SEMICLASSICAL STATES FOR A STATIC SUPERCRITICAL KLEIN-GORDON-MAXWELL-PROCA SYSTEM ON A CLOSED RIEMANNIAN MANIFOLD

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ABSTRACT. We establish the existence of semiclassical states for a nonlinear Klein-Gordon-Maxwell-Proca system in static form, with Proca mass 1, on a closed Riemannian manifold.

Our results include manifolds of arbitrary dimension and allow supercritical nonlinearities. In particular, we exhibit a large class of 3-dimensional manifolds on which the system has semiclassical solutions for every exponent $p \in (2, \infty)$. The solutions we obtain concentrate at closed submanifolds of positive dimension as the singular perturbation parameter goes to cero.

1. INTRODUCTION

Let $(\mathfrak{M}, \mathfrak{g})$ be a closed (i.e. compact and without boundary) smooth Riemannian manifold of dimension $m \geq 2$. Given real numbers $\varepsilon > 0$, $q > 0$, $\omega \in \mathbb{R}$ and $p \in (2, \infty)$, and a real-valued $\mathcal{C}^1$-function α such that $\alpha(x) > \omega^2$ on $\mathfrak{M}$, we consider the system

$$\begin{cases} -\varepsilon^2 \Delta_{\mathfrak{g}} \mathfrak{u} + \alpha(x)\mathfrak{u} = \mathfrak{u}^{p-1} + \omega^2(q\mathfrak{v} - 1)^2 \mathfrak{u} & \text{on } \mathfrak{M}, \\ -\Delta_{\mathfrak{g}} \mathfrak{v} + (1 + q^2\mathfrak{u}^2)\mathfrak{v} = q\mathfrak{u}^2 & \text{on } \mathfrak{M}, \\ \mathfrak{u}, \mathfrak{v} \in H^1_{\mathfrak{g}}(\mathfrak{M}), \quad \mathfrak{u}, \mathfrak{v} > 0. \end{cases} \tag{1.1}$$

The space $H^1_{\mathfrak{g}}(\mathfrak{M})$ is the completion of $\mathcal{C}^\infty(\mathfrak{M})$ with respect to the norm defined by $\|v\|^2_{\mathfrak{g}} := \int_{\mathfrak{M}} (|\nabla_{\mathfrak{g}} v|^2 + v^2) d\mu_{\mathfrak{g}}$.

Solutions to this system correspond to standing waves of a Klein-Gordon-Maxwell-Proca (KGMP) system in static form (i.e. one in which the external Proca field is time-independent) with Proca mass 1.

KGMP-systems are massive versions of the more classical electrostatic Klein-Gordon-Maxwell (KGM) systems: KGM-systems are KGMP-systems with Proca mass 0, i.e. the second equation in (1.1) is replaced by

$$-\Delta_{\mathfrak{g}} \mathfrak{v} + q^2 \mathfrak{u}^2 \mathfrak{v} = q\mathfrak{u}^2.$$

Note that $\mathfrak{v} = 1/q$ solves this last equation and reduces the KGM-system to a single Schrödinger equation in $\mathfrak{u}$. So for the system on a closed manifold the Proca

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formalism is more interesting and more appropriate. We refer to [11] for a detailed discussion on KGMP-systems and their physical meaning.

For $\varepsilon = 1$ existence of solutions to system (1.1), which are stable with respect to the phase ω, was established by Druet and Hebey [7] and Hebey and Truong [10] for manifolds of dimension $m = 3$ and 4, and subcritical ($2 < p < \frac{2m}{m-2}$) or critical ($p = \frac{2m}{m-2}$) nonlinearities, under certain assumptions. For critical systems in dimension 3 Hebey and Wei [11] showed the existence of standing waves with multispike amplitudes, which are unstable with respect to the phase, i.e. they blow up with k singularities as the phase ω aproaches some phase ω_0.

Here we are interested in semiclassical states, i.e. in solutions to system (1.1) for ε small. The existence of semiclassical states for similar systems in flat domains Ω in $\mathbb{R}^m$ has been investigated e.g. in [4, 5, 15]. On closed 3-dimensional manifolds, the existence of semiclassical states to system (1.1), which concentrate at a single point as $\varepsilon \to 0$, was established in [8] and [9] for subcritical exponents $p \in (2, 6)$.

The results we present in this paper apply to manifolds of arbitrary dimension and include supercritical nonlinearities $p > 2^*_m$, where $2^*_m := \frac{2m}{m-2}$ is the critical Sobolev exponent in dimension $m \geq 3$ and $2^*_2 := \infty$. In particular, we shall exhibit a large class of 3-dimensional manifolds on which the system (1.1) has semiclassical solutions for every exponent $p \in (2, \infty)$. The solutions $\mathfrak{u}$ we obtain concentrate at closed submanifolds of $\mathfrak{M}$ of positive dimension. Moreover, for fixed ε, they are stable with respect to the phase in the sense of [7].

Our approach consists in reducing system (1.1) to a system of a similar type on a manifold M of lower dimension but with the same exponent p. This way, if $n := \dim M < \dim \mathfrak{M} =: m$ and $p \in [2^*_m, 2^*_n)$, then p is subcritical for the new system but it is critical or supercritical for the original one. Moreover, solutions of the new system which concentrate at a point in M as $\varepsilon \to 0$ will give rise to solutions of the original system concentrating at a closed submanifold of $\mathfrak{M}$ of dimension $m - n$ as $\varepsilon \to 0$.

This approach was introduced by Ruf and Srikanth in [13], where a Hopf map is used to obtain the reduction. Reductions may also be performed by means of other maps which preserve the Laplace-Beltrami operator, or by considering warped products, or by a combination of both, see [3, 14] and the references therein. We describe these reductions in the following two subsections.

1.1. **Warped products.** If (M, g) and (N, h) are closed smooth Riemannian manifolds of dimensions n and k respectively, and $f : M \to (0, \infty)$ is a $\mathcal{C}^1$-map, the *warped product* $M \times_{f^2} N$ is the cartesian product $M \times N$ equipped with the Riemannian metric $\mathfrak{g} := g + f^2 h$.

For example, if M is a closed Riemannian submanifold of $\mathbb{R}^\ell \times (0, \infty)$, then

$$\mathfrak{M} := \{(y, z) \in \mathbb{R}^\ell \times \mathbb{R}^{k+1} : (y, |z|) \in M\},$$

with the induced euclidian metric, is isometric to the warped product $M \times_{f^2} \mathbb{S}^k$, where $\mathbb{S}^k$ is the standard k-sphere and $f(x_1, \ldots, x_{\ell+1}) = x_{\ell+1}$.

Let $\pi_M : M \times_{f^2} N \to M$ be the projection. A straightforward computation gives the following result, cf. [6].

Proposition 1.1. *Let $\beta : M \to \mathbb{R}$ and $\alpha = \beta \circ \pi_M$. Then $u_\varepsilon, v_\varepsilon : M \to \mathbb{R}$ solve*

(1.2)
$$\begin{cases} -\varepsilon^2 \mathrm{div}_g \left(f^k(x) \nabla_g u\right) + f^k(x)\beta(x)u = f^k(x)u^{p-1} + \omega^2 f^k(x)(qv-1)^2 u & \text{on } M, \\ -\mathrm{div}_g \left(f^k(x) \nabla_g v\right) + f^k(x)\left(1 + qu^2\right) v = q f^k(x) u^2 & \text{on } M, \end{cases}$$

iff $\mathfrak{u}_\varepsilon := u_\varepsilon \circ \pi_M$, $\mathfrak{v}_\varepsilon := v_\varepsilon \circ \pi_M : M \times_{f^2} N \to \mathbb{R}$ *solve*

$$\begin{cases} -\varepsilon^2 \Delta_{\mathfrak{g}} \mathfrak{u} + \alpha(x)\mathfrak{u} = \mathfrak{u}^{p-1} + \omega^2 (q\mathfrak{v} - 1)^2 \mathfrak{u} & \text{on } M \times_{f^2} N, \\ -\Delta_{\mathfrak{g}} \mathfrak{v} + \left(1 + q\mathfrak{u}^2\right) \mathfrak{v} = q\mathfrak{u}^2 & \text{on } M \times_{f^2} N. \end{cases} \tag{1.3}$$

Note that the exponent p is the same for both systems. So if $p \in (2^*_{n+k}, 2^*_n)$ then p is subcritical for (1.2) but supercritical for (1.3). Moreover, if the functions u_ε concentrate at a point $\xi_0 \in M$ as $\varepsilon \to 0$, then the functions $\mathfrak{u}_\varepsilon := u_\varepsilon \circ \pi_M$ concentrate at the submanifold $\pi_M^{-1}(\xi_0) \cong (N, f^2(\xi_0)h)$ as $\varepsilon \to 0$.

1.2. **Harmonic morphisms.** Let $(\mathfrak{M}, \mathfrak{g})$ and (M, g) be closed Riemannian manifolds of dimensions m and n respectively. A *harmonic morphism* is a horizontally conformal submersion $\pi : \mathfrak{M} \to M$ with dilation $\lambda : \mathfrak{M} \to [0, \infty)$ which satisfies

$$(n-2)\mathcal{H}(\nabla_{\mathfrak{g}} \ln \lambda) + (m-n)\kappa^{\mathcal{V}} = 0, \tag{1.4}$$

where $\kappa^{\mathcal{V}}$ is the mean curvature of the fibers of π and $\mathcal{H}$ is the projection of the tangent space of $\mathfrak{M}$ onto the space orthogonal to the fibers, see [1].

So for $n = 2$ a harmonic morphism is just a horizontally conformal submersion $\pi : \mathfrak{M} \to M$ with minimal fibers. Typical examples are the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ whose fiber is $\mathbb{S}^1$, and the induced fibration $\mathbb{R}P^3 \to \mathbb{S}^2$ with fiber $\mathbb{R}P^1$, see [1, Example 2.4.15]. They are, in fact, Riemannian submersions (i.e. $\lambda \equiv 1$).

Harmonic morphisms preserve the Laplace-Beltrami operator, i.e.

$$\Delta_{\mathfrak{g}}(u \circ \pi) = \lambda^2 \left[(\Delta_g u) \circ \pi\right]$$

for every $\mathcal{C}^2$-function $u : M \to \mathbb{R}$. This fact yields the following result.

