfying also the local energy inequality

$$\partial_t\left(\frac{|u|^2}{2}\right) + \operatorname{div}\left(\left(\frac{|u|^2}{2}+p\right)u\right) - \Delta\left(\frac{|u|^2}{2}\right) + |\nabla u|^2 \le 0 \quad \text{in } \mathcal{D}'(]0,T[\times\mathbb{T}^3).$$

Solution satisfying the above inequality (and minimal assumptions on the pressure) are known in literature as *suitable weak solutions* and they are of fundamental importance for at least two reasons: 1) From the theoretical point of view it is known that for these solutions the possible set of singularities has vanishing 1D-parabolic Hausdorff measure, see Scheffer [19] and Caffarelli-Kohn-Nirenberg [11]; 2) The local energy inequality is a sort of entropy condition and, even if it is not enough to prove uniqueness, it seems a natural request to select physically relevant solutions; for this reason the above inequality has to be satisfied by solutions constructed by numerical methods, see Guermond *et al.* [16, 17].

The terminology (and an existence result) for suitable weak solutions can be found in Caffarelli, Kohn, and Nirenberg [11], where a retarded-time approximation method has been used in the construction. Since uniqueness is not known in the class of weak solutions, the question of understanding which are the approximations producing suitable solutions became central, see the papers by Beirão da Veiga [1, 2, 3]. In these papers it has also been raised the question whether the "natural" Faedo-Galerkin methods will produce suitable solutions and the question remained completely unsolved for almost twenty years, when the positive answer, at least for certain finite element spaces, appeared in Guermond [13, 14]. (In the above papers the space discretization is considered, while the time variable is kept continuous). It is

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important to observe that, most of the known regularization procedures to construct Leray-Hopf weak solutions of the Navier-Stokes equations (hyper-viscosity, Leray, Leray-α, Voigt, artificial compressibility…) seem to produce in the limit solutions satisfying the local energy inequality. We started a systematics study of this question and, even if technicalities could be rather different, we obtained several positive answers, see [8, 9, 12]. The technical problems related with discrete (numerical, finite dimensional) approximations are of a different nature. In particular, obtaining the local energy inequality is "formally" based on multiplying the equations by $u\,\phi$, where ϕ is a non-negative bump function, and integrating over in space and time variables. Clearly, if u belongs to a finite dimensional space X_h (say of finite elements), then $u\,\phi$ is not allowed to be used as a test function and one needs to project $u\,\phi$ back on X_h. This is the reason why results obtained in [13, 14] require the use of spaces satisfying a suitable commutator property, which is a local property, see Section 3.1. In particular, the standard Fourier-Galerkin method (which is not local being a spectral approximation) does not satisfy the commutator property and the convergence to a suitable weak solution is still an interesting open problem, see the partial results in Craig *et al.* [10], with interesting links with the global energy equality.

From the numerical point of view another important issue is that of considering also the time discretization, hence going from a semi-discrete scheme to a fully-discrete one. Also regarding this issue few results are available. In [7] it is proved that solutions of space-periodic Navier-Stokes constructed by semi-discretization (in the time variable, with the standard implicit Euler algorithm) are suitable. The argument has been also extended to a general domain in [4] assuming vorticity-based slip boundary conditions, which are important in the vanishing viscosity problem [5, 6]. The case of Dirichlet boundary conditions is still unsolved and one main difficulty is that of proving, at the discrete level, coercive pressure estimates, similar to those obtained by Sohr and Von Wahl [20] in the continuous case.

The aim of this paper is to extend the results from [13] and [7] to a general space-time numerical discretization with a general θ-method in the time-variable and finite elements in the space variables. The extension of the results regarding only on the space or only on the time discretization presents some additional difficulties and it is not just a combination of the previous ones. In particular, the main core of the proof is obtaining appropriate a-priori estimates and using compactness results. Contrary to [13, 7] results obtained here require a more subtle compactness argument for space-time discrete functions and the main theorem is obtained by using a technique borrowed from the treatment of non-homogeneous and compressible fluids and resembling the compensated compactness arguments, see P.L. Lions [15, Lemma 5.1]. Observe that here the simple convergence in the sense of distributions is not enough, contrary to the case of the product density/velocity in the weak formulation of the Navier-Stokes equations with variable density. We also observe that at present the extension to the Dirichlet problem, as in [14], seems a challenging problem; the estimates in the negative spaces obtained in [14] look not enough to handle the discretization in time made with step functions, which cannot be in fractional Sobolev spaces with order larger than one-half.

To set the problem we consider as in [13] two sequences of discrete approximation spaces $\{X_h\}_h \subset H^1_\#$ and $\{M_h\}_h \subset H^1_\#$ which satisfy –among other properties described in Section 3– an appropriate commutator property, see Definition 3.1. Then, given a net $t_m := m\,\Delta t$ we consider the following space-time discretization of the problem (1.1)-(1.2): Set $u_h^0 = P_h(u_0^h)$, where P_h is the projection over X_h. For any $m = 1, ..., N$ and given $u_h^{m-1} \in X_h$ and $p_h^{m-1} \in M_h$, find $u_h^m \in X_h$ and $p^m \in M_h$ such that

$$\begin{aligned} (d_t u_h^m, v_h) + (\nabla u_h^{m,\theta}, \nabla v_h) + b_h(u_h^{m,\theta}, u_h^{m,\theta}, v_h) - (p_{h,}^m, \operatorname{div} v_h) = 0,\\ (\operatorname{div} u_h^m, q_h) = 0, \end{aligned} \tag{1.3}$$

where $u_h^{m,\theta} := \theta\, u_h^m + (1-\theta)\, u_h^{m-1}$, $d_t u^m := \frac{u_h^m - u_h^{m-1}}{\Delta t}$, and $b_h(u_h^{m,\theta}, u_h^{m,\theta}, v_h)$ is a suitable discrete approximation of the non-linear term; see also Quarteroni and Valli [18] and Thomée [22] for general properties of θ-schemes for parabolic equations. We refer to Sections 2-3 for the other notations, definitions, and properties regarding (1.3). As usual in time-discrete problem (see for instance [21]) in order to study the convergence to the solutions of the continuous problem it is useful to rewrite (1.3) on $(0,T)$ as follows:

$$\begin{aligned} \left(\partial_t v_h^{\Delta t}, w_h\right) + \left(\nabla u_h^{\Delta t}, \nabla w_h\right) + b_h\left(u_h^{\Delta t}, u_h^{\Delta t}, w_h\right) - \left(p_h^{\Delta t}, \operatorname{div} q_h\right) &= 0, \\ \left(\operatorname{div} u_h^{\Delta t}, w_h\right) &= 0, \end{aligned} \tag{1.4}$$

where $v_h^{\Delta t}$ is the linear interpolation of $\{u_h^m\}_{m=1}^N$ (over the net $t_m = m\Delta t$), while $u_h^{\Delta t}$ and $p_h^{\Delta t}$ are the time-step functions which on the interval $[t_{m-1}, t_m)$ are equal to $u_h^{m,\theta}$ and p_h^m, respectively.

The main result of the paper is the following, we refer to Section 2 for further details on the notation.

