DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES S
Volume **3**, Number **2**, June **2010**

doi:10.3934/dcdss.2010.3.199

pp. **199–219**

AN ELEMENTARY APPROACH TO THE 3D NAVIER-STOKES EQUATIONS WITH NAVIER BOUNDARY CONDITIONS: EXISTENCE AND UNIQUENESS OF VARIOUS CLASSES OF SOLUTIONS IN THE FLAT BOUNDARY CASE

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Dedicated to V. A. Solonnikov on the occasion of his 75th birthday

Abstract. We study with elementary tools the stationary 3D Navier-Stokes equations in a flat domain, equipped with Navier (slip without friction) boundary conditions. We prove existence and uniqueness of weak, strong, and very-weak solutions in appropriate Banach spaces and most of the result hold true without restrictions on the size of the data. Results are partially known, but our approach allows us to give rather elementary and self-contained proofs.

1. **Introduction.** We consider the stationary system of the Navier-Stokes equations for the unknowns $u = (u^1, u^2, u^3)$ and p

$$\begin{cases} -\nabla \cdot T(u,p) + \nabla \cdot (u \otimes u) = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \end{cases}$$

where the stress tensor is defined by $T(u,p) := \nu(\nabla u + \nabla u^T) - p\,\mathbb{I}$ and $\Omega \subset \mathbb{R}^3$ is a smooth and bounded domain. The equations can be supplemented with Navier (slip without friction) boundary conditions

$$\begin{cases} u \cdot n = \alpha & \text{on } \Gamma, \\ T(u,p) \cdot n - \big(n \cdot T(u,p) \cdot n\big)\, n = \beta & \text{on } \Gamma, \end{cases}$$

where $n = (n^1, n^2, n^3)$ denotes the exterior unit normal vector on the boundary Γ of the domain Ω. This is not the classical Dirichlet boundary value problem with $u = 0$ on Γ, which has been proposed by Stokes [32] and which concerns most of the literature. The Navier's boundary conditions have been introduced by Navier [22] and interesting remarks on their use in certain physical situations can be found, e.g., in Serrin [25] and Beavers and Joseph [3]. Recently, these boundary conditions have also been identified as appropriate for some Large Eddy Simulation models for turbulent flows, see for instance Galdi and Layton [16] (these results with new applications are summarized in [10, 11]). Furthermore, the numerical implementation is studied in, e.g., John [19] and Verfürth [34, 35].

2000 *Mathematics Subject Classification.* Primary: 35Q30; Secondary: 76D03.
Key words and phrases. Stokes and Navier-Stokes equations, Navier's boundary conditions, L^q-theory, very-weak solutions.

The basic existence and regularity results for the linearized problem have been proved in the pioneering paper by Solonnikov and Ščadilov [31]. Recently Beirão da Veiga [4, 5, 6] proved existence results for weak and strong solutions in the L^2-setting. Other related results concerning these boundary conditions can be found in Fujita [14] (problems related to leakage) and in Bae and Jin [1] (for the flat case).

Here, we want to give some results in the flat case and we consider a very simple setting, which corresponds to the "half-space," but without the complications arising from an infinite domain. In particular, this simple setting will make clear how to obtain an elementary approach to the existence of various classes of solutions for this problem. Moreover, the problem in a general bounded domain will be treated in a forthcoming paper [8] and some of the results here are just sketched (for limitation of space) and complete proofs will appear elsewhere.

The main new contribution we give is the proof of existence for *very-weak solutions*, cf. Theorem 8.2, which extends previous results of Galdi, Simader, and Sohr [17] and Kim [20] for the Dirichlet problem. We consider L^q-solutions with $q \geq 3$ and it is important to observe that estimates in the L^q-spaces (instead of the simpler Hilbert setting) are requested to deal with the nonlinear problem: if we consider very-weak solutions belonging just to L^2, the term $u \otimes u$ will belong to $L^1(\Omega)$, which is generally not a good function-space, cf. Eq. (20) (Some results in the L^2-setting for linearized problems have been recently obtained by Borselli [12].) In particular, it is relevant to observe that our approach does not use the results on L^q-strong solutions for the Stokes problem *à la* Cattabriga [13].

1.1. **Setting of the problem.** The main results will concern the linear Stokes problem, and the nonlinear one will be treated by suitable perturbation arguments, by means of: i) fixed point theorems; ii) appropriate approximation of the data of the problem. The basic results regard the solutions of the Stokes problem

$$\begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \end{cases} \tag{1}$$

in the domain $\Omega =]-1, 1[^2 \times]0, 1[$, with flat boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, where

$$\Gamma_i = \left\{ x \in \mathbb{R}^3 : |x_1|, |x_2| < 1,\, x_3 = i \right\} \qquad \text{for} \quad i = 0, 1,$$

while the problem is assumed periodic (with period 2) in the other two directions. We also define $x' := (x_1, x_2)$ and we call "x'-periodic" any function that is periodic with period 2 in both x_1 and x_2. We impose the following Navier's (slip without friction) boundary conditions (N.-bc) on Γ,

$$\begin{cases} -\partial_3 u^1 = a^1 & \text{on } \Gamma_0 \qquad \partial_3 u^1 = 0 \quad \text{on } \Gamma_1, \\ -\partial_3 u^2 = a^2 & \text{on } \Gamma_0 \qquad \partial_3 u^2 = 0 \quad \text{on } \Gamma_1, \\ -u^3 = b & \text{on } \Gamma_0 \qquad u^3 = 0 \quad \text{on } \Gamma_1, \end{cases} \tag{2}$$

where a^i and b are given functions. The Navier's boundary conditions become the above ones (2) since the outer unit vector is $n = (0, 0, (-1)^{i+1})$ on Γ_i and the domain is flat. For simplicity we set homogeneous boundary conditions on Γ_1.

Remark 1. If (u, p) is a solution also $(u + u_0, p + p_0)$ with $u_0 := (c_1, c_2, 0,)$ and $p_0 := c_3$, for $c_i \in \mathbb{R}$ is a solution. Hence to have uniqueness we fix the mean value of u^1, u^2, and p equal to zero. In the general case uniqueness depends on (axial) symmetries of the domain, cf. [4]. In our functional setting Poincaré-Sobolev inequalities hold true.

We give an elementary, but rather complete approach to the study of $L^q(\Omega)$-solutions for the problem (1)-(2), which is based only on the regularity theory for the Poisson equation. The results we prove are not completely new, but in our opinion the proofs we give are of independent interest being very simple and based only on L^q-theory for the Laplace equations (which follows by Calderón-Zygmund theory). The main idea of the paper is to reduce the Stokes problem to the solution of several Poisson problems: This approach is similar to that introduced by Simader and Sohr [27, 28] for the Stokes problem in bounded and exterior domains and we refer to the latter reference for the classical results concerning the Poisson problems. We also use approximation arguments, in order to show existence of L^q-solutions by means of "*perturbations*" of the much simpler L^2-theory.

1.2. **Notation.** We introduce the (standard) notation we will use throughout the paper. The symbol $\|\,.\,\|_{L^q(\Omega)}$, $1 \leq q \leq +\infty$, denotes the usual norm of the Lebesgue space $L^q(\Omega)$. We use the customary Sobolev spaces $W^{k,q}(\Omega)$, $k \in \mathbb{R}$, with norm $\|\,.\,\|_{W^{k,q}(\Omega)}$, Sobolev spaces on the boundary Γ, denoted by $W^{s,q}(\Gamma)$, $s \in \mathbb{R}$, with norm $\|\,.\,\|_{W^{s,q}(\Gamma)}$, and we do not distinguish between scalar, vector, or tensor valued function spaces. The subscript "$_\#$" will denote the subspace with vanishing mean value (or such that duality with $\mathbb{1} := \{f \text{ such that } f(x) = 1 \ a.e.\}$ vanishes), "$_{(\#)}$" will denote vector fields with the first two components with vanishing mean value, while the subscript "$_\sigma$" will denote (weakly) divergence-free vector fields. We use also the following notation to denote suitable duality pairings.

$$
\begin{aligned}
&<,> && \text{is the duality pairing } (W^{2,q'}(\Omega))^*\text{-}W^{2,q'}(\Omega),\\
&<.\,,\,.>_\Gamma && \text{is the duality pairing } W^{-1/q,q}(\Gamma)\text{-}W^{1-1/q',q'}(\Gamma),\\
&<<.\,,\,.>>_\Gamma && \text{is the duality pairing } W^{-1-1/q,q}(\Gamma)\text{-}W^{2-1/q',q'}(\Gamma),
\end{aligned}
$$

where $q' := \frac{q}{q-1}$ is the conjugate exponent of q, $W^{-s,q}(\Gamma) := (W^{s,q'}(\Gamma))^*$, and X^* denotes the topological dual of the Banach space X.

The notions of strong and weak solution for system (1)-(2) are well-known and we recall that of very-weak solution.

