Giuseppe Buttazzo and Bozhidar Velichkov

9 Spectral optimization problems for Schrödinger operators

In this chapter we consider Schrödinger operators of the form $-\Delta+V(x)$ on the Sobolev space $H^1_0(D)$, where D is an open subset of $\mathbb{R}^d$. We are interested in finding optimal potentials for some suitable criteria; the optimization problems we deal with are then written as

$$\min\left\{F(V)\ :\ V\in\mathcal{V}\right\}$$

where F is a suitable cost functional and $\mathcal{V}$ is a suitable class of admissible potentials. For simplicity, we consider the case when D is bounded and $V\geq 0$; under these conditions the resolvent operator of $-\Delta+V(x)$ is compact and the spectrum $\lambda(V)$ of the Schrödinger operator is discrete and consists of an increasing sequence of positive eigenvalues

$$\lambda(V)=\big(\lambda_1(V),\lambda_2(V),\dots\big).$$

This allows us to consider as cost functions the so-called *spectral functionals*, of the form

$$F(V)=\Phi\big(\lambda(V)\big),$$

where Φ is a given function. The cases when D is unbounded or V takes on negative values may provide in general a continuous spectrum and are more delicate to treat; some examples in this framework are considered in [171] and in the references therein.

The largest framework in which Schrödinger operators can be considered is the one where the potentials are *capacitary measures*; these ones are nonnegative Borel measures on D, possibly taking on the value $+\infty$ and vanishing on all sets of capacity zero (we refer to Section 2.2 for the definition of capacity). This framework will be considered in Section 9.1 together with the related optimization problems. We want to stress here that the class of capacitary measures μ is very large and contains both the case of standard potentials $V(x)$, in which $\mu = V\,dx$, as well as the case of classical domains Ω, in which $\mu=+\infty_{D\setminus\Omega}$. By this notation, we intend to reference the measure defined in (9.3).

Optimization problems for domains, usually called *shape optimization problems*, are often considered in the literature; the other chapters in the present volume deal with this kind of problem and in particular with *spectral optimization problems*, in

Giuseppe Buttazzo: Dipartimento di Matematica, Università di Pisa, Largo B. Pontecorvo 5, 56127 Pisa - Italy, E-mail: buttazzo@dm.unipi.it
Bozhidar Velichkov: Laboratoire Jean Kuntzmann, Université de Grenoble et CNRS 38041 Grenoble cedex 09 - France, E-mail: bozhidar.velichkov@imag.fr

which the cost functional depends on the spectrum of the Laplace operator $-\Delta$ on $H_0^1(\Omega)$:

$$F(\Omega) = \Phi\big(\lambda(\Omega)\big)$$

being Ω a domain which varies in the admissible class. For further details on shape optimization problems we refer the reader to the other chapters of this book and to [207], [505], [510]; here we simply recall some key facts. The existence of optimal domains for a problem of the form

$$\min\big\{F(\Omega)\ :\ \Omega \subset D,\ |\Omega| \le m\big\} \tag{9.1}$$

has been obtained under some additional assumptions, that we resume below.

- On the admissible domains Ω, some additional geometrical constraints are imposed, including convexity, uniform Lipschitz condition, uniform exterior cone properties, capacitary conditions, Wiener properties, ...; a detailed analysis of these conditions can be found in the book [207].
- No geometrical conditions are required on the admissible domains Ω but the functional F is assumed to satisfy some monotonicity conditions; in particular it is supposed to be decreasing with respect to set inclusion. The first result in this direction has been obtained in [238] and several generalizations, mainly to the cases where the set D is not bounded, have been made in [206] and in [700].

Without the extra assumptions above, the existence of an optimal shape may fail, in general, as several counterexamples show (see for instance [207]); in these cases the minimizing sequences (Ω_n) for the problem (9.1) converge in the γ-convergence sense (see Definition 9.1) to capacitary measures μ. In Section 9.1 we will see that many problems admit a capacitary measure as an optimal solution; this class is very large and only mild assumptions on the cost functional are required to provide the existence of a solution. In Section 9.2 we restrict our attention to the subclass of *Schrödinger potentials* $V(x)$ that belong to some space $L^p(D)$; we call them *integrable potentials* and we will see that suitable assumptions on the cost functional still imply the existence of an optimal potential. Finally, in Section 9.3 we consider the case of *confining potentials* $V(x)$ that are very large out of a bounded set, or more generally fulfill some integral inequalities of the form $\int_D \psi\big(V(x)\big)\,dx \le 1$ for some suitable integrand ψ.

9.1 Existence results for capacitary measures

In this section we consider a bounded open subset D of $\mathbb{R}^d$ and the class $\mathcal{M}_{cap}(D)$ of all capacitary measures on D, that is the Borel nonnegative measures on D, possibly $+\infty$ valued, that vanish on all sets of capacity zero. The analysis of capacitary measures and of their variational properties was made in [314]; the related optimization problems have been first considered in [237].

The key ingredient we need is the notion of γ-convergence. For a given measure $\mu \in \mathcal{M}_{cap}(D)$ we consider the Schrödinger-like operator $-\Delta + \mu$ defined on $H^1_0(D)$ and its resolvent operator R_μ which associates to every $f \in L^2(D)$ the unique solution $u = R_\mu(f)$ of the PDE

$$-\Delta u + \mu u = f, \qquad u \in H^1_0(D) \cap L^2_\mu(D).$$

The PDE above has to be defined in the weak sense

$$\begin{cases} u \in H^1_0(D) \cap L^2_\mu(D) \\ \int_D \nabla u \nabla \phi \, dx + \int_D u\phi \, d\mu = \int_D f\phi \, dx \qquad \forall \phi \in H^1_0(D) \cap L^2_\mu(D). \end{cases} \tag{9.2}$$

Definition 9.1. *We say that a sequence (μ_n) of capacitary measures γ-converges to a capacitary measure μ if and only if*

$$R_{\mu_n}(f) \to R_\mu(f) \text{ weakly in } H^1_0(D) \qquad \forall f \in L^2(D).$$

In the definition above one can equivalently require that the resolvent operators R_{μ_n} converge to the resolvent operator R_μ in the norm of the space of operators $\mathcal{L}\big(L^2(D); L^2(D)\big)$.

We summarize here below the main properties of the class $\mathcal{M}_{cap}(D)$; we refer for the details to [207].

- Every domain Ω can be seen as a capacitary measure, by taking $\mu = \infty_{D\setminus\Omega}$, or more precisely
$$\mu(E) = \begin{cases} 0 & \text{if } \operatorname{cap}(\Omega \setminus E) = 0 \\ +\infty & \text{if } \operatorname{cap}(\Omega \setminus E) > 0. \end{cases} \tag{9.3}$$
- Every capacitary measure is the γ-limit of a suitable sequence (Ω_n) of (smooth) domains; in other words, the class $\mathcal{M}_{cap}(D)$ is the closure with respect to the γ-convergence, of the class of (smooth) domains D.
- For every sequence (μ_n) of capacitary measures there exists a subsequence (μ_{n_k}) which γ-converges to a capacitary measure μ; in other words the class $\mathcal{M}_{cap}(D)$ is compact with respect to the γ-convergence.
- If μ is a capacitary measure, we may consider the PDE formally written as
$$-\Delta u + \mu u = f, \qquad u \in H^1_0(D). \tag{9.4}$$

  The meaning of the equation above, as specified in (9.2), is in a weak sense, by considering the Hilbert space $H^1_\mu(D) = H^1_0(D) \cap L^2_\mu(D)$ with the norm
$$\|u\|_{H^1_\mu(D)} = \|u\|_{H^1_0(D)} + \|u\|_{L^2_\mu(D)}$$

and defining the solution in the weak sense (9.2). By Lax-Milgram theory, for every $\mu \in \mathcal{M}_{cap}(D)$ and $f \in L^2(D)$ (actually it would be enough to have f in the dual space of $H^1_\mu(D)$) there exists a unique solution $u_{\mu,f}$ of the PDE above. Moreover, if $\mu_n \to \mu$ in the γ-convergence, we have $u_{\mu_n,f} \to u_{\mu,f}$ weakly in in $H^1_0(D)$, hence strongly in $L^2(D)$.

- In order to have the γ-convergence of μ_n to μ it is enough to have the weak convergence in $H^1_0(D)$ of $R_{\mu_n}(1)$ to $R_\mu(1)$; in other words, we need to test the convergence of solutions of the PDEs related to the operators $-\Delta + \mu_n$ only with $f = 1$.
- The space $\mathcal{M}_{cap}(D)$, endowed with the γ-convergence, is metrizable; more precisely, the γ-convergence on $\mathcal{M}_{cap}(D)$ is equivalent to the distance

$$d_\gamma(\mu, \nu) = \|w_\mu - w_\nu\|_{L^2(D)}$$

where w_μ and w_ν are the solutions of the problems

$$-\Delta w_\mu + \mu w_\mu = 1 \text{ on } H^1_\mu(D)\,, \qquad -\Delta w_\nu + \nu w_\nu = 1 \text{ on } H^1_\nu(D)\,.$$

Remark 9.2. *We notice that the definition of γ-convergence of a sequence of capacitary measures μ_n to μ can be equivalently expressed in terms of the Γ-convergence in $L^2(D)$ of the corresponding energy functionals*

$$J_n(u) = \int_D |\nabla u|^2\,dx + \int_D u^2\,d\mu_n$$

to the limit energy

$$J(u) = \int_D |\nabla u|^2\,dx + \int_D u^2\,d\mu.$$

For all details about Γ-convergence theory we refer to [313].

