RANDOM INITIAL CONDITIONS FOR SEMI–LINEAR PDES

DIRK BLÖMKER, GIUSEPPE CANNIZZARO, AND MARCO ROMITO

Abstract. We analyze the effect of random initial conditions on the local well–posedness of semi–linear PDEs, to investigate to what extent recent ideas on singular stochastic PDEs can prove useful in this framework.

In particular, in some cases stochastic initial conditions extend the validity of the fixed–point argument to larger spaces than deterministic initial conditions would allow, but in general it is never possible to go beyond the threshold that is predicted by critical scaling, as in our general class of equations we are not exploiting any special structure present in the equation.

We also give a specific example where the level of regularity for the fixed–point argument reached by random initial conditions is not yet critical, but it is already sharp in the sense that we find infinitely many random initial conditions of slightly lower regularity, where there is no solution at all. Thus criticality cannot be reached even by random initial conditions.

The existence and uniqueness in a critical space is always delicate, but we can consider the Burgers equation in logarithmically sub–critical spaces, where existence and uniqueness hold, and again random initial conditions allow to extend the validity to spaces of lower regularity which are still logarithmically sub-critical.

1. Introduction

This paper is a "proof of concept" that tries to investigate the effect of random initial conditions for the existence of partial differential equations of evolution type. These ideas were pioneered by Bourgain [4, 5], and recently the seminal papers by Burq and Tzvetkov [6, 7] generated a lot of activity. We refer to the recent lecture notes of Tzvetkov [22] for a more detailed account of the literature.

Most, if not all, of the existing results analyse the interesting case of dispersive or hyperbolic equations (with exceptions, see for instance [18]). On the other hand in that case the intrinsic difficulties of the problems examined may hide the limitations and features of the method we are analysing.

Date: July 12, 2018.

2010 *Mathematics Subject Classification.* Primary 35R60.

Key words and phrases. random initial condition, semilinar PDEs.

M. R. acknowledges the support of the Università di Pisa under the *PRA Progetti di Ricerca di Ateneo* (Institutional Research Grants) - Project no. PRA_2016_41 *Fenomeni singolari in problemi deterministici e stocastici ed applicazioni*.

We focus on semi–linear PDEs, because the theory on the linear propagator is well established and do not obfuscate the issues derived by the random initial condition method. Our aim is thus to shed light on its possibilities and limitations.

The main subject of our investigation is a semi–linear PDE with a simple linear operator (think of Laplacian or bi–Laplacian operator), and a polynomial nonlinearity, and we expect that the equation satisfies some kind of scaling invariance. The idea is that this class of equations represents, at first order, a general class of fundamental equations. In other words, we are interested in fundamental characteristics, so we focus on homogeneous nonlinearities that ensure scaling laws.

Scaling invariance gives an indication of the spaces in which we can expect to solve the equation by a fixed point argument. It is a well understood fact (see for instance [11]) that a critical space of initial conditions is a space whose norm is left invariant by the scaling of the equation. Continuity of the nonlinearity in sub–critical spaces (i. e., smaller than a critical space) is not prevented by scaling, and thus in such spaces a fixed point strategy is expected to be successful (when only using multi–linear estimates).

We analyse the problem in the class of (possibly negative) Hölder spaces. On the one hand they provide the largest critical spaces, on the other in such spaces there is no apparent gain in using a Gaussian randomization of the initial condition. Indeed, for a Gaussian random variable, summability for every $p \geq 1$ comes for free once one knows that summability holds for at least one exponent.

A full account of the general strategy considered is given in the next section. In short, we decompose the solution in the linear propagator on the initial condition, referred to as the *rough* term (since for rough initial condition, it should capture all the degrees of irregularity of the solution) and a (hopefully smoother) remainder. We thus obtain a new equation for the remainder coming with an additional summand, given by the nonlinearity computed on the rough term. Thus the main feature of the random initial condition is to tame the "roughness" of the latter and to make it well-defined for a wider range of the parameters. Here regularity/roughness should be understood in terms of singularity at $t = 0$, as all these functions are smooth when $t > 0$.

In the setting we have described, we are thus able to answer a series of questions that we believe are relevant for the subject.

1. **Is a random initial condition useful (in this setting)?** The general strength of the method has already been established in the literature we have cited before. In this setting the method is effective in a series of examples (see the next section), namely we prove a. s. existence of local solutions with respect to suitable Gaussian measures supported over function spaces larger than those available through a standard fixed point argument.
2. **When is it useful?** The validity of the method is graded though by a ratio between the linear and non–linear part of the equation (our parameter δ from Assumption 3.2, that is, roughly speaking, the ratio between the

largest order of derivative of the non–linear term, and the largest order of derivative of the linear term). The larger is the ratio, the lower is the validity.

3. **Are initial distributions supported on spaces of super–critical initial conditions possible?** Unfortunately the method does not allow to prove results for super–critical data. Our analysis on semi–linear PDEs allows to set the analysis on Hölder spaces of negative order, that are essentially the largest critical spaces. We do not get results outside such spaces.

   A simple explanation is that, as already explained, critical spaces are determined by the scaling properties of the problem. By randomizing the initial condition we do not introduce any additional argument that "breaks" the scaling invariance. An argument that breaks the scaling invariance is given in Section 5.4.

   We point out that the situation is in a way different when dealing with dispersive/hyperbolic equations, where the properties we know of the linear propagator do not allow to set the analysis in arbitrary function spaces.
4. **May a second order (or beyond) expansion be useful?** As illustrated in the next section, the randomization is exploited by decomposing the solution in an "irregular" term (the linear propagator computed over the random initial condition) and a "smoother" remainder. The first term should capture the highest degree of irregularity of the solution. It is thus reasonable to believe that whenever the initial condition is "very" irregular, adding further additional terms in a, so-to-say, Taylor expansion, might be helpful. It turns out that in the setting of semi–linear PDEs this is not necessary, since the linear term (think of the Laplace operator) already makes the first term of the expansion super–smooth (the irregularity is read in terms of a singularity in time at $t = 0$). In the setting of dispersive/hyperbolic equations, where the regularization of the linear problem is way milder, additional terms in the expansion may be effective [22, 1].
5. **Is renormalization needed?** In the recent theory on singular stochastic PDEs [15, 14, 13] some stochastic objects can be defined only when taking suitable infinities into account. We do not observe such a phenomenon in full generality since, whenever a stochastic object cannot be defined, the random initial condition cannot fix the problem (see the example in Section 5.5). Moreover, our examples all exhibit parabolic regularization, where for positive times solutions are smooth, and there is in that case no need to use renormalization to define terms.

   Nevertheless, there might be situations in which renormalization could play a role, as the example given in Section 5.4 shows[1]. We notice though that the example is very specific and works only through a special non-linear transformation (Cole-Hopf transform) while the general aim of the present

[1]We thank the anonymous referee for pointing it out.

paper is to investigate general conditions under which the use of a random initial condition can improve the regularity theory for semilinear PDEs.

We notice additionally that for dispersive problem this is in general not the case and renormalization may prove useful (see for instance [20]).

6. **Can we borrow further ideas from the theory of singular stochastic PDEs?** One of the deepest ideas in [15, 14, 13], that goes beyond the global decomposition first considered for such problems in the stochastic setting in [8, 9, 10], is the local description of the degree of irregularity of a solution. In Section 6 we present a result based on the local description to prove local existence for logarithmically sub–critical initial conditions for the one-dimensional Burgers equation. We notice that a local description is useful only when the initial condition has regularity close to the critical level. We believe that this contribution is the main novelty of the paper.

2. The main examples

The examples we consider are equations on the d–dimensional torus, with periodic boundary conditions, of the form

$$\partial_t u = Au + B(u), \tag{2.1}$$

where A is a linear operator and B is a multi–linear operator.

2.1. The general strategy.

We expect that some scaling invariance holds, that is there are σ, τ such that if u is solution of (2.1), then so is

$$(t, x) \mapsto \lambda^\sigma u(\lambda^\tau t, \lambda x). \tag{2.2}$$

2.1.1. *Deterministic initial condition.* We first consider (2.1), with a deterministic initial condition. We expect that the maximal Besov space[2] where we are able to solve our equation (2.1) by means of a fixed point argument is $\mathscr{C}^{-\sigma}$, since the homogeneous version of this space is invariant under the transformation $u \rightsquigarrow \lambda^\sigma u(\lambda\cdot)$.

Assume the non–linear term B is bi–linear and symmetric (we will discuss more general cases in Sections 5.1 and 5.3), and set

$$\mathcal{V}(u_1, u_2)(t) = \int_0^t \mathrm{e}^{A(t-s)}\, B(u_1, u_2)\, ds.$$

A standard fixed point theorem (Theorem 3.5) solves the above equation as

$$u = \eta_1 + \mathcal{V}(u, u),$$

[2] more precisely, the maximal critical space where we are able to solve the equation is $\mathscr{V}^{\alpha,\beta}$, defined in formula (3.4) (see also Remark 3.6), with $\beta = 1 - \delta$, and $\delta > \frac{1}{2}$.

where $\eta_1(t) = \mathrm{e}^{tA}u(0)$. To this end we assume that the nonlinearity is suitably continuous, namely there is $\delta \in [0,1)$ such that [3]

$$\|\mathcal{V}(u_1,u_2)(t)\|_{\mathscr{C}^\alpha} \lesssim \int_0^t (t-s)^{-\delta}\|u_1\|_{\mathscr{C}^\alpha}\|u_2\|_{\mathscr{C}^\alpha}\,ds.$$

By scaling (see Remark 3.3),

$$\delta = 1 - \frac{\alpha+\sigma}{\tau}.$$

The fixed point is solved in spaces $\mathscr{X}_T^{\alpha,\beta}$, defined by means of the norm

$$\|\cdot\|_{\alpha,\beta,T} = \sup_{t\in[0,T]} t^\beta\|\cdot\|_{\mathscr{C}^\alpha},$$

that encode the singularity at $t=0$. The value α must be large enough for $\mathcal{V}$ to satisfy some continuity property. On the other hand the larger is α, the larger β needs to be so to compensate the difference in regularity with the initial condition. We will choose α minimal to minimise the singularity. Theorem 3.5 ensures now the existence of a unique local solution as long as $\|\eta_1\|_{\alpha,\beta,T} \to 0$ as $T\to\infty$, and $\beta < \frac{1}{2}$, $\beta+\delta \leq 1$. If $\delta > \frac{1}{2}$ the result is optimal and includes the critical space. Further improvements are only possible through some additional information, such as for instance a–priori estimates, that break the scaling.

If on the other hand $\delta \leq \frac{1}{2}$, the fixed point theorem is not optimal and we can solve the problem with initial conditions in $\mathscr{C}^r$ only for $r > \alpha - \frac{1}{2}\tau$ (see Remark 3.6). A possible strategy could be to single out η_1, the most singular part of u. Set $u = v + \eta_1$, then

$$v = \mathcal{V}(v,v) + 2\mathcal{V}(v,\eta_1) + \eta_2, \tag{2.3}$$

where $\eta_2 = \mathcal{V}(\eta_1,\eta_1)$. Unfortunately this does not really help without a more detailed understanding of the nonlinearity (see Remark 3.9).

2.1.2. *Random initial condition.* We turn to random initial conditions. For simplicity and to make our point clear, we assume u_0 is a random field on the torus with independent Gaussian Fourier components. Regularity of u_0, as well as of $\eta^{\dagger}(t) := \mathrm{e}^{tA}u_0$ (here we adopt Hairer's notation to make clear that we deal with random objects), is standard and does not give any advantage.

The crucial point is that randomness plays its major role in taming the term $\eta^{\vee} = B(\eta^{\dagger},\eta^{\dagger})$, and in turn $\eta^{\Yup} = \mathcal{V}(\eta^{\dagger},\eta^{\dagger})$, is well defined for a wider range of the parameters (see Remark 4.8). To do these computations, we take some simplifying assumptions, in particular we impose that, at small scales, B is essentially of the type $D^a\big((D^b\cdot)(D^b\cdot)\big)$, where D is a generic differential operator that might be a derivative, a gradient or a divergence.

[3] We adopt the standard notation that '$\lesssim$' denotes an inequality that is true up to a generic constant.

Since the random initial condition has smoothed the singularity of η^{Ψ} at $t = 0$, it is now worthwhile to apply the fixed point strategy to the formulation (2.3). Then, we have to check that the assumptions of Theorem 3.8 are met by the terms $\eta^{\vee}$ and η^{Ψ}. To this aim, Proposition 4.5 and Theorem 4.10 yield all the needed information. We end up with a series of inequalities over the parameters τ, a, b that restrict the possible regularity of the random field.

The first example we consider (surface growth) is one of those for which $\delta > \frac{1}{2}$, thus the deterministic theory is sufficient (as already known from [2]). The second (KPZ) is borderline, since $\delta = \frac{1}{2}$. For the third (Kuramoto–Sivashinsky), the deterministic theory is not sufficient to get initial conditions up to the critical space, and this is only possible with random initial conditions. Finally, in the fourth example (reaction-diffusion), not even random initial conditions are sufficient to catch the critical case.

Notice that in general, when the random initial condition method fails, this is always due to $\eta^{\vee}$ having a singularity in time at $t = 0$ that is too strong. In particular, going further to a second order expansion does not help any more.

In the last part of the paper we shall give some remarks and present some additional examples. Since in the paper we will analyse mass–conservative, symmetric quadratic nonlinearities, roughly speaking of the form $D^a((D^b\cdot)^2)$ that allow for optimal results, in Section 5.1 we will discuss what happens in asymmetric case, while in Section 5.3 we will look at the case of nonlinearities with higher powers, and finally in Section 5.2 we will relax the constraint of mass conservation.