Proposition 1.2. *Assume there exist* $\beta : M \to \mathbb{R}$ *and* $\mu : M \to (0, \infty)$ *such that* $\beta \circ \pi = \alpha$ *and* $\mu \circ \pi = \lambda^2$. *Then* $u_\varepsilon, v_\varepsilon : M \to \mathbb{R}$ *solve the system*

$$\begin{cases} -\varepsilon^2 \Delta_g u + \frac{\beta(x)}{\mu(x)} u = \frac{1}{\mu(x)} u^{p-1} + \frac{\omega^2}{\mu(x)} (qv - 1)^2 u & \text{on } M, \\ -\Delta_g v + \frac{1}{\mu(x)} \left(1 + qu^2\right) v = \frac{q}{\mu(x)} u^2 & \text{on } M, \end{cases} \tag{1.5}$$

iff $\mathfrak{u}_\varepsilon := u_\varepsilon \circ \pi_M$, $\mathfrak{v}_\varepsilon := v_\varepsilon \circ \pi_M : \mathfrak{M} \to \mathbb{R}$ *solve the system*

$$\begin{cases} -\varepsilon^2 \Delta_{\mathfrak{g}} \mathfrak{u} + \alpha(x)\mathfrak{u} = \mathfrak{u}^{p-1} + \omega^2 (q\mathfrak{v} - 1)^2 \mathfrak{u} & \text{on } \mathfrak{M}, \\ -\Delta_{\mathfrak{g}} \mathfrak{v} + \left(1 + q\mathfrak{u}^2\right) \mathfrak{v} = q\mathfrak{u}^2 & \text{on } \mathfrak{M}. \end{cases} \tag{1.6}$$

Again, if $p \in (2^*_m, 2^*_n)$, the system (1.5) is subcritical and the system (1.6) is supercritical and, if the functions u_ε concentrate at a point $\xi_0 \in M$ as $\varepsilon \to 0$, the functions $\mathfrak{u}_\varepsilon := u_\varepsilon \circ \pi_M$ concentrate at the $(m-n)$-dimensional submanifold $\pi_M^{-1}(\xi_0)$ of $\mathfrak{M}$ as $\varepsilon \to 0$.

1.3. **The main result for the general system.** Propositions 1.1 and 1.2 suggest studying a more general KGMP-system.

Let (M, g) be a closed Riemannian manifold of dimension $n = 2$ or 3, $a, b, c \in \mathcal{C}^1(M, \mathbb{R})$ be strictly positive functions, $\varepsilon, q \in (0, \infty)$, $p \in (2, 2^*_n)$, and $\omega \in \mathbb{R}$ be such that $a(x) > \omega^2 b(x)$ on M. We consider the subcritical system

$$\begin{cases} -\varepsilon^2 \mathrm{div}_g \left(c(x) \nabla_g u\right) + a(x)u = b(x)u^{p-1} + b(x)\omega^2 (qv - 1)^2 u & \text{in } M, \\ -\mathrm{div}_g \left(c(x) \nabla_g v\right) + b(x)(1 + q^2 u^2)v = b(x)qu^2 & \text{in } M, \\ u, v \in H^1_g(M), \quad u, v > 0. \end{cases} \tag{1.7}$$

Theorem 1.3. *Let K be a $\mathcal{C}^1$-stable critical set of the function $\Gamma : M \to \mathbb{R}$ given by*

$$\Gamma(x) := \frac{c(x)^{\frac{n}{2}} a(x)^{\frac{p}{p-2}-\frac{n}{2}}}{b(x)^{\frac{2}{p-2}}}.$$

Then, for ε small enough, the system (1.7) has a solution $(u_\varepsilon, v_\varepsilon)$ such that u_ε concentrates at a point $\xi_0 \in K$ as $\varepsilon \to 0$.

Recall that K is a $\mathcal{C}^1$*-stable critical set* of a function $f \in \mathcal{C}^1(M, \mathbb{R})$ if $K \subset \{x \in M : \nabla_g f(x) = 0\}$ and for any $\mu > 0$ there exists $\delta > 0$ such that, if $h \in \mathcal{C}^1(M, \mathbb{R})$ with

$$\max_{d_g(x,K)\leq \mu} |f(x) - h(x)| + |\nabla_g f(x) - \nabla_g h(x)| \leq \delta,$$

then h has a critical point x_0 with $d_g(x_0, K) \leq \mu$. Here d_g denotes the geodesic distance associated to the Riemannian metric g.

1.4. **The main results for the KGMP-system.** Theorem 1.3, together with Propositions 1.1 and 1.2, yields the following results.

Theorem 1.4. *Let $\mathfrak{M}$ be the warped product $M \times_{f^2} N$ of two closed Riemannian manifolds (M, g) and (N, h) with $n := \dim M = 2$ or 3. Set $k := \dim N$, and let $p \in (2, \infty)$ if $n = 2$ and $p \in (2, 6)$ if $n = 3$. Assume there exists $\beta \in \mathcal{C}^1(M, \mathbb{R})$ such that $\alpha = \beta \circ \pi_M$ and let K be a $\mathcal{C}^1$-stable critical set for the function $\Gamma := f^k \beta^{\frac{p}{p-2}-\frac{n}{2}}$ on M. Then, for ε small enough, the KGMP-system (1.1) has a solution $(\mathfrak{u}_\varepsilon, \mathfrak{v}_\varepsilon)$ such that $\mathfrak{u}_\varepsilon$ concentrates at the submanifold $\pi_M^{-1}(\xi_0) \cong (N, f^2(\xi_0)h)$ for some $\xi_0 \in K$ as $\varepsilon \to 0$.*

Theorem 1.5. *Assume there exist a closed Riemannian manifold M with $n := \dim M = 2$ or 3 and a harmonic morphism $\pi : \mathfrak{M} \to M$ whose dilation λ is such that $\mu \circ \pi = \lambda^2$. Assume further that $\alpha = \beta \circ \pi$ with $\beta \in \mathcal{C}^1(M, \mathbb{R})$. Let $p \in (2, \infty)$ if $n = 2$ and $p \in (2, 6)$ if $n = 3$, and let K be a $\mathcal{C}^1$-stable critical set for the function $\Gamma := \beta^{\frac{p}{p-2}-\frac{n}{2}} \mu^{\frac{n}{2}-1}$ on M. Then, for ε small enough, the KGMP-system (1.1) has a solution $(\mathfrak{u}_\varepsilon, \mathfrak{v}_\varepsilon)$ such that $\mathfrak{u}_\varepsilon$ concentrates at the submanifold $\pi^{-1}(\xi_0)$ of $\mathfrak{M}$ for some $\xi_0 \in K$ as $\varepsilon \to 0$.*

This last result applies, in particular, to the standard 3-sphere $\mathfrak{M} = \mathbb{S}^3$ and the real projective space $\mathfrak{M} = \mathbb{R}P^3$ for all $p \in (2, \infty)$ with $\mu = \lambda \equiv 1$, see subsection 1.2.

The rest of the paper is devoted to the proof of Theorem 1.3. In section 2 we reduce the system to a single equation and give the outline of the proof of Theorem 1.3, which follows the well-known Lyapunov-Schmidt reduction procedure. In section 3 we establish the Lyapunov-Schmidt reduction and in section 4 we derive the expansion of the reduced energy functional. Section 5 is devoted to the proof of some technical results.

2. Outline of the proof of Theorem 1.3

2.1. **Reduction to a single equation.** First, we reduce the system to a single equation. To overcome the problems caused by the competition between u and v, using an idea of Benci and Fortunato [2], we consider the map $\Psi : H_g^1(M) \to H_g^1(M)$ defined by the equation

$$-\operatorname{div}_g (c(x)\nabla_g \Psi(u)) + b(x)(1 + q^2u^2)\Psi(u) = b(x)qu^2. \tag{2.1}$$

It follows from standard variational arguments that Ψ is well-defined in $H_g^1(M)$.

Using the maximum principle and regularity theory it is not hard to prove that

$$0 < \Psi(u) < 1/q \qquad \text{for all } u \in H_g^1(M). \tag{2.2}$$

For the proofs of the following two lemmas we refer to [7].

Lemma 2.1. *The map $\Psi : H_g^1(M) \to H_g^1(M)$ is of class $\mathcal{C}^1$, and its differential $V_u := \Psi'(u)$ at u is defined by*

$$-\operatorname{div}_g\left(c(x)\nabla_g V_u[h]\right) + b(x)\left(1+q^2u^2\right)V_u[h] = 2b(x)qu(1-q\Psi(u))h \tag{2.3}$$

for every $h \in H_g^1(M)$. Moreover,

$$0 \le \Psi'(u)[u] \le \frac{2}{q} \qquad \textit{for all } u \in H_g^1(M).$$

Lemma 2.2. *The map $\Theta : H_g^1(M) \to \mathbb{R}$ given by*

$$\Theta(u) := \frac{1}{2}\int_M b(x)(1-q\Psi(u))u^2 d\mu_g$$

is of class $\mathcal{C}^1$ and

$$\Theta'(u)[h] = \int_M b(x)(1-q\Psi(u))^2 uh\, d\mu_g \qquad \textit{for all } u,h \in H_g^1(M).$$

Next, we introduce the functionals $I_\varepsilon, J_\varepsilon, G_\varepsilon : H_g^1(M) \to \mathbb{R}$ given by

$$I_\varepsilon(u) := J_\varepsilon(u) + \frac{\omega^2}{2}G_\varepsilon(u), \tag{2.4}$$

where

$$J_\varepsilon(u) := \frac{1}{2\varepsilon^2}\int_M \left[\varepsilon^2 c(x)|\nabla_g u|^2 + d(x)u^2\right] d\mu_g - \frac{1}{p\varepsilon^2}\int_M b(x)\left(u^+\right)^p d\mu_g$$

with $d(x) := a(x) - \omega^2 b(x)$, and

$$G_\varepsilon(u) := \frac{q}{\varepsilon^2}\int_M b(x)\Psi(u)u^2 d\mu_g.$$

From Lemma 2.2 we deduce that

$$\frac{1}{2}G_\varepsilon'(u)[\varphi] = \frac{1}{\varepsilon^2}\int_M b(x)[2q\Psi(u) - q^2\Psi^2(u)]u\varphi\, d\mu_g.$$

Hence,

$$I_\varepsilon'(u)\varphi = \frac{1}{\varepsilon^2}\int_M \varepsilon^2 c(x)\nabla_g u \nabla_g \varphi + a(x)u\varphi - b(x)(u^+)^{p-1}\varphi - b(x)\omega^2(1-q\Psi(u))^2 u\varphi\, d\mu_g.$$

Therefore, if u is a critical point of the functional I_ε, then u solves the problem

$$\begin{cases} -\varepsilon^2 \operatorname{div}_g\left(c(x)\nabla_g u\right) + (a(x)-\omega^2 b(x))u + \omega^2 q b(x)\Psi(u)(2-q\Psi(u))u = b(x)(u^+)^{p-1}, \\ u \in H_g^1(M). \end{cases} \tag{2.5}$$

If $u \neq 0$ by the maximum principle and regularity theory we have that $u > 0$. Thus the pair $(u, \Psi(u))$ is a solution of the system (1.7). This reduces the existence problem for the system (1.7) to showing that the functional I_ε has a nontrivial critical point.

2.2. **The limit problems.** Theorem 1.3 concerns manifolds of dimensions 2 and 3. To simplify the exposition we shall treat in full detail only the case $n = 2$. Everything can be extended in a straightforward way to the case $n = 3$, except for the estimates in section 5. These estimates, however, were computed in the appendix of [9] for $n = 3$.

