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Elisa Miragliotta[1] and Anna Baccaglini-Frank[2]

[1]University of Ferrara, Department of Mathematics and Computer Science, Ferrara, Italy; elisa.miragliotta@student.unife.it

[2]University of Pisa, Department of Mathematics, Pisa, Italy; anna.baccaglinifrank@unipi.it

We propose a theoretical interpretation of visuo-spatial abilities, as classified in the field of Cognitive Psychology, in the domain of Euclidean Geometry. In this interpretation we make use of Fischbein's theory of figural concepts and of Duval's cognitive apprehensions. Our interpretation lays the foundations for a new theoretical framework that we propose as a tool for qualitative analysis of students' processes of visualization as they carry out geometrical activities. In particular, we present analyses of excerpts from a set of activities designed and proposed in a didactical intervention aimed at strengthening visuo-spatial abilities of a group of students identified as the weakest from a selected 9th grade class of an Italian high school.

Keywords: Geometric reasoning, spatial thinking, visualization, visuo-spatial abilities.

Introduction

Research in the domain of *visualization* and *spatial thinking* has pursued several purposes: understanding the different imaginative strategies used by students (Owens, 1999); studying the effects of teaching practices, aimed at encouraging processes of visualization (Presmeg, 2006); developing theoretical constructs, useful for the interpretation of students' perception of geometric shapes and how this perception improves in learning geometry (Duval, 1995; Fischbein, 1993, Mariotti, 2005). Some ideas in this field have been developed from the psychological studies on *mental imagery*. Since the advent of Cognitive Psychology and contemporary Neuroscience, researchers have been elaborating models to describe processes related to visualizing and using mental imagery, and they have listed sets of *visuo-spatial abilities* involved (e.g., Cornoldi & Vecchi, 2004). However, a shared definition of these abilities does not exist yet. Nor have the fields of Mathematics Education and Cognitive Psychology been able to elaborate common grounds to study visualization processes, in which they are both interested.

In this paper we propose a theoretical interpretation of visuo-spatial abilities, as classified in the field of Cognitive Psychology, in the domain of reasoning in Euclidean Geometry, that was developed as part of a study that has been recently carried out (Miragliotta, 2016; Miragliotta, Baccaglini-Frank & Tomasi, submitted). The study had two main objectives: on the one hand, we attempted to give a theoretical analysis of some visuo-spatial abilities in the context of learning Euclidean Geometry; on the other hand, we used such theoretical interpretation to study the effects of a set of activities proposed (for the most part) using a Dynamic Geometry Environment (DGE) in terms of strengthening the students' visuo-spatial abilities (as it is widely accepted that DGEs yield great potential in fostering processes such as visualization, as well as in mediating, in general, the learning of geometry: e.g., Mariotti, 2005; Baccaglini-Frank, 2010; Leung, Baccaglini-Frank & Mariotti, 2013). In this paper we will concentrate on the description of theoretical analysis of the visuo-spatial abilities considered and on its power as a tool for qualitative analysis of students' behavior as they carry out geometrical activities. As an example of how the framework can be used, we will analyze an excerpt taken from a question (not involving the use of any DGE) of the post-intervention interview of a student from the group of students identified as the weakest of the Italian high school class of involved in the study. Since for what we present in this paper the role of the DGE is marginal, and space quite constrained, we will not discuss visualization in a DGE.

Theoretical background

According to Clements and Battista (1992), *spatial reasoning* "consists of the set of cognitive processes by which mental representation for spatial objects, relationships, and transformations are constructed and manipulated" (ibid., p.420). Referring to Kosslyn (1983), these authors observe that geometrical reasoning requires spatial reasoning, which includes four classes of image processing: generating an image; inspecting an image to answer questions about it; transforming and operating on an image; maintaining an image in the service of some other mental operation. In particular we are interested in processes involving two-dimensional geometric objects.

From the perspective of Cognitive Psychology, generating and processing mental images take place within a complex process of acquisition and use of cognitive abilities, including those denoted *visuo-spatial abilities*. A list of these appears in Cornoldi and Vecchi (2004, p. 16). We elaborated our theoretical interpretation starting from the following set of abilities: *visual organization*, the ability to organize incomplete, not perfectly visible or fragmented patterns; *planned visual scanning*, the ability to scan a visual configuration rapidly and efficiently to reach a particular goal; s*patial orientation*, the ability to perceive and recall a particular spatial orientation or be able to orient oneself generally in space; v*isual reconstructive ability*, the ability to reconstruct a pattern (by drawing or using elements provided) on the basis of a given model; *imagery generation ability*, the ability to generate vivid visuo-spatial mental images quickly; *imagery manipulation ability*, the ability to manipulate a visuospatial mental image in order to transform or evaluate it; *spatial sequential short-term memory*, the ability to remember a sequence of different locations; *visuo-spatial simultaneous short-term memory*, the ability to remember different locations presented simultaneously; *visual memory*, the ability to remember visual information; *long-term spatial memory*, the ability to maintain spatial information over long periods of time.