The γ-convergence is very strong, and so many functionals are γ-lower semicontinuous, or even continuous (see below some important examples). The classes of functionals we are interested in are the following.

Integral functionals. Given a function $f \in L^2(D)$, for every $\mu \in \mathcal{M}_{cap}(D)$ we consider the solution $u_{\mu,f} = R_\mu(f)$ to the elliptic PDE (9.4). The integral cost functionals we consider are of the form

$$F(\mu) = \int_D j\big(x, u_{\mu,f}, \nabla u_{\mu,f}\big)\,dx, \tag{9.5}$$

where $j(x, s, z)$ is a suitable integrand that we assume measurable in the x variable, lower semicontinuous in the s, z variables, and convex in the z variable. Moreover, the function j is assumed to fulfill bounds from below of the form

$$j(x, s, z) \geq -a(x) - c|s|^2,$$

with $a \in L^1(D)$ and c smaller than the first Dirichlet eigenvalue of the Laplace operator $-\Delta$ in D. In particular, the energy $\mathcal{E}_f(\mu)$ defined by

$$\mathcal{E}_f(\mu) = \inf\left\{\frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D u^2\,d\mu - \int_D fu\,dx \;:\; u \in H_0^1(D)\right\}, \tag{9.6}$$

belongs to this class since, integrating its Euler-Lagrange equation by parts, we have

$$\mathcal{E}_f(\mu) = -\frac{1}{2}\int_D f(x)u_{\mu,f}\,dx,$$

which corresponds to the integral functional above with

$$j(x,s,z) = -\frac{1}{2}f(x)s.$$

Thanks to the assumptions above and to the strong-weak lower semicontinuity theorem for integral functionals (see for instance [235]) all functionals of the form (9.5) are γ-lower semicontinuous on $\mathcal{M}_{cap}(D)$.

Spectral functionals. For every capacitary measure $\mu \in \mathcal{M}_{cap}(D)$ we consider the spectrum $\lambda(\mu)$ of the Schrödinger operator $-\Delta+\mu$ on $H_0^1(D)\cap L^2_\mu(D)$. Since D is bounded (it is enough to consider D to be of finite measure), then the operator $-\Delta+\mu$ has a compact resolvent and so its spectrum $\lambda(\mu)$ is discrete:

$$\lambda(\mu) = \big(\lambda_1(\mu), \lambda_2(\mu), \dots\big),$$

where $\lambda_k(\mu)$ are the eigenvalues of $-\Delta+\mu$, counted with their multiplicity. The same occurs if D is unbounded, and the measure μ satisfies some suitable *confinement* integrability properties (see for instance [208]). The spectral cost functionals we may allow are of the form

$$F(\mu) = \Phi\big(\lambda(\mu)\big),$$

for suitable functions $\Phi : \mathbb{R}^{\mathbb{N}} \to (-\infty, +\infty]$. For instance, taking $\Phi(\lambda) = \lambda_k$ we obtain

$$F(\mu) = \lambda_k(\mu).$$

Since a sequence (μ_n) γ-converges to μ if and only if the sequence of resolvent operators (R_{μ_n}) converges in the operator norm convergence of linear operators on $L^2(D)$ to the resolvent operator R_μ, the spectrum $\lambda(\mu)$ is continuous with respect to the γ-convergence, that is

$$\mu_n \to_\gamma \mu \;\Rightarrow\; \lambda_k(\mu_n) \to \lambda_k(\mu) \quad \forall k \in \mathbb{N}.$$

Therefore, the spectral functionals above are γ-lower semicontinuous, provided that the function Φ is lower semicontinuous, in the sense that

$$\lambda_n \to \lambda \text{ in } \mathbb{R}^{\mathbb{N}} \;\Rightarrow\; \Phi(\lambda) \le \liminf_n \Phi(\lambda_n),$$

where $\lambda_n \to \lambda$ in $\mathbb{R}^{\mathbb{N}}$ is intended in the componentwise convergence.

The relation between γ-convergence and weak*-convergence of measures is given in the proposition below.

Proposition 9.3. *Let $\mu_n \in \mathcal{M}_{cap}(D)$ be capacitary and Radon measures weakly* converging to the measure ν and γ-converging to the capacitary measure $\mu \in \mathcal{M}_{cap}(D)$. Then $\mu \le \nu$ in D.*

Proof. It is enough to show that $\mu(K) \le \nu(K)$ whenever K is a compact subset of D. Let u be a nonnegative smooth function with compact support in D such that $u \le 1$ in D and $u = 1$ on K; we have

$$\mu(K) \le \int_D u^2\,d\mu \le \liminf_{n\to\infty} \int_D u^2\,d\mu_n = \int_D u^2\,d\nu \le \nu\big(\{u > 0\}\big).$$

Since u is arbitrary, the conclusion follows from the definition of Borel regularity of the measure ν. □

Remark 9.4. *When $d = 1$, as a consequence of the compact embedding of $H_0^1(D)$ into the space of continuous functions on D, we obtain that any sequence (μ_n) weakly* converging to μ is also γ-converging to μ.*

In several shape optimization problems the class of admissible domains Ω is slightly larger than the class of open sets.

Definition 9.5. *We say that a set $\Omega \subset \mathbb{R}^d$ is quasi-open if for every $\varepsilon > 0$ there exists an open subset $\Omega_\varepsilon \subset \mathbb{R}^d$ such that $\mathrm{cap}(\Omega_\varepsilon \triangle \Omega) < \varepsilon$, where $\triangle$ denotes the symmetric difference of sets.*

Remark 9.6. *It is possible to prove (see for instance [207]) that a set $\Omega \subset D$ is quasi-open if and only if it can be written as*

$$\Omega = \{x \in D \;:\; u(x) > 0\}$$

for a suitable function $u \in H_0^1(D)$. Since Sobolev functions are defined only up to sets of capacity zero, a quasi-open set is defined up to capacity zero sets too.

In many problems the admissible domains Ω are constrained to verify a measure constraint of the form $|\Omega| \le m$; in order to *relax* this constraint to capacitary measures we have to introduce, for every $\mu \in \mathcal{M}_{cap}(D)$, the *set of finiteness* Ω_μ. A precise definition would require the notion of fine topology and finely open sets (see for instance [207]); however, a simpler equivalent definition can be given in terms of the solution $w_\mu = R_\mu(1)$ of the elliptic PDE

$$-\Delta u + \mu u = 1, \qquad u \in H_\mu^1(D).$$

Definition 9.7. *For every $\mu \in \mathcal{M}_{cap}(D)$ we denote by Ω_μ the set of finiteness of μ, defined by*

$$\Omega_\mu = \big\{w_\mu > 0\big\}.$$

By definition, the set Ω_μ is quasi-open, being the set where a Sobolev function is positive. Of course, since the function w_μ is defined only up to sets of capacity zero, the set Ω_μ is defined up to sets of capacity zero too.

Proposition 9.8. *The Lebesgue measure $|\Omega_\mu|$ is γ-lower semicontinuous.*

Proof. This follows from the definition of Ω_μ and from the fact that the γ-convergence $\mu_n \to_\gamma \mu$ is equivalent to the convergence of the solutions $w_{\mu_n} = R_{\mu_n}(1)$ to $w_\mu = R_\mu(1)$ in $L^2(D)$. The conclusion then follows by the Fatou's lemma. □

In summary, thanks to the γ-compactness of the class $\mathcal{M}_{cap}(D)$, the following general existence result holds.

Theorem 9.9. *Let $F : \mathcal{M}_{cap}(D) \to \overline{\mathbb{R}}$ be a γ-lower semicontinuous functional (for instance one of the classes above); then the minimization problem*

$$\min\left\{F(\mu)\ :\ \mu \in \mathcal{M}_{cap}(D),\ |\Omega_\mu| \le m\right\}$$

admits a solution $\mu_{opt} \in \mathcal{M}_{cap}(D)$.

In general, the optimal measure μ_{opt} is not unique; however, in the situation described below, the uniqueness occurs. Consider the optimization problem for the integral functional

$$F(\mu) = \int_D j\big(x, u_{\mu,f}, \nabla u_{\mu,f}\big)\, dx$$

where $f \ge 0$ is a given function in $L^2(D)$. We can write the problem as a double minimization, in μ and in u:

$$\min\left\{\int_D j(x, u, \nabla u)\, dx\ :\ \mu \in \mathcal{M}_{cap}(D),\ u \in H^1_0(D),\ -\Delta u + \mu u = f\right\}.$$

Since $f \ge 0$, by the maximum principle we know that $u \ge 0$ and, at least formally (the rigorous justification can be found in [269]),

$$\mu = \frac{f + \Delta u}{u}\ ,$$

so that we can eliminate the variable μ from the minimization and the optimization problem can be reformulated in terms of the function u only, as

$$\min\left\{\int_D j(x, u, \nabla u)\, dx\ :\ u \in \mathcal{K}\right\},$$

where $\mathcal{K}$ is the subset of $H^1_0(D)$ given by

$$\mathcal{K} = \left\{u \in H^1_0(D)\ :\ f + \Delta u \ge 0\right\}.$$

The inequality $f + \Delta u \geq 0$ has to be formulated in a weak sense, as

$$\int_D f\phi\,dx - \int_D \nabla u \nabla\phi\,dx \geq 0 \qquad \forall\phi \in H_0^1(D),\ \phi \geq 0. \tag{9.7}$$

The set $\mathcal{K}$ is clearly convex and it is easy to see that it is also closed. Hence, as a consequence, if the function $j(x, s, z)$ is *strictly convex* with respect to the pair (s, z), the solution of (9.7) is unique. Thus the solution μ_{opt}, that exists thanks to Theorem 9.9 is also unique. Note that in this case, no measure constraint of the form $|\Omega_\mu| \leq m$ is imposed.