Henceforth, we assume that $\dim M = 2$. We fix $r > 0$ smaller than the injectivity radius of M. We identify the tangent space of M at ξ with $\mathbb{R}^2$ and denote by $B(x,r)$ the ball in $\mathbb{R}^2$ centered at x of radius r and by $B_g(\xi,r)$ the ball in M centered at ξ of radius r, with respect to the distance induced by the Riemannian metric g. The exponential map $\exp_\xi : B(0,r) \to B_g(\xi,r)$ provides local coordenates on M, which are called normal coordinates. We denote by g_ξ the Riemannian metric at ξ given in normal coordinates by the matrix (g_{ij}). We denote the inverse matrix by $(g^{ij}(z)) := (g_{ij}(z))^{-1}$ and write $|g_\xi(z)| := \det(g_{ij}(z))$. Then, we have that

$$g^{ij}(\varepsilon z) = \delta_{ij} + \frac{\varepsilon^2}{2}\sum_{r,k=1}^{n} \frac{\partial^2 g^{ij}}{\partial z_r \partial z_k}(0) z_r z_k + O(\varepsilon^3|z|^3) = \delta_{ij} + o(\varepsilon), \tag{2.6}$$

$$|g(\varepsilon z)|^{\frac{1}{2}} = 1 - \frac{\varepsilon^2}{4}\sum_{i,r,k=1}^{n} \frac{\partial^2 g^{ii}}{\partial z_r \partial z_k}(0) z_r z_k + O(\varepsilon^3|z|^3) = 1 + o(\varepsilon). \tag{2.7}$$

Here δ_{ij} denotes the Kronecker symbol.

For $p \in (2,\infty)$ and $\xi \in M$, set

$$A(\xi) := \frac{a(\xi)}{c(\xi)}, \qquad B(\xi) := \frac{b(\xi)}{c(\xi)}, \qquad ,\gamma(\xi) := \left(\frac{a(\xi)}{b(\xi)}\right)^{\frac{1}{p-2}}.$$

We consider the problem

$$-c(\xi)\Delta V + a(\xi)V = b(\xi)V^{p-1}, \qquad V \in H^1(\mathbb{R}^2),$$

and denote by V^ξ its unique positive spherically symmetric solution. This problem is equivalent to

$$-\Delta V + A(\xi)V = B(\xi)V^{p-1}, \qquad V \in H^1(\mathbb{R}^2).$$

The function V^ξ and its derivatives decay exponentially at infinity. V^ξ can be written as

$$V^\xi(z) = \gamma(\xi)U(\sqrt{A(\xi)}z),$$

where U is the unique positive spherically symmetric solution to

$$-\Delta U + U = U^{p-1}, \qquad U \in H^1(\mathbb{R}^2).$$

For $\xi \in M$ and $\varepsilon > 0$ we define $W_{\varepsilon,\xi} \in H^1_g(M)$ by

$$W_{\varepsilon,\xi}(x) := \begin{cases} V^\xi\left(\frac{1}{\varepsilon}\exp_\xi^{-1}(x)\right)\chi\left(\exp_\xi^{-1}(x)\right) & \text{if } x \in B_g(\xi,r), \\ 0 & \text{otherwise,} \end{cases}$$

where $\chi \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a radial cut-off function such that $\chi(z) = 1$ if $|z| \leq r/2$ and $\chi(z) = 0$ if $|z| \geq r$. Setting $V_\varepsilon(z) := V\left(\frac{z}{\varepsilon}\right)$ and $y := \exp_\xi^{-1} x$ we have that

$$W_{\varepsilon,\xi}(\exp_\xi(y)) = V^\xi\left(\frac{y}{\varepsilon}\right)\chi(y) = V^\xi_\varepsilon(y)\chi(y),$$

so the function $W_{\varepsilon,\xi}$ is simply the function V^ξ rescaled, cut off and read in normal coordinates at ξ in M.

Similarly, for $i=1,2$ we define

$$Z^i_{\varepsilon,\xi}(x)=\begin{cases}\psi^i_\xi\left(\frac{1}{\varepsilon}\exp_\xi^{-1}(x)\right)\chi\left(\exp_\xi^{-1}(x)\right) & \text{if } x\in B_g(\xi,r),\\ 0 & \text{otherwise,}\end{cases}$$

where

$$\psi^i_\xi(\eta)=\frac{\partial}{\partial\eta_i}V^\xi(\eta)=\gamma(\xi)\sqrt{A(\xi)}\frac{\partial U}{\partial\eta_i}(\sqrt{A(\xi)}\eta).$$

The functions ψ^i_ξ are solutions of the linearized equation

$$-\Delta\psi+A(\xi)\psi=(p-1)B(\xi)\left(V^\xi\right)^{p-2}\psi\quad\text{in }\mathbb{R}^2.$$

Proposition 2.3. *There is a positive constant C such that*

$$\left\langle Z^h_{\varepsilon,\xi},Z^k_{\varepsilon,\xi}\right\rangle_\varepsilon=C\delta_{hk}+o(1),$$

as $\varepsilon\to 0$.

Proof. From the Taylor expansions of $g^{ij}(\varepsilon z)$, $|g(\varepsilon z)|^{\frac12}$, $a(\exp_\xi(\varepsilon z))$ and $c(\exp_\xi(\varepsilon z))$ we obtain

$$\begin{aligned}\left\langle Z^h_{\varepsilon,\xi},Z^k_{\varepsilon,\xi}\right\rangle_\varepsilon&=\frac{1}{\varepsilon^2}\int_M\varepsilon^2c(x)\nabla_gZ^h_{\varepsilon,\xi}(x)\nabla_gZ^k_{\varepsilon,\xi}(x)+d(x)Z^h_{\varepsilon,\xi}(x)Z^k_{\varepsilon,\xi}(x)d\mu_g\\
&=\int_{B(0,r/\varepsilon)}\sum_{ij}c(\exp_\xi(\varepsilon z))g^{ij}_\xi(\varepsilon z)\frac{\partial}{\partial z_i}(\psi^h_\xi(z)\chi(\varepsilon z))\frac{\partial}{\partial z_j}(\psi^h_\xi(z)\chi(\varepsilon z))|g_\xi(\varepsilon z)|^{\frac12}dz\\
&\quad+\int_{B(0,r/\varepsilon)}d(\exp_\xi(\varepsilon z))\psi^h_\xi(z)\psi^h_\xi(z)\chi^2(\varepsilon z)|g_\xi(\varepsilon z)|^{\frac12}dz\\
&=c(\xi)\int_{\mathbb{R}^2}\nabla\psi^h_\xi\nabla\psi^h_\xi dz+d(\xi)\int_{\mathbb{R}^2}\psi^h_\xi\psi^k_\xi dz+o(1)=C\delta_{hk}+o(1),\end{aligned}$$

as claimed. □

Next, we compute the derivatives of $W_{\varepsilon,\xi}$ with respect to ξ in normal coordinates. Fix $\xi_0\in M$. We write the points $\xi\in B_g(\xi_0,r)$ as

$$\xi=\xi(y)=\exp_{\xi_0}(y)\qquad\text{with } y\in B(0,r).$$

We define

$$\mathcal{E}(y,x)=\exp^{-1}_{\xi(y)}(x)=\exp^{-1}_{\exp_{\xi_0}(y)}(x),$$

where $x\in B_g(\xi(y),r)$ and $y\in B(0,r)$. Then we can write

$$\begin{aligned}W_{\varepsilon,\xi(y)}(x)&=\gamma(\xi(y))U_\varepsilon(\sqrt{A(\xi(y))}\exp^{-1}_{\xi(y)}(x))\chi(\exp^{-1}_{\xi(y)}(x))\\
&=\tilde\gamma(y)U_\varepsilon(\sqrt{\tilde A(y)}\mathcal{E}(y,x))\chi(\mathcal{E}(y,x))\end{aligned}$$

where $\tilde A(y)=A(\exp_{\xi_0}(y))$ and $\tilde\gamma(y)=\gamma(\exp_{\xi_0}(y))$. Thus we have

$$\begin{aligned}\frac{\partial}{\partial y_s}W_{\varepsilon,\xi(y)}\Big|_{y=0}&=\left(\frac{\partial}{\partial y_s}\tilde\gamma(y)\Big|_{y=0}\right)U\left(\frac{1}{\varepsilon}\sqrt{\tilde A(0)}\mathcal{E}(0,x)\right)\chi(\mathcal{E}(0,x))\\
&\quad+\tilde\gamma(0)U\left(\frac{1}{\varepsilon}\sqrt{\tilde A(0)}\mathcal{E}(0,x)\right)\frac{\partial}{\partial y_s}\chi\left(\mathcal{E}_k(y,x)\right)\Big|_{y=0}\\
&\quad+\tilde\gamma(0)\chi(\mathcal{E}(0,x))\frac{\partial}{\partial y_s}U\left(\frac{1}{\varepsilon}\sqrt{\tilde A(y)}\mathcal{E}(y,x)\right)\Big|_{y=0}.\end{aligned}$$

If $x=\exp_{\xi_0}\varepsilon z$, $\xi_0=\xi(0)$, then $\mathcal{E}(0,x)=\varepsilon z$ and we have

$$\begin{aligned} \frac{\partial}{\partial y_s}W_{\varepsilon,\xi(y)}\Big|_{y=0} &= \left(\frac{\partial}{\partial y_s}\tilde\gamma(y)\Big|_{y=0}\right)U(\sqrt{\tilde A(0)}z)\chi(\varepsilon z) \\ &+\tilde\gamma(0)U\left(\sqrt{\tilde A(0)}z\right)\frac{\partial\chi}{\partial\eta_k}(\varepsilon z)\ \frac{\partial}{\partial y_s}\mathcal{E}_k(y,\exp_{\xi_0}\varepsilon z)\Big|_{y=0} \\ &+\tilde\gamma(0)\chi(\varepsilon z)\frac{\sqrt{\tilde A(0)}}{\varepsilon}\frac{\partial U}{\partial\eta_k}\left(\sqrt{\tilde A(0)}z\right)\ \frac{\partial}{\partial y_s}\mathcal{E}_k(y,\exp_{\xi_0}\varepsilon z)\Big|_{y=0}. \end{aligned} \tag{2.8}$$

We also recall the following Taylor expansions:

$$\frac{\partial}{\partial y_h}\mathcal{E}_k(0,\exp_{\xi_0}\varepsilon z)=-\delta_{hk}+O(\varepsilon^2|z|^2). \tag{2.9}$$

2.3. **Outline of the proof of Theorem 1.3.** Let H_ε denote the Hilbert space $H^1_g(M)$ equipped with the inner product

$$\langle u,v\rangle_\varepsilon:=\frac{1}{\varepsilon^2}\left(\varepsilon^2\int_M c(x)\nabla_g u\nabla_g v\,d\mu_g+\int_M d(x)uv\,d\mu_g\right),$$

which induces the norm

$$\|u\|^2_\varepsilon:=\frac{1}{\varepsilon^2}\left(\varepsilon^2\int_M c(x)|\nabla_g u|^2d\mu_g+\int_M d(x)u^2d\mu_g\right),$$

with $d(x):=a(x)-\omega^2b(x)>0$. Similarly, let L^q_ε be the Banach space $L^q_g(M)$ with the norm

$$|u|_{q,\varepsilon}:=\left(\frac{1}{\varepsilon^2}\int_M|u|^q d\mu_g\right)^{1/q}.$$

Since we are assuming that $\dim M=2$, for each $q\geq 2$ the embedding $H_\varepsilon\hookrightarrow L^q_\varepsilon$ is continuous. In fact, there is a positive constant C, independent of ε, such that

$$|u|_{q,\varepsilon}\leq C\,\|u\|_\varepsilon\qquad\forall u\in H_\varepsilon, \tag{2.10}$$

Moreover, this embedding is compact.

