Light scattering in inhomogeneous Tomonaga-Luttinger liquids

E. Orignac
Laboratoire de Physique de l'École Normale Supérieure de Lyon, CNRS-UMR5672, F-69364 Lyon Cedex 7, France

R. Citro
Dipartimento di Fisica "E. R. Caianiello," Università degli Studi di Salerno, Salerno, Fisciano (Sa) I-84084, Italy and Spin-CNR, Università degli Studi di Salerno, Salerno, Fisciano (Sa) I-84084, Italy

S. De Palo
IOM-CNR and Dipartimento di Fisica Teorica, Università di Trieste, Trieste I-34127, Italy

M.-L. Chiofalo
Department of Mathematics, INFN, University of Pisa, Pisa I-56127, Italy
(Received 14 November 2011; published 25 January 2012)

We derive the dynamical structure factor for an inhomogeneous Tomonaga-Luttinger liquid (TLL) as can be formed in a confined strongly interacting one-dimensional gas. In view of the current experimental progress in the field, we provide a simple analytic expression for the light-scattering cross section, requiring only the knowledge of the density dependence of the ground-state energy as it can be extracted, e.g., from exact or quantum Monte Carlo (QMC) techniques and a Thomas-Fermi description. We apply the result to the case of one-dimensional quantum bosonic gases with dipolar interaction in a harmonic trap, using an energy functional deduced from QMC computations. We find a universal scaling behavior peculiar to the TLL, a signature that can be probed eventually by Bragg spectroscopy in experimental realizations of such systems.

DOI: 10.1103/PhysRevA.85.013634 PACS number(s): 67.85.−d, 71.10.Pm, 67.10.Hk

I. INTRODUCTION

It is well known theoretically that systems of reduced dimensionality, especially in one dimension, present simultaneously enhanced quantum fluctuations and stronger interaction effects that can lead to exotic ground states [1,2]. From the experimental point of view, there are many prototypical one-dimensional systems that range from organic [3,4] or inorganic [5,6] conductors and antiferromagnetic spin chain [7,8] or ladder [9,10] materials to nanoscale systems, such as quantum wires [11,12], carbon nanotubes [13–16], or self-organized Au atomic wires on Ge(001) semiconductor surfaces [17]. More recently, advances in atom-trapping technology has permitted the realization of both fermionic and bosonic one-dimensional systems with unprecedented control [18–21]. The low-energy physics of such one-dimensional systems is well described by the Tomonaga-Luttinger liquid (TLL) theory [1,2,22,23]. In a single component TLL, there is a single gapless branch of excitations with linear dispersion, and the interplay between interactions and quantum fluctuations in the ground state leads to power-law decay of correlations with interaction-dependent exponents. Remarkably, the low-energy theory is fully characterized by two parameters: the velocity u of the linear dispersion excitations and the dimensionless exponent K controlling the decay of all correlations, the corresponding exponents being rational functions of K. In physical systems, several prominent features of TLL have been observed after measuring the spectral function [3,6], the structure factor [8], or the conductivity [13], and more recently, the first quantitative check of TLL physics has appeared for the spin-1/2 ladder material bis(piperidinium) tetrabromocuprate(II) $(C_5H_{12}N)_2CuBr_4$, in an applied magnetic field [10]. However, despite this recent achievement, in many of the physical systems mentioned above, little control can be exerted on the values of u and K and, thus, the Luttinger exponent K is taken as an adjustable parameter [3,13]. This fact prompts the search of more than one signature of TLL physics for a single system.

In the case of systems with strong confinement (e.g., confined quantum gases), excitation properties can be accessed most easily by light spectroscopy techniques, as proposed in the early days of atomic Bose-Einstein condensation [24,25]. For example, the spectral function has been measured recently in trapped Fermi gases by radio-frequency spectroscopy [26], and the dynamical structure factor has been studied successfully by optical Bragg spectroscopy in free and trapped Bose-Einstein condensates [27–31] as well as trapped Fermi gases [32]. Bragg spectroscopy can be based on energy transfer to the system at fixed momenta [33–35] or can permit the study of the full momentum composition of excitations by a coherent momentum transfer mapping [36]. For these reasons, Bragg spectroscopy can be especially useful to investigate the properties of the many phases realizable in these systems, such as Mott insulator, Tonks-Girardeau (TG) gas, or supersolid phases as recently proposed [37–45]. The most recent experimental progress in producing long-lived ground-state polar molecules in a three-dimensional (3D) optical lattice and possibly in one-dimensional (1D) arrays of pancakes and two-dimensional arrays of tubes [46] as well as condensates of dipolar atoms [47,48] opens up wide perspectives in the comprehension of controlled quantum systems with tunable short- and long-range interactions under progressively reduced dimensionality.

As we have more extensively reviewed in Ref. [49], among many possible realizations, quantum dipolar gases in 1D confinement are quite peculiar TLL systems. Here, in fact,

one single parameter drives the crossover from weak to strong interaction regimes where, however, the weakest regime is a TG state, the strongest being a dipolar-density-wave (DDW) state characterized by quasiordering.Based on the above motivations, we derive an analytic expression for light-scattering intensity in the case of a weakly inhomogeneous TLL. This expression is valid within a Thomas-Fermi description where the system can be considered locally homogeneous. The expression requires the knowledge of the density dependence of the ground-state energy of the homogeneous system as can be obtained by, e.g., approximate calculations, exact Bethe-Ansatz technique or quantum Monte Carlo (QMC) simulations. The paper is organized as follows. In Sec. II, after reviewing the calculation of the dynamic structure factor and the inelastic light-scattering cross section of homogeneous TLLs, in Sec. III, we derive the general expression for the inhomogeneous system within the Thomas-Fermi approach in terms of the eigenvalues and eigenfunctions of the hydrodynamic TLL. In Sec. IV, we then specialize the case of 1D quantum bosonic gases with dipolar interaction in a harmonic trap, using our previous QMC findings [49]. Here, the results are discussed explicitly in the various regime while the single parameter built up from density and interaction strength is tuned.