In several situations the optimal measure μ_{opt} given by Theorem 9.9 has more regularity or summability properties than a general element of $\mathcal{M}_{cap}(D)$.This happens in the cases below:

- If the functional F is monotonically increasing with respect to the usual order of measures, and a constraint $|\Omega_\mu| \leq m$ is added, then an optimal measure μ_{opt} that is actually a domain exists, that is $\mu_{opt} = \infty_{D\setminus\Omega}$ for some quasi-open subset Ω of D. This fact should be rigorously justified (see [238]), but the argument consists in the fact that the measure $\infty_{D\setminus\Omega}$ is smaller than μ and has the same set of finiteness; then it provides an optimum for the minimization problem due to the monotonicity of F and to the constraint on the measure of the set of finiteness.
- In [241] the optimization of the *elastic compliance* for a membrane is considered, with the additional constraint that the measure μ has a prescribed total mass. In this case it is shown that μ_{opt} is actually an $L^1(D)$ function, that is no singular parts with respect to the Lebesgue measure occur.

In general, we should not expect that μ_{opt} is a domain or a function with any summability; the following example shows that even in simple and natural problems this does not occur.

Example 9.10. *Let D be a ball of radius R and let $f = 1$; consider the optimization problem for the integral functional*

$$F(\mu) = \int_D |u_{\mu,1} - c|^2\,dx, \tag{9.8}$$

where c is a given constant and $u_{\mu,1}$ denotes as before the solution of the PDE

$$-\Delta u + \mu u = 1, \qquad u \in H_\mu^1(D).$$

By the argument described above the problem can be reformulated in terms of the function u only, as

$$\min\left\{\int_D |u - c|^2\,dx \;:\; u \in \mathcal{K}\right\}$$

where $\mathcal{K}$ is the convex closed subset of $H_0^1(D)$ given by

$$\mathcal{K} = \left\{u \in H_0^1(D) \;:\; \Delta u + 1 \geq 0\right\}.$$

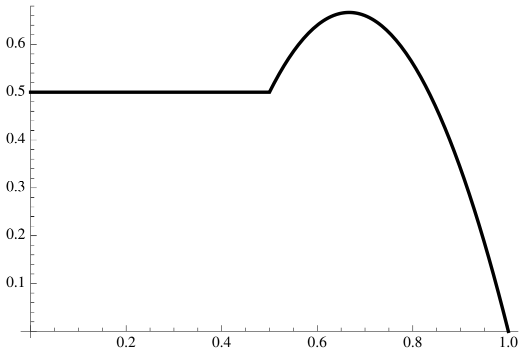

Fig. 9.1. The behavior of an optimal state function $u(r)$.

As we have seen, this auxiliary problem has a unique solution which is radially symmetric. Thus we can write the problem in polar coordinates as

$$\min\left\{\int_0^R |u-c|^2 r^{d-1}\,dx\ :\ u'' + \frac{d-1}{r}u' + 1 \geq 0,\ u(R) = 0\right\}.$$

The minimum problem above can be fully analyzed and its solution is characterized as follows (see [207] for the details).

- *If c is large enough, above a certain threshold $\bar{c}$ that can be computed explicitly, we have for the optimal solution (u, μ)*

$$u(r) = \frac{R^2 - r^2}{2d}, \qquad \textit{hence} \qquad \mu \equiv 0.$$

- *Below the threshold $\bar{c}$ the optimal measure μ is given by*

$$\mu = \frac{1}{c}\mathcal{L}^d \lfloor B_{R_c} + \alpha_c \mathcal{H}^{d-1} \lfloor \partial B_{R_c},$$

  where $\mathcal{L}^d$ denotes the Lebesgue measure in $\mathbb{R}^d$, $\alpha_c > 0$ is a suitable constant, and $R_c < R$ is a suitable radius. The solution u is computed correspondingly, through the equation

$$u'' + \frac{d-1}{r}u' + \mu u = 1.$$

A plot of the behavior of an optimal state function u is given in Figure 9.1.
Note that the functional in (9.8) is not monotonically increasing with respect to μ.

9.2 Existence results for integrable potentials

In this section we consider optimization problems of the form

$$\min\left\{F(V)\ :\ V : D \to [0, +\infty],\ \int_D V^p\,dx \leq 1\right\}, \tag{9.9}$$

where $p > 0$ and $F(V)$ is a cost functional acting on Schrödinger potentials, or more generally on capacitary measures. We assume that F is γ-lower semicontinuous, an assumption that, as we have seen in the previous section, is very mild and verified for most of the functionals of integral or spectral type.

When $p > 1$ a general existence result follows from the following proposition, where we show that the weak $L^1(D)$ convergence (that is the one having $L^\infty(D)$ as the space of test functions) of potentials implies the γ-convergence.

Proposition 9.11. *Let $V_n \in L^1(D)$ converge weakly in $L^1(D)$ to a function V. Then the capacitary measures $V_n\,dx$ γ-converge to $V\,dx$.*

Proof. We have to prove that the solutions $u_n = R_{V_n}(1)$ of the PDE

$$\begin{cases} -\Delta u_n + V_n(x)u_n = 1 \\ u \in H_0^1(D) \end{cases}$$

weakly converge in $H_0^1(D)$ to the solution $u = R_V(1)$ of

$$\begin{cases} -\Delta u + V(x)u = 1 \\ u \in H_0^1(D). \end{cases}$$

Equivalently, as noticed in Remark 9.2, we may prove that the functionals

$$J_n(u) = \int_D |\nabla u|^2\,dx + \int_D V_n(x)u^2\,dx$$

Γ-converge in $L^2(D)$ to the functional

$$J(u) = \int_D |\nabla u|^2\,dx + \int_D V(x)u^2\,dx.$$

Let us prove the Γ-liminf inequality:

$$J(u) \le \liminf_{n\to\infty} J(u_n) \qquad \forall u_n \to u \text{ in } L^2(D).$$

Indeed, if $u_n \to u$ in $L^2(D)$, we have

$$\int_D |\nabla u|^2\,dx \le \liminf_{n\to\infty} \int_D |\nabla u_n|^2\,dx$$

by the lower semicontinuity of the $H^1(D)$ norm with respect to the $L^2(D)$-convergence, and

$$\int_D V(x)u^2\,dx \le \liminf_{n\to\infty} \int_D V_n(x)u_n^2\,dx$$

by the strong-weak lower semicontinuity theorem for integral functionals (see for instance [235]).

Let us now prove the Γ-limsup inequality: there exists $u_n \to u$ in $L^2(D)$ such that

$$J(u) \geq \limsup_{n\to\infty} J(u_n). \tag{9.10}$$

For every $t > 0$ we set

$$u^t = (u \wedge t) \vee (-t);$$

then, by the weak $L^1(D)$ convergence of V_n to V, for every t fixed we have

$$\lim_{n\to\infty} \int_D V_n(x)|u^t|^2\,dx = \int_D V(x)|u^t|^2\,dx.$$

Moreover, letting $t \to \infty$ we have by the monotone convergence theorem

$$\lim_{t\to+\infty} \int_D V(x)|u^t|^2\,dx = \int_D V(x)|u|^2\,dx.$$

Then, by a diagonal argument, we can find a sequence $t_n \to +\infty$ such that

$$\lim_{n\to\infty} \int_D V_n(x)|u^{t_n}|^2\,dx = \int_D V(x)|u|^2\,dx.$$

Taking now $u_n = u^{t_n}$, and noticing that for every $t > 0$

$$\int_D |\nabla u^t|^2\,dx \leq \int_D |\nabla u|^2\,dx,$$

we obtain (9.10) and so the proof is complete. □

The existence of an optimal potential for problems of the form (9.9) is now straightforward.

Theorem 9.12. *Let $F(V)$ be a functional defined for $V \in L^1_+(D)$ (the set of nonnegative functions in $L^1(D)$), lower semicontinuous with respect to the γ-convergence, and let $\mathcal{V}$ be a subset of $L^1_+(D)$, compact for the weak L^1-convergence. Then the problem*

$$\min\left\{F(V)\ :\ V \in \mathcal{V}\right\},$$

admits a solution.

Proof. Let (V_n) be a minimizing sequence in $\mathcal{V}$. By the compactness assumption on $\mathcal{V}$, we may assume that V_n tends to some $V \in \mathcal{V}$ weakly in $L^1(D)$. By Proposition 9.11, we have that V_n γ-converges to V and so, by the semicontinuity of F,

$$F(V) \leq \liminf_{n\to\infty} F(V_n),$$

which gives the conclusion. □

In some cases the optimal potential can be explicitly determined through the solution of a partial differential equation, as for instance in the examples below.