Theorem 1.1. *Let the finite element space (X_h, M_h) satisfy the discrete commutation property, and the technical conditions described in Section 3.1. Let $u_0 \in H^1_{\text{div}}$ and $\theta \in (1/2, 1]$. Let $\{(v_h^{\Delta t}, u_h^{\Delta t}, p_h^{\Delta t})\}_{\Delta t, h}$ be a sequence of solutions of* (1.4) *computed by solving* (1.3). *Then, there exists*

$$(u,p) \in L^\infty(0,T; L^2_{\text{div}}) \cap L^2(0,T; H^1_{\text{div}}) \times L^{4/3}(0,T; L^2_{\#}),$$

such that, up to a sub-sequence, as $(\Delta t, h) \to (0,0)$,

$$\begin{aligned} &v_h^{\Delta t} \to u \text{ strongly in } L^2((0,T)\times\mathbb{T}^3), \\ &u_h^{\Delta t} \to u \text{ strongly in } L^2((0,T)\times\mathbb{T}^3), \\ &\nabla u_h^{\Delta t} \rightharpoonup \nabla u \text{ weakly in } L^2((0,T)\times\mathbb{T}^3), \\ &p_h^{\Delta t} \rightharpoonup p \text{ weakly in } L^{\frac{4}{3}}((0,T)\times\mathbb{T}^3). \end{aligned}$$

Moreover, the couple (u,p) is a suitable weak solution of (1.1)-(1.2) *in the sense of Definition 2.2.*

Remark 1.2. As explicit examples, the MINI and Hood-Taylor elements represents couples (X_h, M_h) of finite element spaces for which the theorem is valid.

The proof of Theorem 1.1 is given in Section 5 and it is based on a compactness argument. We briefly explain the main novelty in the proof: First from the standard discrete energy inequality (Lemma 4.1) only an H^1-bound in space on $u_h^{\Delta t}$ is available but no compactness in time. Note that this is not enough in general to deduce strong convergence of $u_h^{\Delta t}$, which turns out to be necessary to prove even the convergence to a Leray-Hopf weak solution. It is also relevant to observe that in many references (e.g. as in [21, 18]) authors focus on the order of the convergence between the numerical and continuous solution in the L^2-norm. Nevertheless, in our case it is very relevant to obtain the uniform $l^2(H^1_{\text{div}})$ bound on the numerical solution, since this is requested to show convergence to a weak solution in the genuine sense of Leray and Hopf. This explains the limitations on θ, which in this paper are not due to classical stability issues. Hence, in the proof of the main result also the Step 1 (that of proving that the numerical solutions converge to a weak solutions) is original, or at least we did not find this explicitly proved in any reference (In [21] a partial analysis of this point, valid only for certain schemes, is provided). On the other hand, from the equations (1.4) is possible to prove some mild time regularity on $v_h^{\Delta t}$; this will enough to ensure that the product $v_h^{\Delta t}\, u_h^{\Delta t}$ is weakly convergent in $L^1(\mathbb{T}^3)$ to $v\,u$, where v and u are the weak limit of $v_h^{\Delta t}$ and $u_h^{\Delta t}$, respectively. This is the technical point where results *à la* compensated compactness are used. Finally, this additional information combined again with discrete energy inequality allows to infer that $u = v$ and that $u_h^{\Delta t}$ is strongly convergent in $L^2((0,T)\times\mathbb{T}^3)$.

Plan of the paper. In Section 2 we fix the notation that we use in the paper and we recall the main definitions and tools used. In Section 3 we introduce and give some details about the space-time discretization methods. Finally, in Section 4 we prove the main *a priori* estimates needed to study the convergence and finally in Section 5 we prove Theorem 1.1.

2. Notations and Preliminaries

In this section we declare the notation we will use in the paper, we recall the main definitions concerning weak solutions of incompressible Navier-Stokes and also a compactness result.

2.1. **Notations.** We introduce the notations typical of space-periodic problems. The flat three-dimensional torus $\mathbb{T}^3$ is defined by $(\mathbb{R}/2\pi\mathbb{Z})^3$. In the sequel we will use the customary Lebesgue spaces $L^p(\mathbb{T}^3)$ and Sobolev spaces $W^{k,p}(\mathbb{T}^3)$ and we will denote their norms by $\|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}}$ We will not distinguish between scalar and vector valued functions, since it will be clear from the context which one has to be considered. In the case $p=2$, the $L^2(\mathbb{T}^3)$ scalar product is denoted by $(\cdot,\cdot)$ and we use the notation $H^s(\mathbb{T}^3) := W^{s,2}(\mathbb{T}^3)$ and we define, for $s>0$, the dual spaces $H^{-s}(\mathbb{T}^3) = (H^s(\mathbb{T}^3))'$. Moreover, we will consider always subspaces of functions with zero mean value and these will be denoted by

$$L^p_{\#} := \left\{ w \in L^p(\mathbb{T}^3) \quad \int_{\mathbb{T}^3} w\,dx = 0 \right\} \qquad 1 \le p < +\infty,$$

and also

$$H^s_{\#} := H^s(\mathbb{T}^3) \cap L^2_{\#}.$$

As usual in fluid mechanics one has to consider spaces of divergence free vector fields, defined as follows

$$L^2_{\mathrm{div}} := \left\{ w \in (L^2_{\#})^3 : \quad \operatorname{div} w = 0 \right\} \qquad \text{and} \qquad H^s_{\mathrm{div}} := H^s_{\#} \cap L^2_{\mathrm{div}}.$$

Finally, given X a Banach space, $L^p(0,T;X)$ denotes the classical Bochner spaces of X valued functions, endowed with its natural norm, denoted by $\|\cdot\|_{L^p(X)}$. We denote by $l^p(X)$ the discrete counterpart for X-valued sequences $\{x^m\}$, defined on the net $\{m\Delta t\}$ with weighted norm defined by $\|x\|^p_{l^p(X)} := \Delta t \sum_{m=0}^{M} \|x^m\|^p_X$.

2.2. **Weak solutions and suitable weak solutions.** We start by recalling the notion of weak solution (as introduced by Leray and Hopf) for the space periodic setting.

Definition 2.1. *The vector field u is a Leray-Hopf weak solution of* (1.1)-(1.2) *if*

$$u \in L^\infty(0,T;L^2_{\mathrm{div}}) \cap L^2(0,T;H^1_{\mathrm{div}}),$$

and if u satisfies the Navier-Stokes equations (1.1)-(1.2) *in the weak sense, namely the integral equality*

$$\int_0^T \big[(u,\partial_t\phi) - (\nabla u,\nabla\phi) - ((u\cdot\nabla)\,u,\phi) \big]\,dt + (u_0,\phi(0)) = 0, \tag{2.1}$$

holds true for all smooth, periodic, and divergence-free functions $\phi \in C^\infty_c([0,T);C^\infty(\mathbb{T}^3))$ such that $\int_{\mathbb{T}^3} \phi\,dx = 0$. Moreover, the initial datum is attained in the strong L^2-sense, that is

$$\lim_{t\to 0^+} \|u(t)-u_0\|_2 = 0,$$

and the following global energy inequality holds

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2\,ds \le \frac{1}{2}\|u_0\|_2^2, \qquad \textit{for all } t \in (0,T).$$

Suitable weak solutions are a particular subclass of Leray-Hopf weak solutions and the definition is the following.