Definition 1.1 (L^q-very-weak solution for the Stokes system with N.-bc)**.** We say that $u \in L^q(\Omega)$ is a very-weak solution to problem (1)-(2) if the following identities hold true:

$$
-\int_\Omega u\,\Delta\phi\,dx =< f,\phi > + \sum_{i=1}^{2} << a^i,\phi^i >>_{\Gamma_0} - < b,\partial_3\phi^3 >_{\Gamma_0}, \tag{3}
$$

for all $\phi = (\phi^1,\phi^2,\phi^3) \in W^{1,q'}_\sigma(\Omega) \cap W^{2,q'}(\Omega)$ such that $\partial_3\phi^1 = \partial_3\phi^2 = 0$ and $\phi^3 = 0$ on Γ and

$$
\int_\Omega u\,\nabla\psi\,dx =< b,\psi^3 >_{\Gamma_0}, \tag{4}
$$

for all $\psi \in W^{1,q'}(\Omega)$.

Observe that the boundary condition $u \cdot n = -u^3 = b$ on Γ_0 is intended as usual in $W^{-1/q,q}(\Gamma_0)$ by standard trace arguments, see for instance Temam [33]. On the other hand, the precise meaning of the other two boundary conditions, will be explained later on in Section 6.1. We have given the definition of very-weak solution in the flat case, but with minor changes one can also consider a general domain.

Plan of the paper. The paper is organized as follows: we first prove existence (and also estimates in terms of the data) for strong L^q-solutions of the Stokes problem (cf. Theorem 3.1). Next, we consider very-weak solutions of the Poisson problems in general bounded domains (cf. Propositions 1-2) and these results will be used to prove existence of weak-solutions for the Stokes system (cf. Theorem 5.1) as well as in the perturbation arguments. Then, we show existence of very-weak solutions for the Stokes problem (cf. Theorem 6.1). We also study an Oseen system and its adjoint, see Section 7. In Section 8, we use a fixed point argument to show existence of L^3-very-weak solutions for the Navier-Stokes equations with small data. Finally suitable approximations of the data and perturbed problems are studied to remove (at least for existence) limitations on the size of the data, cf. Theorem 8.2.

2. **On the Navier (slip without friction) boundary conditions in the flat case.** In this section we briefly explain how the N.-bc can be treated by elementary tools. We consider the flat case, but the same approach (with more technicalities) can be used also in general domains, by making use of suitable change of coordinates.

In order to understand the main properties of the boundary value problems supplemented with these boundary conditions we briefly explain what happens in the homogeneous case $(a^1, a^2, b) = 0$. First, we take the curl of the Stokes equations (1) and we obtain $\nabla \cdot \omega = 0$ and

$$-\Delta\omega = \operatorname{curl} f \qquad \text{in } \Omega.$$

The boundary conditions for ω are derived by a direct computation from those satisfied by u. In fact on Γ

$$\begin{cases} \omega^1 = \partial_2 u^3 - \partial_3 u^2 & = 0 + 0, \\ \omega^2 = \partial_3 u^1 - \partial_1 u^3 & = 0 + 0, \\ \partial_3\omega^3 = -\partial_1\omega^1 - \partial_2\omega^2 & = 0 + 0, \end{cases}$$

where the first two identities follow since we take tangential derivatives of the constant (on Γ) function u^3 and we use the homogeneous $(2)_{1,2}$; the third one follows from the divergence-free constraint and from the previous ones.

Hence, we have three independent (uncoupled) Poisson problems for the three components of $\omega = (\omega^1, \omega^2, \omega^3)$. The first two problems are supplemented with Dirichlet boundary conditions, while the third one with Neumann boundary conditions. Roughly speaking the use of the N.-bc allows us to use the vorticity equation and this is one of the main advantages with respect to the Dirichlet problem. Some review of these results and further details can be also found in [9].

Once one has considered the vorticity it is possible to study the velocity field by using again regularity results for the Laplace equation. In fact, with the identity

$$\operatorname{curl}\operatorname{curl} u = -\Delta u + \nabla(\nabla \cdot u),$$

we are led to solve the following boundary value problem

$$\begin{cases} -\Delta u = \operatorname{curl}\omega & \text{in } \Omega, \\ \partial_3 u^1 = \partial_3 u^2 = u^3 = 0 & \text{on } \Gamma, \end{cases}$$

Again, we have three uncoupled Poisson problems for the various components of u, which can be solved separately.

Finally, one derives regularity for the pressure by comparison, since (at least) in the sense of distributions

$$\nabla p = f + \Delta u.$$

2.1. **On a related problem.** In the non-flat case the boundary value problems satisfied by ω and u are more complicated, and it is possible to use similar tools when considering the so-called "stress-free" boundary conditions

$$u \cdot n = \omega \times n = 0 \qquad \text{on } \Gamma.$$

(If Γ is flat these conditions are the same of the N.-bc.) If Γ is smooth, but not flat, it is possible to use the vorticity equation, but terms of zero-order appear, see for instance [7, Sec. 2] where a detailed account is given. In particular, in that reference the boundary value problem

$$\begin{cases} -\Delta u = \operatorname{curl} \omega & \text{in } \Omega, \\ u \cdot n = 0 & \text{on } \Gamma, \\ \omega \times n = 0 & \text{on } \Gamma. \end{cases} \tag{5}$$

is studied very carefully by using the fundamental results on Green matrices developed in two papers by Solonnikov [29, 30]. The boundary value problem (5) is of Petrovskĭ type, which means -roughly speaking- that "*... different equations and unknowns have the same differentiability order,*" see [29, p. 126]. Petrovskĭ's systems are an important subclass of Agmon-Douglis-Nirenberg (ADN) elliptic systems, having the same good properties of self-adjoint ADN systems.

3. **Existence of $W^{2,q}$-strong solutions for the Stokes system with N.-bc.** We prove now existence of strong solutions $(u, p) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$ and this will be used in the duality argument needed to show existence of very-weak solutions. Moreover, the compatibility conditions on the data are due to the divergence constraint and to the particular geometric setting, cf. also the introduction in [4].

Theorem 3.1. *Let be given $f \in L^q(\Omega)$, $a^i \in W^{1-1/q,q}(\Gamma_0)$, and $b \in W^{2-1/q,q}(\Gamma_0)$, such that $\int_\Omega f^i + \int_{\Gamma_0} a^i = 0$, for $i = 1, 2$, and $\int_{\Gamma_0} b \, dS = 0$. Then, there exists a unique solution of the Stokes problem (1)-(2) such that $(u, p) \in W^{2,q}_{(\#)}(\Omega) \times W^{1,q}_{\#}(\Omega)$ and*

$$\|u\|_{W^{2,q}(\Omega)} + \|p\|_{W^{1,q}(\Omega)} \leq C \|f, a, b\|_{0,q},$$

for some $C = C(q, \Omega) > 0$, where we set

$$\|f, a, b\|_{0,q} := \|f\|_{L^q(\Omega)} + \sum_{i=1}^{2} \|a^i\|_{W^{1-1/q,q}(\Gamma_0)} + \|b\|_{W^{2-1/q,q}(\Gamma_0)}.$$

Proof. Let us consider first the "easy case," $q \geq 2$ (For the simplified problem in the "cube" see also the related results in Battinelli [2].) Existence of a unique variational solution derives immediately from [5, Thm. 1.2], (based on the Hilbertian variational theory) which states that there exists a unique strong solution such that

$$(u, p) \in W^{2,2}_{(\#)}(\Omega) \times W^{1,2}_{\#}(\Omega).$$

Our aim is to prove better regularity for this weak solution.

3.1. **Regularity for the vorticity field.** We start by proving that the vorticity is smoother than in $\omega \in W^{1,2}(\Omega)$ (and which follows from $u \in W^{2,2}(\Omega)$). We observe that $\omega = \text{curl}\, u$ restricted to the lower boundary satisfies

$$\begin{cases} \omega^1{}_{|\Gamma_0} = -\partial_2 b + a^2 \in W^{1-1/q,q}(\Gamma_0), \\ \omega^2{}_{|\Gamma_0} = -a^1 + \partial_1 b \in W^{1-1/q,q}(\Gamma_0), \\ -\partial_3\omega^3{}_{|\Gamma_0} = -\partial_1 a^2 + \partial_2 a^1 \in W^{-1/q,q}(\Gamma_0), \end{cases}$$

and $\omega^1 = \omega^2 = \partial_3\omega^3 = 0$ on Γ_1. Consequently we have to solve three uncoupled boundary value problems (ω-I)-(ω-II)-(ω-III) for the three components of ω.

$$\begin{cases} -\Delta\omega^1 = [\text{curl}\, f]^1 & \text{in } \Omega, \\ \omega^1 = -\partial_2 b + a^2 & \text{on } \Gamma_0, \\ \omega^1 = 0 & \text{on } \Gamma_1, \end{cases} \qquad (\omega\text{-I})$$

$$\begin{cases} -\Delta\omega^2 = [\text{curl}\, f]^2 & \text{in } \Omega, \\ \omega^2 = -a^1 + \partial_1 b & \text{on } \Gamma_0, \\ \omega^2 = 0 & \text{on } \Gamma_1, \end{cases} \qquad (\omega\text{-II})$$

$$\begin{cases} -\Delta\omega_3 = [\text{curl}\, f]^3 & \text{in } \Omega, \\ -\partial_3\omega^3 = -\partial_1 a^2 + \partial_2 a^1 & \text{on } \Gamma_0, \\ \partial_3\omega^3 = 0 & \text{on } \Gamma_1. \end{cases} \qquad (\omega\text{-III})$$