Example 9.13. *Take $F = -\mathcal{E}_f$, where $\mathcal{E}_f$ is the energy functional defined in (9.6), with f a fixed function in $L^2(D)$, and*

$$\mathcal{V} = \left\{ V \geq 0,\ \int_D V^p\,dx \leq 1 \right\} \qquad \text{with } p > 1. \tag{9.11}$$

Then, the problem we are dealing with is

$$\max_{V\in\mathcal{V}} \mathcal{E}_f(V) = \max_{V\in\mathcal{V}} \min_{u\in H_0^1(D)} \left\{ \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D Vu^2\,dx - \int_D fu\,dx \right\}. \tag{9.12}$$

As we have already seen above, the energy functional can be written, by an integration by parts, as

$$\mathcal{E}_f(V) = -\frac{1}{2}\int_D f(x)R_V(f)\,dx$$

where R_V is the resolvent operator of $-\Delta + V(x)$. Therefore, the functional F is γ-continuous and the existence Theorem 9.12 applies. In order to compute the optimal potential, interchanging the min and the max in (9.12) we obtain the inequality

$$\max_{V\in\mathcal{V}} \min_{u\in H_0^1(D)} \left\{ \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D Vu^2\,dx - \int_D fu\,dx \right\}$$
$$\leq \min_{u\in H_0^1(D)} \max_{V\in\mathcal{V}} \left\{ \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D Vu^2\,dx - \int_D fu\,dx \right\}.$$

The maximization with respect to V is very easy to compute; in fact, for a fixed u, the maximal value is reached at

$$V = \left(\int_D |u|^{2p/(p-1)}\,dx\right)^{-1/p} |u|^{2/(p-1)}, \tag{9.13}$$

so that

$$\min_{u\in H_0^1(D)} \max_{V\in\mathcal{V}} \left\{ \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D Vu^2\,dx - \int_D fu\,dx \right\}$$
$$= \min_{u\in H_0^1(D)} \left\{ \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\left(\int_D |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} - \int_D fu\,dx \right\}.$$

In order to find the optimal potential V_{opt} we have then to solve the auxiliary variational problem

$$\min_{u\in H_0^1(D)} \left\{ \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\left(\int_D |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} - \int_D fu\,dx \right\},$$

and then, by means of its solution $\bar{u}$, recovering V_{opt} from (9.13). The auxiliary variational problem above can be written, via its Euler-Lagrange equation, as the nonlinear PDE

$$-\Delta\bar{u} + C(p,\bar{u})|\bar{u}|^{2/(p-1)}\bar{u} = f, \qquad \bar{u} \in H_0^1(D),$$

with the constant $C(p, \bar{u})$ given by

$$C(p, \bar{u}) = \left(\int_D |\bar{u}|^{2p/(p-1)}\,dx\right)^{-1/p} .$$

The fact that V_{opt} actually solves our optimization problem (9.12) *follows from the fact that $\bar{u} = R_{V_{opt}}(f)$, hence we have*

$$\begin{aligned}
\mathcal{E}_f(V_{opt}) &= \min_{u\in H_0^1(D)} \left\{\frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D V_{opt}u^2\,dx - \int_D fu\,dx\right\}\\
&= \frac{1}{2}\int_D |\nabla \bar{u}|^2\,dx + \frac{1}{2}\int_D V_{opt}\bar{u}^2\,dx - \int_D f\bar{u}\,dx\\
&= \min_{u\in H_0^1(D)} \left\{\frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\left(\int_D |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} - \int_D fu\,dx\right\}\\
&\geq \max_{V\in\mathcal{V}} \mathcal{E}_f(V).
\end{aligned}$$

We notice that, replacing $-\mathcal{E}_f$ by $\mathcal{E}_f$ transforms the maximization problem in (9.12) *into the minimization of $\mathcal{E}_f$ on $\mathcal{V}$, which has the only trivial solution $V \equiv 0$.*

Example 9.14. *More generally, we may consider the optimization problem*

$$\min_{V\in\mathcal{V}} \min_{u\in H_0^1(D)} \left\{\int_D j(x,u)\,dx \;:\; -\Delta u + Vu = f\right\}$$

where the constraint $\mathcal{V}$ is given by (9.11). *If $f \geq 0$ and $j(x,\cdot)$ is decreasing, by the maximum principle the best choice for the potential V is on the boundary of the admissible set $\mathcal{V}$ and we may consider the Lagrangian functional*

$$\int_D \big(j(x,u) + \nabla u \nabla\phi + Vu\phi - f\phi\big)\,dx ,$$

where ϕ is the adjoint state function. Optimizing with respect to V provides the optimal potential

$$V = |u\phi|^{1/(p-1)}\left(\int_D |u\phi|^{p/(p-1)}\,dx\right)^{-1/p}$$

which, combined with the Lagrangian functional above, reduces the problem to the minimization of the functional

$$\int_D \big(j(x,u) + \nabla u\nabla\phi - f\phi\big)\,dx + \left(\int_D |u\phi|^{p/(p-1)}\,dx\right)^{(p-1)/p} .$$

Differentiating with respect to ϕ gives the PDE (to let ϕ go up) for the state function u:

$$-\Delta u - f + C(p,u,\phi)|u\phi|^{1/(p-1)}u = 0 ,$$

where

$$C(p,u,\phi) = \left(\int_D |u\phi|^{p/(p-1)}\,dx\right),$$

while, differentiating with respect to u gives the equation for the adjoint state function ϕ:

$$-\Delta\phi + j'(x,u) + C(p,u,\phi)|u\phi|^{1/(p-1)}\phi = 0\ .$$

Example 9.15. *Similarly to what done in Example 9.13 we may consider F as the functional $-\lambda_1(V)$, where $\lambda_1(V)$ is the first eigenvalue of the Schrödinger operator $-\Delta + V(x)$, given by the minimization*

$$\lambda_1(V) = \min_{u\in H_0^1(D)} \left\{\int_D \left(|\nabla u|^2 + Vu^2\right)dx\ :\ \int_D u^2\,dx = 1\right\}. \tag{9.14}$$

We are then dealing with the optimization problem

$$\max_{V\in\mathcal{V}} \lambda_1(V) = \max_{V\in\mathcal{V}} \min_{u\in H_0^1(D)} \left\{\int_D \left(|\nabla u|^2 + Vu^2\right)dx\ :\ \int_D u^2\,dx = 1\right\}, \tag{9.15}$$

where the constraint $\mathcal{V}$ is as in (9.11). *Arguing as before, we interchange the max and the min above and we end up with the auxiliary problem*

$$\min_{u\in H_0^1(D)} \left\{\int_D |\nabla u|^2\,dx + \left(\int_D |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p}\ :\ \int_D u^2\,dx = 1\right\}.$$

In the same way as before, the optimal potential V_{opt} can be recovered through the solution $\bar u$ of the auxiliary problem above, by taking

$$V_{opt} = \left(\int_D |\bar u|^{2p/(p-1)}\,dx\right)^{-1/p} |\bar u|^{2/(p-1)}.$$

Remark 9.16. *In the case $p < 1$ problem* (9.12) *with the admissible class* (9.11) *does not admit any solution. Indeed, for a fixed real number $\alpha > 0$, take $V_n(x) = n\chi_{\Omega_n}(x)$, where χ_E denotes the characteristic function of the set E (with value 1 on E and 0 outside E) and $\Omega_n \subset D$ are such that the sequence (V_n) converges weakly in $L^1(D)$ to the constant function α. In particular, we have $n|\Omega_n| \to \alpha$ as $n \to \infty$ and so, since $p < 1$, we have*

$$\int_D V_n^p\,dx = n^p|\Omega_n| \to 0 \qquad \text{as } n \to \infty.$$

Therefore, for n large enough, the potentials V_n belong to the admissible class $\mathcal{V}$. By Proposition 9.11 we have $\mathcal{E}_f(V_n) \to \mathcal{E}_f(\alpha)$ and, since α was arbitrary, we obtain

$$\sup_{V\in\mathcal{V}} \mathcal{E}_f(V) \ge \sup_{\alpha\in\mathbb{R}} \mathcal{E}_f(\alpha) = \lim_{\alpha\to+\infty} \mathcal{E}_f(\alpha).$$

The limit on the right-hand side above is zero; on the other hand we have $\mathcal{E}_f(V) \le 0$ for any V. Thus, if a maximal potential V_{opt} exists, it should verify $\mathcal{E}_f(V_{opt}) = 0$ which is impossible.

It remains to consider the maximization problem (9.12) when $p = 1$. In this case the result of Proposition 9.11 cannot be applied because the unit ball of $L^1(D)$ is not weakly compact. However, the existence of an optimal potential still holds, as we show below. It is convenient to introduce the functionals

$$J_p(u) := \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\left(\int_D |u|^{2p/(p-1)}\,dx\right)^{(p-1)/p} - \int_D fu\,dx \qquad \text{if } p > 1$$

$$J_1(u) := \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\|u\|_\infty^2 - \int_D fu\,dx \qquad \text{if } p = 1.$$

Proposition 9.17. *The functionals J_p Γ-converge in $L^2(D)$ to J_1 as $p \to 1$.*

Proof. Let $v_n \in L^2(D)$ be a sequence of positive functions converging in $L^2(D)$ to $v \in L^2(D)$ and let $\alpha_n \to +\infty$. Then we have

$$\|v\|_{L^\infty(D)} \le \liminf_{n\to\infty} \|v_n\|_{L^{\alpha_n}(D)}. \tag{9.16}$$

In fact, suppose first that $\|v\|_{L^\infty} = M < +\infty$ and let $\omega_\varepsilon = \{v > M - \varepsilon\}$, for some $\varepsilon > 0$. Then, we have

$$\liminf_{n\to\infty} \|v_n\|_{L^{\alpha_n}(D)} \ge \lim_{n\to\infty} |\omega_\varepsilon|^{(1-\alpha_n)/\alpha_n} \int_{\omega_\varepsilon} v_n\,dx = |\omega_\varepsilon|^{-1} \int_{\omega_\varepsilon} v\,dx \ge M - \varepsilon,$$

and so, letting $\varepsilon \to 0$, we have

$$\liminf_{n\to\infty} \|v_n\|_{L^{\alpha_n}(\Omega)} \ge M.$$

If $\|v\|_{L^\infty} = +\infty$, then setting $\omega_k = \{v > k\}$, for any $k \ge 1$, and arguing as above, we obtain (9.16).