Definition 2.2. *A pair (u,p) is a suitable weak solution to the Navier-Stokes equation* (1.1) *if u is a Leray-Hopf weak solution, $p \in L^{\frac{4}{3}}(0,T;L^2_{\#})$, and the local energy inequality*

$$\int_0^T \int_{\mathbb{T}^3} |\nabla u|^2 \phi \, dxdt \leq \int_0^T \int_{\mathbb{T}^3} \left[\frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \left(\frac{|u|^2}{2} + p \right) u \cdot \nabla \phi \right] dxdt. \tag{2.2}$$

holds for all $\phi \in C_0^\infty(0,T;C^\infty(\mathbb{T}^3))$ such that $\phi \geq 0$,

Remark 2.3. The definition of suitable weak solution is usually stated with $p \in L^{\frac{5}{3}}((0,T)\times\mathbb{T}^3)$ while in Definition 2.2 $p \in L^{\frac{4}{3}}(0,T;L^2(\mathbb{T}^3))$. This is not an issue since of course we have a bit less integrability in time but we gain a full L^2-integrability in space. We stress that the main property of suitable weak solutions is the fact that they satisfy the local energy inequality (2.2) and weakening the request on pressure does not influence the validity of local regularity results, see for instance discussion in Vasseur [23]

2.3. **A compactness lemma.** In this subsection we recall the main compactness lemma which allows us to prove the strong convergence of the approximations. We remark that it is a particular case of a more general lemma whose statement and proof can be found in [15, Lemma 5.1]. For sake of completeness we give a proof adapted to the case we are interested in.

Lemma 2.4. *Let $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be uniformly bounded in $L^\infty(0,T;L^2(\mathbb{T}^3))$ and let be given $f,g \in L^\infty(0,T;L^2(\mathbb{T}^3))$ such that*

$$\begin{aligned} &f_n \rightharpoonup f \ \textit{weakly in}\ L^2((0,T)\times\mathbb{T}^3), \\ &g_n \rightharpoonup g \ \textit{weakly in}\ L^2((0,T)\times\mathbb{T}^3). \end{aligned} \tag{2.3}$$

Let $p \geq 1$ and assume that

$$\{\partial_t f_n\}_n \subset L^p(0,T;H^{-1}(\mathbb{T}^3)), \qquad \{g_n\}_n \subset L^2(0,T;H^1(\mathbb{T}^3)), \tag{2.4}$$

with uniform (with respect to $n \in \mathbb{N}$) bounds on the norms. Then,

$$f_n\, g_n \rightharpoonup f\, g \ \textit{weakly in}\ L^1((0,T)\times\mathbb{T}^3). \tag{2.5}$$

Proof. By using (2.3), (2.4), and the fact that $L^2(\mathbb{T}^3)$ is compactly embedded in $H^{-1}(\mathbb{T}^3)$ it follows from the Banach space version of Arzelà-Ascoli theorem that

$$f_n \to f \text{ strongly in } C(0,T;H^{-1}(\mathbb{T}^3)). \tag{2.6}$$

From (2.4) it follows that

$$g_n \rightharpoonup g \text{ weakly in } L^2(0,T;H^1(\mathbb{T}^3)). \tag{2.7}$$

Then, (2.6) and (2.7) easily imply that

$$f_n\, g_n \rightharpoonup f\, g \text{ in the sense of distribution on } (0,T)\times\mathbb{T}^3.$$

The L^1-weak convergence in (2.5) follows by noting that the bounds in (2.4) imply that the sequence $\{f_n\, g_n\}_n$ is equi-integrable. ∎

3. Setting of the numerical approximation

In this section we introduce the time and space discretization of the initial value problem (1.1)-(1.2). We start by introducing the space discretization.

3.1. **Space discretization.** For the space discretization we strictly follow the setting considered in [13]. Let $\{X_h\}_{h>0} \subset H^1_\#$ be the discrete space for approximate velocity and $\{M_h\}_{h>0} \subset L^2_\#$ be that of approximate pressure. To avoid further technicalities, we assume as in [13], that $M_h \subset H^1_\#$.

We make the following (technical) assumptions on the spaces X_h and M_h:

(1) For any $v \in H^1_\#$ and for any $q \in L^2_\#$ there exists $\{v_h\}_h$ and $\{q_h\}_h$ with $v_h \in X_h$ and $q_h \in M_h$ such that

$$\begin{aligned} v_h \to v \quad &\text{strongly in } H^1_\# \quad \text{as } h \to 0, \\ q_h \to q \quad &\text{strongly in } L^2_\# \quad \text{as } h \to 0; \end{aligned} \tag{3.1}$$

(2) Let $\pi_h : L^2(\mathbb{T}^3) \to X_h$ be the L^2-projection onto X_h. Then, there exists $c > 0$ independent of h such that,

$$\forall q_h \in M_h \quad \|\pi_h(\nabla q_h)\|_2 \geq c\|q_h\|_2; \tag{3.2}$$

(3) There is c independent of h such that for all $v \in H^1_\#$

$$\begin{aligned} &\|v - \pi_h(v)\|_2 = \inf_{w_h \in X_h} \|v - w_h\|_2 \leq c\,h\|v\|_{H^1}, \\ &\|\pi_h(v)\|_{H^1} \leq c\|v\|_{H^1}; \end{aligned}$$

(4) There exists c independent of h such that

$$\|v_h\|_{H^1} \leq c\,h^{-1}\|v_h\|_2 \quad \forall v_h \in X_h. \tag{3.3}$$

Moreover, we assume that X_h and M_h satisfy the following discrete commutator property.

Definition 3.1. *We say that X_h (resp. M_h) has the discrete commutator property if there exists an operator $P_h \in \mathcal{L}(H^1; X_h)$ (resp. $Q_h \in \mathcal{L}(L^2; M_h)$) such that for all $\phi \in W^{2,\infty}$ (resp. $\phi \in W^{1,\infty}$) and all $v_h \in X_h$ (resp. $q_h \in M_h$)*

$$\|v_h\phi - P_h(v_h\phi)\|_{H^l} \leq ch^{1+m-l}\|v_h\|_{H^m}\|\phi\|_{W^{m+1,\infty}}, \tag{3.4}$$

$$\|q_h\phi - Q_h(q_h\phi)\|_2 \leq ch\|q_h\|_2\|\phi\|_{W^{1,\infty}}, \tag{3.5}$$

for all $0 \leq l \leq m \leq 1$.

Remark 3.2. We want to stress that in the case of spectral method, e.g. the Galerkin methods based on Fourier expansion on the torus, the discrete commutator property fails. This is one of the main obstacles in proving the local energy inequality for weak solutions of (1.1) constructed by the Fourier-Galerkin method.

We recall from [13] that the coercivity hypothesis (3.2), allows us to define the map $\psi_h : H^2_\# \to M_h$ such that, for all $q \in H^2_\#$, the function $\psi_h(q)$ is the unique solution to the problem:

$$(\pi_h(\nabla\psi_h(q)), \nabla r_h) = (\nabla q, \nabla r_h)\,. \tag{3.6}$$

This map has the following properties: there exists c independent of h such that for all $q \in H^2_\#$,

$$\begin{aligned} &\|\nabla(\psi_h(q) - q)\|_2 \leq c\,h\|q\|_{H^2}, \\ &\|\pi_h\nabla\psi_h(q)\|_{H^1} \leq c\|q\|_{H^2}. \end{aligned} \tag{3.7}$$

Let us introduce

$$V_h = \left\{v_h \in X_h : (\operatorname{div} v_h, q_h) = 0 \quad \forall q_h \in L^2(\Omega)\right\}.$$

To have the basic energy estimate we need to modify the non-linear term since V_h is not a subspace of H^1_{div}. Let us define the following

$$nl_h(u, v) := (u \cdot \nabla)\,v + \frac{1}{2}v \operatorname{div} u. \tag{3.8}$$