To be more precise we do not need to solve these problems, but we need only to show suitable *a priori* estimates on their solution, since we know that an unique solution $\omega = \text{curl}\, u$ (belonging *a priori* only to $W^{1,2}(\Omega)$) does exist. Since $f \in L^q(\Omega)$, and it is x'-periodic then

$$\text{curl}\, f \in (W^{1,q'}_0(\Omega))^* \times (W^{1,q'}_0(\Omega))^* \times (W^{1,q'}(\Omega))^*.$$

In fact for $i = 1, 2$

$$< [\text{curl}\, f]^i, \phi >= \int_\Omega f\, [\text{curl}\, \phi]^i\, dx \qquad \forall\, \phi \in W^{1,q'}_0(\Omega),$$

and the boundary integral vanishes due to the fact that ϕ is zero on Γ. Concerning the third component, periodicity implies that

$$< [\text{curl}\, f]^3, \phi >= \int_\Omega f\, [\text{curl}\, \phi]^3\, dx \qquad \forall\, \phi \in W^{1,q'}(\Omega).$$

Further, the compatibility condition is satisfied for the third system (ω-III) because

$$\int_\Omega [\text{curl}\, f]^3\, dx = \int_\Omega (\partial_1 f^2 - \partial_2 f^1)\, dx = -\int_{\Gamma_0} \partial_3\omega^3\, dS = 0$$

holds true by x'-periodicity of f and of a^i.

By using the standard $W^{1,q}$-regularity for the solutions of the Poisson equation (see e.g. [27, 28]) one obtains that $\exists\, C = C(q, \Omega) > 0$ such that

$$\|\omega\|_{W^{1,q}(\Omega)} \leq C\|f, a, b\|_{0,q}.$$

In the case $\frac{6}{5} \leq q < 2$ the proof remains essentially the same. (Observe that we will need $L^{3/2}$-strong-solutions to prove our existence result for $L^3 = L^{(3/2)'}$-very-weak solutions for the Navier-Stokes equations.) Results for $\frac{6}{5} \leq q < 2$ follow by observing that (in three dimensions) $f \in L^{6/5}(\Omega) \hookrightarrow (W^{1,2}(\Omega))^*$, $b \in$

$W^{2-1/q,q}(\Gamma_0) \hookrightarrow W^{1/2,2}(\Gamma_0)$, and also $a^i \in W^{1-1/q,q}(\Gamma_0) \hookrightarrow W^{-1/2,2}(\Gamma_0)$. Hence, one can still use the variational theory to prove existence of a unique weak solution $(u,p) \in W^{1,2}_{(\#)}(\Omega) \times L^2_{\#}(\Omega)$ and use the same arguments to prove regularity of the vorticity of this solution.

3.1.1. *Regularity for second derivatives of the velocity.* Next, we need to prove the estimates for the velocity (in $L^q(\Omega)$ up to second order derivatives), which solves

$$\begin{cases} -\Delta u^1 = [\operatorname{curl}\omega]^1 & \text{in } \Omega, \\ -\partial_3 u^1 = a^1 & \text{on } \Gamma_0, \\ \partial_3 u^1 = 0 & \text{on } \Gamma_1, \end{cases} \tag{u-I}$$

$$\begin{cases} -\Delta u^2 = [\operatorname{curl}\omega]^2 & \text{in } \Omega, \\ -\partial_3 u^2 = a^2 & \text{on } \Gamma_0, \\ \partial_3 u^2 = 0 & \text{on } \Gamma_1, \end{cases} \tag{u-II}$$

$$\begin{cases} -\Delta u^3 = [\operatorname{curl}\omega]^3 & \text{in } \Omega, \\ -u^3 = b & \text{on } \Gamma_0, \\ u^3 = 0 & \text{on } \Gamma_1, \end{cases} \tag{u-III}$$

and observe that the compatibility conditions for (u-I)-(u-II) are automatically satisfied. The standard regularity theory for Poisson problems can be used again on the above uncoupled problems to prove that

$$\|u\|_{W^{2,q}(\Omega)} \le C(\|\omega\|_{W^{1,q}(\Omega)} + \sum_{i=1}^{2} \|a^i\|_{W^{1-1/q,q}(\Gamma_0)} + \|b\|_{W^{2-1/q,q}(\Gamma_0)}).$$

The previous results on the vorticity imply that

$$\|u\|_{W^{2,q}(\Omega)} \le C\|f, a^i, b\|_{0,q},$$

for some $C = C(q,\Omega) > 0$. Finally, the equality in the sense of distributions

$$\nabla p = \Delta u + f \in L^q(\Omega)$$

shows regularity of the first derivatives of the pressure and also the estimate in terms of the data. This ends the proof of the theorem if $q \ge \frac{6}{5}$.

In order to treat also the case $1 < q < \frac{6}{5}$ one cannot apply directly the same tools, but one way to overcome the problem without resorting to more complicated techniques is to study some approximate problems, corresponding to data $\{(f_m, a^i_m, b_m)\}_{m\in\mathbb{N}}$ belonging to $L^2(\Omega) \times W^{1/2,2}(\Gamma) \times W^{3/2,2}(\Gamma)$. (We will see later on in Section 4 a way to construct such approximate data). To these data converging to (f, a^i, b) just in $L^q(\Omega) \times W^{1-1/q,q}(\Gamma_0) \times W^{2-1/q,q}(\Gamma_0)$ one can associate a sequence of weak solutions $\{(u_m, p_m)\}_{m\in\mathbb{N}} \in W^{2,2}_{(\#)}(\Omega) \times W^{1,2}_{\#}(\Omega)$, whose norm clearly cannot be controlled as $m \to \infty$. The same tools as before can be used to show that $\{(u_m, p_m)\}_m$ is bounded uniformly in $W^{2,q}_{(\#)}(\Omega) \times W^{1,q}_{\#}(\Omega)$, in terms of $C\|f, a^i, b\|_{0,q}$. Hence, we can extract a sub-sequence $\{(u_{m_k}, p_{m_k})\}_{k\in\mathbb{N}}$ such that

$$(u_{m_k}, p_{m_k}) \rightharpoonup (u,p) \in W^{2,q}_{(\#)}(\Omega) \times W^{1,q}_{\#}(\Omega).$$

By linearity of the Stokes problem (u,p) is the unique weak solution of the Stokes equation (1)-(2), with the requested regularity, hence a strong solution. □

4. **Very-weak solutions for two Poisson problems.** In this section we consider very-weak solutions for the Poisson problem with various boundary conditions and throughout this section Ω will be a bounded convex domain with smooth boundary Γ.

Results from this section will be used: i) to prove existence of the weak solutions $(u,p) \in W^{1,q}_{(\#)}(\Omega) \times L^q_\#(\Omega)$ of the problem (1)-(2); ii) for some approximation results. Most of the results are known (those concerning the Dirichlet problem can be found also in Kim [20]), but we restate them for completeness and we focus on Neumann problems.

We first consider the Poisson problem with Dirichlet boundary data.

Proposition 1 (See [20]). *Let $1 < q < +\infty$, let be given $f \in (W^{1,q'}_0(\Omega) \cap W^{2,q'}(\Omega))^*$, and let be given $b \in W^{-1/q,q}(\Gamma)$. Then, there exists a unique very-weak solution of the Dirichlet-Poisson problem*

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = b & \text{on } \Gamma, \end{cases}$$

i.e., a function $u \in L^q(\Omega)$ such that

$$-\int_\Omega u\,\Delta\phi\,dx =< f,\phi > - < b, \frac{\partial\phi}{\partial n} >_\Gamma,$$

for all $\phi \in W^{1,q'}_0(\Omega) \cap W^{2,q'}(\Omega)$. Furthermore

$$\|u\|_{L^q(\Omega)} \leq C(\|f\|_{(W^{1,q'}_0(\Omega)\cap W^{2,q'}(\Omega))^*} + \|b\|_{W^{-1/q,q}(\Gamma)}),$$

for some $C = C(q,\Omega) > 0$.

Proof. The proof is based on the usual duality argument, which relies on the regularity of the adjoint problem with zero boundary conditions. □

Remark 2. The trace $u_{|\Gamma} = b$ makes sense in $W^{-1/q,q}(\Gamma)$ even if *a priori* a function $u \in L^q(\Omega)$ does not have a well-defined trace. This follows since if $u \in L^q(\Omega)$ and $\Delta u \in (W^{1,q'}_0(\Omega) \cap W^{2,q'}(\Omega))^*$ imply that the linear functional $u \mapsto \gamma_0 u \in W^{-1/q,q}(\Gamma)$ is well-defined by the following identity

$$< \gamma_0 u, \frac{\partial\phi}{\partial n} >_\Gamma = - < \Delta u, \phi > + \int_\Omega u\,\Delta\phi\,dx,$$

which holds for all $\phi \in W^{2,q'}(\Omega) \cap W^{1,q'}_0(\Omega)$. In addition, if the solution u is smoother (say $u \in W^{1,q}(\Omega)$) then the boundary value $\gamma_0 u$ coincides with the usual one in the trace sense $u_{|\Gamma} \in W^{1-1/q,q}(\Gamma)$.