II. LIGHT-SCATTERING CROSS SECTION IN HOMOGENEOUS TLLS

The dynamic structure factor $S(q,\omega)$ is central in the description of interacting many-body systems. $S(q,\omega)$ is related to the Fourier transform of the imaginary density-density correlation function with the fluctuation-dissipation theorem. Therefore, it is accessible by means of inelastic scattering where density fluctuations are induced in the system and their subsequent relaxation is measured revealing the system characteristics. While inelastic neutron scattering has been the tool for probing the condensate nature of superfluid helium and the roton spectrum [50], inelastic light scattering has been proposed and has been used widely in dilute quantum degenerate gases. Within linear response theory, the scattering cross section σ of light at frequency ω and angle Ω incident on a Bose atomic sample is as follows:

$$\frac{d^2\sigma}{d\Omega\, d\omega} \propto \frac{1}{\pi n}[n_B(\omega)+1]\mathrm{Im}\,\chi(q,\omega) = S(q,\omega), \qquad (1)$$

where $n_B(\omega)$ is the Bose distribution function, $n = N/V$, and $\chi(q,\omega)$ is the Fourier transform of the density-density correlation function,

$$\chi(r,t) = -i\theta(t)\langle[n(r,t),n(0,0)]\rangle. \qquad (2)$$

Earlier experimental studies [28] have shown that condensate properties of atomic cold gases could be studied by means of Bragg scattering yielding high-energy resolution and sensitivity. The system is illuminated by two laser beams of momenta k_1 and k_2 and frequencies ω_1,ω_2 of difference ω that creates a periodic field whose intensity is proportional to $\cos[(k_1-k_2)r-\omega t]$. The external potential couples to the density $n(q)$ of the system where $q = k_1 - k_2$. After using the golden rule, the response of the system to this perturbation is the dynamical structure factor [51]. Light-scattering experiments then directly measure $S(q,\omega)$.

This quantity is then a benchmark against the theoretical descriptions of the systems. For a homogeneous TLL occurring in an interacting 1D system, the dynamic structure factor can be obtained readily [23]. In the following, we briefly sketch the derivation. For a system of interacting spinless particles, either bosons or fermions, the low-energy physics is that of a TLL whose Hamiltonian is

$$H = \int \frac{dx}{2\pi}\left[uK(\pi\Pi)^2 + \frac{u}{K}(\partial_x\phi)^2\right], \qquad (3)$$

with u as the velocity of the excitations and K as the TL exponent. The density operator $n(x)$ is expressed in terms of bosonic operators ϕ,

$$n(x) = n_0 - \frac{1}{\pi}\partial_x\phi + \sum_m A_m \cos\{2m[\phi(x)-\pi n_0 x]\}, \qquad (4)$$

with m as an integer and n_0 as the equilibrium density.

If the wavelength of the incoming light is much larger than the average interparticle distance, we can neglect the contribution of the oscillatory terms in Eq. (4). Using translational invariance, the expression for the density-density response function becomes

$$\chi(x-x',t) = i\frac{\theta(t)}{\pi^2}\langle[\partial_x\phi(x,t),\partial_x\phi(x',0)]\rangle. \qquad (5)$$

Knowing that the time-ordered correlation function $\langle T_\tau[\phi(x,\tau)-\phi(0,0)]^2\rangle = KF_1(x,\tau)$ with $F_1(x,\tau) = \ln\{[x^2+(u|\tau|+a)^2]/a^2\}/2$, the imaginary part of the response function (5) can be obtained [23] as

$$\mathrm{Im}\,\chi(q,\omega) = \frac{q^2}{2\omega}uK[\delta(\omega+u|q|)-\delta(\omega-u|q|)], \qquad (6)$$

leading to the scattered intensity at zero temperature,

$$\frac{d^2\sigma}{d\Omega\, d\omega} \propto S(q,\omega) = \mathrm{sgn}(\omega)\mathrm{Im}\,\chi(q,\omega) = \frac{K|q|}{2}[\delta(\omega+u|q|)+\delta(\omega-u|q|)]. \qquad (7)$$

Expression (7) embodies the symmetry with respect to inversion of velocity u as required by Galilean invariance and evidences the dependence of the light-scattering signal from the ratio q/ω.

III. LIGHT-SCATTERING CROSS SECTION IN INHOMOGENEOUS TLLS

A. Hydrodynamic approach

The presence of an external potential $V(x)$ confining the cold atomic cloud induces density inhomogeneity, and the external light perturbation probing the density-density correlation function introduces time-dependent processes. The treatment of the problem is easier under conditions of weak inhomogeneity and slow processes as they can be met in experiments where external potentials vary in length and time scales longer than the characteristic system quantities, and local equilibrium hydrodynamic behavior sets in. Under these conditions, the gas still can be described by a hydrodynamic

TLL Hamiltonian [49,52–56],

$$H_{\rm TLL} = \int_{-R}^{R} \frac{dx}{2\pi} \left[u(x)K(x)\pi^2 \Pi(x)^2 + \frac{u(x)}{K(x)} [\partial_x \phi(x)]^2 \right]. \tag{8}$$

Here, the boundary conditions imposed are $\phi(-R) = 0$ and $\phi(R) = -\pi N$, with N as the number of particles in the system. The parameters $u(x)$ and $K(x)$ now depend on position. In analogy with the homogeneous case where u and K are related by the expressions $u/K = (\hbar\pi)^{-1}\partial\mu/\partial n$ and by Galilean invariance $uK = \pi\hbar n/m$, one sets

$$u(x)K(x) = \pi \frac{\hbar}{m} n_0(x), \tag{9}$$

$$\frac{u(x)}{K(x)} = \frac{1}{\hbar\pi} \left(\frac{\partial \mu(n)}{\partial n} \right)_{n=n_0(x)}. \tag{10}$$

Once an estimate of the equilibrium density $n_0(x)$ and of the chemical potential $\mu(n)$ is known, this phenomenological approach allows the determination of $u(x)$ and $K(x)$.