Sections 5.4 and 5.5 are somewhat different. In Section 5.4 we see that an argument that breaks the scaling invariance allows for super-critical initial data through renormalisation. Based on an example, we will see in Section 5.5 that the fact that even a random initial condition cannot in general cover all cases up to the critical level (as we shall see in the examples of Sections 2.4 and 2.5) is not a limitation of our proofs.

Finally, in Section 6 we present a result that shows that, when dealing with (almost) critical random initial conditions, a global decomposition in terms of stochastic objects and a remainder term as in (2.3), is not sufficient. Our strategy is to understand the local degree of irregularity of the solution and to exploit this fact to gain a tiny (logarithmic) improvement that allows to close the fixed point argument. This may be seen as a glimpse of the extremely sophisticated ideas introduced in [15, 14, 13]. Notice though that in the aforementioned papers they use, roughly speaking, two fundamental ideas: the first is to understand the most irregular part of the solution – as we have done. The second is to exploit again the probabilistic structure to define the terms in the most irregular part of the solution. This is apparently not needed here.

2.2. **Surface growth.** Consider the following example (see [3] for a general overview),

$$\partial_t u = -\Delta^2 u - \Delta u - \Delta|\nabla u|^2,$$

with periodic boundary conditions and zero mean. The equation (without the lower order term $-\Delta u$) has scaling invariance according to formula (2.2), with exponents $\sigma = 0$ and $\tau = 4$. Thus the *critical* space for fixed point is at the level of $\mathscr{C}^0$ (more precisely, $\mathscr{V}^{-1,1/4}$, defined in formula (3.4)).

Assumption 4.2 holds with values $a = 2$, $b = 1$, and Assumption 3.2 holds with $\alpha = 1$, $\delta = \frac{3}{4}$. Notice that the choice of α is the minimal value that gives sense to the non–linear term. The standard fixed point result, Theorem 3.5, holds sharp for initial conditions in $\mathscr{C}^\gamma$ with $\gamma \geq 0$. The argument yielding the critical space has been given also in [2]. We do not need random initial condition here.

2.3. **KPZ.** Consider the following problem on the torus,

$$\partial_t u = \Delta u - \mathcal{M}|\nabla u|^2, \tag{2.4}$$

subject to periodic boundary conditions and zero mean, where $\mathcal{M}$ is the projector onto the zero mean space, namely

$$(\mathcal{M}z)(x) = z(x) - \int_{\mathbb{T}^d} z(y)\,dy.$$

With additive noise this is a fundamental model in mathematical physics, recently solved by Hairer [15]. The equation has scaling invariance with exponents $\sigma = 0$ and $\tau = 2$. Thus the critical space is at the level of $\mathscr{C}^0$ (more precisely $\mathscr{V}^{0,1/2}$). It can be easily checked that Assumption 4.2 holds with $a = 0$, $b = 1$, and that Assumption 3.2 holds with $\alpha = 1$, $\delta = \frac{1}{2}$, and again α has been chosen to be minimal. Theorem 3.5, holds for initial conditions in $\mathscr{C}^\gamma$, with $\gamma > 0$. Unfortunately the critical space $\mathscr{V}^{0,\frac{1}{2}}$ cannot be captured neither by the deterministic results (Theorem 3.5 and Theorem 3.8), nor by the random initial condition.

2.4. **KS.** Consider the following mass–conservative Kuramoto–Sivashinsky equation

$$\partial_t u = -\Delta^2 u - \Delta u - \mathcal{M}|\nabla u|^2,$$

with periodic boundary conditions and zero mean. The scaling exponents (when the lower order term Δu is neglected) are $\sigma = 2$ and $\tau = 4$, and the critical space for fixed point is $\mathscr{C}^{-2}$. Once more, assumption 4.2 holds with $a = 0$, $b = 1$, while Assumption 3.2 holds with $\alpha = 1$, $\delta = \frac{1}{4}$, where again α is the minimal number of derivatives to give sense to the nonlinearity. Here Theorem 3.5 holds for deterministic initial conditions in $\mathscr{C}^\gamma$, with $\gamma > -1$, which is still smaller than the critical space we have identified.

In the stochastic case we will see in Theorem 3.8 that we can solve the equation for random initial conditions in $\mathscr{C}^\gamma$, with the restriction from Proposition 4.5 and Theorem 4.10. The main obstacle is the regularity or singularity of η^{Υ} at $t = 0$. that leads to $\gamma > \max\{-2, -1 - d/4\}$, and for the mixed term we additionally need $\gamma > \min\{-4/3 - d/6, 3/2 - d/4\}$.

2.5. **Reaction–diffusion.** Consider the following equation with periodic boundary conditions and zero mean,

$$\partial_t u = \Delta u - u + \mathcal{M} u^2,$$

The scaling exponents are $\sigma = 2$, $\tau = 2$ (neglecting as usual the lower order term), with critical space $\mathscr{C}^{-2}$. Assumption 4.2 holds with values $a = 0$, $b = 0$, Assumption 3.2 holds with $\delta = 0$ and the minimal value $\alpha = 0$. Thus Theorem 3.5 applies for initial conditions in $\mathscr{C}^\gamma$ with $\gamma > -1$.

Again in the stochastic case of Theorem 3.8 we can extend this to random initial conditions in $\mathscr{C}^\gamma$, with the restrictions $\gamma > \max\{-2, -1 - d/4\}$ and $\gamma > \min\{-4/3 - d/6, -d/4\}$ due to Proposition 4.5 and Theorem 4.10.

2.6. **A short summary on Besov spaces.** We will work with Besov spaces, which have somewhat maximal regularity in terms of integrability. These Hölder spaces are natural spaces for the regularity of Gaussian random variables.

Besov spaces are defined via Littlewood-Paley projectors. Let χ, ϱ be non–negative smooth radial functions such that

- The support of χ is contained in a ball and the support of ϱ is contained in an annulus;
- $\chi(\xi) + \sum_{j\geq 0} \varrho(2^{-j}\xi) = 1$ for all $\xi \in \mathbf{R}^d$;
- $\operatorname{Supp}(\chi) \cap \operatorname{Supp}(\varrho(2^{-j}\cdot)) = \emptyset$, for $j \geq 1$ and $\operatorname{Supp}(\varrho(2^{-i}\cdot)) \cap \operatorname{Supp}(\varrho(2^{-j}\cdot)) = \emptyset$ when $|i - j| > 1$.

Set $\varrho_j(x) := \varrho(2^{-j}x)$ for all $j \geq 0$ and $\varrho_{-1}(x) := \chi(x)$. The Littlewood-Paley blocks are given in terms of the discrete Fourier transform,

$$\Delta_j u = (2\pi)^{-d} \sum_{k\in\mathbf{Z}^d} \varrho_j(k) \mathscr{F}_{\mathbb{T}^d}(u)(k) e_k(x) = \sum_{k\in\mathbf{Z}^d} \varrho_j(k) u_k e_k.$$

Let $\alpha \in \mathbf{R}$, $p, q \in [1, \infty]$, we define the Besov space $B^\alpha_{p,q}(\mathbb{T}^d)$ as the closure of the space of smooth periodic functions with respect to the norm[4]

$$\|u\|^q_{B^\alpha_{p,q}(\mathbb{T}^d)} := \|(2^{j\alpha} \|\Delta_j u\|_{L^p(\mathbb{T}^d)})_{j\geq -1}\|_{\ell^q}\ .$$

We will mainly deal with the special case $p = q = \infty$, so we introduce the notation $\mathscr{C}^\alpha := B^\alpha_{\infty,\infty}(\mathbb{T}^d)$ and denote by

$$\|u\|_\alpha = \|u\|_{B^\alpha_{\infty,\infty}}$$

its norm.

[4]Recall $\|\{x_j\}_j\|^p_{\ell^p} = \sum_j |x_j|^p$ for a sequence.

2.6.1. *The Bony paraproduct.* The Bony paraproduct $\circledless$ is defined for distributions f, g with Littlewood–Paley blocks $(\Delta_j f)_{j\geq -1}$ and $(\Delta_j g)_{j\geq -1}$ as

$$f \circledless g = \sum_{m\leq n-1} (\Delta_m f)(\Delta_n g).$$

The term $f \circledgtr g$ is then defined as $f \circledgtr g = g \circledless f$, and the *resonant* term is defined by

$$f \circleddash g = \sum_{|m-n|\leq 1} (\Delta_m f)(\Delta_n g),$$

so that, whenever the product makes sense, we can decompose it into $f \cdot g = f \circledless g + f \circleddash g + f \circledgtr g$.

3. The fixed point argument

We outline here an abstract fixed point argument that yields local existence and uniqueness for initial conditions in the scale of Hölder–Besov spaces. The argument is given in two flavours: standard and with rough initial condition. To this end we state some assumptions on the linear and non–linear part of the equation (2.1) that capture the essential features of our examples and that are needed here.

Assumption 3.1 (Schauder estimates)**.** The unbounded operator A generates an analytic semigroup. Moreover there is $\tau > 0$ such that the following estimates hold,

$$\| \mathrm{e}^{tA} u\|_{\alpha+\beta} \leq ct^{-\frac{\beta}{\tau}} \|u\|_\alpha, \tag{3.1}$$

for every $\alpha \in \mathbf{R}$, every $u \in B^\alpha_{\infty,\infty}$, and every $\beta \geq 0$.

Define the integrated nonlinearity,

$$\mathcal{V}(u_1, u_2)(t) = \int_0^t \mathrm{e}^{(t-s)A} B(u_1(s), u_2(s))\, ds.$$

Assumption 3.2. There are $\alpha \in \mathbf{R}$, $\delta \in [0, 1)$, and $c > 0$ such that

$$\|\mathcal{V}(u_1, u_2)(t)\|_\alpha \leq c \int_0^t (t-s)^{-\delta} \|u_1(s)\|_\alpha \|u_2(s)\|_\alpha\, ds \tag{3.2}$$

Remark 3.3. A few remarks on the assumptions above,

- it is fairly easy to check that the exponent on the right–hand side of (3.1) follows by a scaling argument if (2.2) holds;
- likewise, if (2.2) holds (and $\delta > 0$), then again by scaling invariance $\delta = 1 - \frac{\alpha+\sigma}{\tau}$, if the inequalities are optimal;
- there is no apparent gain if we assume different norms for u_1, u_2 on the right–hand side of (3.2). On the contrary, usually this gives a δ that depends on the smallest index of the norm.

Consider the equation (2.1) in its mild formulation,

$$u(t) = \mathrm{e}^{At} u_0 + \mathcal{V}(u,u)(t).$$

3.1. **The standard fixed point argument.** Given u_0, set

$$\eta_1(t) = \mathrm{e}^{At} u_0.$$

We wish to solve by fixed point the problem

$$u = \eta_1 + \mathcal{V}(u,u), \tag{3.3}$$

in the normed space

$$\mathscr{X}_T^{\alpha,\beta} := \{u : \|u\|_{\alpha,\beta,T} := \sup_{t\in[0,T]} t^\beta \|u(t)\|_\alpha < \infty\}.$$

We immediately have the following proposition.

Proposition 3.4. *For $\beta < \frac{1}{2}$ and $\delta + \beta \le 1$,*

$$\|\mathcal{V}(u_1,u_2)\|_{\alpha,\beta,T} \le cT^{1-\beta-\delta}\|u_1\|_{\alpha,\beta,T}\|u_2\|_{\alpha,\beta,T}$$

Proof. Assumption (3.2) ensures

$$t^\beta\|\mathcal{V}(u_1,u_2)(t)\|_\alpha \le ct^\beta\|u_1\|_{\alpha,\beta,T}\|u_2\|_{\alpha,\beta,T}\int_0^t (t-s)^{-\delta}s^{-2\beta}\,ds.$$

Hence, as long as $\beta < \frac{1}{2}$, the statement follows. □

The proposition above allows to verify the following theorem.

Theorem 3.5. *Consider $\beta < 1/2$ and $\beta \le 1-\delta$, and assume $\|\eta_1\|_{\alpha,\beta,T} \to 0$ for $T \to 0$. Then there is $T > 0$ such that the equation (3.3) has a unique fixed point in $\mathscr{X}_T^{\alpha,\beta}$.*

Notice that the initial conditions u_0 to whom the above theorem applies are those such that $\|\eta_1\|_{\alpha,\beta,T} \to 0$ as $T \to 0$. Let us denote by $\mathscr{V}^{\alpha,\beta}$ such a space, namely

$$\mathscr{V}^{\alpha,\beta} = \big\{u_0 : \lim_{T\to 0}\|\eta_1\|_{\alpha,\beta,T} = 0\big\}, \tag{3.4}$$

where we recall that $\eta_1(t) = \mathrm{e}^{At} u_0$. The most interesting case is when $\beta = 1-\delta$, since $\mathscr{V}^{\alpha,\beta}$ becomes critical. Indeed, a simple computation shows that the norm $\|\cdot\|_{\alpha,\beta,T}$ is invariant by the scaling (2.2), in the sense that

$$\sup_{[0,T]} t^\beta[\lambda^\sigma u(\lambda^\tau t, \lambda\cdot)]_\alpha = \lambda^{\sigma+\alpha-\tau\beta}\sup_{[0,\lambda^\tau T]} t^\beta[u(t,\cdot)]_\alpha = \sup_{[0,\lambda^\tau T]} t^\beta[u(t,\cdot)]_\alpha,$$

since, according to Remark 3.3, $\sigma + \alpha = \tau\beta$ and where $[\cdot]_\alpha$ is the semi–norm of $\mathscr{C}^\alpha$.