Then, nl_h is a bi-linear operator

$$nl_h : H^1_\# \times H^1_\# \to H^{-1},$$

where $H^{-1} := (H^1_\#)'$. Moreover, the following estimate holds true

$$\|nl_h(u,v)\|_{H^{-1}} \leq \|u\|_{L^3}\|v\|_{H^1} \quad \forall\, u,v \in H^1_\#. \tag{3.9}$$

Finally, by defining $b_h(u,v,w) := \langle nl_h(u,v), w\rangle_{H^{-1}\times H^1_\#}$, it follows that

$$b_h(u,v,v) = 0 \qquad \forall\, u,\, v \in H^1_{\mathrm{div}} + V_h.$$

Then, the space discretization of (1.1)-(1.2) reads as follows:
Find $u_h \in C(0,T;X_h)$ with $\partial_t u_h \in L^2(0,T;X_h)$ and $p_h \in L^2(0,T;M_h)$ such that, for all $v_h \in X_h$ and $q_h \in M_h$:

$$\begin{aligned}(\partial_t u_h, v_h) + b_h(u_h,u_h,v_h) - (p_h, \operatorname{div} v_h) + (\nabla u_h, \nabla v_h) &= 0,\\ (\operatorname{div} u_h, q_h) &= 0.\end{aligned} \tag{3.10}$$

with the initial datum

$$u_h|_{t=0} = u_0^h,$$

where u_0^h is an approximation of u_0 such that $u_0^h \in X_h$, and

$$u_0^h \to u_0 \text{ strongly in } L^2_{\mathrm{div}} \quad \text{ as } h \to 0.$$

3.2. **Time discretization.** For the time variable t we define the mesh as follows: Given $N \in \mathbb{N}$ the time-step $0 < \Delta t \leq T$ is defined as $\Delta t := T/N$. Accordingly, we define the corresponding net $\{t_m\}_{m=1}^N$ by

$$t_0 := 0 \qquad t_m := m\,\Delta t, \qquad m = 0,\dots,N.$$

To discretize in time the semi-discrete problem (3.10) we consider the following θ-method (cf. [18, § 5.6.2]) for $\theta \in [0,1]$:
Set $u_h^0 = u_0 \in H^1_{\mathrm{div}}$. For any $m = 1,...,N$ given $u_h^{m-1} \in X_h$ and $p_h^{m-1} \in M_h$ find $u_h^m \in X_h$ and $p_h^m \in M_h$ such that

$$\begin{aligned}(d_t u_h^m, v_h) + (\nabla u_h^{m,\theta}, \nabla v_h) + b_h(u_h^{m,\theta}, u_h^{m,\theta}, v_h) - (p_{h,}^m, \operatorname{div} v_h) &= 0,\\ (\operatorname{div} u_h^m, q_h) &= 0,\end{aligned} \tag{3.11}$$

for all $v_h \in X_h$ and for all $q_h \in M_h$. We recall that here $d_t u^m$ is the backward finite-difference approximation for the time-derivative in the interval (t_{m-1}, t_m)

$$\partial_t u_h \sim d_t u^m := \frac{u_h^m - u_h^{m-1}}{\Delta t},$$

and $u_h^{m,\theta} := \theta u_h^m + (1-\theta)u_h^{m-1}$ is the convex combination. With a slight abuse of notation we consider $\Delta t = T/N$ and h, instead of (N,h), as the indexes of the sequences for which we prove the convergence. Then, the convergence will be proved in the limit as $(\Delta t, h)$ both going to zero. We stress that this does not affect the proofs since all the convergences are proved up to sub-sequences.

Once (3.11) is solved, we consider a continuous version useful to study the convergence. To this end we associate to the triple $(u_h^{m,\theta}, u_h^m, p_h^m)$ the functions

$$(v_h^{\Delta t}, u_h^{\Delta t}, p_h^{\Delta t}) : [0,T] \times \mathbb{T}^3 \to \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R},$$

defined as follows:

$$
\begin{aligned}
v_h^{\Delta t}(t) &:= \begin{cases} u_h^{m-1} + \dfrac{t-t_{m-1}}{\Delta t}(u_h^m - u_h^{m-1}) & \text{for } t \in [t_{m-1}, t_m), \\ u_h^N & \text{for } t = t_N, \end{cases} \\
u_h^{\Delta t}(t) &:= \begin{cases} u_h^{m,\theta} & \text{for } t \in [t_{m-1}, t_m), \\ u_h^{N,\theta} & \text{for } t = t_N, \end{cases} \\
p_h^{\Delta t}(t) &:= \begin{cases} p_h^m & \text{for } t \in [t_{m-1}, t_m), \\ p_h^N & \text{for } t = t_N. \end{cases}
\end{aligned}
\tag{3.12}
$$

Then, the discrete equations (3.11) can be rephrased as as follows:

$$
\begin{aligned}
\left(\partial_t v_h^{\Delta t}, w_h\right) + b_h\left(u_h^{\Delta t}, u_h^{\Delta t}, w_h\right) + \left(\nabla u_h^{\Delta t}, \nabla w_h\right) - \left(p_h^{\Delta t}, \operatorname{div} q_h\right) &= 0, \\
\left(\operatorname{div} u_h^{\Delta t}, w_h\right) &= 0,
\end{aligned}
\tag{3.13}
$$

for all $w_h \in L^s(0,T;X_h)$ (with $s \geq 4$) and for all $q_h \in L^2(0,T;M_h)$. We notice that the divergence-free condition comes from the fact that u_h^m is such that

$$
(\operatorname{div} u_h^m, q_h) = 0 \qquad \text{for } m = 1, ..., N, \ \forall\, q_h \in M_h.
$$

4. A priori estimates

In this section we prove the *a priori* estimates that we need to study the convergence of solutions of (3.13) to suitable weak solutions of (1.1)-(1.2). We start with the following discrete energy equality.

Lemma 4.1. *For any $1/2 < \theta \leq 1$, $N \in \mathbb{N}$, and $m = 1, .., N$ the following discrete energy-type equality holds true*

$$
\frac{1}{2}(\|u_h^m\|_2^2 - \|u_h^{m-1}\|_2^2) + \frac{(2\theta-1)}{2}\|u_h^m - u_h^{m-1}\|_2^2 + \Delta t\, \|\nabla u_h^{m,\theta}\|_2^2 = 0. \tag{4.1}
$$

Proof. For any $m = 1, ..., N$ take $w_h = \chi_{[t_{m-1}, t_m)} u_h^{m,\theta} \in L^\infty(0,T;X_h)$ in (3.13). Then,

$$
\left(\frac{u_h^m - u_h^{m-1}}{\Delta t}, u_h^{m,\theta}\right) + \|\nabla u_h^{m,\theta}\|_2^2 = 0,
$$

because since $u_h^{m,\theta} \in X_h$ and $p_h^m \in M_h$ it follows that

$$
b_h(u_h^{m,\theta}, u_h^{m,\theta}, u_h^{m,\theta}) = 0 \qquad \text{and} \qquad (p_h^m, \operatorname{div} u_h^{m,\theta}) = 0.
$$