By using duality arguments we can also consider the Poisson problem with Neumann boundary conditions and prove the following result.

Proposition 2. *Let $1 < q < +\infty$, let be given $f \in (W^{2,q'}(\Omega))^*$, and let be given $a \in W^{-1-1/q,q}(\Gamma)$ such that*

$$< f, \mathbb{1} > + << a, \mathbb{1} >>_\Gamma = 0.$$

Then, there exists a unique very-weak solution of the Neumann-Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = a & \text{on } \Gamma, \end{cases} \tag{6}$$

i.e., a function $u \in L^q_\#(\Omega)$ *such that*

$$-\int_\Omega u\,\Delta\phi\,dx = < f, \phi > + << a, \phi >>_\Gamma, \tag{7}$$

for all $\phi \in W^{2,q'}(\Omega)$ *with* $\frac{\partial \phi}{\partial n} = 0$ *on* Γ. *Furthermore*

$$\|u\|_{L^q(\Omega)} \le C(\|f\|_{(W^{2,q'}(\Omega))^*} + \|a\|_{W^{-1-1/q,q}(\Gamma)}),$$

for some $C = C(q, \Omega) > 0$.

Proof. We give the proof since the same argument will be used also for the Stokes problem and in both case a representation's formula for the solution is used. Let us assume that u is a very-weak solution of (6). Given $\psi \in L^{q'}_\#(\Omega)$ we use as test function in the identity (7) a function $\phi \in W^{2,q'}(\Omega)$ which solves (strongly) the problem

$$\begin{cases} -\Delta\phi = \psi & \text{in } \Omega, \\ \dfrac{\partial \phi}{\partial n} = 0 & \text{on } \Gamma, \end{cases} \tag{8}$$

Then, the linear functional $\psi \mapsto < f, \phi >$ (defined on $L^{q'}_\#$) is continuous since

$$|< f, \phi >| \le \|f\|_{(W^{2,q'}(\Omega))^*}\|\phi\|_{W^{2,q'}(\Omega)} \le C\|f\|_{(W^{2,q'}(\Omega))^*}\|\psi\|_{L^{q'}(\Omega)},$$

by standard regularity results for the Neumann problem (see e.g. [27, 28]). The Riesz representation theorem proves there exists a unique $\widehat{f} \in L^q_\#(\Omega)$ such that

$$< f, \phi > = \int_\Omega \widehat{f}\,\psi\,dx.$$

With the same arguments we can show that there exists a unique $\widehat{a} \in L^q_\#(\Omega)$ such that

$$<< a, \phi >>_\Gamma = \int_\Omega \widehat{a}\,\psi\,dx.$$

Hence, if u satisfies (7) it follows $\int_\Omega u\,\psi\,dx = \int_\Omega \widehat{f}\,\psi\,dx + \int_\Omega \widehat{a}\,\psi\,dx$, and we get the representation's formula in $L^q_\#(\Omega)$

$$u := \widehat{f} + \widehat{a}. \tag{9}$$

Conversely, given f and a satisfying the hypotheses of Proposition 2, the function $u \in L^q_\#(\Omega)$ defined by (9) is a very-weak solution. Furthermore, uniqueness follows in a standard way by linearity of the problem. In fact, if we have two very-weak solutions u_1 and u_2 corresponding to the same data, then $\widetilde{u} := u_1 - u_2 \in L^q_\#(\Omega)$ is a very-weak solution of the homogeneous problem, and consequently

$$\int_\Omega \widetilde{u}\,\Delta\phi = 0 \qquad \forall\,\phi \in W^{2,q'}(\Omega), \text{ with } \frac{\partial \phi}{\partial n} = 0.$$

In particular, taking $\psi \in L^{q'}_\#(\Omega)$ and ϕ solution of (8), then

$$\int_\Omega \widetilde{u}\,\psi\,dx = 0 \qquad \forall\,\psi \in L^{q'}_\#(\Omega),$$

showing that $u_1 = u_2$ in $L^q_\#(\Omega)$. □

We use now the above propositions to prove some approximation results.

Corollary 1. *Let* $1 < q < +\infty$ *and let be given* $f \in (W^{2,q'}(\Omega))^*$*. Then, there exists a sequence* $\{f_k\}_{k\in\mathbb{N}}$ *such that* $f_k \in C_0^\infty(\Omega)$ *and*

$$f_k \to f \qquad \textit{in } (W^{2,q'}(\Omega))^*.$$

Proof. The proof of this result is similar to that in [20, Thm. 10]. We first observe that $W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega) \hookrightarrow W^{2,q'}(\Omega) \Rightarrow (W^{2,q'}(\Omega))^* \hookrightarrow (W^{2,q'}(\Omega) \cap W_0^{1,q'}(\Omega))^*$. Hence, there exists a unique very-weak solution $u \in L^q(\Omega)$ of the Poisson Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \quad u = 0 & \text{on } \Gamma. \end{cases}$$

By usual density arguments in Lebesgue spaces we can prove that there exists a sequence $\{u_k\}_{k\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that $u_k \to u$ in $L^q(\Omega)$. Finally, the requested sequence is defined by

$$f_k := -\Delta u_k.$$

□

Corollary 2. *Let* $1 < q < +\infty$ *and let be given* $a \in W^{-1-1/q,q}(\Gamma)$*. Then, there exists a sequence* $\{a_k\}_{k\in\mathbb{N}}$ *such that* $a_k \in C^\infty(\Gamma)$ *and*

$$a_k \to a \qquad \textit{in } W^{-1-1/q,q}(\Gamma).$$

Proof. Let $u \in L^q_\#(\Omega)$ be the very-weak solution of the Neumann-Poisson problem (and note that $[\text{meas }(\Omega)]^{-1} << a, \mathbb{1} >> \mathbb{1} \in (W^{2,q'}(\Omega))^*$)

$$\begin{cases} -\Delta u = -\dfrac{<< a, \mathbb{1} >>}{\text{meas }(\Omega)} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = a & \text{on } \Gamma. \end{cases}$$

Let now $\rho_\epsilon(x)$ be a standard family of mollifiers and let us follow [33, Ch. 1, Thm 1.1]: since Ω is smooth and bounded it is also locally star-shaped. We can find a finite number of smooth and bounded open sets $\mathcal{O}_j$, with $j = 1, \dots, M$ such that $\Omega \cup_j \mathcal{O}_j$ is an open covering of $\overline{\Omega}$. We consider a partition of unity subordinated to this covering and consisting of the smooth functions ϕ_j, with $j = 0, \dots, M$, such that

$$\mathbb{1} = \phi_0 + \sum_{j=1}^M \phi_j, \quad \text{and support}(\phi_0) \subset\subset \Omega, \text{support}(\phi_j) \subset\subset \mathcal{O}_j.$$

Consequently $u = \phi_0 u + \sum_{j=1}^M \phi_j u$. By extending $\phi_0 u$ by zero outside Ω and denoting it as v, standard results imply that $\rho_\epsilon * v \to v$ and $\Delta(\rho_\epsilon * v) \to \Delta v$ in $L^q(\mathbb{R}^3)$. Moreover supp $(\rho_\epsilon * v) \subset \Omega$ for $\epsilon > 0$ small enough (this is the "interior case"). Consider now $v = \phi_j u$ for some $j = 1, \dots, M$ and observe that $\mathcal{O}_j' = \mathcal{O}_j \cap \Omega$ is star-shaped (modulo a translation) with respect to the origin. Let now σ_λ denote the linear transformation $x \mapsto \lambda x$ and consider the function

$$x \mapsto v(\sigma_\lambda(x)) \qquad \lambda > 1.$$

Since $v(\sigma_\lambda(x)) \to v(x)$ in $\mathcal{O}_j'$ we can take a smooth function ψ_j with compact support in $\sigma_\lambda(\mathcal{O}_j')$ and such that $\psi_j(x) = 1$ for all $x \in \mathcal{O}_j'$. In this way the function $w_j := \psi_j(v(\sigma_\lambda(x)))$ is of compact support in $\mathbb{R}^3$. Then $\rho_\epsilon * w_j \to w_j$ and

$\Delta(\rho_\epsilon * w_j) \to \Delta w_j$ in $L^q(\mathcal{O}'_j)$. With the partition of unity and the transformation σ_λ we can construct a sequence of smooth functions $\{u_k\}_{k\in\mathbb{N}}$ such that

$$\begin{cases} u_k \to u & \text{in } L^q(\Omega), \\ \Delta u_k \to \Delta u & \text{in } L^q(\Omega). \end{cases}$$

We finally set

$$a_k := \frac{\partial u_k}{\partial n}$$

and we observe that

$$\begin{aligned} \big| << a - a_k, \phi_{|\Gamma} >>_\Gamma \big| &\leq \left| \int_\Omega (u - u_k)\Delta\phi\, dx \right| + \left| \int_\Omega (\Delta u - \Delta u_k)\phi\, dx \right| \\ &\leq \|u - u_k\|_{L^q(\Omega)} \|\Delta\phi\|_{L^{q'}(\Omega)} + \|\Delta u - \Delta u_k\|_{L^q(\Omega)} \|\phi\|_{L^{q'}(\Omega)}. \end{aligned}$$

The right-hand side converges to 0 as $k \to +\infty$ for all $\phi \in W^{2,q'}(\Omega)$ such that $\partial\phi/\partial n = 0$ on Γ. We end the proof by observing that by solving a standard Dirichlet bi-harmonic problem (see for instance Simader [26]) each $h \in W^{2-1/q',q'}(\Gamma)$ can be seen as the restriction on the boundary Γ of some $\phi \in W^{2,q'}(\Omega)$ with zero normal derivative. □

5. Existence of $W^{1,q}$-weak solutions for the Stokes system with N.-bc.

The next result we prove is the existence of weak solutions.