Remark 3.6 (Initial conditions in $\mathscr{C}^r$). Another way to understand the computation above is to realize that the larger is β, the larger is the set of initial conditions. To see this we look for the minimal values of γ such that $u_0 \in \mathscr{C}^r$ yields $\eta_1 \in \mathscr{V}^{\alpha,\beta}$ for some β compatible with the assumptions of the above theorem. By (3.1),

$$\|\eta_1\|_{\alpha,\beta,T} \leq T^{\beta - \frac{\alpha - r}{\tau}} \|u_0\|_r, \tag{3.5}$$

thus $\eta_1 \in \mathscr{V}^{\alpha,\beta}$ when $r > \alpha - \beta\tau$. Therefore,

- if $\delta > \frac{1}{2}$, we can take the maximal value $\beta = 1 - \delta$. With this choice of the parameters, if $u_0 \in \mathscr{C}^r$ then $\eta_1 \in \mathscr{V}^{\alpha,\beta}$ for all $r > -\sigma$, and the space $\mathscr{V}^{\alpha,\beta}$ is critical.
- if $\delta \leq \frac{1}{2}$, then we are restricted to $\beta < \frac{1}{2}$. We have that if $u_0 \in \mathscr{C}^r$ then $\eta_1 \in \mathscr{V}^{\alpha,\beta}$, under the sub–optimal condition $r > \alpha - \frac{\tau}{2}$.

Note finally that the estimate (3.5) always rules out the critical regularity, as it can never verify the limit needed in the definition of $\mathscr{V}^{\alpha,\beta}$. This limit in a critical Besov-space seems to be an open question for general operators A.

3.2. **Fixed point with rough initial condition.** Suppose that the initial condition u_0 is too rough to apply the results of the previous section. Set as before $\eta_1(t) = \mathrm{e}^{tA} u_0$ and let

$$v = u - \eta_1.$$

Instead of (3.3), this time we solve the transformed problem

$$v = \mathcal{V}(v, v) + 2\mathcal{V}(v, \eta_1) + \eta_2, \tag{3.6}$$

where we have set $\eta_2 = \mathcal{V}(\eta_1, \eta_1)$. The key argument is that by taking a random distribution over the initial condition, Gaussian for instance, the quadratic term η_2 is well defined, although it could not be in principle defined in general using only the regularity properties of η_1 or, more precisely, its singularity at $t = 0$. We expect that the mixed product $\mathcal{V}(v, \eta_1)$ might be fine, since η_1 is smooth away from 0, and v is zero at 0. Nevertheless, as we have seen in Section 2, the expansion does not allow to cover, in general, the critical case.

The following proposition is a minor modification of Proposition 3.4.

Proposition 3.7. *If $\beta + \gamma < 1$ and $\delta + \gamma \leq 1$, then*

$$\|\mathcal{V}(u_1, u_2)\|_{\alpha,\beta,T} \leq cT^{1-\delta-\gamma} \|u_1\|_{\alpha,\beta,T} \|u_2\|_{\alpha,\gamma,T}.$$

Based on this proposition, we can prove the following theorem.

Theorem 3.8. *Consider $\beta < \frac{1}{2}$, $\beta + \delta \leq 1$, $\gamma + \beta < 1$, and $\delta + \gamma \leq 1$. Given u_0, assume that $\|\eta_1\|_{\alpha,\gamma,T} \to 0$ and $\|\eta_2\|_{\alpha,\beta,T} \to 0$, for $T \to 0$. Then there is $T > 0$ such that the problem (3.6) has a unique fixed point in $\mathscr{X}_T^{\alpha,\beta}$.*

Proof. The proof is again by fixed point argument and very similar to the proof of Theorem 3.5. This time though we are allowed to use a different weight in time for η_1.

For the self-mapping we use

$$\|\mathcal{V}(v,v)+2\mathcal{V}(v,\eta_1)+\eta_2\|_{\alpha,\beta,T}\leq c(\|v\|^2_{\alpha,\beta,T}+\|v\|_{\alpha,\beta,T}\|\eta_1\|_{\alpha,\gamma,T})+\|\eta_2\|_{\alpha,\beta,T}.$$

Likewise, for the contraction property,

$$\begin{aligned}\|\mathcal{V}(v_1,v_1)+2\mathcal{V}(v_1,\eta_1)-\mathcal{V}(v_2,v_2)+2\mathcal{V}(v_2,\eta_1)\|_{\alpha,\beta,T}&\\ =\|\mathcal{V}(v_1+v_2,v_1-v_2)+2\mathcal{V}(v_1-v_2,\eta_1)\|_{\alpha,\beta,T}&\\ \leq c\big(\|v_1+v_2\|_{\alpha,\beta,T}+\|\eta_1\|_{\alpha,\gamma,T}\big)\cdot\|v_1-v_2\|_{\alpha,\beta,T},&\end{aligned}$$

by Propositions 3.4 and 3.7. □

Remark 3.9 (Initial conditions in $\mathscr{C}^r$). In comparison with Remark 3.6, we look for initial conditions $u_0\in\mathscr{C}^r$ such that the assumptions of Theorem 3.8 hold. By the (scaling–wise optimal) estimate (3.2) we have

$$\|\eta_2(t)\|_\alpha\lesssim\int_0^t(t-s)^{-\delta}\|\eta_1(s)\|^2_\alpha\,ds\lesssim t^{1-2\gamma-\delta}\|\eta_1\|_{\alpha,\gamma,T},$$

therefore $\|\eta_2\|_{\alpha,\beta,T}\leq T^{\beta+1-2\gamma-\delta}\|\eta_1\|^2_{\alpha,\gamma,T}$ if $\gamma<\frac{1}{2}$ and $2\gamma+\delta\leq\beta+1$. The same computation of Remark 3.6 yields $\eta_1\in\mathscr{X}^{\alpha,\gamma}$ if $u_0\in\mathscr{C}^r$ and $r\geq\alpha-\gamma\tau$. To minimize the value of r we wish to take γ as large as possible, thus $\gamma\approx\frac{1}{2}$. Unfortunately this yields the condition $r>\alpha-\frac{\tau}{2}$, which is the same of Remark 3.6.

In conclusion the additional expansion has not given, at least for general initial conditions, any additional benefit. We will see in the next section, that this is different for random initial conditions.

4. Stochastic objects

Here we discuss the existence and regularity of the terms appearing in the fixed point arguments of the previous section, when the initial condition u_0 is a random variable with peculiar structure. Here we focus on the case of bi–linear mass–conservative nonlinearity B, we will comment later on the no–moving–frame case and the need of renormalization.

4.1. **Diagonal (simplifying) assumptions.** Here we greatly simplify our problem (2.1), by assuming that the linear operator acts diagonally on the Fourier basis, and that the non-linear operator is a *bona fide* product. The reason is that we wish to exploit in a simple setting the decorrelations of the random initial condition.

Let $(e_k)_{k\in\mathbf{Z}^d}$ be the standard Fourier basis of the torus $\mathbb{T}_d$ of normalized complex exponentials.

Assumption 4.1. For every $k\in\mathbf{Z}^d$, $Ae_k=\lambda_k e_k$, with $\lambda_k\sim-c|k|^\tau$, for some constant $c>0$.

In the sequel we will assume, without loss of generality, that $c=1$.

Assumption 4.2. For each $k, m, n \in \mathbf{Z}^d$, let B_{kmn} denote the product $B_{kmn} = \langle B(e_m, e_n), e_k\rangle$. Then B is such that $B_{kmn} = 0$ if $k \neq m + n$ and

$$B(u, v) = \sum_{k\in\mathbf{Z}^d} \sum_{m+n=k} B_{kmn} u_m v_n e_k$$

for $u = \sum_m u_m e_m$, $v = \sum v_n e_n$. Moreover

- (mass conservation) $B_{0mn} = 0$ for all m, n,
- (regularity) there are numbers $a, b \geq 0$ such that for $m + n = k$,

$$|B_{kmn}| \leq c|k|^a |m|^b |n|^b. \tag{4.1}$$

Remark 4.3. We do not assume that (4.1) is sharp. As already pointed out in Remark 3.3 for the fixed point argument, the asymmetric case does not help. If on the other hand (4.1) is sharp, a elementary scaling argument shows that $\sigma+a+2b = \tau$. Indeed, if a, b are optimal, then roughly speaking $B \approx D^a\big((D^b\cdot)(D^b\cdot)\big)$ that scales as

$$B(u_\lambda, u_\lambda) = \lambda^{2\sigma+a+2b}(B(u, u))_\lambda,$$

where $u_\lambda(t, x) = \lambda^\sigma u(\lambda^\tau t, \lambda x)$. On the other hand $(\partial_t - A)u_\lambda = \lambda^{\tau+\sigma}(\partial_t u - Au)_\lambda$.

Actually the same result could be directly obtained, starting from (4.1), by elementary paraproduct estimates as those in [13]. These estimates would provide also, together with (3.1), a connection with Assumption 3.2.

4.2. **Random initial condition.** For simplicity, we give Gaussian structure to the initial condition. Other distributions are possible though, once one assumes essentially sub–normality and hyper–contraction.

Assumption 4.4. The random variable u_0 is Gaussian with the following representation in Fourier modes,

$$u_0(x) = \sum_{k\in\mathbf{Z}^d} \phi_k \xi_k e_k,$$

where $(\xi_k)_{k\in\mathbf{Z}^d}$ is a family of centred complex valued Gaussian random variables with $\bar{\xi}_k = \xi_{-k}$ for all k, and covariance

$$\mathbb{E}[\xi_{k_1}\bar{\xi}_{k_2}] = \mathbb{1}_{\{k_1=k_2\}}. \tag{4.2}$$

Moreover, $(\phi_k)_{k\in\mathbf{Z}^d}$ is a sequence of "weights" with

- (mass conservation) $\phi_0 = 0$,
- (regularity) there is $\theta \in \mathbf{R}$ such that $|\phi_k| \sim |k|^\theta$.

4.3. **Regularity of the stochastic objects.** Given a random initial condition u_0 as above, we set

$$\eta^{\mathsf{I}}(t) = \mathrm{e}^{At} u_0, \qquad \eta^{\mathsf{V}} = B(\eta^{\mathsf{I}}, \eta^{\mathsf{I}}), \qquad \eta^{\mathsf{Y}}(t) = \mathcal{V}(\eta^{\mathsf{I}}, \eta^{\mathsf{I}})(t) = \int_0^t \mathrm{e}^{(t-s)A}\, \eta^{\mathsf{V}}(s)\, ds.$$

In the rest of the section we study the regularity of η^{I}, η^{V}, and η^{Y} in Hölder–Besov spaces.

4.3.1. *Regularity of u_0.* We start with the regularity of u_0. This follows from standard results. Indeed, by [16, Theorem 6.3], we have that there exists $c > 0$ such that

$$\mathbb{P}[\|\Delta_j u_0\|_{L^\infty} \geq c\sqrt{j}2^{\frac{1}{2}(2\theta+d)j}] \lesssim 2^{-2j}.$$

Then the first Borel–Cantelli lemma ensures that there is a random number C such that a. s.,

$$\|\Delta_j u_0\|_{L^\infty} \leq C\sqrt{j}2^{\frac{1}{2}(2\theta+d)j}. \tag{4.3}$$

In conclusion $\|u_0\|_\alpha$ is almost surely finite (and with exponential moments) as long as $\alpha < -\theta - \frac{d}{2}$. Notice that this holds in general for *sub–normal* independent sequences $(\xi_k)_{k\in\mathbf{Z}^d}$ (since so is for the results in [16]). Here a random variable X is sub–normal if $\mathbb{E}[\mathrm{e}^{\lambda X}] \leq \mathrm{e}^{\lambda^2/2}$.

In the Gaussian case we can completely characterize the regularity of u_0 as follows. An elementary computation shows that $\mathbb{E}[\|u_0\|^2_{H^\alpha}] = \infty$ if $\alpha \geq -\theta - \frac{d}{2}$. Hence by Fernique's theorem $u_0 \notin H^\alpha$ a. s.. Finally, $\mathscr{C}^{\alpha'} \subset H^\alpha$ if $\alpha' > \alpha$, thus $u_0 \notin \mathscr{C}^\alpha$ a. s. for every $\alpha > -\theta - \frac{d}{2}$.

4.3.2. *Regularity of $\eta^{\dagger}$.* We turn to study the regularity of $\eta^{\dagger}$ in terms of spaces $\mathscr{X}_T^{\alpha,\beta}$. The previous considerations and Assumption 3.1 immediately yield the following result.

Proposition 4.5. *If u_0 is as in Assumption 4.4, then $\eta^{\dagger} \in \mathscr{X}_T^{\alpha,\gamma}$ for every $T > 0$ and all α, γ such that*

- $\alpha < -(\theta + \frac{d}{2})$, $\gamma = 0$,
- $\alpha \geq -(\theta + \frac{d}{2})$, $\gamma > \frac{\alpha+\theta+\frac{d}{2}}{\tau}$.

Moreover, using Assumptions 4.1, we immediately obtain

$$\|\eta^{\dagger}_t - \eta^{\dagger}_s\|_\alpha \lesssim s^{-\frac{\epsilon}{\tau}}(t-s)^{\frac{\epsilon}{\tau}}\|u_0\|_\alpha,$$

if $\alpha < -(\theta+\frac{d}{2})$, with $0 < \epsilon < -\alpha-\theta-\frac{d}{2}$, and similarly if $\alpha \geq -(\theta+\frac{d}{2})$. In conclusion we always have $\eta^{\dagger} \in C([0,T];\mathscr{C}^\alpha)$ if $\alpha < -(\theta+\frac{d}{2})$, and $\eta^{\dagger} \in C((0,T];\mathscr{C}^\alpha)$ otherwise.