By using the elementary algebraic identity

$$
(a-b, a) = \frac{|a|^2}{2} - \frac{|b|^2}{2} + \frac{|a-b|^2}{2},
$$

the term involving the discrete derivative reads as follows:

$$
\begin{aligned}
(u_h^m - u_h^{m-1}, u_h^{m,\theta}) &= (u_h^m - u_h^{m-1}, \theta u_h^m + (1-\theta) u_h^{m-1}) \\
&= \theta(u_h^m - u_h^{m-1}, u_h^m) + (1-\theta)(u_h^{m-1} - u_h^m, u_h^{m-1}) \\
&= \frac{\theta}{2}(\|u_h^m\|_2^2 - \|u_h^{m-1}\| + \|u_h^m - u_h^{m-1}\|_2^2) \\
&\quad - \frac{(1-\theta)}{2}(\|u_h^{m-1}\|_2^2 - \|u_h^m\| + \|u_h^m - u_h^{m-1}\|_2^2) \\
&= \frac{1}{2}(\|u_h^m\|_2^2 - \|u_h^{m-1}\|_2^2) + \frac{(2\theta-1)}{2}(\|u_h^m - u_h^{m-1}\|_2^2).
\end{aligned}
$$

Then, multiplying by $\Delta t > 0$, Eq. (4.1) holds true. In addition, summing over m we also get

$$\frac{1}{2}\|u_h^N\|_2^2 + \frac{(2\theta-1)}{2}\sum_{m=0}^{N}\|u_h^m - u_h^{m-1}\|_2^2 + \Delta t\sum_{m=0}^{N}\|\nabla u_h^{m,\theta}\|_2^2 = \frac{1}{2}\|u_h^0\|_2^2,$$

which proves the $l^\infty(L^2_\#) \cap l^2(H^1_\#)$ uniform bound for the sequence $\{u_h^m\}$. ∎

Remark 4.2. Strictly speaking the requirement $\theta > 1/2$ is not required for the proof of Lemma 4.1. However, it is needed in order to deduce the most important *a priori* estimates since it makes the coefficient of the second term from the left-hand side of (4.1) positive. Moreover, since we actually need that term in the convergence proof to a weak solution, we cannot consider the endpoint case $\theta = 1/2$.

The next lemma concerns the regularity of the pressure. We follow the argument in [13] and we notice that we are essentially solving the standard discrete Poisson problem associated to the pressure.

Lemma 4.3. *There exists a constant $c > 0$, independent of Δt and of h, but eventually depending on θ, such that*

$$\|p_h^m\|_2 \le c\left(\|u_h^{m,\theta}\|_{H^1} + \|u_h^{m,\theta}\|_{L^3}\|u_h^{m,\theta}\|_{H^1}\right) \quad \text{for } m = 1, \dots, N. \tag{4.2}$$

Proof. Let $q^m \in H^2_\#$ be the unique solution of the following Poisson problem:

$$(\nabla q^m, \nabla\phi) = (p_h^m, \phi) \qquad \forall\, \phi \in H^1_\#. \tag{4.3}$$

Standard elliptic estimates imply there exists an absolute constant $c > 0$ such that

$$\|q^m\|_{H^2} \le c\|p_h^m\|_2. \tag{4.4}$$

Let us consider $\pi_h\nabla(\psi_h(q^m)) \in X_h$ as a test function in (3.11), then we get

$$\begin{aligned}(d_t u^m, \pi_h\nabla(\psi_h(q^m))) &- (\nabla u_h^{m,\theta}, \nabla\pi_h\nabla(\psi_h(q^m)))\\ &+ b_h(u_h^{m,\theta}, u_h^{m,\theta}, \pi_h\nabla(\psi_h(q^m))) + (p_h^m, \operatorname{div}\pi_h\nabla(\psi_h(q^m))) = 0.\end{aligned}$$

First, by using (3.6) and (4.3) we get

$$(p_h^m, \operatorname{div}\pi_h\nabla(\psi_h(q^m))) = (\nabla p_h^m, \pi_h\nabla(\psi_h(q^m))) = (\nabla p_h^m, \nabla q^m) = (p_h^m, p_h^m) = \|p_h^m\|_2^2.$$

Then, we get

$$\begin{aligned}\|p_h^m\|_2^2 =& (\nabla u_h^{m,\theta}, \nabla\pi_h\nabla(\psi_h(q^m))) - (d_t u^m, \pi_h\nabla(\psi_h(q^m)))\\ &- b_h(u_h^m, u_h^m, \pi_h\nabla(\psi_h(q^m))).\end{aligned}$$

By using (3.7) and (4.4) we have

$$\begin{aligned}\left|\left(\nabla u_h^{m,\theta}, \nabla\pi_h\nabla(\psi_h(q^m))\right)\right| &\le \|\nabla u_h^{m,\theta}\|_2\|\pi_h\nabla(\psi_h(q^m))\|_2\\ &\le C\|\nabla u_h^{m,\theta}\|_2\|q^m\|_{H^2}\\ &\le C\|\nabla u_h^{m,\theta}\|_2\|p_h^m\|_2.\end{aligned}$$

Concerning the term involving the discrete time-derivative we have

$$\left(u_h^m - u_h^{m-1}, \pi_h\nabla(\psi_h(q^m))\right) = \left(u_h^m - u_h^{m-1}, \nabla(\psi_h(q^m))\right) = -\left(\operatorname{div}(u_h^m - u_h^{m-1}), \psi_h(q^m)\right) = 0.$$

Finally, regarding the non-linear term by using (3.9) and (3.7) we have

$$\begin{aligned}|b_h(u_h^{m,\theta}, u_h^{m,\theta}, \pi_h\nabla(\psi_h(q^m)))| &\le \left|\left\langle nl_h(u_h^{m,\theta}, u_h^{m,\theta}), \pi_h\nabla(\psi_h(q^m))\right\rangle\right|\\ &\le C\|u_h^{m,\theta}\|_{L^3}\|u_h^{m,\theta}\|_{H^1}\|p_h^m\|_2.\end{aligned}$$

Then,

$$\|p_h^m\|_2^2 \le c\left(\|u_h^{m,\theta}\|_{L^3}\|u_h^{m,\theta}\|_{H^1} + \|u_h^{m,\theta}\|_{H^1}\right)\|p_h^m\|_2,$$

and (4.2) readily follows. ∎

We are now in position to prove the main a priori estimates on the approximate solutions of (3.13).

Proposition 4.4. *Let be given $u_0 \in L^2_{\mathrm{div}}$ and $1/2 < \theta \leq 1$. Then, there exists a constant $c > 0$, independent of Δt and of h, such that*

a) $\|v_h^{\Delta t}\|_{L^\infty(L^2)} \leq c$,
b) $\|u_h^{\Delta t}\|_{L^\infty(L^2)\cap L^2(H^1)} \leq c$,
c) $\|p_h^{\Delta t}\|_{L^{4/3}(L^2)} \leq c$,
d) $\|\partial_t v_h^{\Delta t}\|_{L^{4/3}(H^{-1})} \leq c$.