Fix $p\in(2,\infty)$. The adjoint operator $i^*_\varepsilon:L^{p'}_\varepsilon\to H_\varepsilon$, $p':=\frac{p}{p-1}$, to the embedding $i_\varepsilon:H_\varepsilon\hookrightarrow L^p_\varepsilon$ is defined by

$$\begin{aligned} u=i^*_\varepsilon(v)\Leftrightarrow\langle u,\varphi\rangle_\varepsilon=\frac{1}{\varepsilon^2}\int_M v\varphi\quad\forall\varphi\in H_\varepsilon \\ \Leftrightarrow-\varepsilon^2\mathrm{div}_g\left(c(x)\nabla_g u\right)+d(x)u=v,\quad u\in H^1_g(M). \end{aligned}$$

One has that

$$\|i^*_\varepsilon(v)\|_\varepsilon\leq C|v|_{p',\varepsilon}\qquad\forall v\in L^{p'}_\varepsilon, \tag{2.11}$$

where the constant C does not depend on ε.

Using the adjoint operator we can rewrite problem (2.5) as

$$u=i^*_\varepsilon\left[b(x)f(u)+\omega^2b(x)g(u)\right],\qquad u\in H_\varepsilon, \tag{2.12}$$

where

$$f(u):=\left(u^+\right)^{p-1}\qquad\text{and}\qquad g(u):=\left(q^2\Psi^2(u)-2q\Psi(u)\right)u.$$

Let

$$K_{\varepsilon,\xi}:=\mathrm{Span}\ \left\{Z^1_{\varepsilon,\xi},Z^2_{\varepsilon,\xi}\right\}$$

and

$$K_{\varepsilon,\xi}^{\perp} := \left\{ \phi \in H_\varepsilon : \left\langle \phi, Z_{\varepsilon,\xi}^{i} \right\rangle_\varepsilon = 0,\ i = 1,2 \right\}.$$

We denote the projections onto these subspaces by

$$\Pi_{\varepsilon,\xi} : H_\varepsilon \to K_{\varepsilon,\xi} \qquad \text{and} \qquad \Pi_{\varepsilon,\xi}^{\perp} : H_\varepsilon \to K_{\varepsilon,\xi}^{\perp}.$$

We look for a solution of (2.5) of the form

$$u_\varepsilon := W_{\varepsilon,\xi} + \phi \qquad \text{with } \ \phi \in K_{\varepsilon,\xi}^{\perp}.$$

This is equivalent to solving the pair of equations

$$\Pi_{\varepsilon,\xi}^{\perp} \left\{ W_{\varepsilon,\xi} + \phi - i_\varepsilon^* \left[b(x) f\left(W_{\varepsilon,\xi} + \phi\right) + \omega^2 b(x) g\left(W_{\varepsilon,\xi} + \phi\right) \right] \right\} = 0, \tag{2.13}$$

$$\Pi_{\varepsilon,\xi} \left\{ W_{\varepsilon,\xi} + \phi - i_\varepsilon^* \left[b(x) f\left(W_{\varepsilon,\xi} + \phi\right) + \omega^2 b(x) g\left(W_{\varepsilon,\xi} + \phi\right) \right] \right\} = 0. \tag{2.14}$$

The first step of the proof of Theorem 1.3 is to solve equation (2.13). More precisely, for any fixed $\xi \in M$ and ε small enough, we will show that there is a function $\phi \in K_{\varepsilon,\xi}^{\perp}$ such that (2.13) holds. To do this we consider the linear operator $L_{\varepsilon,\xi} : K_{\varepsilon,\xi}^{\perp} \to K_{\varepsilon,\xi}^{\perp}$ given by

$$L_{\varepsilon,\xi}(\phi) := \Pi_{\varepsilon,\xi}^{\perp} \left\{ \phi - i_\varepsilon^* \left[b(x) f'\left(W_{\varepsilon,\xi}\right) \phi \right] \right\}.$$

For the proof of the following statement we refer to Lemma 4.1 of [3] (see also Proposition 3.1 of [12]).

Proposition 2.4. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0)$, $\xi \in M$ and $\phi \in K_{\varepsilon,\xi}^{\perp}$,*

$$\left\| L_{\varepsilon,\xi}(\phi) \right\|_\varepsilon \geq C \|\phi\|_\varepsilon.$$

This result allows to use a contraction mapping argument to solve equation (2.13). The following statement is proved in section 3.

Proposition 2.5. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for each $\xi \in M$ and each $\varepsilon \in (0, \varepsilon_0)$, there exists a unique $\phi_{\varepsilon,\xi} \in K_{\varepsilon,\xi}^{\perp}$ which solves equation (2.13). Moreover,*

$$\left\| \phi_{\varepsilon,\xi} \right\|_\varepsilon \leq C\varepsilon.$$

The map $\xi \mapsto \phi_{\varepsilon,\xi}$ is a $\mathcal{C}^1$-map.

The second step is to solve equation (2.14). More precisely, for ε small enough we will find a point ξ in M such that equation (2.14) is satisfied. To this end we introduce the reduced energy function $\widetilde{I}_\varepsilon : M \to \mathbb{R}$ defined by

$$\widetilde{I}_\varepsilon(\xi) := I_\varepsilon\left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}\right),$$

where I_ε is the variational functional defined in (2.4) whose critical points are the solutions to problem (2.5). It is easy to verify that ξ_ε is a critical point of $\widetilde{I}_\varepsilon$ if and only if the function $u_\varepsilon = W_{\varepsilon,\xi_\varepsilon} + \phi_{\varepsilon,\xi_\varepsilon}$ is a critical point of I_ε.

In Lemmas 4.1 and 4.2 we compute the asymptotic expansion of the reduced functional $\tilde{I}_\varepsilon$ with respect to the parameter ε. We prove the following result.

Proposition 2.6. *The expansion*

$$\tilde{I}_\varepsilon(\xi) = C \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} + o(1) = C\Gamma(\xi) + o(1),$$

holds true $\mathcal{C}^1$-uniformly with respect to ξ as $\varepsilon \to 0$, where $C = \left(\frac{1}{2} - \frac{1}{p}\right) \int_{\mathbb{R}^n} U^p dz$.

Using the previous propositions we now prove Theorem 1.3.

Proof of Theorem 1.3. Since K is a $\mathcal{C}^1$-stable critical set for Γ, by Proposition 2.6 $\tilde{I}_\varepsilon$ has a critical point $\xi_\varepsilon \in M$ such that $d_g(\xi_\varepsilon, K) \to 0$ as $\varepsilon \to 0$. Hence, $u_\varepsilon = W_{\varepsilon,\xi_\varepsilon} + \phi_{\varepsilon,\xi_\varepsilon}$ is a solution of (2.5), and the pair $(u_\varepsilon, \Psi(u_\varepsilon))$ is a solution to the system (1.7) such that u_ε concentrates at a point $\xi_0 \in K$ as $\varepsilon \to 0$. □

3. The finite dimensional reduction

This section is devoted to the proof of Proposition 2.5. We denote by

$$\|u\|_g^2 := \int_M \left(|\nabla_g u|^2 + u^2\right) d\mu_g \qquad \text{and} \qquad |u|_{g,q}^q := \int_M |u|^q d\mu_g \tag{3.1}$$

the standard norms in the spaces $H_g^1(M)$ and $L^q(M)$.

Equation (2.13) is equivalent to

$$L_{\varepsilon,\xi}(\phi) = N_{\varepsilon,\xi}(\phi) + S_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi}, \tag{3.2}$$

where

$$N_{\varepsilon,\xi}(\phi) := \Pi_{\varepsilon,\xi}^\perp \left\{ i_\varepsilon^* \left[b(x) \left(f\left(W_{\varepsilon,\xi} + \phi\right) - f\left(W_{\varepsilon,\xi}\right) - f'\left(W_{\varepsilon,\xi}\right)\right)\phi \right] \right\},$$

$$S_{\varepsilon,\xi}(\phi) := \omega^2 \Pi_{\varepsilon,\xi}^\perp \left\{ i_\varepsilon^* \left[b(x) \left(q^2\Psi^2\left(W_{\varepsilon,\xi} + \phi\right) - 2q\Psi\left(W_{\varepsilon,\xi} + \phi\right)\right)\left(W_{\varepsilon,\xi} + \phi\right)\right]\right\},$$

$$R_{\varepsilon,\xi} := \Pi_{\varepsilon,\xi}^\perp \left\{ i_\varepsilon^* \left[b(x) f\left(W_{\varepsilon,\xi}\right)\right] - W_{\varepsilon,\xi} \right\}.$$

In order to solve equation (3.2) we will show that the operator $T_{\varepsilon,\xi} : K_{\varepsilon,\xi}^\perp \to K_{\varepsilon,\xi}^\perp$ defined by

$$T_{\varepsilon,\xi}(\phi) := L_{\varepsilon,\xi}^{-1}\left(N_{\varepsilon,\xi}(\phi) + S_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi}\right)$$

has a fixed point. To this end we prove that $T_{\varepsilon,\xi}$ is a contraction mapping on suitable ball in H_ε. We start with an estimate for $R_{\varepsilon,\xi}$.

Lemma 3.1. *There exist $\varepsilon_0 > 0$ and $C > 0$ such that, for any $\xi \in M$ and any $\varepsilon \in (0, \varepsilon_0)$, the inequality*

$$\left\|R_{\varepsilon,\xi}\right\|_\varepsilon \le C\varepsilon$$

holds true.

Proof. See Lemma 4.2 in [3]. □

Next, we give an estimate for $N_{\varepsilon,\xi}(\phi)$.