Remark 4.6. We see here that a random initial condition does not give any advantage at the level of $\eta^{\dagger}$. Due to the assumptions of Theorems 3.5 and 3.8, $\eta^{\dagger}$ will always be supported over critical spaces.

4.3.3. *Regularity of $\eta^{\vee}$.* The regularity of $\eta^{\vee}$, or more precisely the singularity in time at $t = 0$, is a fundamental step. Here Assumption 4.1 will play a crucial role.

Since $(\xi_k)_{k\in\mathbf{Z}_0^d}$ is a sequence of independent real standard Gaussian random variables, we see immediately that $\eta^{\vee}$ is in the second Wiener chaos. Moreover, as we shall verify below, the 0^{th}–order component is zero, therefore $\eta^{\vee}$ is in the homogeneous second Wiener chaos. To prove that there is no 0^{th}–order component, we recall that the 0^{th}–order component is simply the expectation of $\eta^{\heartsuit}$,

$$\mathbb{E}[\eta^{\heartsuit}(t,x)] = \sum_{k\in\mathbf{Z}^d}\Big(\sum_{m+n=k} B_{kmn}\phi_m\phi_n\,\mathrm{e}^{-t(|m|^\tau+|n|^\tau)}\,\delta_{m+n=0}\Big)e_k(x) =$$
$$= \Big(\sum_{m+n=0} B_{0mn}|\phi_m|^2\,\mathrm{e}^{-2t|m|^\tau}\Big)e_0(x) = 0,$$

by Assumptions 4.2 and 4.4.

For $\beta\geq 0$,

$$\begin{aligned}\mathbb{E}[|\Delta_j\eta^{\heartsuit}|^2] &\lesssim \sum_{|k|\sim 2^j}\sum_{m+n=k}\varrho_j(k)^2|B_{kmn}|^2|\phi_m|^2|\phi_n|^2\,\mathrm{e}^{-2t(|m|^\tau+|n|^\tau)}\\ &\lesssim t^{-2\beta}\sum_{|k|\sim 2^j}|k|^{2a}\sum_{m+n=k}\frac{|m|^{2\theta+2b}|n|^{2\theta+2b}}{(|m|^\tau+|n|^\tau)^{2\beta}}\,.\end{aligned}$$

The sum extended over all m, n such that $m+n=k$ can be decomposed, by symmetry, in two sums over the two sets

$$A_k = \{(m,n) : m+n=k, |m|\geq|n|\geq\frac{1}{2}|k|\}$$

and

$$B_k = \{(m,n) : m+n=k, |n|\leq\frac{1}{2}|k|\leq|m|\}.$$

For the sum over A_k, notice that on A_k we have $\frac{1}{3}|m|\leq|n|\leq|m|$, thus, whatever is the sign of $2\theta+2b$,

$$\sum_{A_k}\frac{|m|^{2\theta+2b}|n|^{2\theta+2b}}{(|m|^\tau+|n|^\tau)^{2\beta}}\lesssim\sum_{|m|\geq\frac{1}{2}|k|}\frac{1}{|m|^{2\beta\tau-4b-4\theta}}\lesssim\frac{1}{|k|^{2\beta\tau-4b-4\theta-d}}.$$

Here we need $2\beta\tau-4b-4\theta>d$, otherwise the sum would diverge.

For the sum over B_k, notice that we also have $|m|\leq\frac{3}{2}|k|$, thus whatever is the sign of $2b+2\theta-2\beta\tau$,

$$\sum_{B_k}\frac{|m|^{2\theta+2b}|n|^{2\theta+2b}}{(|m|^\tau+|n|^\tau)^{2\beta}}\lesssim\sum_{B_k}|m|^{2b+2\theta-2\beta\tau}|n|^{2b+2\theta}\lesssim|k|^{2b+2\theta-2\beta\tau}\sum_{|n|\leq|k|}|n|^{2b+2\theta}\,.$$

It is a standard fact to see that the sum on the right hand side of the formula above behaves as $|k|^{(2b+2\theta+d)\vee 0}$ (and as $\log|k|$ if $2b+2\theta=-d$).

In conclusion we need $2\beta\tau-4b-4\theta>d$, and in that case,

$$\begin{aligned}\mathbb{E}[|\Delta_j\eta^{\heartsuit}|^2] &\lesssim t^{-2\beta}\sum_{|k|\sim 2^j}|k|^{2a-2\beta\tau+2b+2\theta+(2b+2\theta+d)\vee 0}\\ &\lesssim t^{-2\beta}2^{j(2a-2\beta\tau+2b+2\theta+(2b+2\theta+d)\vee 0)},\end{aligned}$$

with a multiplicative correction term of order j (that does not change our conclusions below) in the case $2b + 2\theta = -d$. Therefore, by [13, Lemma A.9] it follows that

$$\sup_t t^{2\beta}\mathbb{E}[\|\eta^{\vee}\|_\alpha^2] < \infty$$

for $\alpha < \beta\tau - \chi_1$, with $\beta\tau > \chi_0$, where

$$\chi_0 = 2b + 2\theta + \tfrac{1}{2}d, \qquad \chi_1 = a + b + \theta + (b + \theta + \tfrac{1}{2}d)_+. \tag{4.4}$$

By hyper–contractivity in the second Wiener chaos [19, 21], the following result holds.

Lemma 4.7. *If $\alpha, \beta \in \mathbf{R}$ are such that*

$$\beta > \beta_0(\alpha) := \Bigl(\frac{\alpha + \chi_1}{\tau}\Bigr) \vee \Bigl(\frac{\chi_0}{\tau}\Bigr)_+,$$

then for every $p \geq 1$,

$$\sup_{t\geq 0} \mathbb{E}[(t^\beta\|\eta^{\vee}\|_\alpha)^p] < \infty\ .$$

Remark 4.8. The advantage of the random initial condition emerges here, as we see that we have a milder singularity at $t = 0$. For comparison, let u_0 be a non–random initial condition and set, as in Section 3, $\eta_1(t) = \mathrm{e}^{tA}u_0$. We wish to find initial conditions where the minimal singularity in time of the Littlewood–Paley block of $B(\eta_1, \eta_1)$ is worse than the one of random initial conditions. To this aim, assume that $B_{kmn} \approx |k|^a|m|^b|n|^b$ and that the Fourier coefficients of u_0 are so that $u_0(k) \approx |k|^\theta$. Then

$$|\Delta_j B(\eta_1, \eta_1) \approx \sum_{|k|\sim 2^j} |k|^a \sum_{m+n=k} |m|^{b+\theta}|n|^{b+\theta}\,\mathrm{e}^{-t(|m|^\tau + n^\tau)},$$

and, for each k,

$$\begin{aligned}\sum_{m+n=k} |m|^{b+\theta}|n|^{b+\theta}\,\mathrm{e}^{-t(|m|^\tau+n^\tau)} &\gtrsim \sum_{A_k} |m|^{b+\theta}|n|^{b+\theta}\,\mathrm{e}^{-t(|m|^\tau+n^\tau)}\\ &\approx \int_{|k|}^{+\infty} \rho^{2b+2\theta+d-1}\,\mathrm{e}^{-2t\rho^\tau}\,d\rho \gtrsim \int_{|k|}^{t^{-1/\tau}} \rho^{2b+2\theta+d-1}\,d\rho \sim t^{-\frac{2b+2\theta+d}{\tau}},\end{aligned}$$

where A_k is as above.

4.3.4. *Regularity of η^{Ψ}.* By means of Assumption 3.1, we can prove that η^{Ψ} is in a $\mathscr{X}_T^{\alpha,\beta}$ space (actually a $\mathscr{V}^{\alpha,\beta}$ space) for suitable values of α, β. We start by stating the following lemma.

Lemma 4.9. *Assume $\frac{\chi_0}{\tau} < 1$. For every $\alpha < 2\tau - \chi_1$, every $\beta > \beta_0(\alpha) - 1$, every $p \geq 1$, and every $T > 0$, there is a number $c_T > 0$ such that*

$$\sup_{t\in[0,T]} \mathbb{E}[(t^\beta\|\eta^{\Psi}\|_\alpha)^p] \leq c_T.$$

Moreover, $c_T \to 0$ as $T \downarrow 0$.

Theorem 4.10. *Under Assumptions 4.1, 4.2, and 4.4, if* $\frac{\chi_0}{\tau} < 1$, *then* $\eta^{\curlyvee} \in \mathscr{X}_T^{\alpha,\beta}$ *for all* $\alpha < 2\tau - \chi_1$, *all* $\beta > \beta_0(\alpha) - 1$, *and all* $T > 0$. *Moreover, for every* $p \geq 1$ *and for the same values of* α, β,

$$\mathbb{E}[\|\eta^{\curlyvee}\|_{\alpha,\beta,T}^p] < \infty.$$

In particular, $\eta^{\curlyvee} \in \mathscr{V}^{\alpha,\beta}$, *a. s., for the same values of* α, β.

Proof. Notice preliminarily that it is sufficient to prove the statement when β is close to $\beta_0(\alpha) - 1$, since for $\epsilon > 0$, $\|\cdot\|_{\alpha,\beta+\epsilon,T} \leq T^{\epsilon}\|\cdot\|_{\alpha,\beta,T}$.

Our strategy to prove the theorem is to find $\gamma \in (0,1)$ and $p \geq 1$ such that $\gamma p > 1$ and $t \mapsto t^{\beta}\eta_t^{\curlyvee}$ is in $W^{\gamma,p}([0,T];\mathscr{C}^{\alpha})$ (with all moments). By Sobolev's embeddings, this concludes the proof of the theorem.

It follows from Lemma 4.9 that

$$\mathbb{E}\Big[\int_0^T \|t^{\beta}\eta^{\curlyvee}\|_{\alpha}^p\,dt\Big] \leq c_T$$

for all $\alpha < 2\tau - \chi_1$ and $\beta > \beta_0(\alpha) - 1$. It remains to analyse the increments.

<u>*Case 1*</u>. Consider first the case $\tau - \chi_1 \leq \alpha < 2\tau - \chi_1$. Here we have $\beta_0(\alpha) - 1 \in [0,1)$, so in view of the initial remark, it is not restrictive to assume that $\beta \in (\beta_0(\alpha) - 1, 1)$. Let $s \leq t \leq T$, then

$$t^{\beta}\eta_t^{\curlyvee} - s^{\beta}\eta_s^{\curlyvee} = (t^{\beta} - s^{\beta})\eta_t^{\curlyvee} + s^{\beta}(\eta_t^{\curlyvee} - \eta_s^{\curlyvee}).$$

Consider the first term. It is elementary to see that for $\lambda \in [0,1]$ (the case $\lambda = 0$ is obvious, the case $\lambda = 1$ follows by Taylor expansion, the intermediate cases by interpolation),

$$t^{\beta} - s^{\beta} \lesssim t^{(1-\lambda)\beta}s^{(\beta-1)\lambda}(t-s)^{\lambda},$$

thus by Lemma 4.9,

$$\begin{aligned}
\mathbb{E}\int_0^T\int_0^t \frac{\|(t^{\beta}-s^{\beta})\eta_t^{\curlyvee}\|_{\alpha}^p}{|t-s|^{1+\gamma p}}\,ds\,dt &\lesssim \int_0^T\int_0^t \frac{t^{\beta p(1-\lambda)-\beta_1 p}s^{\lambda p(\beta-1)}}{|t-s|^{1+(\gamma-\lambda)p}}\mathbb{E}[(t^{\beta_1}\|\eta_t^{\curlyvee}\|_{\alpha})^p]\,ds\,dt\\
&\lesssim \int_0^T t^{\beta p(1-\lambda)-\beta_1 p}\int_0^t \frac{ds}{s^{\lambda p(1-\beta)}|t-s|^{1+(\gamma-\lambda)p}}\,dt\\
&\lesssim \int_0^T t^{-p(\gamma+\beta_1-\beta)}\,dt,
\end{aligned}$$

where $\beta_1 \in (\beta_0(\alpha) - 1, \beta)$, and we need $\lambda p(1-\beta) < 1$, $1 + (\gamma - \lambda)p < 1$, and $p(\gamma + \beta_1 - \beta) < 1$. In the limit $\lambda \downarrow \gamma$ and $\beta_1 \downarrow \beta_0(\alpha) - 1$, we obtain the two conditions

$$\gamma p(1-\beta) < 1, \qquad p\big(\gamma - (\beta + 1 - \beta_0(\alpha))\big) < 1. \tag{4.5}$$

Similar considerations applied to the second term yield additional conditions on γ, p. These can be summarized as follows: given $\beta \in (\beta_0(\alpha) - 1, 1)$, find $\gamma \in (0,1)$

and $p \geq 1$ such that

$$\gamma < \tfrac{1}{1-\beta}\tfrac{1}{p}, \qquad \gamma - \tfrac{1}{p} < \beta + 1 - \beta_0(\alpha), \qquad \gamma - \tfrac{1}{p} > 0,$$
$$\gamma < 2 - \beta_0(\alpha), \qquad \frac{1}{p} > \big(\tfrac{\chi_0}{\tau}\big)_+ - \beta. \tag{4.6}$$

Notice that by the choice of β, we have that $\beta + 1 - \beta_0(\alpha) < 2 - \beta_0(\alpha)$ and $(\chi_0/\tau)_+ - \beta < 1 - \beta < 2 - \beta_0(\alpha)$. Figure 1 shows the non–empty area of all values of γ and $1/p$ that meet all the requirements.