Theorem 5.1. *Let $1 < q < +\infty$ and let be given $f \in (W^{1,q'}(\Omega))^*$, $a^i \in W^{-1/q,q}(\Gamma_0)$, and $b \in W^{1-1/q,q}(\Gamma_0)$ with $< f^i, \mathbb{1} > + < a^i, \mathbb{1} >_{\Gamma_0} = 0$, for $i = 1, 2$, and $\int_{\Gamma_0} b\, dS = 0$. Then, there exists a unique solution $(u, p) \in W^{1,q}_{(\#)}(\Omega) \times L^q_\#(\Omega)$ of the Stokes equations (1) with N.-bc (2) such that*

$$\|u\|_{W^{1,q}(\Omega)} + \|p\|_{L^q(\Omega)} \leq C\|f, a^i, b\|_{-1,q},$$

for some $C = C(q, \Omega) > 0$, where

$$\|f, a^i, b\|_{-1.q} := \|f\|_{(W^{1,q}(\Omega))^*} + \|a^i\|_{W^{-1/q,q}(\Gamma_0)} + \|b\|_{W^{1/q,q}(\Gamma_0)}.$$

Proof. The proof is done by using the same elementary tools as in the proof of existence of strong solutions. We do not give details, but just show the main steps.

By applying the theory of very-weak solutions on the Poisson problems (ω-I)-(ω-II)-(ω-III) we show that $\exists\, C = C(q, \Omega) > 0$ such that

$$\|\omega\|_{L^q(\Omega)} \leq C\|f, a^i, b\|_{-1,q}.$$

The crucial point, which is readily checked by direct computation, is that by the above hypotheses it follows

$$[\text{curl}\, f]^1, [\text{curl}\, f]^2 \in (W_0^{1,q'}(\Omega) \cap W^{2,q'}(\Omega))^* \quad \text{and } [\text{curl}\, f]^3 \in (W^{2,q'}(\Omega))^*.$$

Next, we show by properties of weak solutions of suitable Poisson problems that

$$\|u\|_{W^{1,q}(\Omega)} \leq C(\|\omega\|_{L^q(\Omega)} + \|f, a^i, b\|_{-1,q}) \leq C\|f, a^i, b\|_{-1,q},$$

for some $C = C(q, \Omega) > 0$. In this step we used standard regularity results for the Poisson problem and what is readily checked is that the right-hand side in (u-I)-(u-II)-(u-III) belongs to the correct spaces. The final step concerns the pressure. By comparison and by using well-known theorems on negative norms (see e.g. Nečas [23], Chap 2, § 7) since we fixed the mean value of p, we obtain that

$$\|p\|_{L^q(\Omega)} \leq C\|\nabla p\|_{W^{-1,q}(\Omega)} \leq C(\|\Delta u\|_{W^{-1,q}(\Omega)} + \|f\|_{W^{-1,q}(\Omega)}) \leq C\|f, a^i, b\|_{-1,q},$$

where $W^{-1,q}(\Omega) := (W^{1,q}_0(\Omega))^* \supsetneq (W^{1,q}(\Omega))^*$. $\square$

6. **Existence of L^q-very-weak solutions for the Stokes system with N.-bc.** We study now the problem of the existence of very-weak solutions, cf. Definition 1.1.

Remark 3. We consider for simplicity the problem with $\nabla\cdot u = 0$ (in the weak sense). With the same tools we can also consider the problem with assigned $\nabla\cdot u = k \in (W^{1,q'}(\Omega))^*$ and with the compatibility condition $< b, \mathbb{1} >_{\Gamma_0} = < k, \mathbb{1} >$.

The main result of this section is the following.

Theorem 6.1. *Let for $1 < q < +\infty$ be given $f \in (W^{2,q'}(\Omega))^*$, $a^i \in W^{-1-1/q,q}(\Gamma_0)$, and $b \in W^{-1/q,q}(\Gamma_0)$, with $< f^i, \mathbb{1} > + << a^i, \mathbb{1} >>_{\Gamma_0} = 0$, for $i = 1, 2$, and $< b, \mathbb{1} >_{\Gamma_0} = 0$. Then, there exists a unique $u \in L^q_{(\#)}(\Omega)$ that is a very-weak solution to the problem (1)-(2) such that*

$$\|u\|_{L^q(\Omega)} + \|p\|_{W^{-1,q}(\Omega)} \leq C\|f, a^i, b\|_{q,-2}, \tag{10}$$

for some $C = C(q, \Omega) > 0$, where

$$\|f, a^i, b\|_{q,-2} := \|f\|_{(W^{2,q'}(\Omega))^*} + \sum_{i=1}^{2} \|a^i\|_{W^{-1-1/q,q}(\Gamma_0)} + \|b\|_{W^{-1/q,q}(\Gamma_0)}.$$

Proof. To prove the existence result we derive a suitable representation formula for the very-weak solution, by using duality arguments.

Let us assume that $u \in L^q_{(\#)}(\Omega)$ is a very-weak solution The basic point is to use now a suitable couple of test function $(\phi, \psi) \in W^{2,q'}_\sigma(\Omega) \times W^{1,q'}_\#(\Omega)$ such that

$$\begin{cases} -\Delta\phi - \nabla\psi = v & \text{in } \Omega, \\ \nabla\cdot\phi = 0 & \text{in } \Omega, \\ \partial_3\phi^1 = \partial_3\phi^2 = \phi^3 = 0 & \text{on } \Gamma, \end{cases} \tag{11}$$

for a given $v \in L^{q'}_{(\#)}(\Omega)$, i.e., (ϕ, ψ) is a *strong solution* of the adjoint problem.

By subtracting (4) from (3) we obtain the following equality

$$\int_\Omega u(-\Delta\phi - \nabla\psi)\, dx = < f, \phi > - < b, \partial_3\phi^3 + \psi >_{\Gamma_0} + \sum_{i=1}^{2} << a^i, \phi^i >>_{\Gamma_0}.$$

The linear functional $\mathcal{F}: L^{q'}_{(\#)}(\Omega) \to \mathbb{R}$, defined by

$$v \mapsto (\phi, \psi) \mapsto < f, \phi > - < b, \partial_3\phi^3 + \psi >_{\Gamma_0} + \sum_{i=1}^{2} << a^i, \phi^i >>_{\Gamma_0},$$

satisfies

$$\begin{aligned} |\mathcal{F}(v)| &\leq c\|f, a^i, b\|_{-2,q'}(\|\phi\|_{W^{2,q'}(\Omega)} + \|\psi\|_{W^{1,q'}(\Omega)}). \\ &\leq c\|f, a^i, b\|_{-2,q'}\|v\|_{L^{q'}(\Omega)}, \end{aligned}$$

and this follows immediately by the regularity of the solution (ϕ, ψ) of the adjoint problem. By Riesz representation theorem there exists a unique $u \in L^q_{(\#)}(\Omega)$ such that

$$\mathcal{F}(v) = \int_\Omega u\, v\, dx.$$

Remark 4. One can also show that there exist suitable functions $\widehat{f}$, $\widehat{a^i}$, $\widehat{b}$, by using representation formulas for each term of $\mathcal{F}$, and write a similar expression for u as in [17].