Lemma 3.2. *There exist $\varepsilon_0 > 0$, $C > 0$ and $\widetilde{C} \in (0,1)$ such that, for any $\xi \in M$, $\varepsilon \in (0, \varepsilon_0)$ and $R > 0$, the inequalities*

$$\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C(\|\phi\|_\varepsilon^2 + \|\phi\|_\varepsilon^{p-1}), \tag{3.3}$$

$$\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \le \widetilde{C}\|\phi_1 - \phi_2\|_\varepsilon, \tag{3.4}$$

hold true for $\phi, \phi_1, \phi_2 \in \{\phi \in H_\varepsilon : \|\phi\|_\varepsilon \le R\varepsilon\}$.

Proof. By direct computation we obtain

$$|f'(W_{\varepsilon,\xi} + v) - f'(W_{\varepsilon,\xi})| \le \begin{cases} CW_{\varepsilon,\xi}^{p-3}|v| & 2 < p < 3, \\ C(W_{\varepsilon,\xi}^{p-3}|v| + |v|^{p-2}) & p \ge 3. \end{cases} \tag{3.5}$$

From the mean value theorem and inequality (2.11) we derive

$$\|N_{\varepsilon,\xi}(\phi_1) - N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \le C\left|f'(W_{\varepsilon,\xi} + \phi_2 + t(\phi_1 - \phi_2)) - f'(W_{\varepsilon,\xi})\right|_{\frac{p}{p-2},\varepsilon} \|\phi_1 - \phi_2\|_\varepsilon.$$

Using (3.5) we conclude that

$$C\left|f'(W_{\varepsilon,\xi}+\phi_2+t(\phi_1-\phi_2))-f'(W_{\varepsilon,\xi})\right|_{\frac{p}{p-2},\varepsilon}<1$$

provided $\|\phi_1\|_\varepsilon$ and $\|\phi_2\|_\varepsilon$ are small enough. The same estimates yield (3.3). □

Now we estimate $S_{\varepsilon,\xi}(\phi)$.

Lemma 3.3. *There exists $\varepsilon_0>0$ and $C>0$ such that, for any $\xi\in M$, $\varepsilon\in(0,\varepsilon_0)$ and $R>0$, the inequalities*

$$\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon\leq C\varepsilon, \tag{3.6}$$

$$\|S_{\varepsilon,\xi}(\phi_1)-S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon\leq \ell_\varepsilon\|\phi_1-\phi_2\|_\varepsilon, \tag{3.7}$$

hold true for $\phi,\phi_1,\phi_2\in\{\phi\in H_\varepsilon:\|\phi\|_\varepsilon\leq R\varepsilon\}$, where $\ell_\varepsilon\to 0$ as $\varepsilon\to 0$.

Proof. Let us prove (3.6). From the definition of i^* and inequality (2.11) we derive

$$\begin{aligned}\|S_{\varepsilon,\xi}(\phi)\|_\varepsilon &\leq C\left(\left|\Psi^2\left(W_{\varepsilon,\xi}+\phi\right)\left(W_{\varepsilon,\xi}+\phi\right)\right|_{p',\varepsilon}+\left|\Psi\left(W_{\varepsilon,\xi}+\phi\right)\left(W_{\varepsilon,\xi}+\phi\right)\right|_{p',\varepsilon}\right)\\ &=: I_1+I_2.\end{aligned}$$

For any $t\in(2,\infty)$, setting $s:=\frac{tp'}{t-p'}$ and $\vartheta:=\frac{2}{t'}\in(1,2)$ and applying Lemma 5.3 and Remark 5.2, we obtain

$$\begin{aligned}I_2&\leq C\frac{1}{\varepsilon^{2/p'}}\left(\int_M\left|\Psi\left(W_{\varepsilon,\xi}+\phi\right)\right|^t d\mu_g\right)^{\frac1t}\left(\int_M\left|W_{\varepsilon,\xi}+\phi\right|^s d\mu_g\right)^{\frac1s}\\ &\leq C\frac{1}{\varepsilon^{2/p'}}\left\|\Psi\left(W_{\varepsilon,\xi}+\phi\right)\right\|_g\left(\varepsilon^{\frac2s}\left(\frac{1}{\varepsilon^2}\int_M\left|W_{\varepsilon,\xi}\right|^s d\mu_g\right)^{\frac1s}+|\phi|_{g,s}\right)\\ &\leq C\frac{1}{\varepsilon^{2/p'}}\left(\varepsilon^{\vartheta}+\|\phi\|_\varepsilon^2\right)\left(\varepsilon^{\frac2s}+\|\phi\|_\varepsilon\right)\\ &\leq C\left(\varepsilon^{\vartheta+\frac2s-\frac{2}{p'}}+\varepsilon^{\vartheta+1-\frac{2}{p'}}\right)=C\left(\varepsilon^{\vartheta-\frac2t}+\varepsilon^{\vartheta+1-\frac{2}{p'}}\right)\\ &\leq C\varepsilon\end{aligned}$$

for all $\|\phi\|_\varepsilon\leq R\varepsilon$. From this estimate we deduce that $I_1\leq C\varepsilon$ and, hence, (3.6) follows.

Next, we prove (3.7). From inequality (2.11) we obtain that

$$\begin{aligned}\|S_{\varepsilon,\xi}(\phi_1)-S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon\leq& C\left|\left[\Psi\left(W_{\varepsilon,\xi}+\phi_1\right)-\Psi\left(W_{\varepsilon,\xi}+\phi_2\right)\right]W_{\varepsilon,\xi}\right|_{p',\varepsilon}\\ &+C\left|\left[\Psi^2\left(W_{\varepsilon,\xi}+\phi_1\right)-\Psi^2\left(W_{\varepsilon,\xi}+\phi_2\right)\right]W_{\varepsilon,\xi}\right|_{p',\varepsilon}\\ &+C\left|\Psi\left(W_{\varepsilon,\xi}+\phi_1\right)\phi_1-\Psi\left(W_{\varepsilon,\xi}+\phi_2\right)\phi_2\right|_{p',\varepsilon}\\ &+C\left|\Psi^2\left(W_{\varepsilon,\xi}+\phi_1\right)\phi_1-\Psi^2\left(W_{\varepsilon,\xi}+\phi_2\right)\phi_2\right|_{p',\varepsilon}\\ =&: I_1+I_2+I_3+I_4.\end{aligned}$$

By Remark 5.2 and Lemma 5.4 with $s:=\frac32$, for some $\theta\in(0,1)$ we have that

$$\begin{aligned}I_1^{p'}&\leq\frac{C}{\varepsilon^2}\left(\int_M\left|\Psi'\left(W_{\varepsilon,\xi}+\theta\phi_1+(1-\theta)\phi_2\right)(\phi_1-\phi_2)\right|^p\right)^{\frac{p'}{p}}\left(\frac{1}{\varepsilon^2}\int_M\left|W_{\varepsilon,\xi}\right|^{\frac{p'p}{p-p'}}\right)^{\frac{p-p'}{p}}\varepsilon^{\frac{2(p-p')}{p}}\\ &\leq C\frac{\varepsilon^{\frac{2(p-p')}{p}}}{\varepsilon^2}\left(\varepsilon^{\frac43}+\|\phi_1\|_g+\|\phi_2\|_g\right)^{p'}\|\phi_1-\phi_2\|_g^{p'}\\ &\leq Cl_\varepsilon\|\phi_1-\phi_2\|_\varepsilon^{p'},\end{aligned}$$

for $\|\phi_1\|_\varepsilon, \|\phi_2\|_\varepsilon \le R\varepsilon$, with $l_\varepsilon := \varepsilon^{\frac{p'(p-2)}{p}} \to 0$ as $\varepsilon \to 0$. From the estimate of I_1, recalling that $0 \le \Psi(u) \le \frac{1}{q}$, we derive

$$\begin{aligned} I_2^{p'} &= \frac{1}{\varepsilon^2}\int_M \left|\Psi\left(W_{\varepsilon,\xi}+\phi_1\right)+\Psi\left(W_{\varepsilon,\xi}+\phi_2\right)\right|^{p'}\left|\Psi\left(W_{\varepsilon,\xi}+\phi_1\right)-\Psi\left(W_{\varepsilon,\xi}+\phi_2\right)\right|^{p'}\left|W_{\varepsilon,\xi}\right|^{p'} \\ &\le C I_1^{p'}. \end{aligned}$$

On the other hand, choosing $\vartheta \in (1,2)$ in Lemma 5.3 such that $\vartheta p' > 2$ and applying Lemma 5.4 with $s := \frac{3}{2}$, we obtain

$$\begin{aligned} I_3^{p'} &\le \frac{1}{\varepsilon^2}\int_M \left|\Psi'\left(W_{\varepsilon,\xi}+\theta\phi_1+(1-\theta)\phi_2\right)(\phi_1-\phi_2)\right|^{p'}|\phi_1|^{p'} \\ &\quad + \frac{1}{\varepsilon^2}\int_M \left|\Psi\left(W_{\varepsilon,\xi}+\phi_2\right)\right|^{p'}|\phi_1-\phi_2|^{p'} \\ &\le C\frac{1}{\varepsilon^2}\left(\int_M \left|\Psi'\left(W_{\varepsilon,\xi}+\theta\phi_1+(1-\theta)\phi_2\right)(\phi_1-\phi_2)\right|^{p}\right)^{\frac{p'}{p}}\left(\int_M |\phi_1|^{\frac{p'p}{p-p'}}\right)^{\frac{p-p'}{p}} \\ &\quad + C\frac{1}{\varepsilon^2}\left(\int_M |\phi_1-\phi_2|^{p}\right)^{\frac{p'}{p}}\left(\int_M \left|\Psi\left(W_{\varepsilon,\xi}+\phi_2\right)\right|^{\frac{p'p}{p-p'}}\right)^{\frac{p-p'}{p}} \\ &\le C\frac{1}{\varepsilon^2}\left(\varepsilon^{\frac{4}{3}}+\|\phi_1\|_g+\|\phi_2\|_g\right)^{p'}\|\phi_1-\phi_2\|_g^{p'}\|\phi_1\|_g^{p'} \\ &\quad + C\frac{\varepsilon^{\vartheta p'}}{\varepsilon^2}(1+\|\phi_2\|_\varepsilon^2)\|\phi_1-\phi_2\|_g^{p'} \\ &\le C\left(\frac{\varepsilon^{2p'}}{\varepsilon^2}+\frac{\varepsilon^{\vartheta p'}}{\varepsilon^2}\right)\|\phi_1-\phi_2\|_\varepsilon^{p'} = l_\varepsilon\|\phi_1-\phi_2\|_\varepsilon^{p'}, \end{aligned}$$

for $\|\phi_1\|_\varepsilon, \|\phi_2\|_\varepsilon \le R\varepsilon$, where $l_\varepsilon \to 0$ as $\varepsilon \to 0$.

