Linear wave equations with time-dependent propagation speed and strong damping

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Abstract

We consider a second order linear equation with a time-dependent coefficient $c(t)$ in front of the "elastic" operator. For these equations it is well-known that a higher space-regularity of initial data compensates a lower time-regularity of $c(t)$.

In this paper we investigate the influence of a strong dissipation, namely a friction term which depends on a power of the elastic operator.

What we discover is a threshold effect. When the exponent of the elastic operator in the friction term is greater than $1/2$, the damping prevails and the equation behaves as if the coefficient $c(t)$ were constant. When the exponent is less than $1/2$, the time-regularity of $c(t)$ comes into play. If $c(t)$ is regular enough, once again the damping prevails. On the contrary, when $c(t)$ is not regular enough the damping might be ineffective, and there are examples in which the dissipative equation behaves as the non-dissipative one. As expected, the stronger is the damping, the lower is the time-regularity threshold.

We also provide counterexamples showing the optimality of our results.

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1 Introduction

Let H be a separable real Hilbert space. For every x and y in H, $|x|$ denotes the norm of x, and $\langle x, y\rangle$ denotes the scalar product of x and y. Let A be a self-adjoint linear operator on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Ax, x\rangle \geq 0$ for every $x \in D(A)$, so that for every $\alpha \geq 0$ the power $A^{\alpha}x$ is defined provided that x lies in a suitable domain $D(A^{\alpha})$.

We consider the second order linear evolution equation

$$u''(t) + 2\delta A^{\sigma} u'(t) + c(t)Au(t) = 0, \tag{1.1}$$

with initial data

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{1.2}$$

As far as we know, this equation has been considered in the literature either in the case where $\delta = 0$, or in the case where $\delta > 0$ but the coefficient $c(t)$ is constant. Let us give a brief outline of the previous literature which is closely related to our results.

The non-dissipative case When $\delta = 0$, equation (1.1) reduces to

$$u''(t) + c(t)Au(t) = 0. \tag{1.3}$$

This is the abstract setting of a wave equation in which $c(t)$ represents the square of the propagation speed.

If the coefficient $c(t)$ is Lipschitz continuous and satisfies the strict hyperbolicity condition

$$0 < \mu_1 \leq c(t) \leq \mu_2, \tag{1.4}$$

then it is well-know that problem (1.3)–(1.2) is well-posed in the classic energy space $D(A^{1/2}) \times H$ (see for example the classic reference [16]).

If the coefficient is not Lipschitz continuous, things are more complex, even if (1.4) still holds true. This problem was addressed by F. Colombini, E. De Giorgi and S. Spagnolo in the seminal paper [6]. Their results can be summed up as follows (we refer to section 2 below for the precise functional setting and rigorous statements).

(1) Problem (1.3)–(1.2) has always a unique solution, up to admitting that this solution takes its values in a very large Hilbert space (ultradistributions). This is true for initial data in the energy space $D(A^{1/2}) \times H$, but also for less regular data, such as distributions or ultradistributions.

(2) If initial data are regular enough, then the solution is regular as well. How much regularity is required depends on the time-regularity of $c(t)$. Classic examples are the following. If $c(t)$ is just measurable, problem (1.3)–(1.2) is well-posed in the class of analytic functions. If $c(t)$ is α-Hölder continuous for some $\alpha \in (0, 1)$, problem (1.3)–(1.2) is well-posed in the Gevrey space of order $(1 - \alpha)^{-1}$.

(3) If initial data are not regular enough, then the solution may exhibit a severe derivative loss for all positive times. For example, for every $\alpha \in (0,1)$ there exist a coefficient $c(t)$ which is α-Hölder continuous, and initial data (u_0, u_1) which are in the Gevrey class of order β for every $\beta > (1-\alpha)^{-1}$, such that the corresponding solution to (1.3)–(1.2) (which exists in the weak sense of point (1)) is not even a distribution for every $t > 0$.

In the sequel we call (DGCS)-phenomenon the instantaneous loss of regularity described in point (3) above.