Now, let $u_n \to u$ in $L^2(D)$. Then, by the semicontinuity of the L^2 norm of the gradient, by (9.16), and by the continuity of the term $\int_D uf\,dx$, we have

$$J_1(u) \le \liminf_{n\to\infty} J_{p_n}(u_n),$$

for any decreasing sequence $p_n \to 1$. On the other hand, for any $u \in L^2(D)$, we have $J_{p_n}(u) \to J_1(u)$ as $n \to \infty$ and so, we have the conclusion. □

Lemma 9.18. *Let $D \subset \mathbb{R}^d$ be a bounded open set. Then for every $p \ge 1$ there is a unique minimizer u_p of the functional $J_p : H_0^1(D) \to \mathbb{R}$. Moreover, the following facts hold.*

(a) *There is a constant $C > 0$ such that for every $p > 1$ we have*

$$\|\nabla u_p\|_{L^2(D)} + \|u_p\|_{L^{2p/(p-1)}(D)} \le C\|f\|_{L^2(D)}.$$

(b) *For every open set $\Omega \subset\subset D$, there is a constant C_Ω such that*

$$\|u_p\|_{H^2(\Omega)} \le C_\Omega \|f\|_{L^2(D)}, \quad \textit{for every} \quad p > 1. \tag{9.17}$$

Proof. The existence of a minimizer follows by the direct method in the calculus of variations, while the uniqueness is a consequence of the strict convexity of the functional. Moreover, for every $p > 1$, the minimizer u_p satisfies the Euler-Lagrange PDE

$$-\Delta u_p + c|u_p|^\alpha u_p = f, \qquad u_p \in H^1_0(D) \tag{9.18}$$

where

$$\alpha = \frac{2}{p-1} \qquad \text{and} \qquad c = \left(\int_D |u_p|^{\frac{2p}{p-1}}\right)^{-1/p}.$$

Now (a) follows by multiplying equation (9.18) by u_p and integrating on D. In fact one may simply take the constant C to be the first Dirichlet eigenvalue $\lambda_1(D)$.

In order to prove (b) we use an argument similar to that of the classical elliptic regularity theorem. For $h \in \mathbb{R}$ and $k = 1, \dots, d$, we use the notation

$$\partial_k^h u = \frac{u(x + he_k) - u(x)}{h},$$

and we consider a function $\phi \in C_c^\infty(D)$ such that $\phi \equiv 1$ on Ω. Then we have that for h small enough $\partial_k^h u$ satisfies the following equation on the support of ϕ :

$$-\Delta\partial_k^h u + \frac{c}{h}\big(u(x+he_k)|u(x+he_k)|^\alpha - u(x)|u(x)|^\alpha\big) = \partial_k^h f. \tag{9.19}$$

Multiplying (9.19) by $\phi^2\partial_k^h u$ and taking into account the inequality

$$(X|X|^\alpha - Y|Y|^\alpha)(X - Y) \ge 0, \quad \text{for all} \quad X, Y \in \mathbb{R},$$

we obtain

$$\int_D \nabla(\partial_k^h u)\cdot\nabla(\phi^2\partial_k^h u)\,dx \le \int_D (\partial_k^h f)\phi^2\partial_k^h u\,dx.$$

By a change of variables, the Cauchy-Schwartz and the Poincaré inequalities we get that

$$\begin{aligned}\int_D (\partial_k^h f)\phi^2\partial_k^h u\,dx &= -\int_D f\partial_k^{-h}\big(\phi^2\partial_k^h u\big)\,dx \le \|f\|_{L^2}\|\partial_k^{-h}\big(\phi^2\partial_k^h u\big)\|_{L^2}\\ &\le \|f\|_{L^2}\|\nabla\big(\phi^2\partial_k^h u\big)\|_{L^2} \le C_\phi\|f\|_{L^2}\|\phi\nabla(\partial_k^h u)\|_{L^2},\end{aligned} \tag{9.20}$$

where C_ϕ is a constant depending on ϕ. On the other hand we have

$$\begin{aligned}\int_D \phi^2|\nabla(\partial_k^h u)|^2\,dx &= \int_D \nabla(\partial_k^h u)\cdot\nabla(\phi^2\partial_k^h u)\,dx - 2\int_D \phi(\partial_k^h u)\nabla\phi\cdot\nabla(\partial_k^h u)\,dx\\ &\le C_\phi\|f\|_{L^2}\|\phi\nabla(\partial_k^h u)\|_{L^2} + \frac{1}{2}\int_D \phi^2|\nabla(\partial_k^h u)|^2\,dx + 2\|\nabla\phi\|^2_{L^\infty}\|\nabla u\|^2_{L^2}.\end{aligned} \tag{9.21}$$

Thus, there is a constant $C_{D,\phi}$ depending on D and ϕ such that

$$\|\phi\nabla(\partial_k^h u)\|^2_{L^2} \le C_{D,\phi}\|f\|_{L^2}\|\phi\nabla(\partial_k^h u)\|_{L^2} + C^2_{D,\phi}\|f\|^2_{L^2},$$

which finally gives that

$$\|\partial_k^h(\nabla u)\|_{L^2(\Omega)} \le \|\phi\nabla(\partial_k^h u)\|_{L^2(D)} \le 2C_{D,\phi}\|f\|_{L^2(D)},$$

and since this last ineaqulity is true for every $k = 1, \dots, d$ and every h small enough we get that $u \in H^2(\Omega)$ and

$$\|\nabla^2 u\|_{L^2(\Omega)} \le C_\Omega \|f\|_{L^2(D)},$$

for an appropriate constant C_Ω depending on the function ϕ associated to Ω. □

Proposition 9.19. *Let $D \subset \mathbb{R}^d$ be a bounded open set and $f \in L^2(D)$ a given function. Then there is a unique minimizer $u_1 \in H_0^1(D)$ of the functional $J_1 : H_0^1(D) \to \mathbb{R}$,*

$$J_1(u) = \frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\|u\|^2_{L^\infty(D)} - \int_D uf\,dx.$$

Setting $M = \|u_1\|_{L^\infty}$, $\omega_+ = \{u_1 = M\}$ and $\omega_- = \{u_1 = -M\}$, we have that
(i) $u_1 \in H^2_{loc}(D)$;
(ii) u_1 is the solution of the equation

$$-\Delta u + \frac{1}{M}(\chi_{\omega_+} f - \chi_{\omega_-} f)\,u = f, \qquad u \in H_0^1(D)\,; \tag{9.22}$$

(iii) $\displaystyle\frac{1}{M}\int_{\omega_+} f\,dx - \frac{1}{M}\int_{\omega_-} f\,dx = 1$;
(iv) $f \ge 0$ on ω_+ and $f \le 0$ on ω_-.

Proof. (i) Let u_p be the minimizer of J_p. By Proposition 9.18 we have that the family $\{u_p\}_{p>1}$ is bounded in $H_0^1(D)$. From the estimate (9.17) we have that for every sequence $p_n \to 1$ the solutions u_{p_n} admit a subsequence converging weakly in $H_0^1(D)$ to some $u \in H^2_{loc}(D) \cap H_0^1(D)$. Since by Proposition 9.17 the functionals J_p Γ-converge in $L^2(D)$ to the functional J_1 as $p \to 1$, and since the minimizer of the functional J_1 is unique, we have $u = u_1$ and so, $u_1 \in H^2_{loc}(D) \cap H_0^1(D)$. Moreover, since this happens for every sequence $p_n \to 1$ we have $u_p \to u_1$ in $L^2(D)$ as $p \to 1$.

(ii) Let us define $\omega = \omega_+ \cup \omega_-$. We claim that u_1 satisfies, on D the PDE

$$-\Delta u_1 + \chi_\omega f = f, \qquad u_1 \in H_0^1(D). \tag{9.23}$$

Since $u_1 \in H^2_{loc}(D)$ (9.23) is equivalent to

$$\int_D (-\Delta u_1)\varphi\,dx = \int_D \chi_{D\setminus\omega} f\varphi\,dx, \quad \text{for every} \quad \varphi \in C_c^\infty(D). \tag{9.24}$$

Let $v_{\varepsilon,\delta} = 1 \wedge \left(\frac{1}{\delta}u\big((M-\varepsilon-u)\vee 0\big)\big((u+M-\varepsilon)\vee 0\big)\right)$. Since $u \in H_0^1(D) \cap L^\infty(D)$, we have that $v_{\varepsilon,\delta} \in H_0^1(D)$. By choosing ε and δ appropriately we may construct a sequence $v_n \in H_0^1(D)$ such that

- $v_n \uparrow \chi_{D\setminus\omega}$ as $n \to \infty$;
- $v_n = 0$ a.e. on the (Lebesgue measurable) set $\left\{-M + \frac{1}{n} \le u_1 \le M - \frac{1}{n}\right\}$.

Now since for $t \in \mathbb{R}$ such that $|t| \le \frac{1}{n\|\varphi\|_{L^\infty}}$ we have that $\|u_1 + t\varphi v_n\| \le M$, the minimality of u_1 gives that

$$\frac{1}{2}\int_D |\nabla u_1|^2\,dx - \int_D u_1 f\,dx \le \frac{1}{2}\int_D |\nabla(u_1 + t\varphi v_n)|^2\,dx - \int_D (u_1 + t\varphi v_n) f\,dx,$$

and taking the derivative with respect to t at $t = 0$, we get that

$$\int_D (-\Delta u_1)\varphi v_n\,dx = \int_D \nabla u_1 \cdot \nabla(\varphi v_n)\,dx = \int_D f\varphi v_n,\,dx,$$

and passing to the limit as $n \to \infty$ we obtain (9.24). We can now obtain (9.22) by (9.23) and the fact that $u_1 = M$ on ω_+ and $u = -M$ on ω_-.