To interpret how these general cognitive abilities might come into play during reasoning in the specific context of Euclidean Geometry, we referred to theoretical constructs elaborated in mathematics education to this purpose.

Fischbein's theory of figural concepts

The *Theory of figural concepts* (Fischbein, 1993) describes geometrical figures as follows:

> A geometrical figure may, then, be described as having intrinsically conceptual properties. Nevertheless, a geometrical figure is not a mere concept. It is an image, a visual image. It possesses a property which usual concepts do not possess, namely, it includes the mental representation of space property. […] all the geometrical figures represent mental constructs which possess, simultaneously, conceptual and figural properties. (ibid., pp. 141-142).

According to Fischbein *figural concepts* "reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities - like ideality, abstractness, generality, perfection" (ibid., p. 143); a geometrical figure is made up of two fundamental components: the figural component and the conceptual component. From the developmental point of view, initially the visual aspect is dominant, and gradually the role of formal constraints becomes more important, until the construction of *figural concept* is reached (Mariotti, 2005).

Duval's types of cognitive apprehension

Today the importance of visualization in mathematics is widely recognized. Since several studies have addressed visualization in different ways, we clarify that our interpretation is in line with the definition given by Arcavi (2003).

> *Visualization* is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (ibid., p. 217, emphasis added)

Peculiarities of visualization in geometry have been highlighted by Duval (1995) in describing different approaches to dealing with geometric figures: *cognitive apprehension* stresses that "there are several ways of looking at a drawing or a visual stimulus array" (ibid., p. 143). Duval speaks of four cognitive apprehensions. *Perceptual apprehension* responds to the laws of figural organization and identification of form, and helps to "recognize something (shape, representation of a thing,…) in a plane or in depth" (ibid., p. 145) at first glance. In a perceived figure we can also recognize sub-figures that do not depend on its construction. *Sequential apprehension* "is required whenever one must construct a figure or describe its construction" (ibid., p. 146). Here the sub-figures emerge in a specific order, depending on the geometrical construction, on technical constraints of the instrument used and on mathematical properties. Furthermore, Duval (1995, p. 146) claims that "mathematical properties represented in a drawing cannot be determined through perceptual apprehension", indeed, "a drawing without denomination or hypothesis is an ambiguous representation". So, indications given through speech help us to identify properties of a perceived geometrical figure, thanks to the *discursive apprehension*. Here we are in the domain of deductive reasoning. The apprehension that has a heuristic function in problem solving is the *operative apprehension*. This apprehension depends on different ways of modifying a figure that happen only within the figural register and that are independent from mathematical knowledge.

Each type of apprehension seems to be related to different cognitive processes that could be accomplished through the coordination of different visuo-spatial abilities as we hypothesise below.

Grounding for a new visuo-spatial abilities framework

While maintaining the classification proposed by Cognitive Psychology, we selected a subset of visuo-spatial abilities and provided an interpretation in the specific context of geometrical reasoning. We used the *Theory of Figural Concepts* to interpret the terms "model" and "image" as follows: *image* is the figural component of a geometric figure; *model* is a synonym of figural concept in which image and concept realize their dialectic. Since our interpretation aims at being a stronger lens for analyzing students' processes than the visuo-spatial abilities as described in the Cognitive Psychology literature, there is not always a one-to-one correspondence with such abilities.

- *Visual organization* is the ability to recognize figural concepts from incomplete or not perfectly visible representations.

Visual organization seems to be an ability that intervenes in tasks that require the recognition of figures within another figure, or in the recognition of a simple figure within a more complex figure. This ability echoes Duval's *perceptual apprehension*.

- *Visual scanning* is the ability to recognize the properties of a figure starting from its representation.

This representation can be static or dynamic. It depends on the task and on the context in which it is proposed. For example, in the case of a dynamic figure in a DGE, *visual scanning* is involved in the recognition of properties that are invariant under dragging (see Leung et al., 2013). This ability echoes Duval's *perceptual apprehension*, but we also recognize in this ability some aspects of his *sequential* and *discursive apprehension*. For example, when observing a quadrilateral obtained through steps of a specific construction starting from two perpendicular lines, we can notice that quadrilateral seems to have a right angle. However, to recognize the property "having a right angle" only observing the figure on the screen, one needs to look at its written geometrical construction and deduce that the point one vertex is at the intersection of two perpendicular lines.