The dissipative case with constant coefficients If $\delta > 0$ and $c(t)$ is a constant function (equal to 1 without loss of generality), equation (1.1) reduces to

$$u''(t) + 2\delta A^{\sigma} u'(t) + Au(t) = 0. \tag{1.5}$$

Equations of this type have been considered in mathematical literature from different points of view. The original work [1] introduced more general equations of the form

$$u''(t) + Bu'(t) + Au(t) = 0, \tag{1.6}$$

where the dissipation operator B is comparable to a fractional power A^{σ} of the elastic operator in the sense of inner products. This generality accommodates many physically relevant boundary conditions. The model (1.6) has been rigorously investigated in the late 80s in a series of papers [2, 3, 4], where it was shown that $\sigma = 1/2$ determines a threshold effect: for $\sigma \in [1/2, 1]$ equation (1.6) generates an analytic contraction semigroup on the classical energy space, while for $\sigma \in (0, 1/2)$ the semigroup is in general not analytic but just Gevrey-class of specific order depending on σ, again on the classical energy space. In some sense, these models exhibit both hyperbolic and parabolic features, with parabolicity prevailing in the range $\sigma \in [1/2, 1]$. We refer to [15] for a summary of this theory, and to [11] for a proof of analyticity or Gevrey regularity of the semigroup in the simpler case where $B = A^{\sigma}$. The general assumptions in these papers are that the operator A is strictly positive, $\sigma \in [0, 1]$, and the phase space is $D(A^{1/2}) \times H$ and sometimes $H \times H$ when $\sigma = 1$.

On a different side, the community working on dispersive equations considered equation (1.5) in the concrete case where $\sigma \in [0, 1]$ and $Au = -\Delta u$ in $\mathbb{R}^n$ or in special classes of unbounded domains. They proved energy decay and dispersive estimates, but exploiting in an essential way the spectral properties of the Laplacian in those domains. The interested reader is referred to [12, 13, 14, 21] and to the references quoted therein.

Finally, equation (1.5) was considered in [10] in full generality, namely for every $\sigma \geq 0$ and every nonnegative self-adjoint operator A. As in previous literature, two different regimes appeared. In the subcritical regime $\sigma \in [0, 1/2]$, problem (1.5)–(1.2) is well-posed in the classic energy space $D(A^{1/2}) \times H$ or more generally in $D(A^{\alpha+1/2}) \times D(A^{\alpha})$ with $\alpha \geq 0$. In the supercritical regime $\sigma \geq 1/2$, problem (1.5)–(1.2) is well-posed in $D(A^{\alpha}) \times D(A^{\beta})$ if and only if

$$1 - \sigma \leq \alpha - \beta \leq \sigma. \tag{1.7}$$

This means that in the supercritical regime different choices of the phase space are possible, even with $\alpha - \beta \neq 1/2$.

The dissipative case with time-dependent coefficients As far as we know, the case of a dissipative equation with a time-dependent propagation speed had not been considered yet. The main question we address in this paper is the extent to which the dissipative term added in (1.1) prevents the (DGCS)-phenomenon of (1.3) from happening. We discover a composite picture, depending on σ.

- In the subcritical regime $\sigma \in [0, 1/2]$, if the strict hyperbolicity assumption (1.4) is satisfied, well-posedness results do depend on the time-regularity of $c(t)$ (see Theorem 3.2). Classic examples are the following.
  - If $c(t)$ is α-Hölder continuous for some exponent $\alpha > 1 - 2\sigma$, then the dissipation prevails, and problem (1.1)–(1.2) is well-posed in the classic energy space $D(A^{1/2}) \times H$ or more generally in $D(A^{\beta+1/2}) \times D(A^{\beta})$ with $\beta \geq 0$.
  - If $c(t)$ is no more than α-Hölder continuous for some exponent $\alpha < 1 - 2\sigma$, then the dissipation can be neglected, so that (1.1) behaves exactly as the non-dissipative equation (1.3). This means well-posedness in the Gevrey space of order $(1 - \alpha)^{-1}$ and the possibility to produce the (DGCS)-phenomenon for less regular data (see Theorem 3.10).
  - The case with $\alpha = 1 - 2\sigma$ is critical and also the size of the Hölder constant of $c(t)$ compared with δ comes into play.
- In the supercritical regime $\sigma > 1/2$ the dissipation prevails in an overwhelming way. In Theorem 3.1 we prove that, if $c(t)$ is just in $L^{\infty}((0, +\infty))$ and satisfies just the degenerate hyperbolicity condition

$$0 \leq c(t) \leq \mu_2, \tag{1.8}$$

  then (1.1) behaves as (1.5). This means that problem (1.1)–(1.2) is well-posed in $D(A^{\alpha}) \times D(A^{\beta})$ if and only if (1.7) is satisfied, the same result obtained in [10] in the case of a constant coefficient.