(iii) Since u_1 is the minimizer of J_1, we have

$$J_1\big((1+\varepsilon)u_1\big) - J_1(u_1) \ge 0 \qquad \forall \varepsilon \in \mathbb{R}.$$

Taking the derivative of this difference at $\varepsilon = 0$, we obtain

$$\int_D |\nabla u_1|^2\,dx + M^2 = \int_D f u_1\,dx.$$

On the other hand, by (9.23), we have

$$\int_D |\nabla u_1|^2\,dx + \int_\omega f u_1\,dx = \int_D f u_1\,dx.$$

Now since $u_1 = M$ on ω_+ and $u_1 = -M$ on ω_-, we obtain

$$M^2 = \int_\omega f u_1\,dx = M\int_{\omega_+} f\,dx - M\int_{\omega_-} f\,dx.$$

(iv) Consider the function $w_\varepsilon = \big(M - \varepsilon - u\big) \vee 0$ which vanishes on the set $\{u \ge M - \varepsilon\}$ and is strictly positive on the set $\{u < M - \varepsilon\}$. For any non-negative $\varphi \in C_c^\infty(D)$ we have that for t small enough $\|u_1 + t\varphi\|_{L^\infty} < M$. Therefore, the optimality of u_1 gives

$$\begin{aligned}
0 &\le \lim_{t\to 0^+} \frac{J_1(u_1 + t\varphi w_\varepsilon) - J_1(u_1)}{t}\\
&= \int_D \nabla u_1 \cdot \nabla(\varphi w_\varepsilon)\,dx - \int_D f\varphi w_\varepsilon\,dx\\
&= \int_D (-\Delta u_1 - f)\varphi w_\varepsilon\,dx.
\end{aligned}$$

Since the last inequality holds for any $\varphi \ge 0$ and any $\varepsilon > 0$ we get that

$$-\Delta u_1 - f \ge 0 \quad \text{almost everywhere on} \quad \{u_1 < M\}.$$

On the other hand, $\Delta u_1 = 0$ almost everywhere on $\omega_- = \{u = -M\}$, and so we obtain that $f \le 0$ on ω_-. Arguing in the same way, and considering test functions supported on $\{u_1 \ge -M + \varepsilon\}$, we can prove that $f \ge 0$ on ω_+. □

Theorem 9.20. *Let $D \subset \mathbb{R}^d$ be a bounded open set, let $p = 1$, and let $f \in L^2(D)$. Then there is a unique solution to problem*

$$\max\left\{\mathcal{E}_f(V) \;:\; V \ge 0,\ \int_D V\,dx \le 1\right\}, \tag{9.25}$$

given by

$$V_1 = \frac{1}{M}\left(\chi_{\omega_+}f - \chi_{\omega_-}f\right),$$

where $M = \|u_1\|_{L^\infty(D)}$, $\omega_+ = \{u_1 = M\}$, $\omega_- = \{u_1 = -M\}$, being $u_1 \in H_0^1(D) \cap L^\infty(D)$ the unique minimizer of the functional J_1.

Proof. For any $u \in H_0^1(D)$ and any $V \ge 0$ with $\int_D V\,dx \le 1$ we have

$$\int_D u^2 V\,dx \le \|u\|_{L^\infty}^2 \int_D V\,dx \le \|u\|_{L^\infty}^2.$$

Thus we obtain the inequality

$$\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega u^2 V\,dx - \int_\Omega uf\,dx \le J_1(u), \quad \text{for every} \quad u \in H_0^1(D),$$

and taking the minimum with respect to u we get

$$\mathcal{E}_f(V) \le \min_{u \in H_0^1(D)} J_1(u),$$

which finally gives

$$\max\left\{\mathcal{E}_f(V) \;:\; V \ge 0,\ \int_D V\,dx \le 1\right\} \le \min_{u \in H_0^1(D)} J_1(u) = J_1(u_1),$$

where u_1 is the minimizer of J_1. By Proposition 9.19 we have that u_1 satisfies the equation

$$-\Delta u_1 + V_1 u_1 = f, \qquad u_1 \in H_0^1(D), \tag{9.26}$$

where we set

$$V_1 := \frac{1}{M}\left(\chi_{\omega_+}f - \chi_{\omega_-}f\right).$$

By Proposition 9.19 (iii) we have that $\int_D V_1 u_1^2\,dx = M^2$ and so

$$J_1(u_1) = \mathcal{E}_f(V_1).$$

Moreover, again by (iii) and (iv) we obtain that $V_1 \ge 0$ and $\int_D V_1\,dx = 1$, which concludes the proof. □

Example 9.21. *When f is a general $H^{-1}(D)$ function, the result of Theorem 9.20 may fail in the sense that problem (9.25) may admit an optimal solution which is merely a capacitary measure. Take for instance $f = \mathcal{H}^{d-1}\lfloor S$ where $S \subset D$ is a regular $(d-1)$-dimensional surface. In this case the energy $\mathcal{E}_f(V)$ has the form*

$$\mathcal{E}_f(V) = \min\left\{\frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\int_D Vu^2\,dx - \int_S u\,d\mathcal{H}^{d-1} \;:\; u \in H_0^1(D)\right\}.$$

By the results of Section 9.1 the maximization problem

$$\max\left\{\mathcal{E}_f(\mu) \;:\; \mu \in \mathcal{M}_{cap}(D),\ \int_D d\mu = 1\right\},$$

admits a solution μ_{opt} which is a capacitary measure. Repeating the proof of Theorem 9.20 we obtain the auxiliary variational problem

$$\mathcal{E}_f(\mu_{opt}) = \min\left\{\frac{1}{2}\int_D |\nabla u|^2\,dx + \frac{1}{2}\|u\|_{L^\infty}^2 - \int_S u\,d\mathcal{H}^{d-1} \;:\; u \in H_0^1(D)\right\}.$$

Denoting by u its unique solution and by M the maximum of u, we obtain that the optimal capacitary measure μ_{opt} is supported by the set $\{u = M\}$, this is contained in S (since the function u is subharmonic on $D \setminus S$) and so μ_{opt} is singular with respect to the Lebesgue measure. Moreover μ_{opt} has the form

$$\mu_{opt} = \frac{1}{M}\mathcal{H}^{d-1}\lfloor\{u = M\}.$$

The result in the following Theorem was proved in [356] (see also [505, Theorem 8.2.4]). We present it in a slightly different form as a simple consequence of Proposition 9.19. We recall the notation $\lambda_1(V)$ introduced in (9.14) for the first eigenvalue related to the potential V.

Theorem 9.22. *Let $D \subset \mathbb{R}^d$ be a bounded open set. Then there exists a unique solution to the maximization problem*

$$\max\left\{\lambda_1(V) \;:\; V \geq 0,\ \int_D V\,dx \leq 1\right\}, \tag{9.27}$$

given by

$$V_1 = \lambda\chi_\omega,$$

where $\omega = \left\{u_1 = \|u_1\|_{L^\infty(D)}\right\}$ and $u_1 \in H_0^1(D)$ solves the auxiliary variational problem

$$\lambda = \min\left\{\int_D |\nabla u|^2\,dx + \|u\|_{L^\infty(D)}^2 \;:\; u \in H_0^1(D),\ \int_D u^2\,dx = 1\right\}. \tag{9.28}$$

Proof. We first notice that due to the compact inclusion $H_0^1(D) \subset L^2(D)$ and the semicontinuity of the norm of the gradient there is a solution $u_\lambda \in H_0^1(D)$ of the problem

(9.28). We now set $f = \lambda u_\lambda$. Since for every $u \in H_0^1(D) \setminus \{0\}$ we have that

$$\min_{t\in\mathbb{R}} \left\{ \frac{t^2}{2} \int_D |\nabla u|^2\, dx + \frac{t^2}{2} \|u\|_{L^\infty}^2 - t \int_D uf\, dx \right\} = -\frac{\frac{1}{2}\left(\int_D uf\, dx\right)^2}{\int_D |\nabla u|^2\, dx + \|u\|_{L^\infty}^2},$$

we obtain that the minimizer of the functional J_1 corresponding to the function f is also the minimizer of the functional

$$J(u) = \frac{\int_D |\nabla u|^2\, dx + \|u\|_{L^\infty}^2}{\left(\int_D uf\, dx\right)^2}.$$

On the other hand, for every $u \in H_0^1(D)$ we have

$$\frac{\int_D |\nabla u|^2\, dx + \|u\|_{L^\infty}^2}{\left(\int_D uf\, dx\right)^2} \geq \frac{\int_D |\nabla u|^2\, dx + \|u\|_{L^\infty}^2}{\|u\|_{L^2}^2 \|f\|_{L^2}^2}$$
$$\geq \frac{\int_D |\nabla u_\lambda|^2\, dx + \|u_\lambda\|_{L^\infty}^2}{\|u_\lambda\|_{L^2}^2 \|f\|_{L^2}^2} = J(u_\lambda),$$

which proves that u_λ is the minimizer of J_1. Thus u_λ satisfies the equation

$$-\Delta u_\lambda + V_1 u_\lambda = \lambda u_\lambda, \qquad u_\lambda \in H_0^1(D), \qquad \int_D u_\lambda^2\, dx.$$

where V_1 is such that

$$V_1 \geq 0, \qquad \int_D V_1\, dx = 1 \qquad \text{and} \qquad \int_D V_1 u_\lambda^2\, dx = \|u_\lambda\|_{L^\infty}^2.$$

Thus we have that

$$\lambda_1(V_1) = \min\left\{ \int_D |\nabla u|^2\, dx + \|u\|_{L^\infty}^2 \;:\; u \in H_0^1(D),\ \int_D u^2\, dx = 1 \right\},$$

On the other hand for every $V \geq 0$ such that $\int_D V\, dx = 1$ we have

$$\int_D |\nabla u|^2\, dx + \int_D u^2 V\, dx \leq \int_D |\nabla u|^2\, dx + \|u\|_{L^\infty}^2, \quad \text{for every} \quad u \in H_0^1(D),$$

which after taking the minimum with respect to u gives

$$\lambda_1(V) \leq \lambda = \lambda_1(V_1),$$

which proves that V_1 is a solution of (9.27).