The response function (5) in the case of the Hamiltonian (8) can be calculated using the decomposition,

$$\phi(x) = -\pi \frac{\int_{-R}^{x} dx' \frac{K(x')}{u(x')}}{\int_{-R}^{R} dx' \frac{K(x')}{u(x')}} N + \sum_n \sqrt{\frac{\pi}{2\omega_n}} (a_n^\dagger + a_n)\varphi_n(x). \tag{11}$$

Here, $[a_n, a_n^\dagger] = \delta_{n,m}$, and the first term comes from the addition of N particles in the system. The functions φ_n satisfy the eigenvalue equation,

$$-\omega_n^2 \varphi_n = u(x)K(x)\partial_x \left(\frac{u(x)}{K(x)} \partial_x \varphi_n \right), \tag{12}$$

with boundary conditions $\varphi_n(\pm R) = 0$, and the normalization,

$$\int dx \frac{\varphi_n(x)\varphi_m(x)}{u(x)K(x)} = \delta_{n,m}. \tag{13}$$

The influence of the trapping potential enters Eq. (12) via the equations for $u(x)$ and $K(x)$ (9). The density-density response function, thus, can be expressed as

$$\chi(x,x',t) = \theta(t) \sum_n \frac{1}{\pi\omega_n} \frac{d\varphi_n}{dx} \frac{d\varphi_n}{dx'} \sin(\omega_n t). \tag{14}$$

Taking the Fourier transforms with respect to x and x' and the Laplace transform with respect to t, we find

$$\chi(q,z) = \sum_n \frac{q^2 |\hat{\varphi}_n(q)|^2}{2\pi\omega_n} \left(\frac{1}{z+\omega_n} - \frac{1}{z-\omega_n} \right), \tag{15}$$

where $\mathrm{Im}(z) > 0$. Finally, taking the limit $z \to \omega + i0_+$, we obtain

$$\mathrm{Im}\,\chi(q,\omega+i0_+) = \frac{q^2}{2\omega} \sum_n |\hat{\varphi}_n(q)|^2 [\delta(\omega-\omega_n) + \delta(\omega+\omega_n)]. \tag{16}$$

Equation (16) maintains the structure of its homogeneous counterpart (6).

The density-density response function can be determined whenever the density dependence of the ground-state energy per unit length $e(n)$ or of the chemical potential $\mu(n) = (\frac{\partial e}{\partial n})|_{n=n(x)}$ is known. An especially simple situation is realized when $e(n) \propto n^{\gamma+2}$. That type of dependence of energy on density corresponds to several limiting cases of 1D TLL systems. For example, in the Lieb-Liniger gas [57,58], there are two well-understood limits. At low density or strong repulsion, the gas behaves as a hard-core boson gas [59] with $\gamma = 1$, while at high density or weak repulsion, the Bogoliubov approximation applies and gives an energy density proportional to n^2 so that $\gamma = 0$. The study of the crossover between these two limits requires the Bethe-Ansatz computation of the ground-state energy density [57]. A similar situation occurs in the case of dipolar gases. For low densities, the energy per unit length $e(n)$ has the $\gamma = 1$ behavior typical of hard-core bosons, whereas, for high density, it has the $\gamma = 2$ behavior of a crystal of classical dipoles, and a DDW manifests [49]. As density increases, the system crosses over from the low-density hard-core boson gas to the high-density DDW.

In the model with $e(n) = gn^{\gamma+2}$ and in the case of harmonic trapping potential $V(x) = m\Omega_0^2 x^2/2$, the eigenvalues ω_n of Eq. (11) can be found exactly, and the functions φ_n are expressible in terms of Gegenbauer polynomials [56,60] as

$$\varphi_n(x) = A_n \left(1 - \frac{x^2}{R^2} \right)^{\alpha+1/2} C_n^{(\alpha+1)} \left(\frac{x}{R} \right), \tag{17}$$

$$\omega_n^2 = \frac{u_0^2}{R^2} (n+1)(n+2\alpha+1). \tag{18}$$

Here, u_0 and K_0 are the TL parameters corresponding to the density at the trap center,

$$A_n = \sqrt{\frac{u_0 K_0}{R} \frac{n!(n+\alpha+1)}{\pi\Gamma(n+2\alpha+2)}} 2^{\alpha+1/2}\Gamma(1+\alpha), \tag{19}$$

and $\alpha = (\gamma+1)^{-1} - 1/2$. In particular, in the case of a hard-core Bose gas when $\gamma = 1$, $\alpha = 0$ and the Gegenbauer polynomials reduce to Chebyshev polynomials [61]. In order to calculate the scattered light intensity, we need the Fourier transform of the φ_n's. Using Eq. (7.321) of Ref. [62], we obtain

$$|\hat{\varphi}_n(q)|^2 = 2u_0 K_0 R(n+\alpha+1) \frac{\Gamma(n+2\alpha+2)}{\Gamma(n+1)} \times \frac{J_{n+\alpha+1}^2(qR)}{(qR)^{2\alpha+2}}, \tag{20}$$

where the J_m are the Bessel function of the first kind of parameter m [61]. Thus,

$$\mathrm{Im}\,\chi(q,\omega+i0_+) = \frac{u_0 K_0}{R\omega} \sum_n (n+\alpha+1) \frac{\Gamma(n+2\alpha+2)}{\Gamma(n+1)} \times \frac{J_{n+\alpha+1}^2(qR)}{(qR)^{2\alpha}} [\delta(\omega-\omega_n) + \delta(\omega+\omega_n)]. \tag{21}$$

Equation (21) shows the main features of the scattered light intensity. This is a set of discrete peaks, whose weight is a function of qR, and whose spacing reduces with increasing the trap size $R \to \infty$.

B. Approach via density-functional theory with local-density approximation

In the present section, we derive an approximate expression for the dynamical structure factor of an inhomogeneous 1D TLL, reverting to the density functional theory (DFT) accompanied by a local-density approximation (LDA). In the following, we sketch the main concepts and derivation. Through the Hohenberg and Kohn theorem, DFT establishes that the ground-state energy of a system, subjected to an external potential $V(x)$, is a functional $E_g[n(x)] = E[n(x)] + \int_{-\infty}^{+\infty} n(x)V(x)dx$ of the density $n(x)$ where $E[n(x)]$ embodies the kinetic and exchange-correlation parts. The equilibrium density profile is determined by the variational condition,

$$\frac{\delta E_g[n(x)]}{\delta n(x)} = \mu, \tag{22}$$

stating that equilibrium corresponds to a minimum of the energy against changes in the particle density, whereas, the total number of particles is fixed through the (density-dependent) chemical potential μ. Equation (22) reminds us of the Thomas-Fermi equilibrium condition in noninteracting systems, and, in fact, the DF sets a one-to-one correspondence between the ground-state energies of an interacting system and its noninteracting analog. Whenever an analytic expression of $\mu(n)$ is available, inversion of the equation of state (22) allows the determination of the equilibrium density $n_0(x)$.