The function u determined by Riesz representation theorem turns out to be a very-weak solution to (1)-(2). In fact, given $(\phi, \Psi) \in W^{2,q'}_{\sigma}(\Omega) \times W^{1,q'}(\Omega)$ with ϕ satisfying the homogeneous N.-bc on Γ, we set $v = -\Delta\phi - \nabla\Psi$ and $\psi := \Psi - |\Omega|^{-1} \int_\Omega \Psi \, dx$. It follows that

$$\int_\Omega u(-\Delta\phi - \nabla\psi)\, dx$$
$$= L(v) = < f, \phi > - < b, \partial_3\phi^3 + \psi >_{\Gamma_0} + \sum_{i=1}^{2} << a^i, \phi^i >>_{\Gamma_0} .$$

This argument proves both existence of a L^q-very-weak solution and also the estimate in terms of the data. Uniqueness follows in a standard way, since if we have two L^q-very-weak solutions u_1 and u_2 corresponding to the same data, then $\widetilde{u} := u_1 - u_2$ is a very-weak solution with homogeneous data, hence

$$\int_\Omega \widetilde{u}\,(-\Delta\phi - \nabla\psi)\, dx = 0,$$

for all (ϕ, ψ) as above. Then, given $v \in L^{q'}_{(\#)}(\Omega)$ and taking (ϕ, ψ) as the solution to (11) we obtain

$$\int_\Omega \widetilde{u}\, v\, dx = 0, \qquad \forall\, v \in L^{q'}_{(\#)}(\Omega),$$

and consequently $\widetilde{u} \equiv 0$. Finally, by using a test function in $W^{2,q'}_0(\Omega)$, by de Rham theorem there exists a distribution $p \in W^{-1,q}(\Omega) := (W^{1,q'}_0(\Omega))^*$ such that

$$-\Delta u + \nabla p = f \qquad \text{in } W^{-2,q}(\Omega),$$

and satisfying

$$\|p\|_{W^{-1,q}(\Omega)} \leq C\|f, a^i, b\|_{-2,q},$$

for some $C = C(q, \Omega) > 0$. □

Remark 5. One can also say, in a manner equivalent to Definition 1.1 that the couple $(u, p) \in L^q(\Omega) \times W^{-1,q}(\Omega)$ is a very-weak solution to the Stokes equations (1) with N.-bc (2).

6.1. **On the meaning of the boundary conditions.** A function $u \in L^q(\Omega)$ does not have *a priori* a well-defined traces. For a L^q-very-weak solution the definition of the normal component is clear since both u and $\nabla \cdot u \in L^q(\Omega)$, hence for simplicity we can suppose $-u^3 = b = 0$ on Γ_0. The terms $\partial_3 u^1$ and $\partial_3 u^2$ restricted to the boundary require a precise treatment. In particular, the formula of integration by parts implies that

$$-\sum_{i=1}^{2} << \frac{\partial u^i}{\partial n}, \phi^i >>_\Gamma = \int_\Omega u\, \Delta\phi\, dx + < f, \phi >,$$

for all $\phi \in W^{2,q'}_\sigma(\Omega)$, such that $\partial_3\phi^1 = \partial_3\phi^2 = \phi^3 = 0$ on Γ. The expression defines a linear functional γ_1 on $W^{2-1/q',q'}(\Gamma_0)$ by

$$\gamma_1(h) := \int_\Omega u\, \Delta(Eh)\, dx + < f, Eh >,$$

where Eh is any extension of h such that $Eh \in W^{2,q'}_\sigma(\Omega)$, and $\partial_3[Eh]^1 = \partial_3[Eh]^2 = [Eh]^3 = 0$ on Γ. The definition is independent of the extension Eh, since if Eh and $\widetilde{E}h$ are admissible extensions, then the function $Eh - \widetilde{E}h$ can be used as test function, showing that

$$\int_\Omega u\,\Delta(Eh - \widetilde{E}h)\,dx + < f, Eh - \widetilde{E}h >= 0.$$

Hence, that the operator is γ_1 is well-defined (and that for smooth functions it will be the standard restriction of the normal derivative). Next, one has to show that for any $(h^1, h^2, 0) \in W^{2-1/q',q'}(\Gamma_0)$ one can find a divergence-free extension Eh satisfying all the boundary conditions and vanishing on Γ_1. The vector $\phi \in W^{2,q'}(\Omega)$ (see for instance Simader [26]) solution of the Dirichlet bi-harmonic problem

$$\begin{cases} \Delta^2\phi = 0 & \text{in } \Omega, \\ \phi = (h^1, h^2, 0) & \text{on } \Gamma_0, \\ -\partial_3\phi = (0, 0, \partial_1 h^1 + \partial_2 h^2) & \text{on } \Gamma_0, \\ \phi = \partial_3\phi = 0 & \text{on } \Gamma_1, \end{cases}$$

satisfies $\nabla\cdot\phi \in W^{1,q'}_0(\Omega)$. Next, we solve the divergence equation $\nabla\cdot b = \nabla\cdot\phi$ in $W^{2,q'}_0(\Omega)$ by means of Bogovskiĭ formula. Finally the function $Eh := \phi - b$ is the requested extension with zero divergence.

7. Existence of L^q-very-weak solutions for an Oseen system with N.-bc.

In this section we consider, for $u_1 \in L^3(\Omega)$ the following Oseen problem or also "perturbed (linear) Stokes" equations

$$\begin{cases} -\Delta u + \nabla\cdot(u_1 \otimes u) + \nabla p = f & \text{in } \Omega, \\ \nabla\cdot u = 0 & \text{in } \Omega, \end{cases} \tag{12}$$

with the N.-bc (2). Existence of various classes of solutions for this system will be the core of the perturbation arguments needed to prove existence of very-weak solutions for the full Navier-Stokes system with arbitrary data. In order to prove existence of very-weak solutions for the problem (12)-(2) we need to study its adjoint equation

$$\begin{cases} -\Delta U - u_1\cdot\nabla U - \nabla P = f & \text{in } \Omega, \\ \nabla\cdot U = 0 & \text{in } \Omega, \end{cases} \tag{13}$$

with homogeneous N.-bc. In the sequel we will need the following lemma (see, e.g., Galdi [15, §VIII]).

Lemma 7.1. *Suppose that $u \in L^3(\Omega)$. If $v \in W^{1,q}(\Omega)$ with $1 \le q < 3$ then $u\,v \in L^q(\Omega)$ and*

a) $\quad \exists\, C = C(q,\Omega) > 0: \qquad \|u\,v\|_{L^q(\Omega)} \le C\|u\|_{L^3(\Omega)}\|v\|_{W^{1,q}(\Omega)}.$

In addition, for all $\epsilon > 0$

b) $\quad \exists\, C_\epsilon = C_\epsilon(\epsilon, q, u, \Omega) > 0: \qquad \|u\,v\|_{L^q(\Omega)} \le \epsilon\,\|v\|_{W^{1,q}(\Omega)} + C_\epsilon\|v\|_{L^q(\Omega)}.$

In order to prove existence of very-weak solutions for (12) the first result we need is the existence of strong solutions, for (13) with N.-bc (2).

Proposition 3. *Let be given $u_1 \in L^3(\Omega)$ such that $\nabla \cdot u_1 = 0$ in Ω and $u_1^3 = 0$ on Γ (in the sense of $W^{-1/3,3}(\Gamma)$). Then, for any $\frac{6}{5} \le q < 3$ and $f \in L^q_{(\#)}(\Omega)$ there exists a unique strong solution $(U, P) \in W^{2,q}_{(\#)}(\Omega) \times W^{1,q}_{\#}(\Omega)$ of (13) such that*

$$\|U\|_{W^{2,q}(\Omega)} + \|P\|_{W^{1,q}(\Omega)} \le C\|f\|_{L^q(\Omega)},$$

for some $C = C(q, \Omega, u_1) > 0$.

Proof. In order to apply the previous results we study the following problem

$$\begin{cases} -\Delta\overline{U} - \nabla\overline{P} = f + u_1 \cdot \nabla U & \text{in } \Omega, \\ \nabla \cdot U = 0 & \text{in } \Omega, \end{cases} \tag{14}$$

with homogeneous N.-bc. The basic theory developed in Section 3 implies that for any $U \in W^{2,q}_{(\#)}(\Omega)$ there exists a unique strong solution $(\overline{U}, \overline{P}) \in W^{2,q}_{(\#)}(\Omega) \times W^{1,q}_{\#}(\Omega)$ such that

$$\|\overline{U}\|_{W^{2,q}(\Omega)} + \|\overline{P}\|_{W^{1,q}(\Omega)} \le C\|f + u_1 \cdot \nabla U\|_{L^q(\Omega)}.$$

By using Lemma 7.1-(a), the linear mapping $\mathcal{L}$ from $W^{2,q}(\Omega)$ into itself defined by $\mathcal{L} : U \to \overline{U}$ turns out to be continuous since

$$\|\mathcal{L}U_1 - \mathcal{L}U_2\|_{W^{2,q}(\Omega)} \le C\|u_1\|_{L^3(\Omega)}\|U_1 - U_2\|_{W^{2,q}(\Omega)} \qquad \forall\, U_1,\, U_2 \in W^{2,q}(\Omega).$$

Moreover, $\mathcal{L}$ is compact. In fact, given a bounded sequence $\{U_k\}_{k\in\mathbb{N}} \subseteq W^{2,q}(\Omega)$ one can take a (relabeled) sub-sequence $\{U_k\}_{k\in\mathbb{N}}$ strongly convergent in $W^{1,q}(\Omega)$ to some $U \in W^{2,q}(\Omega)$. Lemma 7.1-(b) implies that for any $\epsilon > 0$ there exists $C_\epsilon = C(\epsilon, u_1, q, \Omega)$, independent of $\{U_k\}_{k\in\mathbb{N}}$, such that

$$\|\mathcal{L}U_k - \mathcal{L}U\|_{W^{2,q}(\Omega)} \le \epsilon\, \|U_k - U\|_{W^{2,q}(\Omega)} + C_\epsilon\|U_k - U\|_{W^{1,q}(\Omega)}.$$