Moreover, we also have the following estimate

$$\int_0^T \|v_h^{\Delta t} - u_h^{\Delta t}\|_2^2 \, dt \leq \Delta t \left(\frac{1}{3} - \theta + \theta^2 \right) \sum_{m=1}^{N} \|u_h^m - u_h^{m-1}\|_2^2. \tag{4.5}$$

Proof. The bound in $L^\infty(0,T;L^2_\#) \cap L^2(0,T;H^1_\#)$ for $v_h^{\Delta t}$ follows from (3.12) and Lemma 4.1, as well as the bounds on $u_h^{\Delta t}$ in b). The bound on the pressure $p_h^{\Delta t}$ follows again from (3.12) and Lemma 4.3. Finally, the bound on the time derivative of $v_h^{\Delta t}$ follows by (3.13) and a standard comparison argument. Concerning (4.5), by using the definitions in (3.12) we get for $t \in [t_{m-1}, t_m)$

$$\begin{aligned} v_h^{\Delta t} - v_h^{\Delta t} &= \theta\, u_h^m + (1-\theta)\, u_h^{m-1} - u_h^{m-1} - \frac{t - t_{m-1}}{\Delta t}(u_h^m - u_h^{m-1}) \\ &= \left(\theta - \frac{t - t_{m-1}}{\Delta t} \right) \left(u_h^m - u_h^{m-1} \right). \end{aligned}$$

Then, evaluating the integrals, we have

$$\begin{aligned} \int_0^T \|v_h^{\Delta t} - v_h^{\Delta t}\|_2^2 \, dt &= \sum_{m=1}^{N} \|u_h^m - u_h^{m-1}\|_2^2 \int_{t_{m-1}}^{t_m} \left(\theta - \frac{t - t_{m-1}}{\Delta t} \right)^2 dt \\ &\leq \Delta t \left(\frac{1}{3} - \theta + \theta^2 \right) \sum_{m=1}^{N} \|u_h^m - u_h^{m-1}\|_2^2, \end{aligned}$$

ending the proof. ∎

5. Proof of the main theorem

In this section we prove Theorem 1.1. We split the proof in the two main steps.

Proof of Theorem 1.1. We first prove the convergence of the numerical sequence to a Leray-Hopf weak solution and then we prove that the weak solution constructed is suitable, namely it satisfies the local energy inequality (2.2).

Step 1: Convergence towards a Leray-Hopf weak solution. We start by observing that from a simple density argument, the test functions considered in (2.1) can be chosen in the space $L^s(0,T;H^1_{\mathrm{div}}) \cap C^1(0,T;L^2_{\mathrm{div}})$, with $s \geq 4$. In particular, by using (3.1) for any $w \in L^s(0,T;H^1_{\mathrm{div}}) \cap C^1(0,T;L^2_{\mathrm{div}})$ such that $w(T,x) = 0$ we can find a sequence $\{w_h\}_h \subset L^s(0,T;H^1_\#) \cap C(0,T;L^2_\#)$ such that

$$\begin{aligned} &w_h \to w \text{ strongly in } L^s(0,T;H^1_\#) \quad \text{as } h \to 0, \\ &w_h(0) \to w(0) \text{ strongly in } L^2_\# \quad \text{as } h \to 0. \end{aligned} \tag{5.1}$$

Let $\{(v_h^{\Delta t}, v_h^{\Delta t}, p_h^{\Delta t})\}_{(\Delta t,h)}$, defined as in (3.12), be a family of solutions of (3.13). By Proposition 4.4-a) we have that

$$\{v_h^{\Delta t}\}_{(\Delta t,h)} \subset L^\infty(0,T;L^2_\#), \qquad \text{with uniform bounds on the norms.}$$

Then, by standard compactness arguments there exists $v \in L^\infty(0,T;L^2_\#)$, such that (up to a sub-sequence)

$$v_h^{\Delta t} \rightharpoonup v \text{ weakly in } L^2(0,T;L^2_\#) \quad \text{as } (\Delta t, h) \to (0,0). \tag{5.2}$$

Again by using Proposition 4.4 b), there exists $u \in L^\infty(0,T;L^2_\#)$ such that (up to a sub-sequence)

$$\begin{aligned} u_h^{\Delta t} &\rightharpoonup u \text{ weakly in } L^2(0,T;L^2_\#) \quad \text{as } (\Delta t, h) \to (0,0),\\ u_h^{\Delta t} &\rightharpoonup u \text{ in } L^2(0,T;H^1_\#) \quad \text{as } (\Delta t, h) \to (0,0). \end{aligned} \tag{5.3}$$

Moreover, by using (3.1), for any $q \in L^2(0,T;L^2_\#)$ we can find a sequence $\{q_h\}_h \subset L^2(0,T;L^2_\#)$ such that $q_h \in L^2(0,T;M_h)$ and

$$q_h \to q \text{ strongly in } L^2(0,T;L^2_\#) \quad \text{as } h \to 0.$$

Then, by using (5.3) and (3.13) we have that

$$0 = \int_0^T \left(\nabla \cdot u_h^{\Delta t}, q_h\right) dt \to \int_0^T (\nabla \cdot u, q)\, dt \quad \text{as } (\Delta t, h) \to (0,0),$$

hence u is divergence-free, belonging to H^1_{div}. Let us consider (4.5), then

$$\int_0^T \|v_h^{\Delta t} - u_h^{\Delta t}\|_2^2\, dt = \Delta t \left(\frac{1}{3} - \theta + \theta^2\right) \sum_{m=1}^N \|u^m - u^{m-1}\|_2^2 \le c\, \Delta t, \tag{5.4}$$

where in the last inequality we used Lemma 4.1. Hence, the integral $\int_0^T \|v_h^{\Delta t} - u_h^{\Delta t}\|_2^2\, dt$ vanishes as $\Delta t \to 0$. Then, by using (5.2) and (5.3) it easily follows that $v = u$.

At this point we note that Proposition 4.4 b) and d) imply that (with uniform bounds)

$$\{\partial_t v_h^{\Delta t}\}_{(\Delta t,h)} \subset L^{\frac{4}{3}}(0,T;H^{-1}) \quad \text{and} \quad \{u_h^{\Delta t}\}_{(\Delta t,h)} \subset L^2(0,T;H^1_\#).$$

Then, by using Lemma 2.4 and the fact that $u = v$ we get that

$$u_h^{\Delta t}\, v_h^{\Delta t} \rightharpoonup |u|^2 \text{ weakly in } L^1((0,T) \times \mathbb{T}^3) \quad \text{as } (\Delta t, h) \to (0,0). \tag{5.5}$$

In particular, by using (5.4) and (5.5) we have that

$$\begin{aligned} v_h^{\Delta t} &\to u \text{ strongly in } L^2(0,T;L^2_\#) \quad \text{as } (\Delta t, h) \to (0,0),\\ u_h^{\Delta t} &\to u \text{ strongly in } L^2(0,T;L^2_\#) \quad \text{as } (\Delta t, h) \to (0,0). \end{aligned} \tag{5.6}$$

Concerning the pressure term the uniform bound in Proposition 4.4 d) ensures the existence of $p \in L^{\frac{4}{3}}(0,T;L^2_\#)$ such that (up to a sub-sequence)

$$p_h^{\Delta t} \rightharpoonup p \text{ weakly in } L^{\frac{4}{3}}(0,T;L^2_\#) \quad \text{as } (\Delta t, h) \to (0,0). \tag{5.7}$$

Then, by using (5.1) and (5.2) we have that

$$\begin{aligned} \lim_{(\Delta t,h)\to(0,0)} \int_0^T (\partial_t v_h^{\Delta t}, w_h)\, dt &= \lim_{(\Delta t,h)\to(0,0)} \left(- \int_0^T (v_h^{\Delta t}, \partial_t w_h)\, dt + (u_0, w_h(0)) \right) \\ &\quad - \int_0^T (u, \partial_t w)\, dt + (u_0, w(0)), \end{aligned}$$

Next, by using (5.3) we also get

$$\lim_{(\Delta t,h)\to(0,0)} \int_0^T (\nabla v_h^{\Delta t}, \nabla w_h)\, dt = \int_0^T (\nabla u, \nabla w)\, dt.$$

By (5.1), (5.7), and the fact that w is (weakly) divergence-free we obtain

$$\int_0^T (p_h^{\Delta t}, \operatorname{div} w_h)\, dt \to 0 \qquad \text{as } (\Delta t, h) \to (0,0).$$