Whereas, Eq. (22) is exact, the actual determination of the functional $E[n(x)]$ needs approximations. Under the conditions of shallow confinement, we safely can use the LDA. Here, the functional $E[n(x)]$ is replaced by

$$E^{\mathrm{LDA}}[n(x)] = \int e^{\mathrm{hom}}[n(x)]n(x)dx, \tag{23}$$

where $e^{\mathrm{hom}}(n)$ is the energy per particle of the homogeneous system with density n.

Differentiating $E^{\mathrm{LDA}}[n(x)] = E_g[n(x)] + \int dx[V(x) - \lambda]n(x)$ with respect to $n(x)$, λ being a Lagrange multiplier fixing the total number of particles, one obtains the condition for the local chemical potential,

$$\mu[n(x)] = V(R) - V(x), \tag{24}$$

where the local chemical potential is defined by the functional derivative,

$$\mu(n) = \frac{\delta E}{\delta n(x)} = \left(\frac{n\,\partial e^{\mathrm{hom}}(n)}{\partial n}\right)_{n=n_0(x)}. \tag{25}$$

If an analytic expression of $\mu(n)$ is given, Eq. (25) would allow finding $n(x)$ by inverting the relation Eq. (24). The energy $e^{\mathrm{hom}}(n)$ can be obtained after perturbation theory or by exact calculations, such as Bethe-Ansatz or else by computational QMC methods.

We now turn to the problem of determining the dynamical structure factor of the inhomogeneous system. In this respect, we follow the reasoning in Refs. [45,63] and imagine slicing it into small segments of length Δx where the density $n_0(x)$ can be considered uniform and, thus, sum together all the contributions (7) of the different segments. The dynamical structure factor of the inhomogeneous system then would be approximated by

$$S(q,\omega) = \int \frac{dx}{2R} S_{\mathrm{hom}}(q,\omega,n_0(x)). \tag{26}$$

$S_{\mathrm{hom}}(q,\omega,n)$ is given by Eq. (7) where, now, the TL parameters $u = u(n)$ and $K = K(n)$ depend on density.

With the help of Eq. (7), we obtain

$$S(q,\omega) = \frac{|q|}{4R}\int_{-R}^{R} dx\, K(n_0(x))\{\delta[\omega - u(n_0(x))|q|) + \delta(\omega + u(n_0(x))|q|]\}. \tag{27}$$

Introducing $x^*(\omega/|q|)$ such that $\omega = u(n(x^*))|q|$, we can rewrite

$$S(q,\omega) = \frac{K(n_0(x^*))}{2R\left|\frac{du}{dn}\right|_{n=n_0(x^*)}\left|\frac{dn_0}{dx}\right|_{x=x*}}. \tag{28}$$

Since the compressibility is a positive quantity, the chemical potential is an increasing function of the density. Moreover, for a trapping potential that is an increasing function of position, from Eq. (24), the density is seen to decrease with position. Thus, when the velocity is an increasing function of density, the solution x^* turns out to be unique.

The quantity $\frac{dn_0}{dx}$ can be obtained by differentiating the relation (24) with respect to x, i.e.,

$$\left(\frac{d^2e}{dn^2}\right)_{n=n_0(x)}\frac{dn_0}{dx} + \frac{dV}{dx} = 0. \tag{29}$$

Therefore, we can write

$$S(q,\omega) = \frac{K(n_0(x^*))\left|\frac{d^2e}{dn^2}\right|_{n=n_0(x^*)}}{2R\left|\frac{du}{dn}\right|_{n=n_0(x^*)}\left|\frac{dV}{dx}\right|_{x=x*}}. \tag{30}$$

We now use the relation $u^2(n) = \frac{n}{m}\frac{d^2e}{dn^2}$ obtained from Eq. (9) and rewrite Eq. (30) as

$$S(q,\omega) = \frac{\pi\hbar}{R\left|\frac{dV}{dx}\right|_{x=x^*}}\frac{n_0(x^*)}{\left|1 + n_0(x^*)\frac{e'''(n_0(x^*))}{e''(n_0(x^*))}\right|}, \tag{31}$$

with the notations $e'(x) = de/dn$, $e''(n) = d^2e/dn^2$, and $e'''(n) = d^3e/dn^3$.

Formula (31) represents the main result of this paper. It gives an analytical expression for the light-scattering cross section of an inhomogeneous TLL once the ground-state energy as a function of the density is known, e.g., by an exact analytical (Bethe-Ansatz) or via numerical simulations (QMC). Remarkably, Eq. (31) predicts that $S(q,\omega)$ is only a function of $\omega/|q|$. In fact, this is the specific signature of TLL behavior in shallow-trapped 1D Bose systems as can be measured by Bragg spectroscopy.

In order to illustrate the relevant features and to make the connection with Eq. (21) obtained via the hydrodynamic approach of Sec. III A, we now treat the case of harmonic trapping. In this case, $dV/dx = m\Omega_0^2 x$, and using Eq. (24), we have $m\Omega_0^2|x^*| = \sqrt{2m\Omega_0^2[e'(n_0(0)) - e'(\rho^*)]}$ where we

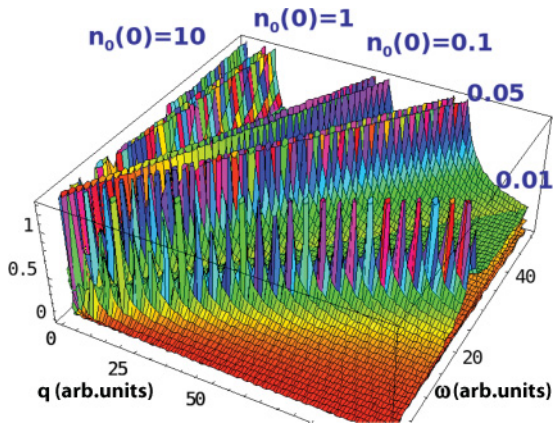

FIG. 1. (Color online) TLL model with $e(nr_0) \propto n^\gamma$ and $\gamma = 2$ in a harmonic trap. $S(q,\omega)$ in arbitrary units in the (ω,q) plane and different densities at the trap center.