This implies that

$$\limsup_{k\to+\infty} \|\mathcal{L}U_k - \mathcal{L}U\|_{W^{2,q}(\Omega)} \le 2\epsilon \sup_k \|U_k\|_{W^{2,q}(\Omega)}.$$

The arbitrariness of $\epsilon > 0$ shows that $\mathcal{L}U_k \to \mathcal{L}U$ in $W^{2,q}(\Omega)$. To apply the Leray-Schauder theory it is enough to prove that if $U^{(\lambda)} \in W^{2,q}(\Omega)$ is a solution of

$$U^{(\lambda)} = \lambda\, \mathcal{L}U^{(\lambda)}$$

for $0 \le \lambda \le 1$ then the *a priori* estimate holds true

$$\|\mathcal{L}U^{(\lambda)}\|_{W^{2,q}(\Omega)} \le C\|f\|_{L^q(\Omega)}, \tag{15}$$

for some constant independent of λ. If $U^{(\lambda)}$ satisfies

$$\begin{cases} -\Delta U^{(\lambda)} - \nabla P^{(\lambda)} = \lambda f + \lambda u_1 \cdot \nabla U^{(\lambda)} & \text{in } \Omega, \\ \nabla \cdot U^{(\lambda)} = 0 & \text{in } \Omega, \end{cases} \tag{16}$$

with homogeneous N.-bc, then Lemma 7.1-(b) implies that

$$\begin{aligned} \|U^{(\lambda)}\|_{W^{2,q}(\Omega)} &\le C\lambda\big(\|f\|_{L^q(\Omega)} + \|u_1 \cdot \nabla U^{(\lambda)}\|_{L^q(\Omega)}\big) \\ &\le \frac{1}{2}\|U^{(\lambda)}\|_{W^{2,q}(\Omega)} + C\big(\|f\|_{L^q(\Omega)} + \|U^{(\lambda)}\|_{W^{1,q}(\Omega)}\big). \end{aligned}$$

Next, we use the interpolation inequality $\|v\|_{W^{1,q}(\Omega)} \le \delta\|v\|_{W^{2,q}(\Omega)} + C_\delta\|v\|_{W^{1,2}(\Omega)}$, and we obtain an estimate on the $W^{1,2}$-norm of the solution by "testing" the equation $U^{(\lambda)} = \lambda\mathcal{L}U^{(\lambda)}$ with $U^{(\lambda)}$ (the restriction $q \ge \frac{6}{5}$ comes from the embedding

$L^{6/5}(\Omega) \hookrightarrow (W^{1,2}(\Omega))^*$). With some integration by parts (it is at this point that the hypothesis $u_1^3 = 0$ on Γ is used) we get

$$\|U^{(\lambda)}\|^2_{W^{1,2}(\Omega)} \leq C\|f\|_{L^q(\Omega)}\|U^{(\lambda)}\|_{W^{1,2}(\Omega)}.$$

This finally shows (15) with C independent of $\lambda \in [0,1]$.

Leray-Schauder theory implies existence of a solution $U \in W^{2,q}(\Omega)$ of (13) and the previous inequalities with $\lambda = 1$ give the estimate on the solution in terms of the data, which imply also uniqueness of such strong solution. □

Slightly modifying the proof one can replace the assumption $u_1^3 = 0$ on the boundary, with "$\|u_1\|_{L^3}$ small enough." The same arguments imply also existence of weak solutions, and the results on strong solutions can be used directly to prove existence of very-weak solutions for (12).

Theorem 7.2. *Under the same assumptions of Proposition 3, for each $q \in]\frac{3}{2}, +\infty[$ there exists a unique L^q-very-weak solution of (12), such that*

$$\|u\|_{L^q(\Omega)} + \|p\|_{W^{-1,q}(\Omega)} \leq C\|f, a^i, b\|_{-2,q},$$

for some $C = C(q, u_1, \Omega) > 0$.

Proof. The proof is based on the same duality argument and on the estimates for strong solutions of the adjoint problem, and is left to the reader. □

Remark 6. With the same tools one can treat also the Stokes system perturbed by $\nabla\cdot(u_1 \otimes u + u \otimes u_2)$, with u_1, $u_2 \in L^3(\Omega)$. For technical reasons, to prove the same results concerning strong and weak solutions one has also to consider the problem with zero flux and one has to impose that $\nabla \cdot u_1 \in (W^{1,3/2}(\Omega))^*$ and $u_2 \in L^3(\Omega)$ are small enough (see also [20, Lemma 4]). Moreover, if $u \in L^q(\Omega)$, $\frac{3}{2} < q < +\infty$, is a very-weak solution of the Stokes system perturbed by $\nabla \cdot (u_1 \otimes u + u \otimes u_2)$, then smallness of $\|\nabla \cdot u_1\|_{(W^{1,3/2}(\Omega))^*}$ and of $\|u_2\|_{L^3(\Omega)}$ implies uniqueness. This observation will be useful to study uniqueness of very-weak solutions of the Navier-Stokes equations.

8. On L^3-very-weak solutions for Navier-Stokes equations with N.-bc.

Finally, we consider the Navier-Stokes equations. It is worth to add to the references before mentioned also the papers by Serre [24] and Giga [18] in which Navier-Stokes equations with non-regular (Dirichlet) data have been considered. We give now the definition of very-weak solution.

Definition 8.1 (L^q-very-weak solution for the Navier-Stokes system with N.-bc)**.** We say that $u \in L^q(\Omega)$ is an L^q-very-weak solution to the boundary value problem

$$\begin{cases} -\Delta u + \nabla \cdot (u \otimes u) + \nabla p = f & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ -\partial_3 u^1 = a^1 & \text{on } \Gamma_0, \\ -\partial_3 u^2 = a^2 & \text{on } \Gamma_0, \\ u^3 = 0 & \text{on } \Gamma_0, \\ -\partial_3 u^1 = -\partial_3 u^2 = u^3 = 0 & \text{on } \Gamma_1, \end{cases} \tag{17}$$

if the following identities hold true:

$$-\int_\Omega u\,\Delta\phi\,dx = \int_\Omega u \otimes u\,\nabla\phi\,dx + < f, \phi > + << a^i, \phi^i >>_{\Gamma_0},$$

for all $\phi \in W^{1,q'}_\sigma(\Omega) \cap W^{2,q'}(\Omega)$ such that $\partial_3\phi^1 = \partial_3\phi^2 = 0$ and $\phi^3 = 0$ on Γ and if

$$\int_\Omega u\,\nabla\psi\,dx = 0, \qquad \forall\,\psi \in W^{1,q'}(\Omega).$$

The main result of the paper is the following.

Theorem 8.2. *Let be given $f \in (W^{2,3/2})^*$ and $a^i \in W^{-1-1/3,3}(\Gamma_0)$ such that $< f^i, \mathbb{1} > + << a^i, \mathbb{1} >>_{\Gamma_0} = 0$ for $i = 1,2$. Then, there exists a L^3-very-weak solution to the problem (17) such that*

$$\|u\|_{L^3(\Omega)} + \|p\|_{W^{-1,3}(\Omega)} \le C\|f, a^i\|_{-2,3}, \tag{18}$$

for some $C = C(\Omega) > 0$ and the solution is unique if $\|f, a^i\|_{-2,3}$ is small enough.

Remark 7. We observe that we are considering the problem with zero flux. This is motivated by the fact that the term $\nabla\cdot(u\otimes u)$ must satisfy at least

$$\nabla\cdot(u\otimes u) \in (W^{2,3/2}(\Omega))^* \subsetneq (W^{1,3/2}_0(\Omega)\cap W^{2,3/2}(\Omega))^*.$$

In the case of the Dirichlet problem treated in [17, 20] the test functions are vanishing at the boundary and one can freely integrate by parts, showing that the duality $< \nabla\cdot(u\otimes u), \phi >$ is well-defined if $\phi \in W^{1,3/2}_0(\Omega)\cap W^{2,3/2}(\Omega)$. For the problem with Navier boundary conditions the lack of vanishing tangential part of the test-functions seems to require the restriction $u\cdot n_{|\Gamma=\Gamma_0\cup\Gamma_1} = 0$.

The proof of Theorem 8.2 is based on an existence result for small data and on a perturbation argument. We first prove the result for small data, similar to [17].

Lemma 8.3. *Let the same hypotheses as in Theorem 8.2 be satisfied. There exists a constant $\delta^*(\Omega) > 0$ such that if*

$$\|f, a^i\|_{-2,3} \le \delta^*,$$

then there exists a unique L^3-very-weak solution of the problem (17), with

$$\|u\|_{L^3(\Omega)} + \|p\|_{W^{-1,3}(\Omega)} \le C^*\delta,$$

for some $C^* = C^*(\Omega) > 0$.