The second issue we address in this paper is the further space-regularity of solutions for positive times, since a strong dissipation is expected to have a regularizing effect similar to parabolic equations. This turns out to be true provided that the assumptions of our well-posedness results are satisfied, and in addition $\sigma \in (0, 1)$. Indeed, we prove that in this regime $u(t)$ lies in the Gevrey space of order $(2 \min\{\sigma, 1 - \sigma\})^{-1}$ for every $t > 0$. We refer to Theorem 3.8 and Theorem 3.9 for the details. This effect had already been observed in [17] in the dispersive case.

We point out that the regularizing effect is maximum when $\sigma = 1/2$ (the only case in which solutions become analytic with respect to space variables) and disappears when $\sigma \geq 1$, meaning that a stronger overdamping prevents smoothing.

Overview of the technique The spectral theory reduces the problem to an analysis of the family of ordinary differential equations

$$u''_\lambda(t) + 2\delta\lambda^{2\sigma}u'_\lambda(t) + \lambda^2 c(t)u_\lambda(t) = 0. \tag{1.9}$$

When $\delta = 0$, a coefficient $c(t)$ which oscillates with a suitable period can produce a resonance effect so that (1.9) admits a solution whose oscillations have an amplitude which grows exponentially with time. This is the primordial origin of the (DGCS)-phenomenon for non-dissipative equations. When $\delta > 0$, the damping term causes an exponential decay of the amplitude of oscillations. *The competition between the exponential energy growth due to resonance and the exponential energy decay due to dissipation originates the threshold effect we observed.*

When $c(t)$ is constant, equation (1.9) can be explicitly integrated, and the explicit formulae for solutions led to the sharp results of [10]. Here we need the same sharp estimates, but without relying on explicit solutions. To this end, we introduce suitable energy estimates.

In the supercritical regime $\sigma \geq 1/2$ we exploit the following σ-adapted "Kovaleskyan energy"

$$E(t) := |u'_\lambda(t) + \delta\lambda^{2\sigma}u_\lambda(t)|^2 + \delta^2\lambda^{4\sigma}|u_\lambda(t)|^2. \tag{1.10}$$

In the subcritical regime $\sigma \leq 1/2$ we exploit the so-called "approximated hyperbolic energies"

$$E_\varepsilon(t) := |u'_\lambda(t) + \delta\lambda^{2\sigma}u_\lambda(t)|^2 + \delta^2\lambda^{4\sigma}|u_\lambda(t)|^2 + \lambda^2 c_\varepsilon(t)|u_\lambda(t)|^2, \tag{1.11}$$

obtained by adding to (1.10) an "hyperbolic term" depending on a suitable smooth approximation $c_\varepsilon(t)$ of $c(t)$, which in turn is chosen in a λ-dependent way. Terms of this type are the key tool introduced in [6] for the non-dissipative equation.

Future extensions We hope that this paper could represent a first step in the study of strongly dissipative equations with variable coefficients, both linear and nonlinear. For the time being, the theory developed in this paper allowed the authors to improve in [9] the classic results by K. Nishihara [18, 19] for Kirchhoff equations, whose linearization has indeed a variable propagation speed depending only on time. Next step in this direction could be considering a coefficient $c(x,t)$ depending both on time and space variables, in view of applications to more general quasilinear equations.

In a different direction, the subcritical case $\sigma \in [0, 1/2]$ with degenerate hyperbolicity assumptions remains open and could be the subject of further research, in the same way as [7] was the follow-up of [6].