have set $\rho^* = n_0(x^*)$ and $u(\rho^*) = \omega/|q|$. Equation (31), thus, simplifies into

$$S(q,\omega) = \frac{\pi\hbar\rho^*}{R\sqrt{2m\Omega_0^2[e'(n_0(0)) - e'(\rho^*)]}\left|1 + \rho^* \frac{e'''(\rho^*)}{e''(\rho_*)}\right|}. \quad (32)$$

We now check the consistency of the result (31) with Eq. (21) by explicitly calculating Eq. (32) for the model $e(n) \propto n^{\gamma+2}$. Equation (32) then reads

$$S(q,\omega) = \frac{\pi\hbar}{(\gamma+1)m\Omega_0^2R^2}\left(\frac{m\Omega_0^2R^2}{2g(\gamma+2)}\right)^{1/(\gamma+1)} \times \frac{\left(\frac{\omega}{u_0q}\right)^{2/(\gamma+1)}}{\sqrt{1-\left(\frac{\omega}{u_0q}\right)^2}}, \quad (33)$$

where we have defined $u_0 = u(n_0(0))$ as the velocity of excitations in a uniform system having a density equal to that at the trap center. We first notice that the dynamical structure factor in Eq. (33) makes the characteristic already embodied in the structure of Eq. (32) explicit, namely, that $S(q,\omega)$ depends on wave vectors and frequencies solely through their ratio ω/q. Second, the formula (33) with $\gamma = 1$ agrees with the result of Ref. [63] in the limiting $\omega \gg q^2/2$ case. Finally, in the Appendix, we show, by inspection, that the LDA approximation (33) is fully recovered from expression (21).

Figure 1 displays the 3D plot of $S(q,\omega)$ resulting from the use of Eq. (33) in the (ω,q) plane while varying the densities at the trap center. $S(q,\omega)$ is a set of discrete peaks whose position varies linearly with ω/q, and such linear behavior is independent of the interaction strength.

Before proceeding to apply Eq. (32) to a dipolar 1D Bose gas, we start by commenting on the found correspondence between hydrodynamic and DFT-LDA approaches on a more general footing. For normal Fermi systems [64] with an extension to Bose superfluids [65], it is known well that the treatment of dynamical processes in interacting inhomogeneous systems do require the development the current DFT where invariance conditions render the energy to be a functional of the current besides density. In this case, it was demonstrated that the analog of the LDA leads to Navier-Stokes equations (Landau-Khalatnikov two-fluid equations for superfluids) where viscosities, densities, and currents (normal and superfluid) have a microscopic expression in terms of Kubo relations and low-frequency response functions as they can be calculated in the homogeneous system at the local densities and currents. Such a general view is reflected by the present result. In the TLL free-harmonic Hamiltonian where the interactions are embodied effectively in u and K, the Navier-Stokes equations indeed become the simple hydrodynamic relations of Sec. III A. On the other hand, in the DFT and LDA approaches of Sec. III B, the treatment explicitly uses the two mappings: from an interacting to a noninteracting system (DFT) and from inhomogeneous to homogeneous (LDA).

IV. ONE-DIMENSIONAL BOSE GASES COUPLED VIA DIPOLAR INTERACTIONS

In this section, we specialize the case of a 1D dipolar gas in a harmonic trapping potential. We first recall the main results known for the homogeneous system and then apply Eq. (32) to determine the scattered light intensity. The system is characterized by the strength of the interactions C_{dd}, resulting from either magnetic $C_{dd} = \mu_0\mu_d^2$ or electric $C_{dd} = d^2/\epsilon_0$ dipoles, where μ_d and d are the magnetic and electric dipole moments and μ_0 and ϵ_0 are the vacuum permittivities. An effective Bohr radius can be defined from C_{dd} as $r_0 \equiv MC_{dd}/(2\pi\hbar^2)$ and the Hamiltonian in effective Rydberg units $\mathrm{Ry}^* = \hbar^2/(2Mr_0^2)$ is

$$H = (nr_0)^2\left[-\sum_i \frac{\partial^2}{\partial x_i^2} + (nr_0)\sum_{i<j}\frac{1}{|x_i - x_j|^3}\right], \quad (34)$$

where lengths are expressed in $1/n$ units. The physics of the model is specified entirely by the dimensionless coupling parameter nr_0 so that, in the high-density limit, the system becomes strongly correlated and a quasiordered state occurs where the potential energy dominates.

The ground-state energy $e(n)$ of this model was determined by means of the reptation QMC method in Ref. [49]. In the low $nr_0 \to 0$ limit, it reproduces the TG state energy per particle of a free-spinless Fermi gas, whose energy per particle is $e_{\mathrm{TG}}(n) = \pi^2(nr_0)^2/3\,\mathrm{Ry}^*$. In the large $nr_0 \to \infty$ limit of high-density dipoles, it reproduces the DDW state where $e_{\mathrm{DDW}}(n) = \zeta(3)(nr_0)^3\,\mathrm{Ry}^*$ and $\zeta(3) = 1.202\,05$. The QMC thermodynamic energy per particle in Rydberg units can be represented as an analytical function of nr_0,

$$e_p(nr_0) = \frac{\zeta(3)(nr_0)^4 + a(nr_0)^e + b(nr_0)^f + c(nr_0)^{(2+g)}}{1+nr_0} + \frac{\pi^2}{3}\frac{(nr_0)^2}{1+d(nr_0)^g}. \quad (35)$$

The fitting coefficients, yielding a reduced $\chi^2 \simeq 5$, are as follows: $a = 3.1(1)$, $b = 3.2(2)$, $c = 4.3(4)$, $d = 1.7(1)$, $e = 3.503(4)$, $f = 3.05(5)$, and $g = 0.34(4)$.

Thus, the Bragg intensity is obtained easily by Eq. (32) once the value of ρ^* is determined.