Proof. To employ the Banach fixed point theorem (cf. [17, 20]) we rewrite $(17)_1$ as follows

$$-\Delta u + \nabla p = f - \nabla\cdot(u\otimes u),$$

and, for a given $v \in L^q_\sigma(\Omega)$ (such that $v^3 = 0$ on Γ), we solve the Stokes problem

$$\begin{cases} -\Delta u + \nabla p = f - \nabla\cdot(v\otimes v) & \text{in } \Omega,\\ \nabla\cdot u = 0 & \text{in } \Omega,\end{cases} \tag{19}$$

with the same N.-bc as in (17). If $v^3 = 0$ on Γ then

$$< -\nabla\cdot(v\otimes v), \phi > = \int_\Omega (v\otimes v)\nabla\phi\,dx. \tag{20}$$

hence, by Hölder inequality and the Sobolev embedding $W^{2,q'}(\Omega) \hookrightarrow W^{1,q/(q-2)}(\Omega)$, (which holds for $q \ge 3$ in three dimensions) we get

$$|< -\nabla\cdot(v\otimes v), \phi >| \le \|v\|^2_{L^q(\Omega)}\|\nabla\phi\|_{L^{\frac{q}{q-2}}(\Omega)} \le c\|v\|^2_{L^q(\Omega)}\|\phi\|_{W^{2,q'}(\Omega)}.$$

This explains the the choice $q = 3$, since this is the smallest such exponent for which the fixed point has chance to work (similar results for larger values of q can be proved.) By solving (19) we define the mapping $u = Tv$ from the set

$$X := \Big\{ v \in L^3_{(\#)}(\Omega),\ \nabla \cdot v = 0 \text{ in } \Omega,\ \text{and } v^3 = 0 \text{ on } \Gamma \Big\}$$

into itself in such a way that $u = Tv$ is the (unique) L^3-very-weak solution of the problem (19) with the same boundary conditions as in (17). Next, given $v_1, v_2 \in X$, the estimates for the very-weak solution of the Stokes system in terms of data imply

$$\begin{aligned}
&\|Tv\|_{L^3(\Omega)} \le C^* \big(\|f, a^i\|_{-2,3} + \|v\|^2_{L^3(\Omega)} \big),\\
&\|Tv_1 - Tv_2\|_{L^3(\Omega)} \le C^* (\|v_1\|_{L^3(\Omega)} + \|v_2\|_{L^3(\Omega)}) \|v_1 - v_2\|_{L^3(\Omega)},
\end{aligned}$$

for some $C^* = C^*(\Omega) > 0$. We deduce that if $\|f, a^i\|_{-2,3} \le (2\sqrt{2}C^*)^{-2}$ and if $v, v_1, v_2 \in X$ have norm bounded by $(4C^*)^{-1}$, then

$$\begin{aligned}
&\|Tv\|_{L^3(\Omega)} \le 2C^* \|f, a^i\|_{-2,3} = (4C^*)^{-1},\\
&\|Tv_1 - Tv_2\|_{L^3(\Omega)} \le C^* \left(\frac{2}{4C^*} \right) \|v_1 - v_2\|_{L^3(\Omega)} = \frac{1}{2} \|v_1 - v_2\|_{L^3(\Omega)},
\end{aligned}$$

hence T is a contraction on the closed ball $\overline{B(0, (4C^*)^{-1})} \subset X$. This proves the existence of a unique fixed point, which is the solution to (17). This proves also the *a priori* estimate, provided that $\delta^* := (2\sqrt{2}C^*)^{-2}$, where C^* is the constant in Theorem 6.1 for $q = 3$. □

We can now prove the main result of the paper.

Proof of theorem 8.2. Similarly to Marušić-Paloka [21] and Kim [20] we show how to remove smallness of data, at least for existence. Let $\delta^* > 0$ be the constant in Lemma 8.3. By using Corollary 1-2 there exist f_δ and a^i_δ such that

$$f - f_\delta \in (W^{1,2}(\Omega))^* \quad a^i - a^i_\delta \in W^{-1/2,2}(\Gamma_0), \quad \text{and} \quad \|f_\delta, a^i_\delta\|_{-2,3} = \delta \le \delta^*.$$

We consider the system with "small data"

$$\begin{cases}
-\Delta u_\delta + \nabla \cdot (u_\delta \otimes u_\delta) + \nabla p_\delta = f_\delta & \text{in } \Omega,\\
\nabla \cdot u_\delta = 0 & \text{in } \Omega,\\
-\partial_3 u^1_\delta = a^1_\delta & \text{on } \Gamma_0,\\
-\partial_3 u^2_\delta = a^2_\delta & \text{on } \Gamma_0,\\
-u^3_\delta = 0 & \text{on } \Gamma_0,\\
\partial_3 u^1_\delta = \partial_3 u^2_\delta = u^3_\delta = 0 & \text{on } \Gamma_1,
\end{cases} \tag{NS$_\delta$}$$

and by Lemma 8.3, this system has a unique L^3-very-weak solution (u_δ, p_δ) with

$$\|u_\delta\|_{L^3(\Omega)} + \|p_\delta\|_{W^{-1,3}(\Omega)} \le C^* \delta.$$

Then, $(u, p) = (u_\delta + v_\delta, p_\delta + \pi_\delta)$ is a L^3-very-weak solution of (17) if and only if v_δ is a very-weak solution of the following system

$$\begin{cases} -\Delta v_\delta + \nabla \cdot (v_\delta \otimes v_\delta + u_\delta \otimes v_\delta + v_\delta \otimes u_\delta) + \nabla \pi_\delta = f - f_\delta & \text{in } \Omega, \\ \nabla \cdot v_\delta = 0 & \text{in } \Omega, \\ -\partial_3 v_\delta^1 = a^1 - a_\delta^1 & \text{on } \Gamma_0, \\ -\partial_3 v_\delta^2 = a^2 - a_\delta^2 & \text{on } \Gamma_0, \\ v_\delta^3 = 0 & \text{on } \Gamma_0, \\ \partial_3 v_\delta^1 = \partial_3 v_\delta^2 = v_\delta^3 = 0 & \text{on } \Gamma_1. \end{cases} \tag{21}$$

Since data are not small -but now more regular- we show (standard variational L^2-theory, together with a fixed point argument in $W^{1,2}_{(\#),\sigma}(\Omega)$) existence of a $W^{1,2}$-weak-solution (hence of a L^q-very-weak solution for $1 < q \leq 6$) for the nonlinear system (21). We show just the *a priori* estimate for (21), since the rest will follow by using the standard Galerkin approximation method, together with the results in [5]. By using v_δ as test function in (21) we obtain

$$\|\nabla v_\delta\|^2_{L^2(\Omega} \leq \|f - f_\delta\|_{(W^{1,2}(\Omega))^*} \|\nabla v_\delta\|_{L^2(\Omega)} + | << a^1 - a^1_\delta, v^1_\delta >>_{\Gamma_0} | \\ + | << a^2 - a^2_\delta, v^2_\delta >>_{\Gamma_0} | + \left| \int_\Omega (v_\delta \cdot \nabla) v_\delta \, u_\delta \, dx \right|.$$

In order to absorb the last term from the right-hand side into the left-hand side, we use the following estimate

$$\left| \int_\Omega (v_\delta \cdot \nabla) v_\delta \, u_\delta \, dx \right| \leq \|v_\delta\|_{L^6(\Omega)} \|\nabla v_\delta\|_{L^2(\Omega)} \|u_\delta\|_{L^3(\Omega)} \leq c\, C^* \,\delta \|\nabla v_\delta\|^2_{L^2(\Omega)},$$

with a constant depending only on the Sobolev inequality $\|v_\delta\|_{L^6} \leq c\|\nabla v_\delta\|_{L^2}$ (which holds true in our setting). Hence, if δ is small enough (such that $c\, C^* \,\delta = 1/2$) we can absorb this term on the left-hand side to obtain the *a priori* estimate

$$\|\nabla v_\delta\|_{L^2(\Omega)} \leq C \|f - f_\delta, a^i - a^i_\delta\|_{-1,2}.$$

Uniqueness of very-weak solutions for small data can be proved by using the same tools employed in the analysis of Section 7. In fact, let us suppose that there exists u_1, $u_2 \in L^3(\Omega)$ very-weak solutions corresponding to the same small data. It follows that $\|u_1\|_{L^3(\Omega)}, \|u_2\|_{L^3(\Omega)} < C^*\delta$, if the data are such that $\|f, a^i\|_{-3,2} \leq \delta$. By considering the equation for the difference $\widetilde{u} = u_1 - u_2$

$$-\Delta \widetilde{u} + \nabla \cdot (u_1 \otimes \widetilde{u} + \widetilde{u} \otimes u_2) + \nabla \widetilde{p} = 0$$

(with vanishing divergence and homogeneous data) by using the same approach used in the proof the theorems of Section 7 one can easily show (usual duality arguments) that if δ is small enough, then $\widetilde{u} = 0$. □

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Received February 2009; revised June 2009.

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