Finally, from the estimate of I_2 we derive $I_4^{p'} \le C I_3^{p'}$ Collecting the previous estimates we obtain (3.7). □

Proof of Proposition 2.5. From Proposition 2.4 we deduce

$$\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon \le C\left(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|S_{\varepsilon,\xi}(\phi)\|_\varepsilon + \|R_{\varepsilon,\xi}\|_\varepsilon\right)$$

and

$$\|T_{\varepsilon,\xi}(\phi_1)-T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \le C\|N_{\varepsilon,\xi}(\phi_1)-N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon + C\|S_{\varepsilon,\xi}(\phi_1)-S_{\varepsilon,\xi}(\phi_2)\|_\varepsilon.$$

Lemmas 3.1, 3.2 and 3.3 imply that $T_{\varepsilon,\xi}$ is a contraction in the ball centered at 0 of radius $R\varepsilon$ in $K_{\varepsilon,\xi}^\perp$, for a suitable constant R. Hence, $T_{\varepsilon,\xi}$ has a unique fixed point.

In order to prove that the map $\xi \mapsto \phi_{\varepsilon,\xi}$ is $\mathcal{C}^1$ we apply the implicit function theorem to the $\mathcal{C}^1$-function $G : M \times H_\varepsilon \to H_\varepsilon$ defined by

$$\begin{aligned} G(\xi,u) := \Pi_{\varepsilon,\xi}^\perp &\left\{W_{\varepsilon,\xi}+\Pi_{\varepsilon,\xi}^\perp u - i_\varepsilon^*\left[b(x)f\left(W_{\varepsilon,\xi}+\Pi_{\varepsilon,\xi}^\perp u\right)+\omega^2 b(x)g\left(W_{\varepsilon,\xi}+\Pi_{\varepsilon,\xi}^\perp u\right)\right]\right\} \\ &+ \Pi_{\varepsilon,\xi}u. \end{aligned}$$

Note that $G(\xi, \phi_{\varepsilon,\xi}) = 0$. Next we show that the linearized operator $\frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon,\xi}) : H_\varepsilon \to H_\varepsilon$ defined by

$$\begin{aligned}
&\frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon,\xi})(u) \\
&= \Pi^\perp_{\varepsilon,\xi}\left\{\Pi^\perp_{\varepsilon,\xi}(u) - i^*_\varepsilon\left[b(x)f'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\Pi^\perp_{\varepsilon,\xi}(u) + \omega^2 b(x)g'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\Pi^\perp_{\varepsilon,\xi}(u)\right]\right\} \\
&\qquad + \Pi_{\varepsilon,\xi}(u)
\end{aligned}$$

is invertible, provided ε is small enough. For any ϕ with $\|\phi\|_\varepsilon \le C\varepsilon$ we have that

$$\begin{aligned}
&\left\|\frac{\partial G}{\partial u}(\xi, \phi_{\varepsilon,\xi})(u)\right\|_\varepsilon \ge C\|\Pi_{\varepsilon,\xi}(u)\|_\varepsilon \\
&\qquad + C\left\|\Pi^\perp_{\varepsilon,\xi}\left\{\Pi^\perp_{\varepsilon,\xi}(u) - i^*_\varepsilon\left[f'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\Pi^\perp_{\varepsilon,\xi}(u) + \omega^2 g'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\Pi^\perp_{\varepsilon,\xi}(u)\right]\right\}\right\|_\varepsilon \\
&\ge C\|\Pi_{\varepsilon,\xi}(u)\|_\varepsilon + C\left\|L_{\varepsilon,\xi}\left(\Pi^\perp_{\varepsilon,\xi}(u)\right)\right\|_\varepsilon \\
&\qquad - C\left\|\Pi^\perp_{\varepsilon,\xi}\left\{i^*_\varepsilon\left[(f'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi}))\Pi^\perp_{\varepsilon,\xi}(u)\right]\right\}\right\|_\varepsilon \\
&\qquad - C\left\|\Pi^\perp_{\varepsilon,\xi}\left\{i^*_\varepsilon\left[\omega^2 g'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\Pi^\perp_{\varepsilon,\xi}(u)\right]\right\}\right\|_\varepsilon \\
&\ge C\|\Pi_{\varepsilon,\xi}(u)\|_\varepsilon + C\left\|\Pi^\perp_{\varepsilon,\xi}(u)\right\|_\varepsilon - o(1)\left\|\Pi^\perp_{\varepsilon,\xi}(u)\right\|_\varepsilon \\
&\ge C\|u\|_\varepsilon.
\end{aligned}$$

Indeed, by (3.5) we have

$$\begin{aligned}
\left\|\Pi^\perp_{\varepsilon,\xi}\left\{i^*_\varepsilon\left[(f'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi}))\Pi^\perp_{\varepsilon,\xi}(u)\right]\right\}\right\|_\varepsilon &\le C\left(\|\phi\|^{p-2}_\varepsilon + \|\phi\|_\varepsilon\right)\left\|\Pi^\perp_{\varepsilon,\xi}(u)\right\|_\varepsilon \\
&= o(1)\left\|\Pi^\perp_{\varepsilon,\xi}(u)\right\|_\varepsilon.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left\|\Pi^\perp_{\varepsilon,\xi}\left\{i^*_\varepsilon\left[\omega^2 g'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\Pi^\perp_{\varepsilon,\xi}(u)\right]\right\}\right\|_\varepsilon \\
&\le C\left|(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\left(2q - 2q^2\Psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\right)\Psi'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\left[\Pi^\perp_{\varepsilon,\xi}(u)\right]\right|_{p',\varepsilon} \\
&\qquad + C\left|\left[2q\Psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) - q^2\Psi^2(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\right]\Pi^\perp_{\varepsilon,\xi}(u)\right|_{p',\varepsilon} \\
&:= I_1 + I_2.
\end{aligned}$$

From Lemma 5.4 we derive

$$\begin{aligned}
I_1 &\le \frac{C}{\varepsilon^{\frac{2}{p'}}}\left|W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}\right|_{g,2}\left|\Psi'(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\Pi^\perp_{\varepsilon,\xi}(u)\right|_{g,\frac{4p'}{2-p'}}\left|2q - 2q^2\Psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\right|_{g,\frac{4p'}{2-p'}} \\
&\le C\frac{1}{\varepsilon^{\frac{2}{p'}}}\varepsilon(\varepsilon^{\frac{4}{3}} + \varepsilon)\left\|\Pi^\perp_{\varepsilon,\xi}u\right\|_g \le \varepsilon^{2-\frac{2}{p'}}\left\|\Pi^\perp_{\varepsilon,\xi}u\right\|_g = o(1)\left\|\Pi^\perp_{\varepsilon,\xi}u\right\|_g,
\end{aligned}$$

and, since $0 \le \Psi(u) \le 1/q$, from Lemma 5.3 with $\vartheta p' > 2$ we get

$$\begin{aligned}
I_2 &\le \frac{C}{\varepsilon^{\frac{2}{p'}}}\left|\Pi^\perp_{\varepsilon,\xi}u\right|_{g,p}\left|\Psi(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi})\right|_{g,\frac{p'p}{p-p'}} \\
&\le C\frac{\varepsilon^\vartheta}{\varepsilon^{\frac{2}{p'}}}\left(1 + \|\phi_{\varepsilon,\xi}\|^2_\varepsilon\right)\left\|\Pi^\perp_{\varepsilon,\xi}u\right\|_g = o(1)\left\|\Pi^\perp_{\varepsilon,\xi}u\right\|_g
\end{aligned}$$

This concludes the proof. □

4. The reduced energy

This section is devoted to the proof of Proposition 2.6.

Lemma 4.1. *The following estimate*

$$\widetilde{I}_\varepsilon(\xi) = I_\varepsilon\left(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}\right) = I_\varepsilon\left(W_{\varepsilon,\xi}\right)+o(1) = J_\varepsilon\left(W_{\varepsilon,\xi}\right)+\frac{\omega^2}{2}G_\varepsilon\left(W_{\varepsilon,\xi}\right)+o\left(1\right) \tag{4.1}$$

holds true $\mathcal{C}^0$-uniformly with respect to ξ as ε goes to zero. Moreover, setting $\xi(y) := \exp_\xi(y)$, $y\in B(0,r)$, we have that

$$\begin{aligned}\left(\frac{\partial}{\partial y_h}\widetilde{I}_\varepsilon(\xi(y))\right)_{|y=0} &= \left(\frac{\partial}{\partial y_h}I_\varepsilon\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)\right)_{|y=0}\\ &= \left(\frac{\partial}{\partial y_h}I_\varepsilon\left(W_{\varepsilon,\xi(y)}\right)\right)_{|y=0}+o(1)\\ &= \left(\frac{\partial}{\partial y_h}J_\varepsilon\left(W_{\varepsilon,\xi(y)}\right)\right)_{|y=0}+\frac{\omega^2}{2}\left(\frac{\partial}{\partial y_h}G_\varepsilon\left(W_{\varepsilon,\xi(y)}\right)\right)_{|y=0}+o\left(1\right),\end{aligned}$$

$\mathcal{C}^0$-uniformly with respect to ξ as ε goes to zero.

Proof. In Lemma 5.1 of [3] we have proved the following two estimates:

$$J_\varepsilon\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)-J_\varepsilon\left(W_{\varepsilon,\xi(y)}\right)=o(1),$$

$$\left(J'_\varepsilon\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)-J'_\varepsilon\left(W_{\varepsilon,\xi(y)}\right)\right)\left[\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right]=o(1).$$

To complete the proof we shall prove the the following three estimates:

$$G_\varepsilon\left(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}\right)-G_\varepsilon\left(W_{\varepsilon,\xi}\right)=o(1), \tag{4.2}$$

$$\left[G'_\varepsilon\left(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0}\right)-G'_\varepsilon\left(W_{\varepsilon,\xi_0}\right)\right]\left[\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right]=o(1), \tag{4.3}$$

$$\left(J'_\varepsilon\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)+\frac{\omega^2}{2}G'_\varepsilon\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)\right)\left[\frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\right]=o(1). \tag{4.4}$$

We start with (4.2). For some $\theta\in[0,1]$ we have

$$\begin{aligned}&G_\varepsilon\left(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}\right)-G_\varepsilon\left(W_{\varepsilon,\xi}\right)\\ &=\frac{1}{\varepsilon^2}\int_M b(x)\left[\Psi\left(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}\right)\left(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}\right)^2-\Psi\left(W_{\varepsilon,\xi}\right)\left(W_{\varepsilon,\xi}\right)^2\right]\\ &=\frac{1}{\varepsilon^2}\int_M b(x)\Psi'\left(W_{\varepsilon,\xi}+\theta\phi_{\varepsilon,\xi}\right)\left[\phi_{\varepsilon,\xi}\right]\left(W_{\varepsilon,\xi}\right)^2\\ &\quad+\frac{1}{\varepsilon^2}\int_M b(x)\Psi\left(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi}\right)\left[2\phi_{\varepsilon,\xi}W_{\varepsilon,\xi}+\phi^2_{\varepsilon,\xi}\right]\end{aligned}$$

Since $\left\|\phi_{\varepsilon,\xi}\right\|_\varepsilon\le C\varepsilon$, from Lemma 5.4 we obtain (4.2).