Last but not least, it could be interesting to extend our theory in order to include more general dissipation operators as in the original models (1.6). To this end, on the one hand our counterexamples provide a bound to what can be proved even in this general case. On the other hand, a proof of the well-posedness results with more general dissipation operators could require new ideas, since our techniques seem to be tailored on the case where B is a power of A, or at least B commutes with A.

As for the counterexamples, which we consider the main contribution of this paper, we hope that they could finally dispel the diffuse misconception according to which dissipative hyperbolic equations are more stable, and hence definitely easier to handle. Now we know that a friction term below a suitable threshold is substantially ineffective, opening the door to pathologies such as the (DGCS)-phenomenon, exactly as in the non-dissipative case.

Structure of the paper This paper is organized as follows. In section 2 we introduce the functional setting and we recall the classic existence results from [6]. In section 3 we state our main results. In section 4 we provide a heuristic description of the competition between resonance and decay. In section 5 we prove our existence and regularity results. In section 6 we present our examples of (DGCS)-phenomenon.

2 Notation and previous results

Functional spaces Let H be a separable Hilbert space. Let us assume that H admits a countable complete orthonormal system $\{e_k\}_{k\in\mathbb{N}}$ made by eigenvectors of A. We denote the corresponding eigenvalues by λ_k^2 (with the agreement that $\lambda_k \geq 0$), so that $Ae_k = \lambda_k^2 e_k$ for every $k \in \mathbb{N}$. In this case every $u \in H$ can be written in a unique way in the form $u = \sum_{k=0}^{\infty} u_k e_k$, where $u_k = \langle u, e_k\rangle$ are the Fourier components of u. In other words, the Hilbert space H can be identified with the set of sequences $\{u_k\}$ of real numbers such that $\sum_{k=0}^{\infty} u_k^2 < +\infty$.

We stress that this is just a simplifying assumption, with substantially no loss of generality. Indeed, according to the spectral theorem in its general form (see for example Theorem VIII.4 in [20]), one can always identify H with $L^2(M,\mu)$ for a suitable measure space (M,μ), in such a way that under this identification the operator A acts as a multiplication operator by some measurable function $\lambda^2(\xi)$. All definitions and statements in the sequel, with the exception of the counterexamples of Theorem 3.10, can be easily extended to the general setting just by replacing the sequence $\{\lambda_k^2\}$ with the function $\lambda^2(\xi)$, and the sequence $\{u_k\}$ of Fourier components of u with the element $\widehat{u}(\xi)$ of $L^2(M,\mu)$ corresponding to u under the identification of H with $L^2(M,\mu)$.

The usual functional spaces can be characterized in terms of Fourier components as follows.

Definition 2.1. Let u be a sequence $\{u_k\}$ of real numbers.

- *Sobolev spaces.* For every $\alpha \geq 0$ it turns out that $u \in D(A^\alpha)$ if

$$\|u\|_{D(A^\alpha)}^2 := \sum_{k=0}^{\infty} (1+\lambda_k)^{4\alpha} u_k^2 < +\infty. \tag{2.1}$$

- *Distributions.* We say that $u \in D(A^{-\alpha})$ for some $\alpha \geq 0$ if

$$\|u\|_{D(A^{-\alpha})}^2 := \sum_{k=0}^{\infty} (1+\lambda_k)^{-4\alpha} u_k^2 < +\infty. \tag{2.2}$$

- *Generalized Gevrey spaces.* Let $\varphi : [0, +\infty) \to [0, +\infty)$ be any function, let $r \geq 0$, and let $\alpha \in \mathbb{R}$. We say that $u \in \mathcal{G}_{\varphi,r,\alpha}(A)$ if

$$\|u\|_{\varphi,r,\alpha}^2 := \sum_{k=0}^{\infty} (1 + \lambda_k)^{4\alpha} u_k^2 \exp\big(2r\varphi(\lambda_k)\big) < +\infty. \tag{2.3}$$

- *Generalized Gevrey ultradistributions.* Let $\psi : [0, +\infty) \to [0, +\infty)$ be any function, let $R \geq 0$, and let $\alpha \in \mathbb{R}$. We say that $u \in \mathcal{G}_{-\psi,R,\alpha}(A)$ if

$$\|u\|_{-\psi,R,\alpha}^2 := \sum_{k=0}^{\infty} (1 + \lambda_k)^{4\alpha} u_k^2 \exp\big(-2R\psi(\lambda_k)\big) < +\infty. \tag{2.4}$$