- *Visual reconstructive ability* is the ability to reconstruct, in a given representation, the figural component of a figural concept, starting from written or verbal instructions, or staring from partial representations.

For instance, the reconstruction could be realized following a sequence of construction steps given explicitly, using appropriate tools (ruler and compass, primitives in DGE, …), otherwise it could be realized planning these construction steps. It involves the ability to correctly visualize the relationships between the elementary figural units involved (such as points on lines, perpendicular lines) following the steps of a geometric construction or creating a new construction. This ability echoes Duval's *sequential apprehension* and his *discursive apprehension*. The *visual reconstructive ability* seems to intervene, for example, when carrying out the construction steps of a known geometric figure; when completing the steps of an incomplete construction; when following the steps of a given geometric construction.

- *Imagery generation ability* is the ability to instantly mentally reproduce the figural component of a figural concept recovering it from memory or generating it anew.

This ability seems to intervene when one is asked to visualize a geometric concept, for example, while imagining a a sequence of construction steps. Coupled with *long-term spatial memory*, this ability seems to be involved in the retrieval of the prototypes (that is, in Kosslyn's terms, a "stored model of a shape") of geometric shapes and of their properties. Coupled with *spatial sequential short-term memory*, it seems to intervene in the identification of particular geometric loci.

- *Imagery manipulation ability* is the ability to use the properties of a figural concept or to manipulate figural aspects of a figural concept, taking into account the theoretical relationships between elementary figural units of which it is composed.

This ability is involved in tasks that require mental manipulation of a figure in order to transform it into a new one. This ability echoes Duval's *operative apprehension*, but also differs from it. The mental manipulations on figure are tightly connected to the figure's conceptual component. Indeed, to manipulate a figure maintaining given properties, strong conceptual control over it is required, as highlighted also by Arcavi (2003), who emphasizes, as well, the high cognitive demand involved:

> Seeing the unseen may refer to the development and use of an intervening conceptual structure which enables us to see through the same visual display. (ibid, p. 234)
>
> When visualization acts upon conceptually rich images (or in Fischbein's words when there are intervening conceptual structures), the cognitive demand is certainly high. (ibid, p.235).

- *Spatial sequential short-term memory*: this ability seems to be present in various processes of geometric reasoning; here we consider it, in particular, as the ability to remember different configurations assumed by the figural component of a figural concept during an observed or imagined manipulation.
- *Long-term spatial memory* in our interpretation this refers, in particular, to the ability to maintain in long-term memory the figural components of a figural concept.

The last two abilities are involved in solving geometric tasks and are always used in combination with other visuo-spatial abilities. For example, combined with the *imagery generation ability*, *spatial sequential short-term memory* seems to be involved in tasks that require recognizing a particular geometrical locus. Combined with the *imagery manipulation ability*, *spatial sequential short-term memory* seems to be involved in tasks that require remembering the configurations assumed by a figure during an imagined manipulation.

When a solver faces a geometrical problem, s/he interacts with visual or mental images in different ways; a process that seems to occur frequently is imagining the consequence of a (mental) manipulation of the figure. Such process can be carried out through the use of the various abilities listed above that expert solvers combine in an immediate and automatic way. So consider this as an ability in its own right, that we will call *geometric prediction*, intending the identification of certain properties or configurations of a new figure, arising from a process of manipulation. This process appears to be coherent with respect to the notion of *anticipatory image* (Piaget & Inhelder, 1966), which suggests an ability to make predictions, orienting both perception and imagination, in the presence of a specific goal.

Visuo-spatial abilities framework as a tool of analysis

In this section we use the framework to analyze an excerpt taken from a question of the post-intervention interview of a student; he was part of a group of students in the 9th grade (students aged 14-15) class of an Italian scientific high school (Applied Science option), identified through a pre-test as having low performance on geometry tasks heavily involving visualization processes; the teaching intervention lasted five lessons and had been carried out using open problems, mostly proposed in the DGE *GeoGebra*. The post-intervention interview involved tasks both in the context of a DGE and with only pen and paper (if requested by the student). In the excerpt the student is solving a task proposed outside the DGE setting. The analysis has the aim of showing the power of the framework in identifying the proposed visuo-spatial abilities and showing how they can come into play, shedding light onto visualization processes.

Activity: the student is given the following task and allowed to use paper and pencil:

> Imagine a quadrilateral. Focus on the midpoint of each side. Trace the segments that join the midpoints of consecutive sides. What can you tell me about the figure that is formed?

Below is an excerpt describing what the student says [and does].