Next, we prove (4.3). For some $\theta \in [0,1]$ we have

$$
\begin{aligned}
&[G'_\varepsilon(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0}) - G'_\varepsilon(W_{\varepsilon,\xi_0})]\left[\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right]\\
&\le \frac{q}{2\varepsilon^2}\left|\int_M b(x)\left\{[2\Psi(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})-\Psi(W_{\varepsilon,\xi_0})]-\left[q\Psi^2(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})-q\Psi^2(W_{\varepsilon,\xi_0})\right]\right\}\right.\\
&\qquad\qquad \left.\cdot W_{\varepsilon,\xi_0}\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right|\\
&\quad+\left|\frac{q}{2\varepsilon^2}\int_M 2b(x)\left[\Psi(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})-q\Psi^2(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})\right]\phi_{\varepsilon,\xi_0}\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right|\\
&\le \left|\frac{q}{2\varepsilon^2}\int_M 2b(x)\Psi'(W_{\varepsilon,\xi_0}+\theta\phi_{\varepsilon,\xi_0})(\phi_{\varepsilon,\xi_0})W_{\varepsilon,\xi_0}\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right|\\
&\quad+\left|\frac{q}{\varepsilon^2}\int_M b(x)\Psi(W_{\varepsilon,\xi_0}+\theta\phi_{\varepsilon,\xi_0})\Psi'(W_{\varepsilon,\xi_0}+\theta\phi_{\varepsilon,\xi_0})(\phi_{\varepsilon,\xi_0})W_{\varepsilon,\xi_0}\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right|\\
&\quad+\left|\frac{q}{\varepsilon^2}\int_M b(x)\Psi(W_{\varepsilon,\xi_0})\phi_{\varepsilon,\xi_0}\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right|\\
&\quad+\left|\frac{q}{\varepsilon^2}\int_M b(x)\Psi'(W_{\varepsilon,\xi_0}+\theta\phi_{\varepsilon,\xi_0})(\phi_{\varepsilon,\xi_0})\phi_{\varepsilon,\xi_0}\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right|\\
&\quad+\left|\frac{q}{2\varepsilon^2}\int_M b(x)\Psi^2(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})(\phi_{\varepsilon,\xi_0})\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right|\\
&:= I_1+I_2+I_3+I_4+I_5
\end{aligned}
$$

From Lemma 5.4, Remark 5.2 and equations (2.8), (2.9), (2.6), (2.7), recalling that $\left\|\phi_{\varepsilon,\xi(y)}\right\|_\varepsilon \le C\varepsilon$, we get

$$
\begin{aligned}
I_1 &\le C\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}\left(\int_M[\Psi'(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})(\phi_{\varepsilon,\xi_0})]^3\right)^{\frac13}\left(\frac{1}{\varepsilon^2}\int_M W^3_{\varepsilon,\xi_0}\right)^{\frac13}\left(\frac{1}{\varepsilon^2}\int_M\left[\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right]^3\right)^{\frac13}\\
&\le C\varepsilon^{\frac43}\left(\int_{\mathbb{R}^2}\left[\sum_{k=1}^2\left|\frac{1}{\varepsilon}\frac{\partial U}{\partial z_k}(z)\chi(\varepsilon z)+\left(\chi(\varepsilon z)+\frac{\partial\chi}{\partial z_k}(\varepsilon z)\right)U(z)\right|\right]^3 dz\right)^{\frac13}\\
&\le C\varepsilon^{\frac43}\frac{1}{\varepsilon} = O(\varepsilon^{\frac13})
\end{aligned}
$$

In a similar way, using Lemma 5.4 and embedding the first and the second term in L^6 and the third one in $L^{3/2}$, we get

$$
I_4 \le C\frac{1}{\varepsilon^2}[\varepsilon^{4/3}\left\|\phi_{\varepsilon,\xi}\right\|_\varepsilon + \left\|\phi_{\varepsilon,\xi}\right\|^2_\varepsilon]\left\|\phi_{\varepsilon,\xi}\right\|_\varepsilon \varepsilon^{\frac43-1} = O(\varepsilon^{\frac43}).
$$

For I_3 by Lemma 5.3 we have

$$
\begin{aligned}
I_3 &\le C\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}\left(\int_M [\Psi(W_{\varepsilon,\xi_0})]^3\right)^{\frac{1}{3}}\left(\frac{1}{\varepsilon^2}\int_M \phi^3_{\varepsilon,\xi_0}\right)^{\frac{1}{3}}\left(\frac{1}{\varepsilon^2}\int_M \left[\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|y=0}\right]^3\right)^{\frac{1}{3}}\\
&\le C\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}\|\Psi(W_{\varepsilon,\xi_0})\|_g\|\phi_{\varepsilon,\xi_0}\|_\varepsilon\left(\int_{\mathbb{R}^2}\left[\sum_{k=1}^2\left|\frac{1}{\varepsilon}\frac{\partial U}{\partial z_k}(z)\chi(\varepsilon z)+\left(\chi(\varepsilon z)+\frac{\partial \chi}{\partial z_k}(\varepsilon z)\right)U(z)\right|\right]^3 dz\right)^{\frac{1}{3}}\\
&\le C\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}\varepsilon^{\frac{5}{3}}\varepsilon\frac{1}{\varepsilon}=O(\varepsilon)
\end{aligned}
$$

and, from the estimate for I_3, since $0<\Psi(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})<1/q$, we obtain

$$I_5\le CI_3=O(\varepsilon).$$

Finally, we prove (4.4). Following the proof of Lemma 5.1 in [3], we need only to prove that

$$\left|G'_\varepsilon\left(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}\right)[Z^l_{\varepsilon,\xi(y)}]\right|=o(1),$$

that is

$$\left|\frac{1}{\varepsilon^2}\int_M\left[\Psi(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})-q\Psi^2(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})\right](W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})Z^l_{\varepsilon,\xi(y)}\right|=o(1).$$

We have

$$
\begin{aligned}
&\left|\frac{1}{\varepsilon^2}\int_M\left[\Psi(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})-q\Psi^2(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})\right](W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})Z^l_{\varepsilon,\xi(y)}\right|\\
&\le\frac{C}{\varepsilon^2}\int_M\left|\Psi(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})Z^l_{\varepsilon,\xi(y)}\right|\\
&\qquad+\frac{C}{\varepsilon^2}\int_M\left|\Psi^2(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})Z^l_{\varepsilon,\xi(y)}\right|:=I_1+I_2.
\end{aligned}
$$

By Proposition 2.3, we have that $\|Z^l_{\varepsilon,\xi(y)}\|_\varepsilon=O(1)$. So, by Lemma 5.3 and Remark 5.2, we have

$$
\begin{aligned}
I_1&\le C\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}\left(\int_M[\Psi(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})]^3\right)^{\frac{1}{3}}\left(\frac{1}{\varepsilon^2}\int_M(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})^3\right)^{\frac{1}{3}}\left(\frac{1}{\varepsilon^2}\int_M|Z^l_{\varepsilon,\xi(y)}|^3\right)^{\frac{1}{3}}\\
&\le C\frac{\varepsilon^{\frac{4}{3}}}{\varepsilon^2}\|\Psi(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})\|_g\left(\|W_{\varepsilon,\xi_0}\|_{3,\varepsilon}+\|\phi_{\varepsilon,\xi_0}\|_\varepsilon\right)\|Z^l_{\varepsilon,\xi(y)}\|_\varepsilon=O(\varepsilon).
\end{aligned}
$$

Again, as $0<\Psi(W_{\varepsilon,\xi_0}+\phi_{\varepsilon,\xi_0})<1/q$, we obtain

$$I_2\le CI_1=O(\varepsilon).$$

This concludes the proof. □

Lemma 4.2. *The expansion*

$$I_\varepsilon(W_{\varepsilon,\xi})=\left(\frac{1}{2}-\frac{1}{p}\right)\frac{c(\xi)^{\frac{n}{2}}a(\xi)^{\frac{p}{p-2}-\frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}}\int_{\mathbb{R}^n}U^p dz+o(1)$$

holds true $\mathcal{C}^1$-uniformly with respect to $\xi\in M$.

Proof. In Lemma 5.2 of [3] we proved that

$$J_\varepsilon(W_{\varepsilon,\xi}) = \left(\frac{1}{2} - \frac{1}{p}\right) \frac{c(\xi)^{\frac{n}{2}} a(\xi)^{\frac{p}{p-2} - \frac{n}{2}}}{b(\xi)^{\frac{2}{p-2}}} \int_{\mathbb{R}^n} U^p dz + O(\varepsilon).$$

Hence, it suffices to show now that $|G_\varepsilon(W_{\varepsilon,\xi})| = o(1)$, $\mathcal{C}^1$-uniformly with respect to $\xi \in M$.

Regarding the $\mathcal{C}^0$-convergence, by Remark 5.2 and Lemma 5.3, we have that

$$\begin{aligned}
|G_\varepsilon(W_{\varepsilon,\xi})| &\leq \frac{C}{\varepsilon^2} \int_M \Psi(W_{\varepsilon,\xi}) W_{\varepsilon,\xi}^2 d\mu_g \\
&\leq C \frac{\varepsilon}{\varepsilon^2} \left(\int_M \Psi(W_{\varepsilon,\xi})^2 \right)^{\frac{1}{2}} \left(\frac{1}{\varepsilon^2} \int_M W_{\varepsilon,\xi}^4 \right)^{\frac{1}{2}} \\
&\leq C \frac{1}{\varepsilon} \|\Psi(W_{\varepsilon,\xi})\|_g \leq \frac{\varepsilon^{\frac{5}{3}}}{\varepsilon} = O(\varepsilon^{\frac{2}{3}}).
\end{aligned}$$

Regarding the $\mathcal{C}^1$-convergence observe that

$$\begin{aligned}
\left| \frac{\partial}{\partial y_h} G_\varepsilon(W_{\varepsilon,\xi}) \Big|_{y=0} \right| &\leq \left| \frac{C}{\varepsilon^2} \frac{\partial}{\partial y_h} \int_M \Psi(W_{\varepsilon,\xi(y)}) W_{\varepsilon,\xi(y)}^2 \Big|_{y=0} d\mu_g \right| \\
&\leq \left| \frac{C}{\varepsilon^2} \int_M \Psi(W_{\varepsilon,\xi(y)}) 2 W_{\varepsilon,\xi(y)} \left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} d\mu_g \right| \\
&\quad + \left| \frac{C}{\varepsilon^2} \int_M W_{\varepsilon,\xi(y)}^2 \Psi'(W_{\varepsilon,\xi(y)}) \left[\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \Big|_{y=0} \right] d\mu_g \right| \\
&:= I_1 + I_2.
\end{aligned}$$