Concerning the non-linear term, let $s \geq 4$ and s' and s^* be real numbers such that

$$\frac{1}{s} + \frac{1}{s'} = 1, \quad \text{and} \quad \frac{1}{s} + \frac{1}{s^*} = \frac{1}{2}. \tag{5.8}$$

By using (5.6), (5.3), a standard interpolation argument, and Proposition 4.4 b) it follows that

$$u_h^{\Delta t} \to u \text{ strongly in } L^{s^*}(0,T; L^3_\#) \qquad \text{as } (\Delta t, h) \to (0,0),$$

and by (3.8) with a standard compactness argument

$$nl_h\left(u_h^{\Delta t}, u_h^{\Delta t}\right) \rightharpoonup u \cdot \nabla u, \text{ in } L^{s'}(0,T; H^{-1}) \qquad \text{as } (\Delta t, h) \to (0,0).$$

Then, by using also (5.1) it follows that

$$\int_0^T b_h(u_h^{\Delta t}, u_h^{\Delta t}, w_h)\, dt \to \int_0^T \big((u \cdot \nabla)\, u, w\big)\, dt \qquad \text{as } (\Delta t, h) \to (0,0).$$

Finally, the energy inequality follows by Lemma 4.1, by using the lower semicontinuity of the L^2-norm with respect to the weak convergence.

Step 2: Proof of the Local Energy Inequality. In order to conclude the proof of Theorem 1.1 we need to prove that the Leray-Hopf weak solution constructed in Step 1 is suitable. According to Definition 2.2 this requires just to prove that the local energy inequality. To this end, let us consider a smooth, periodic in the space variable function $\phi \geq 0$, vanishing for $t = 0, T$, and use as test function $P_h(u_h^{\Delta t}\phi)$ in the momentum equation in (3.13).

We first consider the term involving the time derivative, which we handle as follows:

$$\begin{aligned}
\int_0^T \left(\partial_t v_h^{\Delta t}, P_h(u_h^{\Delta t}\phi)\right)\, dt &= \int_0^T \left(\partial_t v_h^{\Delta t}, P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi + u_h^{\Delta t}\phi\right)\, dt \\
= \int_0^T \left(\partial_t v_h^{\Delta t}, u_h^{\Delta t}\phi\right) dt &+ \int_0^T \left(\partial_t v_h^{\Delta t}, P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi\right)\, dt =: I_1 + I_2.
\end{aligned}$$

Concerning the term I_1 we have that

$$\begin{aligned}
I_1 &= \int_0^T (\partial_t v_h^{\Delta t}, (v_h^{\Delta t} - v_h^{\Delta t} + u_h^{\Delta t})\, \phi)\, dt \\
&= \int_0^T (\partial_t v_h^{\Delta t}, v_h^{\Delta t})\, \phi\, dt + \int_0^T (\partial_t v_h^N, (u_h^{\Delta t} - v_h^{\Delta t})\, \phi)\, dt \\
&=: I_{11} + I_{12}.
\end{aligned}$$

Let us first consider I_{11}. By splitting the integral over $[0,T]$ as the sum of integrals over $[t_{m-1}, t_m]$ and, by integrating by parts, we get

$$\begin{aligned}
\int_0^T (\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi)\, dt &= \sum_{m=1}^N \int_{t_{m-1}}^{t_m} (\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi)\, dt = \sum_{m=1}^N \int_{t_{m-1}}^{t_m} (\frac{1}{2}\partial_t |v_h^{\Delta t}|^2, \phi)\, dt \\
= \frac{1}{2}\sum_{m=1}^N (|u_h^m|^2, \phi(t_m, x)) &- (|u_h^{m-1}|^2, \phi(t_{m-1}, x)) - \sum_{m=1}^N \int_{t_{m-1}}^{t_m} (\frac{1}{2}|v_h^{\Delta t}|^2, \partial_t \phi)\, dt,
\end{aligned}$$

where we used that $\partial_t v_h^{\Delta t}(t) = \frac{u_h^m - u_h^{m-1}}{\Delta t}$, for $t \in [t_{m-1}, t_m[$. Next, since the sum telescopes and ϕ is with compact support in $(0,T)$ we get

$$\int_0^T (\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi)\, dt = -\int_0^T (\frac{1}{2}|v_h^{\Delta t}|^2, \partial_t \phi)\, dt.$$

By the strong convergence of $v_h^{\Delta t} \to u$ in $L^2(0,T;L^2_\#)$ we can conclude that

$$\lim_{(\Delta t,h)\to(0,0)} \int_0^T (\partial_t v_h^{\Delta t}, v_h^{\Delta t}\phi)\,dt = -\int_0^T \big(\frac{1}{2}|u|^2, \partial_t\phi\big)\,dt.$$

Then, we consider the term I_{12}. Since $u_h^{\Delta t}$ is constant on the interval $[t_{m-1}, t_m[$ we can write

$$\begin{aligned}
&\int_0^T (\partial_t v_h^{\Delta t}, (u_h^{\Delta t} - v_h^{\Delta t})\,\phi)\,dt = -\sum_{m=1}^N \int_{t_{m-1}}^{t_m} (\partial_t(v_h^{\Delta t} - u_h^{\Delta t}), (v_h^{\Delta t} - u_h^{\Delta t})\,\phi)\,dt\\
&= -\sum_{m=1}^N \int_{t_{m-1}}^{t_m} \left(\partial_t \frac{|v_h^{\Delta t} - u_h^{\Delta t}|^2}{2}, \phi\right)\,dt\\
&= \sum_{m=1}^N \int_{t_{m-1}}^{t_m} \left(\frac{|v_h^{\Delta t} - u_h^{\Delta t}|^2}{2}, \partial_t\phi\right)\,dt,\\
&- \sum_{m=1}^N \left(\frac{|v_h^{\Delta t}(t_m) - u_h^{\Delta t}(t_m)|^2}{2}, \phi(t_m)\right) - \left(\frac{|v_h^{\Delta t}(t_{m-1}) - u_h^{\Delta t}(t_{m-1})|^2}{2}, \phi(t_m)\right),
\end{aligned}$$

where in the last line we have used the fact we do not have boundary terms because the sum telescopes and ϕ has compact support in $(0,T)$. Then, since $u_h^{\Delta t} - v_h^{\Delta t}$ vanishes (strongly) in $L^2(0,T;L^2_\#)$, we get that $I_{12} \to 0$ as $(\Delta t, h) \to (0,0)$.

We have that the $I_2 \to 0$ as $(\Delta t, h) \to (+\infty, 0)$. Indeed, by the discrete commutator property (3.4), Proposition 4.4, and the inverse inequality (3.3) we can infer

$$\begin{aligned}
\big|I_2\big| &= \left|\int_0^T \big(\partial_t v_h^{\Delta t}, P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi\big)\;dt\right|\\
&\leq \int_0^T \|\partial_t v_h^{\Delta t}\|_{H^{-1}} \|P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi\|_{H^1} dt\\
&\leq ch\|\partial_t v_h^{\Delta t}\|_{L^{\frac{4}{3}}(H^{-1})} \|u_h^{\Delta t}\|_{L^4(H^1)}\\
&\leq ch^{\frac{1}{2}}\|\partial_t v_h^{\Delta t}\|_{L^{\frac{4}{3}}(H^{-1})} \|u_h^{\Delta t}\|_{L^\infty(L^2)} \|u_h^{\Delta t}\|_{L^2(H^1)} \leq c\,h^{\frac{1}{2}}.
\end{aligned}$$

Hence, also this term vanishes as $h \to 0$ and this concludes the considerations for the term involving the time-derivative.