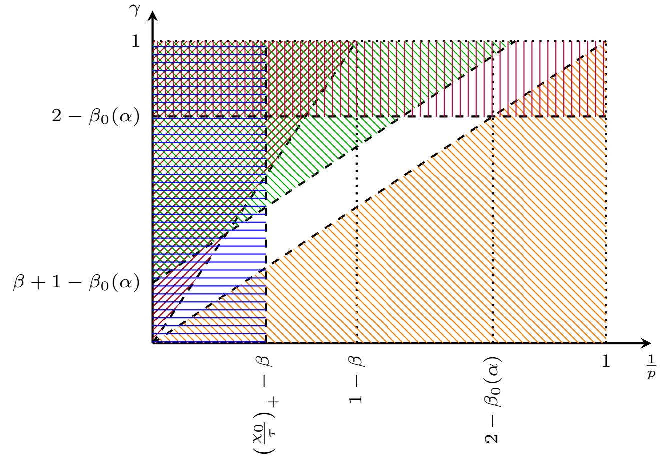

FIGURE 1. The white area contains all values $(\gamma, 1/p)$ that satisfy (4.6).

Case 2. Assume $\alpha < \tau - \chi_1$, then $\beta_0(\alpha) < 1$ and, due to the initial remark, we can assume $\beta < 0$. This time we decompose the increment as

$$t^\beta \eta_t^{\curlyvee} - s^\beta \eta_s^{\curlyvee} = t^\beta(\eta_t^{\curlyvee} - \eta_s^{\curlyvee}) + (t^\beta - s^\beta)\eta_s^{\curlyvee}.$$

Similar estimates as above yield the following conditions on p, γ: given $\beta \in (\beta_0(\alpha) - 1, 0)$, find $\gamma \in (0, 1)$ and $p \geq 1$ such that

$$\beta_0(\alpha)p < 1, \qquad p\big(\gamma - (\beta + 1 - \beta_0(\alpha))\big) < 1. \tag{4.7}$$

□

5. ADDITIONAL EXAMPLES

5.1. **Non–symmetric nonlinearity.** Our Assumption 4.2 (as well as Assumption 3.2 in the case of an optimal inequality) means essentially that B is, at small scales, like $D^a((D^b u)^2)$. If this is not the case, the inequalities on which we base our analysis are not optimal and the results are at most as good as those in the symmetric case. A first order expansion though is still sufficient.

Consider for instance the following one–dimensional problem

$$\partial_t u = Au + uu_{xx}.$$

We can write $uu_{xx} = \frac{1}{2}(u^2)_{xx} - (u_x)^2$, and notice that the three terms $B(u,u) := uu_{xx}$, $B_1(u,u) := (u^2)_{xx}$, and $B_2(u,u) = (u_x)^2$ scale with the same scaling, with $\sigma = \tau - 2$. Thus, using the theory detailed in these pages, we can solve the problem in $\mathscr{X}_T^{1,\beta}$, for a suitable β. This is an optimal choice for B_2, but not for B_1. This discrepancy explains the non–optimal results in such cases.

5.2. **The case without mass conservation.** We have worked so far under the assumption of mass conservation, namely that the solution averages to zero in the spatial domain. In this section we wish to briefly show that the general case follows likewise, without too much hassle when mass conservation does not hold.

Consider B quadratic, and let U be solution of

$$\partial_t U = AU + B(U,U).$$

Decompose $U = \xi + u$, where ξ is the space average of U, and u has spatial mean zero. Recall that $\mathcal{M}$ is the projection onto the zero mass space, so that $u = \mathcal{M}U$. Assume we work under Assumptions 4.1 and 4.2 (this time including the zero modes), then the equations for u and ξ are

$$\begin{cases} \dot{\xi} = \mathcal{M}^\perp B(u,u) + B(\xi,\xi), \\ \partial_t u = Au + \mathcal{M}B(u,u) + 2\mathcal{M}B(u,\xi), \end{cases}$$

since $\mathcal{M}B(\xi,\xi) = 0$ and $\mathcal{M}^\perp B(u,\xi) = 0$.

We first notice that, if the initial condition has infinite mean, there is in general no hope to have a finite mean at positive times. We thus consider in the rest of this section the case of an initial condition with finite mean.

Assume, to fix ideas, that the numbers a, b are integers. We notice that if $a \geq 1$, then $\mathcal{M}B = B$, $\mathcal{M}^\perp B = 0$, and $B(\xi,\xi) = 0$, while if $b \geq 1$ then immediately $B(\xi,\cdot) = 0$. Moreover, $\mathcal{M}B(\mathcal{M}\cdot,\mathcal{M}\cdot)$ satisfies our original Assumption 4.2 (that is, a nonlinearity that preserves the mass). We have three cases.

- If $a \geq 1$, then ξ is a finite constant (in space and time) and the equation of u is of the same kind we have studied so far, with the addition of the term of lower order $\mathcal{M}B(u,\xi)$ that does not change our analysis.
- If $a = 0$, $b \geq 1$, the equation for u decouples from ξ, and is of the same kind we have studied so far. Once u is known, then ξ can be computed by the equation $\dot{\xi} = \mathcal{M}^\perp B(u,u)$. An additional difficulty is that if we solve the problem for u in $\mathscr{X}_T^{\beta,b}$, then we cannot ensure that $\mathcal{M}^\perp B(u,u)$ is well defined. Indeed, for instance in the one–dimensional case (this is only to avoid ambiguity in the understanding of the generic term D^b),

$$\mathcal{M}^\perp B(u,u) \sim \Big(\sum_{m+n=0} m^b n^b u_m u_n\Big) e_0 \sim \|u\|_{H^b}^2,$$

and $\mathscr{C}^b = B_{\infty,\infty}^b$ is not in $H^b = B_{2,2}^b$.

- Likewise, if $a = b = 0$, the equation for u contains the lower order term $\mathcal{M}B(u, \xi)$, while the equation for ξ contains the polynomial term $B(\xi, \xi)$ and again
$$\mathcal{M}^{\perp}B(u, u) = \Big(\sum_{m+n=0} u_m u_n\Big) e_0 \sim \|u\|_{L^2}$$
with $L^2 = B^0_{2,2}$.

In the last two cases a easy workaround is to solve the problem in $\mathscr{X}_T^{b+\epsilon,\beta}$, since for $\alpha \geq 0$, $p \geq 1$, and $\epsilon > 0$,
$$\mathscr{C}^{\alpha+\epsilon} = B^{\alpha+\epsilon}_{\infty,\infty} \subset B^{\alpha}_{2,2}.$$

5.3. **Higher powers in the nonlinearity.** The overall picture provided by quadratic nonlinearities does not change for non–linear terms with higher powers. Indeed, assume B is m–linear, with $m > 2$, then under an assumption analogous to (3.2), we see that if $\delta > \frac{1}{m}$ then Theorem 3.5 is enough for initial conditions up to (and including) the critical space. If $\delta < \frac{1}{m}$ the random initial condition method becomes effective and allows to solve the initial value problem for rougher initial conditions (but not as rough as the critical space in general). We observe that also in the multi–linear case a first order expansion is sufficient, because the method fails for integrability of the analogous of $\eta^{\vee}$ before failing due to the smallness of (the analogous of) η^{Y} in a suitable space.

Likewise, if we relax the condition of mass conservation we can still solve the problem without having divergences (so in the language of [15], there is no need to include renormalization in the analysis).

5.4. **An example with renormalization.** In this section we briefly discuss a case when one can prove an existence result for supercritical renormalized random initial conditions. The case is very specific and makes use of a nonlinear transformation.

Consider the KPZ equation, which is the following PDE,
$$\partial_t h = \Delta h + |\nabla h|^2 \tag{5.1}$$
on the two dimensional torus $\mathbb{T}_2$, whose critical space is at the level of $\mathscr{C}^0$.

Let X be a Gaussian Free Field on the two dimensional torus, which is known to belong to $\mathscr{C}^{-\kappa}$ for any $\kappa > 0$. We will now construct the solution of (5.1) started with γX, where γ is a positive constant strictly less than $\sqrt{2}$, and show that, in order to do so, it is necessary to suitably renormalise the initial condition.

Let ϱ be a smooth compactly supported function, $\varrho^\epsilon(\cdot) := \epsilon^{-2}\varrho(\cdot/\epsilon)$ for $\epsilon > 0$, and $X^\epsilon := X \star \varrho^\epsilon$. Let h^ϵ be the solution to (5.1) such that $h^\epsilon_0(\cdot) := h^\epsilon(0, \cdot) := \gamma X^\epsilon(\cdot)$. By Cole-Hopf transform, $u^\epsilon := \mathrm{e}^{h^\epsilon}$ solves
$$\partial_t u^\epsilon = \Delta u^\epsilon, \qquad u^\epsilon(0) = \mathrm{e}^{\gamma X^\epsilon}. \tag{5.2}$$

In [12], it is shown that the measure $\mathrm{e}^{\gamma X^\epsilon}$ converges if and only if it is suitably *renormalized*. More precisely, define the measure μ^ϵ as

$$\mu^\epsilon(dz) := \mathrm{e}^{\gamma X^\epsilon(z)}\,\epsilon^{\frac{1}{2}\gamma^2}\,dz, \tag{5.3}$$

then as ϵ goes to 0, μ^ϵ almost surely converges weakly to a random measure μ such that, if A is a set of positive Lebesgue measure, then $\mu(A) > 0$ almost surely. The previous implies that, upon setting $\tilde{u}^\epsilon(t,x) := u^\epsilon(t,x)\epsilon^{\gamma^2/2}$, we have

$$\tilde{u}^\epsilon(t,x) = \int_{\mathbb{T}^2} P_t(x-y)\,\mu^\epsilon(dy) \xrightarrow{\epsilon\downarrow 0} \int_{\mathbb{T}^2} P_t(x-y)\,\mu(dy) =: u(t,x), \tag{5.4}$$

where P_t is the usual heat kernel. Now, u is almost surely bounded (actually smooth) and strictly positive at every strictly positive time (the last follows by the fact that μ does not vanish on sets of positive Lebesgue measure).

Getting back to our original problem, thanks to the strict positivity of u, one can simply define $h(t,x) := \log u(t,x)$ as the solution of (5.1). Moreover, we have that

$$\tilde{h}^\epsilon(t,x) := \log(\tilde{u}^\epsilon(t,x)) = \log(u^\epsilon(t,x)) - \frac{1}{2}\gamma^2 \log \epsilon^{-1} \tag{5.5}$$

converges to h as ϵ goes to 0 and $\tilde{h}^\epsilon$ is the solution to (5.1) started with a renormalized initial condition given by $\gamma X^\epsilon - \frac{\gamma^2}{2}\log\epsilon^{-1}$.

5.5. **A counterexample.** Consider the following problem on $[-\pi,\pi]$ with periodic boundary conditions, and zero mean,

$$\begin{cases} \partial_t u = u_{xx} + (u \star u)_x, & x \in [-\pi,\pi], t \geq 0, \\ u \text{ is a odd function,} \end{cases} \tag{5.6}$$

where $\star$ denotes convolution on $(-\pi,\pi)$. The equation has scaling invariance, with $\tau = 2$, $\sigma = 2$, thus the critical space is at the level of $\mathscr{C}^{-2}$.

In the rest of this section we show that we can find (infinitely many) Gaussian initial conditions Ξ that are in $\mathscr{C}^{-\frac{3}{2}-}$ a. s., but such that there is no solution of the above problem with initial condition Ξ.

The problem has a very simple formulation in Fourier coordinates. A mean zero periodic odd function on $[-\pi,\pi]$ has the Fourier expansion

$$u(x) = \sum_{k\in\mathbf{Z}} u_k\,\mathrm{e}^{\mathrm{i}kx} = 2\sum_{k=1}^\infty \xi_k \sin kx,$$

with $u_k = -\mathrm{i}\xi_k$, $\xi_k \in \mathbf{R}$, and $\xi_{-k} = -\xi_k$ for all k. The equation, in terms of the new variables $(\xi_k)_{k\geq 1}$, is

$$\frac{d}{dt}\xi_k = -k^2\xi_k + k\xi_k^2, \qquad k \geq 1.$$

Each equation can be explicitly integrated, and one can easily see that each component ξ_k may blow up at the finite time

$$\tau_k = -\frac{1}{k^2}\log\Bigl(1 - \frac{k^2}{k\xi_k(0)}\Bigr) \tag{5.7}$$

and we set $\tau_k = \infty$ if the argument in the logarithm in (5.7) is negative, or when the formula for τ_k gives a negative number. Elementary computations show that $\tau_k < \infty$ when $\xi_k(0) > k$.

We have the following trichotomy

- $\inf_{k\geq 1}\tau_k = 0$: no local existence for (5.6),
- $\inf_{k\geq 1}\tau_k > 0$ and finite: local existence for (5.6),
- $\inf_{k\geq 1}\tau_k = \infty$: global existence for (5.6).

In view of the probabilistic argument, we notice that $\inf_k \tau_k > 0$ if and only if there is $\epsilon > 0$ such that $\tau_k \geq \epsilon$ eventually.

5.5.1. *Random initial condition.* We consider as initial condition a Gaussian random field $\Xi(x) = \sum_{k\geq 1}\xi_k \sin kx$ with independent ξ_k with Gaussian law $\mathcal{N}(0,\sigma_k^2)$.

Lemma 5.1. *If there are $\lambda > \sqrt{2}$ and $\epsilon > 0$ such that*

$$\sigma_k \leq \frac{k}{\lambda\sqrt{\log k}(1-\mathrm{e}^{-\epsilon k^2})}, \qquad k \geq 1,$$

then $\inf_{k\geq 1}\tau_k > 0$, a. s. for the problem with initial condition $-\mathrm{i}\Xi$. Moreover $\Xi \in \mathscr{C}^{-\frac{3}{2}-}$.