Remark 2.2. If $\varphi_1(x) = \varphi_2(x)$ for every $x > 0$, then $\mathcal{G}_{\varphi_1,r,\alpha}(A) = \mathcal{G}_{\varphi_2,r,\alpha}(A)$ for every admissible value of r and α. For this reason, with a little abuse of notation, we consider the spaces $\mathcal{G}_{\varphi,r,\alpha}(A)$ even when $\varphi(x)$ is defined only for $x > 0$. The same comment applies also to the spaces $\mathcal{G}_{-\psi,R,\alpha}(A)$.

The quantities defined in (2.1) through (2.4) are actually norms which induce a Hilbert space structure on $D(A^\alpha)$, $\mathcal{G}_{\varphi,r,\alpha}(A)$, $\mathcal{G}_{-\psi,R,\alpha}(A)$, respectively. The standard inclusions

$$\mathcal{G}_{\varphi,r,\alpha}(A) \subseteq D(A^\alpha) \subseteq H \subseteq D(A^{-\alpha}) \subseteq \mathcal{G}_{-\psi,R,-\alpha}(A)$$

hold true for every $\alpha \geq 0$ and every admissible choice of φ, ψ, r, R. All inclusions are strict if α, r and R are positive, and the sequences $\{\lambda_k\}$, $\{\varphi(\lambda_k)\}$, and $\{\psi(\lambda_k)\}$ are unbounded.

We observe that $\mathcal{G}_{\varphi,r,\alpha}(A)$ is actually a so-called *scale of Hilbert spaces* with respect to the parameter r, with larger values of r corresponding to smaller spaces. Analogously, $\mathcal{G}_{-\psi,R,\alpha}(A)$ is a scale of Hilbert spaces with respect to the parameter R, but with larger values of R corresponding to larger spaces.

Remark 2.3. Let us consider the concrete case where $I \subseteq \mathbb{R}$ is an open interval, $H = L^2(I)$, and $Au = -u_{xx}$, with periodic boundary conditions. For every $\alpha \geq 0$, the space $D(A^\alpha)$ is actually the usual Sobolev space $H^{2\alpha}(I)$, and $D(A^{-\alpha})$ is the usual space of distributions of order 2α.

When $\varphi(x) := x^{1/s}$ for some $s > 0$, elements of $\mathcal{G}_{\varphi,r,0}(A)$ with $r > 0$ are usually called Gevrey functions of order s, the case $s = 1$ corresponding to analytic functions. When $\psi(x) := x^{1/s}$ for some $s > 0$, elements of $\mathcal{G}_{-\psi,R,0}(A)$ with $R > 0$ are usually called Gevrey ultradistributions of order s, the case $s = 1$ corresponding to analytic functionals. In this case the parameter α is substantially irrelevant because the exponential term is dominant both in (2.3) and in (2.4).

For the sake of consistency, with a little abuse of notation we use the same terms (Gevrey functions, Gevrey ultradistributions, analytic functions and analytic functionals) in order to denote the same spaces also in the general abstract framework. To be more precise, we should always add "with respect to the operator A", or even better "with respect to the operator $A^{1/2}$".

Continuity moduli Throughout this paper we call *continuity modulus* any continuous function $\omega : [0, +\infty) \to [0, +\infty)$ such that $\omega(0) = 0$, $\omega(x) > 0$ for every $x > 0$, and moreover

$$x \to \omega(x) \text{ is a nondecreasing function,} \tag{2.5}$$

$$x \to \frac{x}{\omega(x)} \text{ is a nondecreasing function.} \tag{2.6}$$

A function $c : [0, +\infty) \to \mathbb{R}$ is said to be ω-continuous if

$$|c(a) - c(b)| \leq \omega(|a - b|) \qquad \forall a \geq 0,\ \forall b \geq 0. \tag{2.7}$$

More generally, a function $c : X \to \mathbb{R}$ (with $X \subseteq \mathbb{R}$) is said to be ω-continuous if it satisfies the same inequality for every a and b in X.