Concerning the viscous term, by adding and subtracting $\nabla u_h^{\Delta t}\phi$ we obtain the following three terms:

$$\begin{aligned}
(\nabla u_h^{\Delta t}, \nabla P_h(u_h^{\Delta t}\phi)) &= (\nabla u_h^{\Delta t}, \nabla(u_h^{\Delta t}\phi)) + (\nabla u_h^{\Delta t}, \nabla(P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi))\\
&= (|\nabla u_h^{\Delta t}|^2, \phi) - (\frac{1}{2}|u_h^{\Delta t}|^2, \Delta\phi) + R_{visc},
\end{aligned}$$

where the "viscous remainder" R_{visc} is defined as

$$R_{visc} := \big(\nabla u_h^{\Delta t}, \nabla[P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi]\big),$$

Since $u_h^{\Delta t}$ converges to u weakly in $L^2(0,T;H^1_\#)$ and strongly in $L^2(0,T;L^2_\#)$, by integrating over $(0,T)$ we can infer the following two results:

$$\begin{aligned}
\liminf_{(\Delta t,h)\to(0,0)} \int_0^T (|\nabla u_h^{\Delta t}|^2, \phi)\,dt &\geq \int_0^T (|\nabla u|^2, \phi)\,dt,\\
\int_0^T (\frac{1}{2}|u_h^{\Delta t}|^2, \Delta\phi)\,dt &\to \int_0^T (\frac{1}{2}|u|^2, \Delta\phi)\,dt,
\end{aligned}$$

where in the first inequality we used that ϕ is non-negative. For the remainder R_{visc}, by using again the discrete commutator property from Definition 3.1, we have that

$$\left|\int_0^T R_{visc}\,dt\right| \leq c\,h\int_0^T \|\nabla u_h^{\Delta t}\|^2\,dt \to 0 \qquad \text{as } (\Delta t, h)\to(0,0).$$

We consider now the nonlinear term b_h. We have

$$\begin{aligned} b_h(u_h^{\Delta t}, u_h^{\Delta t}, P_h(u_h^{\Delta t}\phi)) &= b_h(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{N}\phi) + b_h(u_h^{\Delta t}, u_h^{\Delta t}, P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi) \\ &= b_h(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{\Delta t}\phi) + R_{nl}. \end{aligned} \tag{5.9}$$

The "nonlinear remainder" $R_{nl} := b_h(u_h^{\Delta t}, u_h^{\Delta t}, P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi)$ can be estimated by using (3.9), the discrete commutator property, and (3.3). Indeed, we have

$$\begin{aligned} |R_{nl}| &\leq \|nl_h(u_h^{\Delta t}, u_h^{\Delta t})\|_{H^{-1}}\|P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi\|_{H^1} \\ &\leq c\,h\|u_h^{\Delta t}\|_3\|u_h^{\Delta t}\|_{H^1}\|u_h^{\Delta t}\|_{H^1} \\ &\leq c\,h\|u_h^{\Delta t}\|_2^{1/2}\|u_h^{\Delta t}\|_{H^1}^{1/2}\|u_h^{\Delta t}\|_{H^1}^2, \end{aligned} \tag{5.10}$$

hence, by integrating in time

$$\int_0^T R_{nl}\,dt \to 0 \quad \text{as } (\Delta t, h)\to(0,0).$$

The definition of nl_h in (3.8) allows us to handle the first term on the right hand side in (5.9) as follows.

$$\begin{aligned} b_h(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{\Delta t}\phi) &= \left((u_h^{\Delta t}\cdot\nabla)\,u_h^{\Delta t}, u_h^{\Delta t}\phi\right) + \frac{1}{2}\left((u_h^{\Delta t}\,\operatorname{div} u_h^{\Delta t}, u_h^{\Delta t}\phi\right) \\ &= \left((u_h^{\Delta t}\cdot\nabla)\,\frac{1}{2}|u_h^{\Delta t}|^2 + \frac{1}{2}|u_h^{\Delta t}|^2\operatorname{div} u_h^{\Delta t}, \phi\right) \\ &= \left(\operatorname{div}\left(u_h^{\Delta t}\frac{1}{2}|u_h^{\Delta t}|^2\right), \phi\right) = -\left(u_h^{\Delta t}\frac{1}{2}|u_h^{\Delta t}|^2, \nabla\phi\right). \end{aligned}$$

Then, for $4 < s \leq 6$ and s^* as in (5.8) it follows that

$$u_h^{\Delta t}\frac{1}{2}|u_h^{\Delta t}|^2 \to u\frac{1}{2}|u|^2 \quad \text{strongly in } L^{s^*/3}(0,T;L^1), \qquad \text{as } (\Delta t, h)\to(0,0),$$

and one shows that

$$\int_0^T b_h(u_h^{\Delta t}, u_h^{\Delta t}, u_h^{\Delta t}\phi)\,dt \to -\int_0^T \left(u\frac{1}{2}|u|^2, \nabla\phi\right)dt \qquad \text{as} \qquad (\Delta t, h)\to(0,0).$$

The last term we consider is that involving the pressure. By integrating by parts we have

$$(p_h^{\Delta t}, \operatorname{div} P_h(u_h^{\Delta t}\phi)) \quad = (p_h^{N} u_h^{\Delta t}, \nabla\phi) + R_{p1} + R_{p2}. \tag{5.11}$$

where the two "pressure remainders" are defined as follows

$$R_{p1} := \left(p_h^{\Delta t}, \operatorname{div}(P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi)\right) \qquad \text{and} \qquad R_{p2} := \left(\phi\, p_h^{\Delta t}, \operatorname{div} u_h^{\Delta t}\right).$$

By using again the discrete commutator property (3.5) and (3.3) we easily get

$$\begin{aligned} |R_{p1}| &\leq \|p_h^{\Delta t}\|_2\|P_h(u_h^{\Delta t}\phi) - u_h^{\Delta t}\phi\|_{H^1} \\ &\leq c\,h\|p_h^{\Delta t}\|_2\|u_h^{\Delta t}\|_{H^1} \end{aligned}$$

and then, by integrating in time,

$$\left|\int_0^T R_{p1}\,dt\right| \leq c\,h^{\frac{1}{2}}\|p_h^{\Delta t}\|_{L^{\frac{4}{3}}(L^2)}\|u_h^{\Delta t}\|_{L^2(H^1)}^{\frac{1}{2}}\|u_h^{\Delta t}\|_{L^\infty(L^2)}^{\frac{1}{2}},$$

which implies

$$\int_0^T R_{p1}\,dt \to 0 \quad \text{as } (\Delta t, h)\to(0,0).$$

The term R_{p2} can be treated in the same way but now using the discrete commutation property for the projector over Q_h

$$\begin{aligned}|R_{p2}| &\leq c\|Q_h(p_h^{\Delta t}\phi) - \phi p_h^{\Delta t}\|_2 \|u_h^{\Delta t}\phi\|_{H^1}\\ &\leq c\, h^{\frac{1}{2}} \|p_h^{\Delta t}\|_{L^{\frac{4}{3}}(L^2)} \|u_h^{\Delta t}\|_{L^2(H^1)}^{\frac{1}{2}} \|u_h^{\Delta t}\|_{L^\infty(L^2)}^{\frac{1}{2}}\end{aligned}$$

and finally this implies that

$$\int_0^T R_{p2}\, dt \to 0 \qquad \text{as } (\Delta t, h) \to (0,0),$$

hence collecting all terms we have finally proved the local energy inequality (2.2). ∎

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