Proof. The first part follows immediately by a Borel–Cantelli argument, since

$$\sum_{k=1}^{\infty}\mathbb{P}[\tau_k \leq \epsilon] < \infty.$$

Indeed,

$$\begin{aligned}\mathbb{P}[\tau_k \leq \epsilon] = \mathbb{P}\Bigl[\xi_k \geq \frac{k}{(1-\mathrm{e}^{-\epsilon k^2})}\Bigr] = \mathbb{P}\Bigl[Z \geq \frac{k}{\sigma_k(1-\mathrm{e}^{-\epsilon k^2})}\Bigr] \leq \\ \leq \mathbb{P}[Z \geq \lambda\sqrt{\log k}] \lesssim \frac{1}{\lambda\sqrt{\log k}}\,\mathrm{e}^{-\frac{1}{2}\lambda^2\log k} = \frac{1}{\lambda k^{\frac{1}{2}\lambda^2}\sqrt{\log k}},\end{aligned}$$

where Z is a real standard Gaussian random variable. Therefore the series above converges since $\frac{1}{2}\lambda^2 > 1$ by the choice of λ.

To prove that $\Xi \in \mathscr{C}^{-\frac{3}{2}-}$ we use Kolmogorov's continuity theorem. Indeed, let $E = (-\Delta)^{-1}\Xi$ (notice that the Laplace operator is invertible on the subspace of zero mean functions), then

$$\mathbb{E}[|E(x)-E(y)|^2] = \sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^4}(\sin kx - \sin ky)^2 \leq$$

$$\leq \frac{1}{\lambda^2(1-\mathrm{e}^{-\epsilon})^2} \sum_{k=1}^{\infty} \frac{(1\wedge k|x-y|)^2}{k^2 \log k} \lesssim |x-y|.$$

Since E is Gaussian, we deduce that $E \in \mathscr{C}^{\frac{1}{2}-}$ and therefore $\Xi \in \mathscr{C}^{-\frac{3}{2}-}$. □

On the other hand, with the same regularity, we can provide an initial condition that gives non–existence.

Lemma 5.2. *Set*

$$\sigma_k = \frac{k}{\sqrt{2\log k}(1-\mathrm{e}^{-k^2\epsilon_k})},$$

with $\epsilon_k \downarrow 0$. Then $\tau_k \leq \epsilon_k$ infinitely often, a. s. In particular $\inf_k \tau_k = 0$ a. s. and there is no solution with initial condition $-\mathrm{i}\Xi$ with probability one. Moreover, if $\inf_k k^2\epsilon_k > 0$, then $\Xi \in \mathscr{C}^{-\frac{3}{2}-}$, a. s.

Proof. For the first part we use again a Borel–Cantelli argument. As above,

$$\mathbb{P}[\tau_k \leq \epsilon_k] = \mathbb{P}[Z \geq \sqrt{2\log k}] \gtrsim \frac{1}{\sqrt{2\log k}}\,\mathrm{e}^{-\log k} = \frac{1}{k\sqrt{2\log k}},$$

but this time the series diverges and $\tau_k \leq \epsilon_k$ for infinitely many k with probability one.

The regularity follows as in the previous lemma, since for $E = (-\Delta)^{-1}\Xi$,

$$\mathbb{E}[|E(x)-E(y)|^2] = \sum_{k=1}^{\infty} \frac{\sigma_k^2}{k^4}(\sin kx - \sin ky)^2 \leq$$

$$\leq \frac{1}{2(1-\mathrm{e}^{-\delta})^2} \sum_{k=1}^{\infty} \frac{(1\wedge k|x-y|)^2}{k^2} \lesssim |x-y|.$$

where $\delta = \inf_k k^2\epsilon_k$. □

6. A logarithmically sub–critical result

In this section we discuss the existence of solutions with random initial conditions in the critical case. We focus, as a standing example, on the Burgers equation in dimension $d=1$, which is the equation for the derivative of the solution of KPZ,

$$\partial_t u - u_{xx} = (u^2)_x. \tag{6.1}$$

Notice that we have not changed the parameter δ from Assumption 3.2. The critical space on the other hand is (clearly) shifted by one derivative.

6.1. **Setting of the problem.**

6.1.1. *Random initial data.* We consider random initial data

$$u_0 = \sum_{k\in\mathbf{Z}^d} \phi_k \xi_k e_k,$$

where $(\xi_k)_{k\in\mathbf{Z}^d}$ are centred complex valued Gaussian random variables such that $\bar{\xi}_k = \xi_{-k}$ for all k and with covariance as in (4.2), and $(\phi_k)_{k\in\mathbf{Z}^d}$ are a sequence of weights such that $\phi_0 = 0$ (mass conservation), and

$$|\phi_k| \sim |k|^{\theta}(\log(1+|k|))^{-\nu-\frac{1}{2}}, \tag{6.2}$$

with $\theta = 1 - \frac{d}{2} = \frac{1}{2}$.

Using [16, Theorem 6.3] as in formula (4.3), we see that

$$\|\Delta_j u_0\|_\infty \leq C j^{-\nu} 2^j,$$

for a random constant C. Thus $\nu = 0$ corresponds to critical initial data, and $\nu > 0$ to logarithmically sub–critical initial data.

By Proposition 4.5 it follows that

$$\|\Delta_j \eta^{\dagger}\|_\infty \leq C j^{-\nu} 2^j \,\mathrm{e}^{-2^{2j}t}, \tag{6.3}$$

for the same random constant C as above.

6.1.2. *The solution space.* We will solve the problem as in Section 3.2. We set $u = v + \eta^{\dagger}$ and consider the problem

$$\partial_t v - \partial_{xx} v = (v^2)_x + 2(v\eta^{\dagger})_x + \eta^{\vee}, \tag{6.4}$$

The term η^{Ψ}, obtained by applying the heat kernel to $\eta^{\vee}$, has enough regularity for what we will do. The troublemaker is $(v\eta^{\dagger})_x$, since given the regularity of v and $\eta^{\vee}$, the singularity in time at $t = 0$ is not integrable. Before illustrating how to circumvent the problem, we introduce the space where the problem will be solved.

Define the space $\mathscr{C}^{\alpha}_{\kappa}$ as the closure of smooth functions with respect to the norm

$$\|u\|_{(\alpha,\kappa)} := \sup_{j\geq -1} (1+|j|^{\kappa}) 2^{\alpha j} \|\Delta_j u\|_\infty.$$

This is as the space $\mathscr{C}^{\alpha}_{\kappa}$, but with a logarithmically corrected growth. We state a few properties of these spaces we shall need later. To this end, define a tamed logarithm $\ell : (0,\infty) \to \mathbf{R}$ as

$$\ell(t) = \log(\tfrac{1}{t} \vee 2).$$

Lemma 6.1. *The following properties hold,*

- *if $\alpha > 0$ and $\kappa \in \mathbf{R}$, or if $\alpha = 0$ and $\kappa > 1$, then $\mathscr{C}^{\alpha}_{\kappa} \cdot \mathscr{C}^{\alpha}_{\kappa} \subset \mathscr{C}^{\alpha}_{\kappa}$,*
- *$\mathscr{C}^{\alpha+\epsilon} \subset \mathscr{C}^{\alpha}_{\kappa} \subset \mathscr{C}^{\alpha}$, for every $\epsilon > 0$,*
- *if $\alpha' < \alpha$ and any κ, κ', or if $\alpha = \alpha'$ and $\kappa \geq \kappa'$, then for every $t > 0$ and $u \in \mathscr{C}^{\alpha'}_{\kappa'}$,*

$$\|\,\mathrm{e}^{t\Delta} u\|_{(\alpha,\kappa)} \lesssim t^{-\frac{1}{2}(\alpha-\alpha')} \ell(t)^{\kappa-\kappa'} \|u\|_{(\alpha',\kappa')}.$$

Proof. For the first property, if $u, v \in \mathscr{C}^\alpha_\kappa$, with $\|u\|_{(\alpha,\kappa)} \leq 1$, $\|v\|_{(\alpha,\kappa)} \leq 1$,

$$\begin{aligned}\|\Delta_j(u \circledless v)\|_\infty &\approx \Big\|\Delta_j\Big(\sum_{m=-1}^{j-2}(\Delta_m u)(\Delta_n v)\Big)\Big\|_\infty \lesssim \\ &\approx \sum_{m=-1}^{j-2}\|\Delta_m u\|_\infty\|\Delta_j v\|_\infty \lesssim j^{-\kappa}2^{-\alpha j}\sum_{m=-1}^{j-2} m^{-\kappa}2^{-\alpha m} \lesssim j^{-\kappa}2^{-\alpha j},\end{aligned}$$

and

$$\|\Delta_j(u \circleddash v)\|_\infty \lesssim \sum_{m=j}^{\infty}\|\Delta_m u\|_\infty\|\Delta_m v\|_\infty \lesssim \sum_{m=j}^{\infty} m^{-2\kappa}2^{-2\alpha m} \lesssim j^{-\kappa}2^{-\alpha j}.$$

The second property is immediate by the definition of norms. For the third, using [17, Proposition 2.4],

$$\begin{aligned}&j^\kappa 2^{\alpha j}\|\Delta_j(\mathrm{e}^{t\Delta} u)\|_\infty \lesssim j^\kappa 2^{\alpha j}\,\mathrm{e}^{-2^{2j}t}\,\|\Delta_j u\|_\infty = \\ &\quad = (j^{\kappa'}2^{\alpha' j}\|\Delta_j u\|_\infty) j^{\kappa-\kappa'}2^{(\alpha-\alpha')j}\,\mathrm{e}^{-2^{2j}t} \lesssim H_{\kappa-\kappa',\alpha-\alpha',2}(t)\|u\|_{(\alpha',\kappa')},\end{aligned}$$

and the conclusion follows from Lemma 6.8. Here the quantity $H_{\kappa-\kappa',\alpha-\alpha',2}$ is defined in (6.11). ☐

6.2. **A "classical" case.** Let us solve first a fixed point theorem for

$$\partial_t u = \Delta u + (u^2)_x,$$

with a norm better suited for the critical level,

$$\|u\|_\bullet := \sup_{t\leq T} t^{\frac12}\ell(t)^a\|u(t)\|_{(0,\kappa)},$$

with $\kappa > 1$. Then by Lemma 6.1 we obtain

$$\begin{aligned}\|\mathscr{V}(u)\|_{(0,\kappa)} &\leq \int_0^t \|\,\mathrm{e}^{(t-s)\Delta}(u^2)_x\|_{(0,\kappa)}\,ds \lesssim \int_0^t \|\,\mathrm{e}^{(t-s)\Delta}\,u^2\|_{(1,\kappa)}\,ds \\ &\lesssim \|u\|_\bullet^2\int_0^t (t-s)^{-\frac12}s^{-1}\ell(s)^{-2a}\,ds \lesssim t^{-\frac12}\ell(t)^{1-2a}\|u\|_\bullet^2,\end{aligned}$$

where in the last step we used that if $\beta \in (0,1)$ and $a \geq 0$, or $\beta = 1$ and $a > 1$ by an elementary computation we have

$$\int_0^t (t-s)^{-\frac12}s^{-\beta}\ell(s)^{-a}\,ds \lesssim \begin{cases} t^{\frac12-\beta}\ell(t)^{-a}, & \beta < 1, \\ t^{-\frac12}\ell(t)^{1-a}, & \beta = 1.\end{cases}$$

Therefore, if $a > 1$,

$$\|\mathscr{V}(u)\|_\bullet \leq \ell(T)^{-(a-1)}\|u\|_\bullet^2.$$

Consider the initial condition. By Lemma 6.1,

$$\|\,\mathrm{e}^{t\Delta}\,u(0)\|_{(0,\kappa)} \lesssim t^{-\frac12}\ell(t)^{\kappa-\kappa'}\|u(0)\|_{-1,\kappa'}$$

thus

$$\| \mathrm{e}^{t\Delta} u(0)\|_{\bullet} \lesssim \ell(T)^{\kappa-\kappa'+a}\|u(0)\|_{(-1,\kappa')}$$

if $\kappa - \kappa' + a < 0$, that is $\kappa' > \kappa + a$. This allows to prove a fixed point theorem with initial condition in $\mathscr{C}^{-1}_{\kappa'}$.

In view of a comparison with the results in the next sections, consider an initial condition with

$$\|\Delta_j u(0)\|_\infty \sim j^{-\nu}2^j,$$

then

$$\|u(0)\|_{(-1,\kappa')} = \sup_j j^{\kappa'}2^{-j}\|\Delta_j u(0)\|_\infty \sim \sup_j j^{\kappa'-\nu},$$

is finite if $\nu \geq \kappa'$. Hence we have $\nu \geq \kappa' > \kappa + a > 2$. We will find, for random initial conditions in Section 6.3 below, the condition $\nu > 1$, and in Section 6.4 the condition $\nu > \frac{1}{2}$. Both guarantee less regularity than in the deterministic case.