In order to prove the uniqueness of the solution it is sufficient to check that there is a unique solution to the problem (9.28). In fact suppose that u_1 and u_2 are two distinct solutions of (9.28) and denote $M_i = \|u_i\|_{L^\infty}$, $\omega_i = \{u_i = M_i\}$ and $V_i = \lambda\chi_{\omega_i}$, for $i = 1, 2$. We consider now the potential $V = \frac{V_1 + V_2}{2}$. Since the function $V \to \lambda_1(V)$ is the infimum of a family of linear functions we know that it is concave and so, V is also a solution of (9.27). Now since V is optimal, we have that for every $A, B \subset \omega_1 \cup \omega_2$ with $|A| = |B|$,

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \lambda_1\big(V + \varepsilon(\chi_A - \chi_B)\big) = 0.$$

Since the first eigenvalue is simple and the family of operators $-\Delta + V + \varepsilon(\chi_A - \chi_B)$ is analytic with respect to ε, we have that the functions $\varepsilon \mapsto \lambda_1\big(V + \varepsilon(\chi_A - \chi_B)\big)$ and $\varepsilon \mapsto u_\varepsilon$, where u_ε is the solution of

$$\begin{cases} -\Delta u_\varepsilon + \big(V + \varepsilon(\chi_A - \chi_B)\big)u_\varepsilon = \lambda_1\big(V + \varepsilon(\chi_A - \chi_B)\big)u_\varepsilon \\ u_\varepsilon \in H_0^1(D), \quad \int_D u_\varepsilon^2\,dx = 1 \end{cases}$$

are analytic. Taking the derivatives in ε at $\varepsilon = 0$ we obtain

$$\frac{d}{d\varepsilon} u_\varepsilon = u'$$

with

$$\begin{cases} -\Delta u' + Vu' + (\chi_A - \chi_B)u_0 = \lambda_1'(V)u_0 + \lambda_1(V)u' \\ u' \in H_0^1(D), \quad \int_D u' u_0\,dx = 0. \end{cases}$$

Multiplying both sides by u_0 and integrating by parts we get

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \lambda_1\big(V + \varepsilon(\chi_A - \chi_B)\big) = \int_A u_0^2\,dx - \int_B u_0^2\,dx.$$

Since A and B are arbitrary we get that u_0 is a (positive, by the maximum principle) constant on $\omega_1 \cup \omega_2$ and since $u_0 \in H^2_{loc}(D)$ we obtain that

$$Vu_0 = -\Delta u_0 + V_0 u_0 = \lambda u_0 \quad \text{on} \quad \omega_1 \cup \omega_2,$$

and as a consequence $V = \lambda$ on $\omega_1 \cup \omega_2$ which gives that $\omega_1 = \omega_2$, $V_1 = V_2$ and $u_1 = u_2$. □

Remark 9.23. *The proof above is constructed for the maximization of the first eigenvalue $\lambda_1(V)$ on the class*

$$\Big\{V \geq 0, \int_D V\,dx \leq 1\Big\}.$$

It would be interesting to consider the analogous maximization problem for $\lambda_k(V)$ on the same class of potentials

9.3 Existence results for confining potentials

In this section we consider the potential optimization problem

$$\min\left\{F(V)\ :\ V\in\mathcal{V}\right\}\tag{9.29}$$

where the functional F is as in the sections above and the admissible class $\mathcal{V}$ is given by

$$\mathcal{V}=\left\{V:D\to[0,+\infty]\ :\ V\text{ Lebesgue measurable, }\int_D\Psi(V)\,dx\le 1\right\}\tag{9.30}$$

and depends on a function $\Psi : [0,+\infty]\to[0,+\infty]$. On the function Ψ we make the following assumptions:

a) $\Psi:[0,+\infty]\to[0,+\infty]$ is an injective function;
b) there exist $p>1$ such that the function $s\mapsto\Psi^{-1}(s^p)$ is convex.

The assumptions above on the function Ψ are for instance satisfied by the following functions:

– $\Psi(s)=s^{-p}$, for any $p>0$;
– $\Psi(s)=e^{-\alpha s}$, for any $\alpha>0$.

and justify the terminology *“confining potentials”* we used. Indeed, large potentials turn out to be admissible.

The result showing the existence of an optimal potential in this case is as follows.

Theorem 9.24. *Let $D\subset\mathbb{R}^d$ be a bounded open set and let $\Psi:[0,+\infty]\to[0,+\infty]$ be a function satisfying the conditions a) and b) above. Let $F:\mathcal{M}_{cap}(D)\to\overline{\mathbb{R}}$ be a cost functional such that:*
i) F is lower semicontinuous with respect to the γ-convergence;
ii) F is increasing, that is

$$F(\mu_1)\le F(\mu_2)\text{ whenever }\mu_1\le\mu_2.$$

Then the optimization problem (9.29) *has a solution, where the admissible class $\mathcal{V}$ is given by* (9.30).

Proof. Let $V_n\in\mathcal{V}$ be a minimizing sequence for problem (9.29). Then the functions $v_n:=\left(\Psi(V_n)\right)^{1/p}$ are bounded in $L^p(D)$ and so, up to a subsequence, we may assume that v_n converges weakly in $L^p(D)$ to some function v. We will prove that the potential $V:=\Psi^{-1}(v^p)$ is optimal for the problem (9.29). Since v_n converges to v weakly in $L^p(D)$ we have

$$\int_D\Psi(V)\,dx=\int_D v^p\,dx\le\liminf_n\int_D v_n^p\,dx=\liminf_n\int_D\Psi(V_n)\,dx\le 1,$$

which shows that $V \in \mathcal{V}$. It remains to prove that

$$F(V) \le \liminf_n F(V_n).$$

By the compactness of the γ-convergence on the class $\mathcal{M}_{cap}(D)$, we can suppose that, up to a subsequence, V_n γ-converges to some capacitary measure $\mu \in \mathcal{M}_{cap}(D)$. Since F is assumed γ-lower semicontinuous, we have

$$F(\mu) \le \liminf_{n\to\infty} F(V_n). \tag{9.31}$$

We will show that $F(V) \le F(\mu)$, which, together with (9.31) will conclude the proof. By the definition of γ-convergence, we have that for any $u \in H_0^1(D)$, there is a sequence $u_n \in H_0^1(D)$ which converges to u in $L^2(D)$ and is such that

$$\begin{aligned}\int_D |\nabla u|^2\,dx + \int_D u^2\,d\mu &= \lim_{n\to\infty} \int_D |\nabla u_n|^2\,dx + \int_D u_n^2 V_n\,dx \\ &= \lim_{n\to\infty} \int_D |\nabla u_n|^2\,dx + \int_D u_n^2 \Psi^{-1}(v_n^p)\,dx \\ &\ge \int_D |\nabla u|^2\,dx + \int_D u^2 \Psi^{-1}(v^p)\,dx \\ &= \int_D |\nabla u|^2\,dx + \int_D u^2 V\,dx.\end{aligned} \tag{9.32}$$

The inequality in (9.32) is due to the $L^2(D)$ lower semicontinuity of the Dirichlet integral and to the strong-weak lower semicontinuity of integral functionals (see for instance [235]), which follows by the assumption b) on the function Ψ. Thus, for any $u \in H_0^1(D)$, we have

$$\int_D u^2\,d\mu \ge \int_D u^2 V\,dx,$$

which implies $V \le \mu$. Since F was assumed to increase monotonically, we obtain $F(V) \le F(\mu)$, which concludes the proof. □

Just like in the previous section, in some special cases, the solution to the optimization problem (9.29) can be computed explicitly through the solution to some auxiliary variational problem. This occurs for instance when $F(V) = \lambda_1(V)$ or when $F(V) = \mathcal{E}_f(V)$, with $f \in L^2(D)$. In fact, by the variational formulation

$$\lambda_1(V) = \min\left\{\int_D |\nabla u|^2 dx + \int_D u^2 V\,dx \;:\; u \in H_0^1(D),\; \int_D u^2\,dx = 1\right\},$$

we can rewrite the optimization problem (9.29) for $F(V) = \lambda_1(V)$ as

$$\begin{aligned}&\min_{V\in\mathcal{V}} \min_{\|u\|_2=1} \left\{\int_D |\nabla u|^2\,dx + \int_D u^2 V\,dx\right\} \\ &= \min_{\|u\|_2=1} \min_{V\in\mathcal{V}} \left\{\int_D |\nabla u|^2\,dx + \int_D u^2 V\,dx\right\}.\end{aligned} \tag{9.33}$$