Student: It is a quadrilateral, which… which looks like a rhombus, so to speak. [Initially he closes his eyes. Then he places four finger tips (two thumbs and two indexes) on the desk to form what looks like a square, and then, moving along two parallel lines in opposite directions, a non-square rectangle. He drags his fingers back and forth between these two positions.] If quadrilateral is a square it forms a rhombus with congruent diagonals, but if is a random figure…I mean, it depends on the figure. It changes depending on how the points are placed.

Interviewer: Draw it. What are you drawing?

Student: Four scattered points. [He draws (freehand) a quadrilateral with different sides, as shown in Figure 1]

Interviewer: Can you say more about the figure that is formed?

Student: It is a quadrilateral. Mmm…it is a parallelogram!

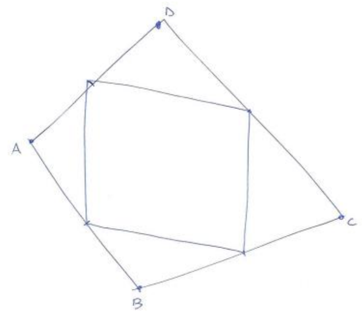

Figure 1 The student's drawing

In addition to what the student says, the excerpt is interesting also for what he does, which gives further insight into visuo-spatial abilities he may be using. After the first answer, he keeps his eyes closed and moves his fingers on the desk. This seems to suggest that a purely mental process is taking place, and the gesture on the desk seems to be a windows onto this process. In order to answer the question, first of all, we would say the student is using the *imagery generation ability* for imagining the first configuration. To this end he needs to recall a prototype of the quadrilateral that is as general as possible (this involves the *imagery generation ability* and *long-term spatial memory*); then he needs to visualize the required elementary figural units (*imagery generation ability*) and go through the steps of the construction (*imagery reconstructive ability*). Now, the student's use of his fingers on the desk is an extremely insightful window onto processes he could be enacting. Our interpretation is that he is using the *imagery manipulation ability*, helping himself with an external image (the quadrilateral with vertexes at his four finger tips) that he can act upon. What is visible of this manipulation are the positions (and their continuous change) of the vertexes. As he moves his fingers (forming what look like various rectangles) he is using *geometric prediction*, possibly aided by *visual scanning*, to visualize the quadrilateral with vertexes at the midpoints of the sides of the manipulated quadrilateral. This interpretation is supported by the fact that the student moves his fingers on his desk seamlessly, he never lifts them up from the surface, and then he selects a position which is coherent with respect to the configuration that he wants to (mentally) observe, and starts to move fingers again. The student seems to be able to manipulate the figure in a manner that goes beyond the kind of transformation described by *operative* *apprehension*. Indeed, the manipulation recalls much more dragging of the vertices, as can be accomplished in a DGE. This cognitive effect could have been promoted by the kind of problems proposed within the DGE during the activity sessions. The student seems to be looking for extra external support for his *imagery manipulation* and *geometric prediction* abilities.

Moreover, this excerpt is very interesting because of what the student then decides to draw on the sheet of paper when invited to so do. Although he has only mentioned the case in which the quadrilateral is a square and realized with his fingers various cases of it being a rectangle, he draws a much more general convex quadrilateral. This behavior supports our previous hypothesis that the student seems to need external support for his *imagery manipulation* and *geometric prediction* abilities. On paper it is as if he gains confidence, possibly because the cognitive load from the conceptual control he would need to exercise over the general figure is lowered this way. Once he sees the general quadrilateral and sketches the midpoint quadrilateral he recognizes (*visual scanning* and *conceptual control*) a parallelogram.

Conclusion

The fields of Mathematics Education and Cognitive Psychology share various research interests; one of these is the identification and classification of strategies and processes involved in visualization. According to Cognitive Psychology, generating and processing mental images take place within a complex process of acquisition and use of abilities, including those denoted *visuo-spatial abilities*. Attempting to interpret visuo-spatial abilities in the context of geometrical reasoning could be beneficial to both fields. In our attempt to give a theoretical interpretation of some visuo-spatial abilities in the context of learning Euclidean Geometry, we used theoretical constructs from the field of Mathematics Education, which led to the introduction of an ability different from the basic visuo-spatial ones, *geometric prediction*, and they also led to highlighting the fundamental contribution, in solving geometric tasks, of *geometric conceptual control* over figures.

This interpretation, which can be seen as groundwork for a new theoretical framework, has allowed us to: (1) design an educational intervention aimed at strengthening visuo-spatial abilities of a group of students identified as the weakest in a selected class; (2) gain insight, through qualitative analysis, into students' geometrical reasoning. We believe that this kind of research can provide new insight into students' difficulties in learning Geometry, and be used to design educational material for strengthening students' visuo-spatial abilities.

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