6.3. **A "classical" case, with random initial condition.** Let $\mathscr{Y}^{\kappa,\beta}_T$ be the space defined as $\mathscr{X}^{0,\beta}$, but with the $\mathscr{C}^0$ norm replaced by the $\mathscr{C}^0_\kappa$ norm. By Theorem 4.10 we know that $\eta^{\curlyvee\curlyvee} \in \mathscr{X}^{0,\beta}_T$ (actually $\mathscr{V}^{0,\beta}$) for $\beta > \beta_0(0) - 1 = \frac{1}{4}$. The same argument shows that $\eta^{\curlyvee\curlyvee} \in \mathscr{V}^{2\epsilon,\beta}$ for $\beta > \frac{1}{4} + \epsilon$, with $\epsilon > 0$, thus by Lemma 6.1 $\eta^{\curlyvee\curlyvee} \in \mathscr{Y}^{\kappa,\beta}_T$ for $\beta > \frac{1}{4}$ and $\kappa \geq 0$, with

$$\|\eta^{\curlyvee\curlyvee}\|_{\mathscr{Y}^{\kappa,\beta}_T} \leq C_{\curlyvee\curlyvee} g_T, \tag{6.5}$$

where $C_{\curlyvee\curlyvee}$ is a random constant and $g_T \lesssim T^\epsilon$ for small enough $\epsilon > 0$ (depending on the value of β).

Moreover the previous lemma ensures that for $\beta \in (\frac{1}{4}, \frac{1}{2})$ and $\kappa > 1$,

$$\|\mathcal{V}(v,v)\|_{\mathscr{Y}^{\kappa,\beta}_T} \lesssim g_T\|v\|^2_{\mathscr{Y}^{\kappa,\beta}_T}, \tag{6.6}$$

with $g_T \lesssim T^\epsilon$ as above. This shows that the term $(v\eta^{\curlyvee})_x$ is the "troublemaker", as is the term that so far prevents us to apply a fixed point theorem to problem (6.4) in $\mathscr{Y}^{\kappa,\beta}_T$

In this section we analyse the term $(v\eta^{\curlyvee})_x$, and show that if $\nu > 1$, then $\mathscr{V}(v \olessthan \eta^{\curlyvee})$ is well defined and the fixed point strategy can be completed.

Proposition 6.2. *Consider a random initial condition as in Assumption 4.4, with coefficients as in (6.2). If $\nu > 1$, then there is a random time T, with $T > 0$ a. s., such that problem (6.4) has a unique solution in $\mathscr{Y}^{\kappa,\beta}_T$, where $\beta \in (\frac{1}{4}, \frac{1}{2})$, and $\kappa \in (1, \nu]$.*

Proof. Let $v \in \mathscr{Y}^{\kappa,\beta}_T$ with $\|v\|_{\mathscr{Y}^{\kappa,\beta}_T} \leq 1$. By (6.3) we have

$$\|\Delta_j(v \ogreaterthan \eta^{\curlyvee})\|_\infty \lesssim \|\Delta_j v\|_\infty \sum_{n=0}^{j-2} \|\Delta_n \eta^{\curlyvee}\|_\infty \lesssim j^{-\kappa}t^{-\beta}\sum_{n=0}^{j-2} n^{-\nu}2^n \,\mathrm{e}^{-2^{2n}t},$$

where we have used the fact that, to compute $\Delta_j(v \circledcirc \eta^{\dagger})$, the relevant modes of v are those at levels $m \approx j$ (for simplicity of computations we have only considered $m = j$, but due to the estimates we have on $\Delta_j v$ and $\Delta_j \eta^{\dagger}$, the result is the same up to a multiplicative constant). Thus

(6.7)
$$\begin{aligned} t^{\beta} j^{\kappa} \Big\| \Delta_j \int_0^t \mathrm{e}^{(t-s)\Delta} (v \circledcirc \eta^{\dagger})_x \, ds \Big\|_{\infty} &\lesssim t^{\beta} j^{\kappa} \int_0^t 2^j \, \mathrm{e}^{-2^{2j}(t-s)} \, \|\Delta(v \circledcirc \eta^{\dagger})\|_{\infty} \, ds \\ &\lesssim t^{\beta} \sum_{n=2}^{j-2} n^{-\nu} 2^n \int_0^t 2^j \, \mathrm{e}^{-2^{2j}(t-s)} \, \mathrm{e}^{-2^{2n} s} \, s^{-\beta} \, ds \\ &\lesssim \sqrt{t} \sum_{n=2}^{j-2} n^{-\nu} 2^n \, \mathrm{e}^{-2^{2n} t} \\ &= \sqrt{t} G_{-\nu,1,2}(t), \end{aligned}$$

where $G_{-\nu,1,2}$ is defined in (6.11). From Lemma 6.8 we know that $\sqrt{t} G_{-\nu,1,2}(t)$ is bounded if $\nu \geq 0$, and converges to 0 as $t \to 0$ if $\nu > 0$. We notice that in particular we do not need the assumption on κ here.

Likewise, using again the regularity of $\eta^{\dagger}$ given by (6.3), we derive

$$\begin{aligned} \|\Delta_j(v \circleddash \eta^{\dagger})\|_{\infty} \lesssim \sum_{n=j}^{\infty} \|\Delta_n v\|_{\infty} \|\Delta_n \eta^{\dagger}\|_{\infty} \lesssim \\ \lesssim j^{-\kappa} t^{-\beta} \sum_{n=j}^{\infty} n^{-\nu} 2^n \, \mathrm{e}^{-2^{2n} t} \lesssim j^{-\kappa} t^{-\beta} G_{-\nu,1,2}(t). \end{aligned}$$

Thus, by Lemma 6.8,

(6.8)
$$\begin{aligned} t^{\beta} j^{\kappa} \Big\| \Delta_j \int_0^t \mathrm{e}^{(t-s)\Delta} (v \circleddash \eta^{\dagger})_x \, ds \Big\|_{\infty} &\lesssim t^{\beta} j^{\kappa} \int_0^t 2^j \, \mathrm{e}^{-2^{2j}(t-s)} \, \|\Delta(v \circleddash \eta^{\dagger})\|_{\infty} \, ds \\ &\lesssim t^{\beta} \int_0^t 2^j \, \mathrm{e}^{-2^{2j}(t-s)} \, s^{-\beta} G_{-\nu,1,2}(s) \, ds \\ &\lesssim t^{\beta} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\beta} G_{-\nu,1,2}(s) \, ds \end{aligned}$$

and, as before, it is sufficient to assume that $\nu > 0$.

Finally, since $\kappa > 1$, by (6.3),

$$\|\Delta_j(v \circledcirc \eta^{\dagger})\|_{\infty} \lesssim \|\Delta_j \eta^{\dagger}\|_{\infty} \sum_{n=0}^{j-2} \|\Delta_n v\|_{\infty} \lesssim j^{-\nu} 2^j \, \mathrm{e}^{-2^{2j} t} \sum_{n=0}^{j-2} n^{-\kappa} t^{-\beta} \lesssim j^{-\nu} 2^j \, \mathrm{e}^{-2^{2j} t} \, t^{-\beta},$$

therefore

$$
\begin{aligned}
t^\beta j^\kappa \Big\| \Delta_j \int_0^t \mathrm{e}^{(t-s)\Delta}(v \circledless \eta^{\dagger})_x\,ds \Big\|_\infty &\lesssim t^\beta j^\kappa \int_0^t 2^j\,\mathrm{e}^{-2^{2j}(t-s)}\,\|\Delta_j(v \circledless \eta^{\dagger})\|_\infty\,ds \\
&\lesssim j^{\kappa-\nu}2^{2j}t\,\mathrm{e}^{-2^{2j}t} \\
&\lesssim tH_{\kappa-\nu,2,2}(t).
\end{aligned} \tag{6.9}
$$

The quantity $H_{\kappa-\nu,2,2}(t)$, whose definition is given in formula (6.11), is such that $tH_{\kappa-\nu,2,2}(t)$ is bounded for $\kappa \le \nu$, and $tH_{\kappa-\nu,2,2}(t) \to 0$ for $\kappa < \nu$, by Lemma 6.8. ☐

6.4. **Local description.** Consider the case $\nu \le 1$. From the proof of Proposition 6.2, we see that there is a random constant C, independent of $T \le 1$, such that

$$\|\mathscr{V}(v \circledgtr \eta^{\dagger})\|_{\mathscr{Y}_T^{\kappa,\beta}} \le C g_{\nu,T}\|v\|_{\mathscr{Y}_T^{\kappa,\beta}}$$

for every $v \in \mathscr{Y}_T^{\kappa,\beta}$, with $g_{\nu,T} \downarrow 0$ as $T \downarrow 0$, and in the above formula by $\mathscr{V}(v \circledgtr \eta^{\dagger})$ we mean that only the part $v \circledgtr \eta^{\dagger}$ of the product $v\eta^{\dagger}$ appears in $\mathscr{V}$. Thus the irregularity of a solution of (6.4) is due to the term $\mathscr{V}(v \circledless \eta^{\dagger})$.

For a given $v \in \mathscr{Y}^{\kappa,\beta}$, set

$$R(v) := \mathscr{V}(v,v) + \eta^{\dagger\dagger} + 2\mathscr{V}(v \circledgtr \eta^{\dagger})$$

Lemma 6.3. *Let u_0 be a random field as in Assumption 4.4, with coefficients as in (6.2), and $\nu \in (\frac{1}{2},1]$. If $\kappa \in (1,2\nu)$ and $v, v' \in \mathscr{Y}_T^{\kappa,\beta}$, then*

$$t^\beta j^\kappa \ell(t)^\nu \|\Delta_j R(v)\|_\infty \lesssim (1+\|v\|_{\mathscr{Y}_T^{\kappa,\beta}})\|v\|^2_{\mathscr{Y}_T^{\kappa,\beta}}.$$

Moreover,

$$\|\mathscr{V}(R(v) \circledless \eta^{\dagger})\|_{\mathscr{Y}_T^{\kappa,\beta}} \lesssim \ell(T)^{\kappa-2\nu}(1+\|v\|_{\mathscr{Y}_T^{\kappa,\beta}})\|v\|_{\mathscr{Y}_T^{\kappa,\beta}}.$$

and

$$\|\mathscr{V}(R(v) \circledless \eta^{\dagger}) - \mathscr{V}(R(v') \circledless \eta^{\dagger})\|_{\mathscr{Y}_T^{\kappa,\beta}} \lesssim \ell(T)^{\kappa-2\nu}(1+\|v+v'\|_{\mathscr{Y}_T^{\kappa,\beta}})\|v-v'\|_{\mathscr{Y}_T^{\kappa,\beta}}.$$

Proof. The first statement follows by (6.5), (6.6), (6.7), and (6.8). For the second statement, for v such that $\|v\|_{\mathscr{Y}_T^{\kappa,\beta}} \le 1$ and by (6.3),

$$
\begin{aligned}
t^\beta j^\kappa \|\Delta_j \mathscr{V}(R(v) \circledless \eta^{\dagger})\|_\infty &\approx t^\beta j^\kappa \int_0^t 2^j\,\mathrm{e}^{-2^{2j}(t-s)}\,\|\Delta_j(R(v) \circledless \eta^{\dagger})\|_\infty\,ds \\
&\approx t^\beta j^\kappa \int_0^t 2^j\,\mathrm{e}^{-2^{2j}(t-s)}\,j^{-\nu}2^j\,\mathrm{e}^{-2^{2j}s}\sum_{n=0}^{j-2} s^{-\beta}\ell(s)^{-\nu}n^{-\kappa} \\
&\lesssim tH_{\kappa-\nu,2,2}(t)\ell(t)^{-\nu} \\
&\lesssim \ell(T)^{\kappa-2\nu},
\end{aligned}
$$

using Lemma 6.8 (where the definition of the quantity $H_{\kappa-\nu,2,2}(t)$ is also given), since we have chosen $\kappa < 2\nu$. The third statement follows likewise. ☐

Our original equation, can be written as

$$v = 2\mathscr{V}(v \varolessthan \eta^{\dagger}) + R(v),$$

where, as we have seen, $R(v)$ is an essentially "smooth" perturbation. The above equality represents both our equation and a decomposition of the solution in its regular and irregular part. We thus replace v with its decomposition in the irregular part of the equation, to get

$$\begin{aligned} v &= 2\mathscr{V}(v \varolessthan \eta^{\dagger}) + R(v) \\ &= 2\mathscr{V}\big(((2\mathscr{V}(v \varolessthan \eta^{\dagger}) + R(v)) \varolessthan \eta^{\dagger})\big) + R(v) \\ &= 4\mathscr{V}(\mathscr{V}(v \varolessthan \eta^{\dagger}) \varolessthan \eta^{\dagger}) + 2\mathscr{V}(R(v) \varolessthan \eta^{\dagger}) + R(v)\ . \end{aligned}$$

Theorem 6.4. *Let u_0 be a random field as in Assumption 4.4, with coefficients as in (6.2). If $\nu \in (\frac{1}{2}, 1]$, then there is a random time T, with $T > 0$ a. s., such that the equation*

$$v = 4\mathscr{V}(\mathscr{V}(v \varolessthan \eta^{\dagger}) \varolessthan \eta^{\dagger}) + 2\mathscr{V}(R(v) \varolessthan \eta^{\dagger}) + R(v) \tag{6.10}$$

has a unique solution in $\mathscr{Y}_T^{\kappa,\beta}$, where $\beta \in (\frac{1}{4}, \frac{1}{2})$, and $\kappa \in (1, 2\nu]$.