In Fig. 2, we report the scaling behavior of $S(q,\omega)$ vs $\omega/(u_0q)$ for different densities at the trap center n_0. Larger n_0r_0

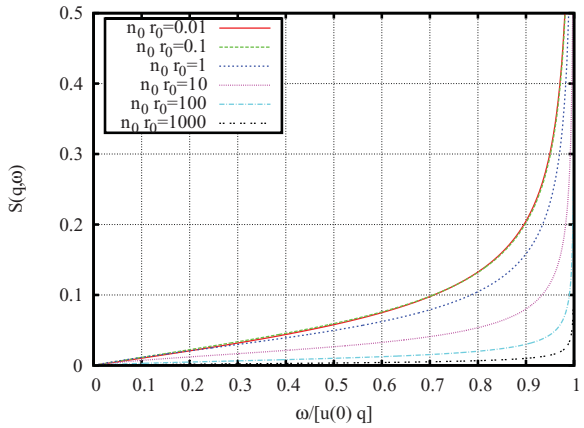

FIG. 2. (Color online) One-dimensional dipolar Bose gas confined in a harmonic trap with $e(n)$ as determined by QMC simulations. $S(q,\omega)$ (arb. units) vs $\omega/(u_0 q)$ for different densities n_0 at the trap center. The values of $n_0 r_0$ running from 0.01 to 1000 are indicated in the legend.

indicate stronger coupling interactions, crossing over from TG to DDW states. The linear behavior in the low ω/q regime is striking, the slope continuously increasing with decreasing $u(0)$ and, thus, n_0. In the TG limit, the tail of $S(q,\omega)$ is insensitive to changes in the density at the center of the trap, and, in fact, the curves with $n_0 r_0 = 0.01$ and 0.1 do coincide. The comparison with the TG gas ($\gamma = 1$ and $n_0 r_0 = 0.01$) and the DDW case ($\gamma = 2$ and $n_0 r_0 = 10^3$) is seen better in Fig. 3 where $S(q,\omega)\sqrt{e'(n_0)}/n_0$ is plotted as a function of $\omega/(u_0 q)$. One can notice that a crossover takes place in the intermediate densities regime. Viewed on the log-log scale, the plot evidences how a measure of the $S(q,\omega)$ tail toward small ω/q would provide a way to determine the interaction regime. A peculiarity of the TLL behavior is the power-law trend when $\omega/(u_0 q = 1)$ is approached. A detailed study of the power-law nonanalyticity for a trapped Bose gas can be found in Ref. [45].

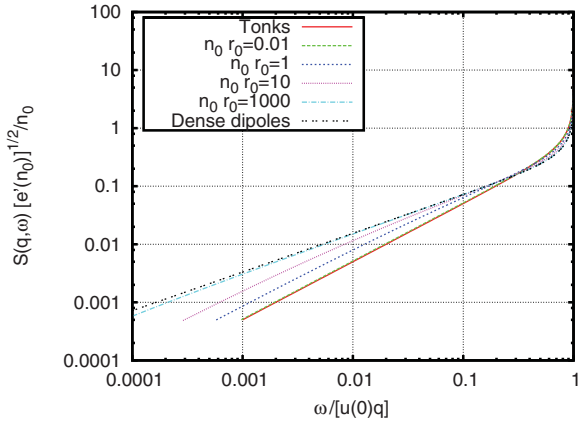

FIG. 3. (Color online) The same as Fig. 2, but on a log-log scale. The comparison with the TG limit gas ($\gamma = 1$) and the dense dipole limit corresponding to a DDW ($\gamma = 2$) is shown in evident manner.

V. CONCLUSIONS

We have derived the dynamical structure factor for an inhomogeneous TLL as it can occur in a confined strongly interacting 1D gas. In view of the current experimental progress in the field, we have provided an easy-to-use and simple analytical expression for the light-scattering cross section, Eq. (31), valid within a LDA.

The analytical expression (31) predicts that $S(q,\omega)$ only is a function of $\omega/|q|$ and is the central result of this paper. In fact, this is the specific signature of TLL behavior in shallow-trapped 1D Bose systems along with a power-law behavior when $\omega/(u_0 q)$ is approached as can be measured by Bragg spectroscopy.

Expression (31) is validated by the independent derivation (21) by means of a hydrodynamic approach, which is reported in detail in the Appendix. The connection between the two approaches is the second result of this paper and is a consequence of the more general current DFT [64,65] applied to the conditions of the present paper.

We, thus, remark that expression (31), in principle, can be applied to the many 1D systems cited in the introductory material, once the trapping potential is known together with the ground-state energy as a function of the density, e.g., by means of perturbative, exact, or computational methods applied to the homogeneous system. Extension of the present method to include additional local perturbations coupling to the density could be used to investigate the propagation of local-density fluctuations.

Finally, we have applied our findings to the case of 1D quantum bosonic gases with dipolar interactions, using the harmonic profile typical of experiments in this field, accompanied by our previous QMC data for the energy per particle. We find a universal scaling behavior peculiar to the TLL [49], a signature that eventually can be probed by Bragg spectroscopy in ongoing experimental realizations of such systems [46].

ACKNOWLEDGMENTS

M.-L.C. would like to thank Debbie Jin for illuminating discussions on the experiments [46] and their potential follow-up.

APPENDIX: JUSTIFICATION OF THE LDA FORMULA

In order to justify the approximate formulas, it is more convenient to work with the integrated intensity,

$$I(q,\Omega) = \int_0^{\Omega} S(q,\omega) d\omega, \tag{A1}$$

since the δ functions in the sum (21) contribute as step functions in $I(q,\Omega)$ giving more regular expressions.

Using the approximation (33), we expect

$$I(q,\Omega) = \frac{\pi \hbar u_0 n_0 |q|}{2m\Omega_0^2 R^2} \frac{2\alpha+1}{\alpha+1} \left(\frac{\Omega}{u_0 q}\right)^{2(\alpha+1)} \times {}_2F_1\left[\alpha+1, \frac{1}{2}; \alpha+2; \left(\frac{\Omega}{u_0 q}\right)^2\right], \tag{A2}$$

where ${}_2F_1$ is the Gauss hypergeometric function. Using the expression (21), instead, we obtain the exact expression,

$$I(q,\omega_N) \propto \sum_{n<N} \frac{2(n+\alpha+1)\Gamma(n+2\alpha+2)}{n!} \times \frac{J^2_{n+\alpha+1}(qR)}{\sqrt{(n+1)(n+2\alpha+1)}}, \tag{A3}$$

where N is such that $\omega_N = \Omega$. In order to check the consistency between Eqs. (A2) and (A3), we can work on the sums in Eq. (A3). We expect that the sum is dominated by the terms having $n \gg 1$. Using Eq. (9.3.2) in Ref. [61], we expect that, for $qR < n$, $J^2_{n+\alpha+1}(qR)$ is an exponentially small quantity with n. For $qR > n$, however, Eq. (9.3.3) in Ref. [61] suggests that

$$J^2_{n+\alpha+1}(qR) \simeq \frac{2}{\pi(n+\alpha+1)\tan\beta} \cos^2[(n+\alpha+1) \times (\tan\beta - \beta) - \pi/4], \tag{A4}$$

where $\cos\beta = qR/(n+\alpha+1)$. Elementary trigonometry gives the approximation,

$$J^2_{n+\alpha+1}(qR) \simeq \frac{(n+\alpha+1)}{\pi\sqrt{(qR)^2-(n+\alpha+1)^2}} \times \left\{1+\sin\left[2\sqrt{(qR)^2-(n+\alpha+1)^2} - (n+\alpha+1)\arccos\left(\frac{n+\alpha+1}{qR}\right)\right]\right\}. \tag{A5}$$