The minimization with respect to V is easy to compute; in fact, if Ψ is differentiable with Ψ' invertible, then the minimum with respect to V in (9.33) is achieved for

$$V = (\Psi')^{-1}(\Lambda_u u^2), \tag{9.34}$$

where Λ_u is a constant such that

$$\int_D \Psi\left((\Psi')^{-1}(\Lambda_u u^2)\right)\,dx = 1.$$

Thus, the solution to the problem on the right hand side of (9.33) is given by the solution to the auxiliary variational problem

$$\min\left\{\int_D |\nabla u|^2\,dx + \int_D u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx \;:\; u \in H_0^1(D),\ \int_D u^2\,dx = 1\right\}. \tag{9.35}$$

Analogously, in the case of the Dirichlet energy $F(V) = \mathcal{E}_f(V)$, we obtain that the optimal potential is given by (9.34), where this time u is a solution to the auxiliary variational problem

$$\min\left\{\int_D \frac{1}{2}|\nabla u|^2\,dx + \int_D \frac{1}{2}u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx - \int_D fu\,dx \;:\; u \in H_0^1(D)\right\}. \tag{9.36}$$

Example 9.25. *Consider the case $\Psi(s) = s^{-p}$ with $p > 0$. Following the argument illustrated above we may conclude that the optimal potentials for the functionals $F(V) = \lambda_1(V)$ and $F(V) = \mathcal{E}_f(V)$ are given by*

$$V = \left(\int_D |u|^{2p/(p+1)}\,dx\right)^{1/p} u^{-2/(p+1)},$$

where u is the minimizer of the auxiliary variational problems (9.35) *and* (9.36) *respectively. We also note that, in this case*

$$\int_D u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx = \left(\int_D |u|^{2p/(p+1)}\,dx\right)^{(1+p)/p}$$

and so the auxiliary variational problems (9.35) *and* (9.36) *give rise to the nonlinear PDEs*

$$-\Delta u + C_1(p,u)|u|^{-2/(p+1)}u = \lambda u \qquad u \in H_0^1(D)$$
$$-\Delta u + C_1(p,u)|u|^{-2/(p+1)}u = f \qquad u \in H_0^1(D)$$

respectively, where the constant $C(p,u)$ is given by

$$C(p,u) = \left(\int_D |u|^{2p/(p+1)}\,dx\right)^{1/p}.$$

Example 9.26. *Consider the case $\Psi(x) = e^{-\alpha x}$ with $\alpha > 0$. Again, the same argument we used above shows that the optimal potentials for the functionals $F(V) = \lambda_1(V)$ and $F(V) = \mathcal{E}_f(V)$ are given by*

$$V = \frac{1}{\alpha}\left(\log\left(\int_D u^2\,dx\right) - \log\left(u^2\right)\right),$$

where u is the minimizer of the auxiliary variational problems (9.35) *and* (9.36) *respectively. We also note that, in this case*

$$\int_D u^2(\Psi')^{-1}(\Lambda_u u^2)\,dx = \frac{1}{\alpha}\left(\int_D u^2\,dx \int_D \log\left(u^2\right)\,dx - \int_D u^2\log\left(u^2\right)\,dx\right)$$

and so the auxiliary variational problems (9.35) *and* (9.36) *give rise to the nonlinear PDEs*

$$-\Delta u + \frac{1}{\alpha}\left(C_2(u) + C_3(u)\frac{1}{u^2} - 2\log|u| - 1\right)u = \lambda u \qquad u \in H_0^1(D)$$
$$-\Delta u + \frac{1}{\alpha}\left(C_2(u) + C_3(u)\frac{1}{u^2} - 2\log|u| - 1\right)u = f \qquad u \in H_0^1(D)$$

respectively, where the constants $C_2(u)$ and $C_3(u)$ are given by

$$C_2(u) = 2\int_D \log|u|\,dx \qquad C_3(u) = \int_D u^2\,dx.$$

The function $\Psi(s) = e^{-\alpha s}$ in the constraint (9.30) can be used to simulate and approximate a volume constraint in a shape optimization problem of the form

$$\min\left\{F(\Omega)\ :\ \Omega \subset D,\ |\Omega| \le 1\right\}$$

in which the main unknown is a domain $\Omega \subset D$, or equivalently a potential of the form $V = \infty_{D\setminus\Omega}$, written as a capacitary measure. Taking the cost functional $F(V) = \mathcal{E}_f(V)$ and replacing the constraint (9.30) by the addition of a Lagrange multiplier term, we obtain the problem

$$\min\left\{\mathcal{E}_f(V) + \Lambda\int_D e^{-\alpha V}\,dx\ :\ V \ge 0\right\}, \tag{9.37}$$

where Λ is a Lagrange multiplier and the potential V now varies among the nonnegative Borel measurable functions on D. As before, we note that the problem (9.37) is equivalent to

$$\min\left\{\int_D\left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}Vu^2 - fu + \Lambda e^{-\alpha V}\right)dx\ :\ u \in H_0^1(D),\ V \ge 0\right\}. \tag{9.38}$$

Fixing $u \in H_0^1(D)$ and minimizing with respect to V leads to the problem

$$\min\left\{\int_D Vu^2\,dx + \Lambda\int_D e^{-\alpha V}\,dx\ :\ V \ge 0\right\},$$

whose solution V can be obtained through the relation

$$u^2 - \Lambda\alpha e^{-\alpha V} = 0 \qquad \text{on } \{V(x) > 0\}.$$

We note that on the set where $u^2 \ge \Lambda\alpha$ we necessarily have that $V = 0$. On the other hand, if $u^2 < \Lambda\alpha$, then by the optimality of V, we have that $V > 0$. Finally, the optimal potential V can be identified in terms of u by

$$V(x) = 0 \vee \left(-\frac{1}{\alpha}\log\frac{u^2}{\Lambda\alpha}\right). \tag{9.39}$$

Replacing the expression above in (9.38), we obtain the auxiliary problem

$$\min_{u\in H_0^1(D)}\Big\{\frac{1}{2}\int_D|\nabla u|^2\,dx-\frac{1}{2\alpha}\int_{\{u^2<\Lambda\alpha\}}u^2\log\Big(\frac{u^2}{\Lambda\alpha}\Big)\,dx$$
$$-\int_D fu\,dx+\Lambda|\{u^2\ge\Lambda\alpha\}|+\frac{1}{\alpha}\int_{\{u^2<\Lambda\alpha\}}u^2\,dx\Big\},$$

which can be equivalently written as

$$\min_{u\in H_0^1(D)}\Big\{\frac{1}{2}\int_D|\nabla u|^2\,dx-\frac{1}{2\alpha}\int_{\{u^2\le\Lambda\alpha\}}u^2\log\Big(\frac{u^2}{\Lambda\alpha}\Big)\,dx$$
$$-\int_D fu\,dx+\Lambda|\{u^2>\Lambda\alpha\}|+\frac{1}{\alpha}\int_{\{u^2\le\Lambda\alpha\}}u^2\,dx\Big\}. \tag{9.40}$$

Note that the function

$$h_\alpha(s)=\begin{cases}\dfrac{s^2}{2\alpha}\Big(2-\log\Big(\dfrac{s^2}{\Lambda\alpha}\Big)\Big) & \text{if } s^2\le\Lambda\alpha\\ \Lambda & \text{if } s^2>\Lambda\alpha\end{cases}$$

is lower semicontinuous and nonnegative, which provides the necessary lower semicontinuity and coercivity to apply the direct methods of the calculus of variations and conclude that the auxiliary problem (9.40) has a solution $u_\alpha\in H_0^1(D)$. Moreover, on the quasi-open set $\{u^2>\Lambda\alpha\}$, we have $-\Delta u=f$. Let us denote by J_α the cost functional appearing in (9.40), that is

$$\begin{aligned}J_\alpha(u)&=\frac{1}{2}\int_D|\nabla u|^2\,dx+\frac{1}{2\alpha}\int_{\{u^2\le\Lambda\alpha\}}u^2\Big[2-\log\Big(\frac{u^2}{\Lambda\alpha}\Big)\Big]\,dx+\Lambda|\{u^2>\Lambda\alpha\}|\\&=\frac{1}{2}\int_D|\nabla u|^2\,dx+\int_D h_\alpha(u)\,dx.\end{aligned}$$

As $\alpha\to0$ the functions h_α increase and converge to the function

$$h(s)=\begin{cases}\Lambda & \text{if } s>0\\ 0 & \text{if } s=0\end{cases}$$

Then the functionals J_α Γ-converges in $L^2(D)$, as $\alpha\to0$, to the functional

$$J(u)=\frac{1}{2}\int_D|\nabla u|^2\,dx+\int_D h(u)\,dx=\frac{1}{2}\int_D|\nabla u|^2\,dx+\Lambda|\{u\ne0\}|.$$

By the properties of the Γ-convergence this implies the convergence of the solutions u_α of (9.40) and hence, thanks to the relation (9.39), of the optimal potentials V_α for (9.37) to a limit potential of the form

$$V(x)=\begin{cases}+\infty & \text{if } u(x)=0\\ 0 & \text{if } u(x)\ne0,\end{cases}$$

where u is a solution to the limit problem

$$\min\left\{\frac{1}{2}\int_D |\nabla u|^2\,dx - \int_D fu\,dx + \Lambda|\{u \neq 0\}| \ :\ u \in H_0^1(D)\right\}.$$

This limit problem is indeed a shape optimization problem written in terms of the state function u; several results on the regularity of the optimal domains are known (see for instance [25], [187], [189], as well as Chapter 3 of the present book).

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