Proof. Everything boils down to an estimate of $\mathscr{V}(\mathscr{V}(v \varolessthan \eta^{\dagger}) \varolessthan \eta^{\dagger})$. All other terms are taken care of by Lemma 6.3. Consider $v \in \mathscr{Y}_T^{\kappa,\beta}$, with $\|v\|_{\mathscr{Y}_T^{\kappa,\beta}} \leq 1$. The estimate (6.9) yields

$$\|\Delta_j \mathscr{V}(v \varolessthan \eta^{\dagger})\|_\infty \lesssim j^{-\nu} 2^{2j} t^{1-\beta}\, \mathrm{e}^{-2^{2j}t}\,.$$

Thus, by the regularity of $\eta^{\dagger}$ given in (6.3) and by Lemma 6.8,

$$\begin{aligned} \|\Delta_j(\mathscr{V}(v \varolessthan \eta^{\dagger}) \varolessthan \eta^{\dagger})\|_\infty &\lesssim j^{-\nu} 2^j\, \mathrm{e}^{-2^{2j}t} \sum_{n=0}^{j-2} \|\Delta_n \mathscr{V}(v \varolessthan \eta^{\dagger})\|_\infty \\ &\lesssim j^{-\nu} 2^j\, \mathrm{e}^{-2^{2j}t} \sum_{n=0}^{j-2} n^{-\nu} 2^{2n}\, \mathrm{e}^{-2^{2n}t}\, t^{1-\beta} \\ &= j^{-\nu} 2^j\, \mathrm{e}^{-2^{2j}t}\, t^{1-\beta} H_{-\nu,2,2}(t) \\ &\lesssim j^{-\nu} 2^j\, \mathrm{e}^{-2^{2j}t}\, t^{-\beta} \ell(t)^{-\nu}, \end{aligned}$$

(see (6.11) for the definition of $H_{-\nu,2,2}(t)$) and therefore,

$$\begin{aligned} t^\beta j^\kappa \|\Delta_j \mathscr{V}(\mathscr{V}(v \varolessthan \eta^{\dagger}) \varolessthan \eta^{\dagger})\|_\infty &\lesssim t^\beta j^{\kappa-\nu} 2^{2j} \int_0^t \mathrm{e}^{-2^{2j}(t-s)}\, s^{-\beta}\, \mathrm{e}^{-2^{2j}s}\, \ell(s)^{-\nu}\, ds \\ &\lesssim t j^{\kappa-\nu} 2^{2j}\, \mathrm{e}^{-2^{2j}t}\, \ell(t)^{-\nu} \\ &\leq t H_{\kappa-\nu,2,2}(t) \ell(t)^{-\nu} \\ &\lesssim \ell(T)^{\kappa-2\nu}\ . \end{aligned}$$

This is sufficient to prove a fixed point theorem, with existence time dependent on the random constants in the above estimate and in Lemma 6.3. □

Remark 6.5. We wish to point out that the necessity of a local description to set up a problem amenable to a fixed point argument as we have done above, emerges only for initial conditions in (almost) critical spaces. Indeed, we know (see Remark 3.6) that if $\delta > \frac{1}{2}$, then Theorem 3.5 is sufficient to find solutions with initial conditions in critical spaces. The challenge for random initial conditions rests in the case $\delta \leq \frac{1}{2}$. If $\delta = \frac{1}{2}$ the only open case is the critical case and can be sorted out as we have done in this section.

Consider the case $\delta < \frac{1}{2}$. It is not difficult to see that, as long as we require that the initial condition is sub–critical and η^{Ψ} is well defined (that is $\eta^{\vee}$ has a integrable singularity at $t = 0$), then the term $\mathscr{V}(v, \eta^{\dagger})$ makes sense in the right space and Theorem 3.8 provides a solution. The case of initial conditions in critical spaces is a different story. Here we need again the methods we have illustrated in this section. The computations are very similar.

Finally, we wish to discuss to what extent the solution provided by Theorem 6.4 is a solution of problem (6.4), and in turn of problem (6.1).

Proposition 6.6. *Under the assumptions of Theorem 6.4 above, if v is the solution defined on $[0, T]$ provided by the above–mentioned theorem, then (6.4) holds in $\mathscr{X}^{0,\beta}_{T'}$ for some a. s. positive random time $T' \leq T$.*

Proof. We give a quick sketch. If we define the norm

$$\|\cdot\|_{\kappa,\nu,\beta,T} := \sup_{[0,T]} t^{\beta}\ell(t)^{\nu}\|\Delta_j \cdot\|_{\infty}\ ,$$

then the arguments in the proofs of Lemma 6.3 and Theorem 6.4 show that

- $\|w\|_{\kappa,\nu,\beta,T} < \infty \Longrightarrow \|\mathscr{V}(w \olessthan \eta^{\dagger})\|_{\kappa,2\nu-\kappa,\beta,T} < \infty$,
- $w \in \mathscr{Y}^{\kappa,\beta}_{T} \Longrightarrow \|\mathscr{V}(\mathscr{V}(w \olessthan \eta^{\dagger}) \olessthan \eta^{\dagger})\|_{\kappa,2\nu-\kappa,\beta,T} < \infty$.

Thus, a solution of (6.10) satisfies $\|v\|_{\kappa,2\nu-\kappa,\beta,T} < \infty$.

Let now $(u_0^n)_{n\geq 1}$ be a sequence of smooth random fields (obtained for instance from u_0 by convolution) such that $u_0^n \to u_0$, in the sense that $\sup_j j^{-1/2}2^{-j}\|\Delta_j(u_0^n - u_0)\|_{\infty} \to 0$, with similar convergence for $\eta_n^{\dagger}$ and η_n^{Ψ} (in the appropriate norms), where $\eta^{\dagger}$ and η^{Ψ} are the stochastic objects derived from u_0^n. If, for every n, v^n is the solution of (6.4) (with $\eta_n^{\dagger}$ and η_n^{Ψ}), then there is an a. s. positive random time T such that $\sup_n \|v^n\|_{\kappa,2\nu-\kappa,\beta,T} < \infty$. Set $w_n = v_n - v$, $p_n = \eta_n^{\dagger} - \eta^{\dagger}$, $q_n = \eta_n^{\Psi} - \eta^{\Psi}$, then

$$R_n(v_n) - R(v) = \mathscr{V}(v_n + v, w_n) + q_n + 2\mathscr{V}(w_n \circledgeq \eta_n^{\dagger}) + 2\mathscr{V}(v \circledgeq p_n),$$

where R_n is the remainder with $\eta_n^{\dagger}$ and η_n^{Ψ}. and, using estimate similar to those in the proof of Lemma 6.3, we see that $\|R_n(v_n) - R(v)\|_{\kappa,\nu,\beta,T} \to 0$. Likewise, since

$$\begin{aligned} w_n = 4\mathscr{V}(\mathscr{V}(w_n \olessthan \eta_n^{\dagger}) \olessthan \eta_n^{\dagger}) + 4\mathscr{V}(\mathscr{V}(v \olessthan p_n) \olessthan \eta_n^{\dagger}) + \\ + 4\mathscr{V}(\mathscr{V}(v \olessthan \eta_n^{\dagger}) \olessthan p_n) + 2\mathscr{V}((R_n(v_n) - R(v)) \olessthan \eta_n^{\dagger}) + \\ + 2\mathscr{V}(R(v) \olessthan p_n) + R_n(v_n) - R(v), \end{aligned}$$

we have that $\|w_n\|_{\kappa,2\nu-\kappa,\beta,T} \to 0$ (using also estimates as those in Theorem 6.4).

Now

$$v_n = \mathscr{V}(v_n, v_n) + 2\mathscr{V}(v_n, \eta_n^{\dagger}) + \eta_n^{\dagger\dagger} = R_n(v_n) + 2\mathscr{V}(v_n \varolessthan \eta_n^{\dagger})$$

and it remains to show that the term $\mathscr{V}(v_n \varolessthan \eta_n^{\dagger})$ converges. Indeed,

$$t^\beta \|\Delta_j \mathscr{V}(w_n \varolessthan \eta^{\dagger})\|_\infty \approx$$

$$\approx \|w_n\|_{\kappa,2\nu-\kappa,\beta,T} t^\beta \sqrt{j} 2^{2j} \int_0^t \mathrm{e}^{-2^{2j}(t-s)} \,\mathrm{e}^{-2^{2j}s} \sum_{n=0}^{j-2} s^{-\beta} n^{-\kappa} \ell(s)^{-\nu}\, ds \lesssim$$

$$\lesssim t H_{\frac{1}{2},2,2}(t)\ell(t)^{-\nu} \|w_n\|_{\kappa,2\nu-\kappa,\beta,T} \lesssim \|w_n\|_{\kappa,2\nu-\kappa,\beta,T},$$

and likewise for $\mathscr{V}(v_n, p_n)$. □

Remark 6.7. We remark that an attempt to run a fixed point in the space defined by the norm $\|\cdot\|_{\kappa,\nu,\beta,T}$ would fail when trying to prove the self mapping property for the term $\mathscr{V}(v \varolessthan \eta^{\dagger})$.

We conclude with an elementary analytical lemma.

Lemma 6.8. *Set for every $\nu \in \mathbf{R}$, $p \geq 0$, $\tau > 0$, and $t > 0$,*

$$G_{\nu,p,\tau}(t) = \sum_{n=1}^\infty n^\nu 2^{pn} \,\mathrm{e}^{-2^{\tau n} t}, \qquad H_{\nu,p,\tau}(t) = \sup_{n\geq 1} n^\nu 2^{pn} \,\mathrm{e}^{-2^{\tau n} t}. \tag{6.11}$$

Then for $p > 0$ and $\nu \in \mathbf{R}$,

$$H_{\nu,p,\tau} \leq G_{\nu,p,\tau}(t) \lesssim t^{-\frac{p}{\tau}} \ell(t)^\nu.$$

Moreover, if $p = 0$, $\nu \in \mathbf{R}$,

$$H_{\nu,0,\tau} \leq \ell(t)^{\nu_+}.$$

References

[1] Bényi Árpád, Tadahiro Oh, and Oana Pocovnicu, *Higher order expansions for the probabilistic local Cauchy theory of the cubic nonlinear Schrödinger equation on* $\mathbb{R}^3$, arXiv:1709.01910, 2017.

[2] Dirk Blömker and Marco Romito, *Local existence and uniqueness in the largest critical space for a surface growth model*, NoDEA Nonlinear Differential Equations Appl. **19** (2012), no. 3, 365–381. [MR2926303]

[3] ———, *Stochastic PDEs and Lack of Regularity*, Jahresber. Dtsch. Math.-Ver. **117** (2015), no. 4, 233–286. [MR3414523]

[4] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. **166** (1994), no. 1, 1–26. [MR1309539]

[5] Jean Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. **176** (1996), no. 2, 421–445. [MR1374420]

[6] Nicolas Burq and Nikolay Tzvetkov, *Random data Cauchy theory for supercritical wave equations I. Local theory*, Invent. Math. **173** (2008), no. 3, 449–475. [MR2425133]

[7] ———, *Random data Cauchy theory for supercritical wave equations II. A global existence result*, Invent. Math. **173** (2008), no. 3, 477–496. [MR2425134]

[8] Giuseppe Da Prato and Arnaud Debussche, *Two-dimensional Navier-Stokes equations driven by a space-time white noise*, J. Funct. Anal. **196** (2002), no. 1, 180–210. [MR1941997]

[9] ———, *Strong solutions to the stochastic quantization equations*, Ann. Probab. **31** (2003), no. 4, 1900–1916. [MR2016604]

[10] Giuseppe Da Prato, Arnaud Debussche, and Luciano Tubaro, *A modified Kardar-Parisi-Zhang model*, Electron. Comm. Probab. **12** (2007), 442–453. [MR2365646]

[11] *Dispersive wiki:critical*, 2006, http://wiki.math.toronto.edu/DispersiveWiki/index.php/Critical.

[12] Bertrand Duplantier and Scott Sheffield, *Liouville quantum gravity and KPZ*, Invent. Math. **185** (2011), no. 2, 333–393. [MR2819163]

[13] Massimiliano Gubinelli, Peter Imkeller, and Nicolas Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi **3** (2015), e6, 75. [MR3406823]

[14] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014), no. 2, 269–504. [MR3274562]

[15] Martin Hairer, *Solving the KPZ equation*, Ann. of Math. (2) **178** (2013), no. 2, 559–664. [MR3071506]

[16] Jean-Pierre Kahane, *Some random series of functions*, second ed., Cambridge Studies in Advanced Mathematics, vol. 5, Cambridge University Press, Cambridge, 1985. [MR833073]

[17] Jean-Christophe Mourrat, Hendrik Weber, and Weijun Xu, *Construction of Φ^4_3 diagrams for pedestrians*, From particle systems to partial differential equations, Springer Proc. Math. Stat., vol. 209, Springer, Cham, 2017, pp. 1–46. [MR3746744]

[18] Andrea R. Nahmod, Nataša Pavlović, and Gigliola Staffilani, *Almost sure existence of global weak solutions for supercritical Navier-Stokes equations*, SIAM J. Math. Anal. **45** (2013), no. 6, 3431–3452. [MR3131480]

[19] Edward Nelson, *The free Markoff field*, J. Functional Analysis **12** (1973), 211–227. [MR0343816]

[20] Tadahiro Oh and Nikolay Tzvetkov, *Quasi-invariant Gaussian measures for the two-dimensional defocusing cubic nonlinear wave equation*, arXiv:1703.10718, 2017.

[21] Barry Simon, *The $P(\phi)_2$ Euclidean (quantum) field theory*, Princeton University Press, Princeton, N.J., 1974, Princeton Series in Physics. [MR0489552]

[22] Nikolay Tzvetkov, *On Hamiltonian partial differential equations with random initial conditions*, Lecture Notes for the CIME-EMS summer school in applied mathematics *Singular random dynamics*, notes available at arXiv:1704.01191, 2016.

Institut für Mathematik, Universität Augsburg, D-86135 Augsburg, Germany
E-mail address: dirk.bloemker@math.uni-augsburg.de

Imperial College London, Department of Mathematics, 180 Queen's Gate, London SW7 2AZ, United Kingdom
E-mail address: g.cannizzaro@imperial.ac.uk

Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I–56127 Pisa, Italia
E-mail address: marco.romito@unipi.it
URL: http://people.dm.unipi.it/romito