Now, from Remark 5.2, Lemma 5.3, and the estimates (2.8) and (2.9), we derive

$$\begin{aligned}
I_1 &\leq C \frac{\varepsilon^{\frac{8}{5}}}{\varepsilon^2} \left(\int_M \Psi(W_{\varepsilon,\xi(y)})^5 \right)^{\frac{1}{5}} \left(\frac{1}{\varepsilon^2} \int_M W_{\varepsilon,\xi(y)}^{\frac{5}{2}} \right)^{\frac{2}{5}} \left(\frac{1}{\varepsilon^2} \int_M \left(\left(\frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right) \Big|_{y=0} \right)^{\frac{5}{2}} \right)^{\frac{2}{5}} \\
&\leq C \frac{\varepsilon^{\frac{8}{5}}}{\varepsilon^2} \varepsilon^{\frac{8}{5}} \frac{1}{\varepsilon} = o(1).
\end{aligned}$$

On the other hand, from Remark 5.2, the proof of Lemma 5.4, and the estimates (2.8) and (2.9), for some $t \in (1, 3/2)$ we obtain

$$\begin{aligned}
I_2 &\le C\frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2}\left(\frac{1}{\varepsilon^2}\int_M W_{\varepsilon,\xi(h)}^{2t}\right)^{\frac{1}{t}}\left(\int_M\left(\Psi'(W_{\varepsilon,\xi(y)})\left[\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(h)}\Big|_{y=0}\right]\right)^{t'}\right)^{\frac{1}{t'}}\\
&\le C\frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2}\left\|\Psi'(W_{\varepsilon,\xi(y)})\left[\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(h)}\Big|_{y=0}\right]\right\|_g\\
&\le C\frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2}\varepsilon^{\frac{4}{3}}\left|\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(h)}\Big|_{y=0}\right|_{g,6}\\
&\le C\frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2}\varepsilon^{\frac{4}{3}}\varepsilon^{\frac{1}{3}}\left(\frac{1}{\varepsilon^2}\int_M\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(h)}\Big|_{y=0}\right)^6\right)^{\frac{1}{6}}\\
&\le C\frac{\varepsilon^{\frac{2}{t}}}{\varepsilon^2}\varepsilon^{\frac{4}{3}}\varepsilon^{\frac{1}{3}}\frac{1}{\varepsilon} = C\varepsilon^{\frac{2}{t}-\frac{4}{3}} = o(1).
\end{aligned}$$

This concludes the proof. □

5. Some estimates involving Ψ

We start by pointing out the following facts.

Remark 5.1. *There exists a constant $C > 0$ such that, for every $\varphi \in H_g^1(M)$ and every $0 < \varepsilon < 1$, we have*

$$\begin{aligned}
C\|\varphi\|_g^2 &= C\int_M \left(|\nabla_g\varphi|^2 + \varphi^2\right) d\mu_g\\
&\le \int_M\left(c(x)|\nabla_g\varphi|^2 + \frac{d(x)}{\varepsilon^2}\varphi^2\right) d\mu_g = \|\varphi\|_\varepsilon^2.
\end{aligned}$$

Remark 5.2. *The following estimates*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left|W_{\varepsilon,\xi}\right|_{g,p}^p \le C|U|_p^p, \qquad p \ge 2,$$

$$\lim_{\varepsilon\to 0}\left|\nabla_g W_{\varepsilon,\xi}\right|_{g,2}^2 \le C|\nabla U|_2^2$$

hold true uniformly with respect to $\xi \in M$.

Abusing notation we write

$$\|u\|_g^2 = \int_M \left(c(x)|\nabla_g\varphi|^2 + b(x)u^2\right) d\mu_g.$$

This norm is equivalent to the standard norm (3.1) of $H_g^1(M)$. From equations (2.1), (2.2) and (2.3) we obtain

$$\begin{aligned}
\|\Psi(u)\|_g^2 &= \int_M b(x)qu^2\Psi(u)d\mu_g - \int_M b(x)q^2u^2\left(\Psi(u)\right)^2 d\mu_g \qquad (5.1)\\
&\le C\int_M u^2\Psi(u)d\mu_g,
\end{aligned}$$

$$\begin{aligned}\|\Psi'(u)\,[h]\,\|_g^2 &= \int_M 2b(x)qu(1-q\Psi(u))h\Psi'(u)\,[h]\,d\mu_g \\ &\quad - \int_M b(x)q^2u^2\left(\Psi'(u)\,[h]\right)^2 d\mu_g \\ &\leq C\int_M |u|\,|h|\,|\Psi'(u)\,[h]|\,d\mu_g,\end{aligned} \tag{5.2}$$

for all $u, h \in H_g^1(M)$.

Lemma 5.3. *Given $\vartheta \in (1,2)$ there is a constant $C > 0$ such that the inequality*

$$\|\Psi(W_{\varepsilon,\xi}+\varphi)\|_g \leq C(\varepsilon^\vartheta + \|\varphi\|_g^2)$$

holds true for every $\varphi \in H_g^1(M)$, $\xi \in M$ and small enough $\varepsilon > 0$.

Proof. Let $t \in (2,\infty)$ be such that $\frac{2}{t'} = \vartheta$ where t' is the exponent conjugate to t. From inequality (5.1) we obtain

$$\begin{aligned}\|\Psi(W_{\varepsilon,\xi}+\varphi)\|_g^2 &\leq C\left(\int_M \left[\Psi\left(W_{\varepsilon,\xi}+\varphi\right)\right]^t d\mu_g\right)^{1/t}\left(\int_M (W_{\varepsilon,\xi}+\varphi)^{2t'}\right)^{1/t'} \\ &\leq C\|\Psi(W_{\varepsilon,\xi}+\varphi)\|_g \left|W_{\varepsilon,\xi}+\varphi\right|_{g,2t'}^2.\end{aligned}$$

Thus, by Remark 5.2,

$$\begin{aligned}\|\Psi(W_{\varepsilon,\xi}+\varphi)\|_g &\leq C\left(\varepsilon^{2/t'}\left(\frac{1}{\varepsilon^2}\int_M W_{\varepsilon,\xi}^{2t'}\right)^{1/t'} + \left(\int_M \varphi^{2t'}\right)^{1/t'}\right) \\ &\leq C(\varepsilon^\vartheta + \|\varphi\|_g^2),\end{aligned}$$

as claimed. □

Lemma 5.4. *Given $s \in (1,2)$ there is a constant $C > 0$ such that the inequality*

$$\|\Psi'(W_{\varepsilon,\xi}+k)[h]\|_g \leq C\|h\|_g\left(\varepsilon^{\frac{2}{s}} + \|k\|_g\right)$$

holds true for every $k, h \in H_g^1(M)$, $\xi \in M$ and small enough $\varepsilon > 0$.

Proof. From inequality (5.2) we obtain,

$$\begin{aligned}\|\Psi'(W_{\varepsilon,\xi}+k)[h]\|_g^2 &\leq C\int_M |W_{\varepsilon,\xi}+k|\,|h|\,|\Psi'(W_{\varepsilon,\xi}+k)\,[h]|\,d\mu_g \\ &\leq C\left(\int_M |W_{\varepsilon,\xi}|\,|h|\,|\Psi'(W_{\varepsilon,\xi}+k)\,[h]|\,d\mu_g + \int_M |k|\,|h|\,|\Psi'(W_{\varepsilon,\xi}+k)\,[h]|\,d\mu_g\right) \\ &=: I_1 + I_2.\end{aligned}$$

Set $t := 2s' \in (4,\infty)$, where s' is the conjugate exponent to s. Using Remark 5.2 we conclude that

$$\begin{aligned}I_1 &\leq C\left|\Psi'(W_{\varepsilon,\xi}+k)[h]\right|_{g,t}|h|_{g,t}\left|W_{\varepsilon,\xi}\right|_{g,s} \\ &= C\|\Psi'(W_{\varepsilon,\xi}+k)[h]\|_g\|h\|_g\varepsilon^{\frac{2}{s}}\left(\frac{1}{\varepsilon^2}\int_M W_{\varepsilon,\xi}^s\right)^{1/s} \\ &= C\|\Psi'(W_{\varepsilon,\xi}+k)[h]\|_g\|h\|_g\varepsilon^{\frac{2}{s}}.\end{aligned}$$

Since

$$I_2 \leq C\left|\Psi'(W_{\varepsilon,\xi}+k)[h]\right|_{g,3}|h|_{g,3}\left|k\right|_{g,3} \leq C\|\Psi'(W_{\varepsilon,\xi}+k)[h]\|_g\|h\|_g\|k\|_g,$$

the claim follows. □

Lemma 5.5. *Consider the functions*

$$\tilde{v}_{\varepsilon,\xi}(z) := \begin{cases} \Psi(W_{\varepsilon,\xi})\left(\exp_\xi(\varepsilon z)\right) & \text{for } z \in B(0, r/\varepsilon), \\ 0 & \text{for } z \in \mathbb{R}^2 \smallsetminus B(0, r/\varepsilon). \end{cases}$$

Then, for any $\vartheta \in (1,2)$, there exists a constant $C > 0$, independent of ε, ξ, such that

$$\left|\tilde{v}_{\varepsilon,\xi}(z)\right|_{L^2(\mathbb{R}^3)} \le C\varepsilon^{\vartheta-1},$$
$$\left|\nabla\tilde{v}_{\varepsilon,\xi}(z)\right|_{L^2(\mathbb{R}^3)} \le C\varepsilon^{\vartheta}.$$

Proof. After a change of variables we have that

$$\int_{B_g(\xi,r)} |\nabla\Psi(W_{\varepsilon,\xi})|^2 + |\Psi(W_{\varepsilon,\xi})|^2 d\mu_g$$
$$= \varepsilon^2 \int_{B(0,r/\varepsilon)} |g_\xi(\varepsilon z)|^{1/2} \left(\sum_{ij} g_\xi^{ij}(\varepsilon z) \frac{1}{\varepsilon^2} \frac{\partial \tilde{v}_{\varepsilon,\xi}(z)}{\partial z_i} \frac{\partial \tilde{v}_{\varepsilon,\xi}(z)}{\partial z_i} + \tilde{v}^2_{\varepsilon,\xi}(z) \right) dz.$$

Thus

$$\|\Psi(W_{\varepsilon,\xi})\|_g^2 \ge C(|\nabla\tilde{v}_{\varepsilon,\xi}|^2_{L^2(\mathbb{R}^3)} + \varepsilon^2 \left|\tilde{v}_{\varepsilon,\xi}\right|^2_{L^2(\mathbb{R}^3)}).$$

This, combined with Lemma 5.3, gives

$$|\nabla\tilde{v}_{\varepsilon,\xi}|_{L^2(\mathbb{R}^3)} + \varepsilon \left|\tilde{v}_{\varepsilon,\xi}\right|_{L^2(\mathbb{R}^3)} \le C\varepsilon^{\vartheta},$$

as claimed. □

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