Dropping the term oscillating with n in Eq. (A5), we use as an approximation,

$$J^2_{n+\alpha+1}(qR) \simeq \frac{\theta(qR-n-\alpha-1)}{\pi\sqrt{(qR)^2-(n+\alpha+1)^2}}. \tag{A6}$$

We also can approximate

$$\frac{\Gamma(n+2\alpha+2)}{\Gamma(n+1)} \simeq n^{2\alpha+1}, \tag{A7}$$

$$\frac{n+\alpha+1}{\sqrt{(n+1)n+2\alpha+1}} \simeq 1, \tag{A8}$$

so that the sum in Eq. (A3) can be approximated by

$$\frac{2}{\pi} \sum_{n<\min(N,qR)} \frac{n^{2\alpha+1}}{\sqrt{(qR)^2-(n+\alpha+1)^2}}. \tag{A9}$$

Finally, by approximating the sum (A9) by an integral, we find

$$\sum_{n<\min(N,qR)} \frac{n^{2\alpha+1}}{\sqrt{(qR)^2-(n+\alpha+1)^2}} \simeq \int_0^{\min(N,qR)} \frac{du\, u^{2\alpha+1}}{\sqrt{(qR)^2-u^2}}. \tag{A10}$$

Using Eq. (6.6.1) in Ref. [61], we have (for $qR < N$)

$$\int_0^{\min(N,qR)} \frac{du\, u^{2\alpha+1}}{\sqrt{(qR)^2-u^2}} = \frac{1}{2}(qR)^{2\alpha+1} B_{(\Omega/qR\Omega_0)^2}(\alpha+1,1/2), \tag{A11}$$

where $B_x(a,b)$ is the incomplete Beta function [61]. With Eq. (6.6.8) of Ref. [61], we can check that Eq. (A2) agrees with the obtained approximate expression (A11).

[1] C. M. Varma, Z. Nussinov, and W. van Saarloos, Phys. Rep. **361**, 267 (2002).

[2] H. J. Schulz, in *Mesoscopic Quantum Physics, Les Houches, Session LXI*, edited by E. Akkermans, G. Montambaux, J. L. Pichard, and J. Zinn-Justin (Elsevier, Amsterdam, 1995), p. 533.

[3] C. Bourbonnais and D. Jérome, in *Advances in Synthetic Metals, Twenty Years of Progress in Science and Technology*, edited by P. Bernier, S. Lefrant, and G. Bidan (Elsevier, New York, 1999), p. 206.

[4] C. Bourbonnais and D. Jérome, in *The Physics of Organic Superconductors and Conductors*, edited by A. G. Lebed, Springer Series in Materials Science Vol. 110 (Springer, Heidelberg, 2008), p. 357.

[5] T. Mizokawa, K. Nakada, C. Kim, Z.-X. Shen, T. Yoshida, A. Fujimori, S. Horii, Y. Yamada, H. Ikuta, and U. Mizutani, Phys. Rev. B **65**, 193101 (2002).

[6] F. Wang, S.-K. Mo, J. W. Allen, H.-D. Kim, J. He, R. Jin, D. Mandrus, A. Sekiyama, M. Tsunekawa, and S. Suga, Phys. Rev. B **74**, 113107 (2006).

[7] P. R. Hammar, M. B. Stone, D. H. Reich, C. Broholm, P. J. Gibson, M. M. Turnbull, C. P. Landee, and M. Oshikawa, Phys. Rev. B **59**, 1008 (1999).

[8] B. Lake, D. A. Tennant, C. D. Frost, and S. E. Nagler, Nat. Mater. **4**, 329 (2005).

[9] E. Dagotto, Rep. Prog. Phys. **62**, 1525 (1999).

[10] M. Klanjšek, H. Mayaffre, C. Berthier, M. Horvatić, B. Chiari, O. Piovesana, P. Bouillot, C. Kollath, E. Orignac, R. Citro, and T. Giamarchi, Phys. Rev. Lett. **101**, 137207 (2008).

[11] O. M. Auslaender, A. Yacoby, R. de Picciotto, K. W. Baldwin, L. N. Pfeiffer, and K. W. West, Science **295**, 825 (2002).

[12] M. Hilke, D. C. Tsui, M. Grayson, L. N. Pfeiffer, and K. W. West, Phys. Rev. Lett. **87**, 186806 (2001).

[13] M. Bockrath, D. H. Cobden, J. Lu, A. G. Rinzler, R. E. Smalley, L. Balents, and P. L. Mceuen, Nature (London) **397**, 598 (1999).

[14] H. Ishii *et al.*, Nature (London) **426**, 540 (2003).

[15] B. Gao, A. Komnik, R. Egger, D. C. Glattli, and A. Bachtold, Phys. Rev. Lett. **92**, 216804 (2004).

[16] J. Lee, S. Eggert, H. Kim, S.-J. Kahng, H. Shinohara, and Y. Kuk, Phys. Rev. Lett. **93**, 166403 (2004).

[17] C. Blumenstein, J. Schafer, S. Mietke, S. Meyer, A. Dollinger, M. Lochner, X. Y. Cui, L. Patthey, R. Matzdorf, and R. Claessen, Nat. Phys. **7**, 776 (2011).

[18] B. Paredes, A. Widera, V. Murg, O. Mandel, S. Folling, I. Cirac, G. Shlyapnikov, T. Hansch, and I. Bloch, Nature (London) **429**, 277 (2004).

[19] T. Kinoshita, T. Wenger, and D. Weiss, Science **305**, 5687 (2004).
[20] Y.-a Liao, A. S. C. Rittner, T. Paprotta, W. Li, G. B. Partridge, R. G. Hulet, S. K. Baur, and E. J. Mueller, Nature (London) **467**, 567 (2010).
[21] M. Cazalilla, R. Citro, T. Giamarch, E. Orignac, and M. Rigol, Rev. Mod. Phys. **83**, 1405 (2011).
[22] J. Voit, Rep. Prog. Phys. **58**, 977 (1995).
[23] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, 2004).
[24] J. Javanainen, Phys. Rev. Lett. **75**, 1927 (1995).
[25] R. Graham and D. Walls, Phys. Rev. Lett. **76**, 1774 (1996).
[26] J. T. Stewart, J. Gaebler, and D. S. Jin, Nature (London) **454**, 744 (1998).
[27] M. Kozuma, L. Deng, E. W. Hagley, J. Wen, R. Lutwak, K. Helmerson, S. L. Rolston, and W. D. Phillips, Phys. Rev. Lett. **82**, 871 (1999).
[28] J. Stenger, S. Inouye, A. P. Chikkatur, D. M. Stamper-Kurn, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. **82**, 4569 (1999).
[29] D. M. Stamper-Kurn, A. P. Chikkatur, A. Görlitz, S. Inouye, S. Gupta, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. **83**, 2876 (1999).
[30] R. Ozeri, N. Katz, J. Steinhauer, and N. Davidson, Rev. Mod. Phys. **77**, 187 (2005).
[31] S. B. Papp, J. M. Pino, R. J. Wild, S. Ronen, C. E. Wieman, D. S. Jin, and E. A. Cornell, Phys. Rev. Lett. **101**, 135301 (2008).
[32] G. Veeravalli, E. Kuhnle, P. Dyke, and C. J. Vale, Phys. Rev. Lett. **101**, 250403 (2008).
[33] X. Du, S. Wan, E. Yesilada, C. Ryu, D. J. Heinzen, Z. Liang, and B. Wu, New J. Phys. **12**, 083025 (2010).
[34] N. Fabbri, D. Clément, L. Fallani, C. Fort, M. Modugno, K. M. R. van der Stam, and M. Inguscio, Phys. Rev. A **79**, 043623 (2009).
[35] D. Clément, N. Fabbri, L. Fallani, C. Fort, and M. Inguscio, Phys. Rev. Lett. **102**, 155301 (2009).
[36] P. T. Ernst, S. Götze, J. S. Krauser, K. Pyka, D.-S. Lühmann, D. Pfannkuche, and K. Sengstock, Nat Phys. **6**, 56 (2010).
[37] C. Menotti, M. Krämer, L. Pitaevskii, and S. Stringari, Phys. Rev. A **67**, 053609 (2003).
[38] R. Roth and K. Burnett, J. Phys. B **37**, 3893 (2004).
[39] W. Hofstetter, J. I. Cirac, P. Zoller, E. Demler, and M. D. Lukin, Phys. Rev. Lett. **89**, 220407 (2002).
[40] D. van Oosten, D. B. M. Dickerscheid, B. Farid, P. van der Straten, and H. T. C. Stoof, Phys. Rev. A **71**, 021601 (2005).
[41] A. M. Rey, P. B. Blakie, G. Pupillo, C. J. Williams, and C. W. Clark, Phys. Rev. A **72**, 023407 (2005).
[42] G. Pupillo, A. M. Rey, and G. G. Batrouni, Phys. Rev. A **74**, 013601 (2006).
[43] L. Mathey, I. Danshita, and C. W. Clark, Phys. Rev. A **79**, 011602 (2009).
[44] I. Titvinidze, M. Snoek, and W. Hofstetter, Phys. Rev. Lett. **100**, 100401 (2008).
[45] V. N. Golovach, A. Minguzzi, and L. I. Glazman, Phys. Rev. A **80**, 043611 (2009).
[46] A. Chotia, B. Neyenhuis, S. A. Moses, B. Yan, J. P. Covey, M. Foss-Feig, A. M. Rey, D. S. Jin, and J. Ye, e-print arXiv:1110.4420.
[47] T. Lahaye, T. Koch, B. Fröhlich, M. Fattori, J. Metz, A. Griesmaier, S. Giovanazzi, and T. Pfau, Nature (London) **448**, 672 (2007).
[48] M. Lu, N. Q. Burdick, S. H. Youn, and B. L. Lev, Phys. Rev. Lett. **107**, 190401 (2011).
[49] R. Citro, S. De Palo, E. Orignac, P. Pedri, and M.-L. Chiofalo, New J. Phys. **10**, 045011 (2008).
[50] P. E. Sokol, Can. J. Phys. **65**, 1393 (1987).
[51] A. Brunello, F. Dalfovo, L. Pitaevskii, S. Stringari, and F. Zambelli, Phys. Rev. A **64**, 063614 (2001).
[52] D. L. Maslov and M. Stone, Phys. Rev. B **52**, R5539 (1995).
[53] I. Safi and H. J. Schulz, Phys. Rev. B **52**, R17040 (1995).
[54] R. Fazio, F. W. J. Hekking, and D. E. Khmelnitskii, Phys. Rev. Lett. **80**, 5611 (1998).
[55] D. S. Petrov, G. V. Shlyapnikov, and J. T. M. Walraven, Phys. Rev. Lett. **85**, 3745 (2000).
[56] D. Petrov, D. Gangardt, and G. Shlyapnikov, J. Phys. IV **116**, 5 (2004).
[57] E. H. Lieb and W. Liniger, Phys. Rev. **130**, 1605 (1963).
[58] E. H. Lieb, Phys. Rev. **130**, 1616 (1963).
[59] M. Girardeau, J. Math. Phys. **1**, 516 (1960).
[60] C. Menotti and S. Stringari, Phys. Rev. A **66**, 043610 (2002).
[61] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).
[62] A. Gradshteyn and R. Ryzhik, *Tables of Integrals Series and Products* (Academic, New York, 1980).
[63] P. Vignolo, A. Minguzzi, and M. P. Tosi, Phys. Rev. A **64**, 023421 (2001).
[64] G. Vignale, C. A. Ullrich, and S. Conti, Phys. Rev. Lett. **79**, 4878 (1997).
[65] M. L. Chiofalo, A. Minguzzi, and M. P. Tosi, Physica B **254**, 188 (1998).