Previous results We are now ready to recall the classic results concerning existence, uniqueness, and regularity for solutions to problem (1.1)–(1.2). We state them using our notations which allow general continuity moduli and general spaces of Gevrey functions or ultradistributions.

Proofs are a straightforward application of the approximated energy estimates introduced in [6]. In that paper only the case $\delta = 0$ is considered, but when $\delta \geq 0$ all new terms have the "right sign" in those estimates.

The first result concerns existence and uniqueness in huge spaces such as analytic functionals, with minimal assumptions on $c(t)$.

Theorem A (see [6, Theorem 1]). *Let us consider problem (1.1)–(1.2) under the following assumptions:*

- *A is a self-adjoint nonnegative operator on a separable Hilbert space H,*
- *$c \in L^1((0, T))$ for every $T > 0$ (without sign conditions),*
- *$\sigma \geq 0$ and $\delta \geq 0$ are two real numbers,*
- *initial conditions satisfy*

$$(u_0, u_1) \in \mathcal{G}_{-\psi,R_0,1/2}(A) \times \mathcal{G}_{-\psi,R_0,0}(A)$$

for some $R_0 > 0$ and some $\psi : (0, +\infty) \to (0, +\infty)$ such that

$$\limsup_{x\to+\infty} \frac{x}{\psi(x)} < +\infty.$$

Then there exists a nondecreasing function $R : [0, +\infty) \to [0, +\infty)$, with $R(0) = R_0$, such that problem (1.1)–(1.2) admits a unique solution

$$u \in C^0\left([0, +\infty); \mathcal{G}_{-\psi,R(t),1/2}(A)\right) \cap C^1\left([0, +\infty); \mathcal{G}_{-\psi,R(t),0}(A)\right). \tag{2.8}$$

Condition (2.8), with the range space increasing with time, simply means that

$$u \in C^0\left([0,\tau];\mathcal{G}_{-\psi,R(\tau),1/2}(A)\right) \cap C^1\left([0,\tau];\mathcal{G}_{-\psi,R(\tau),0}(A)\right) \qquad \forall \tau \geq 0.$$

This amounts to say that scales of Hilbert spaces, rather than fixed Hilbert spaces, are the natural setting for this problem.

In the second result we assume strict hyperbolicity and ω-continuity of the coefficient, and we obtain well-posedness in a suitable class of Gevrey ultradistributions.

Theorem B (see [6, Theorem 3]). *Let us consider problem (1.1)–(1.2) under the following assumptions:*

- *A is a self-adjoint nonnegative operator on a separable Hilbert space H,*
- *the coefficient $c : [0,+\infty) \to \mathbb{R}$ satisfies the strict hyperbolicity assumption (1.4) and the ω-continuity assumption (2.7) for some continuity modulus $\omega(x)$,*
- *$\sigma \geq 0$ and $\delta \geq 0$ are two real numbers,*
- *initial conditions satisfy*

$$(u_0, u_1) \in \mathcal{G}_{-\psi,R_0,1/2}(A) \times \mathcal{G}_{-\psi,R_0,0}(A)$$

for some $R_0 > 0$ and some function $\psi : (0,+\infty) \to (0,+\infty)$ such that

$$\limsup_{x\to+\infty} \frac{x}{\psi(x)}\,\omega\left(\frac{1}{x}\right) < +\infty. \tag{2.9}$$

Let u be the unique solution to the problem provided by Theorem A.
Then there exists $R > 0$ such that

$$u \in C^0\left([0,+\infty),\mathcal{G}_{-\psi,R_0+Rt,1/2}(A)\right) \cap C^1\left([0,+\infty),\mathcal{G}_{-\psi,R_0+Rt,0}(A)\right).$$

The third result we recall concerns existence of regular solutions. The assumptions on $c(t)$ are the same as in Theorem B, but initial data are significantly more regular (Gevrey spaces instead of Gevrey ultradistributions).

Theorem C (see [6, Theorem 2]). *Let us consider problem (1.1)–(1.2) under the following assumptions:*

- *A is a self-adjoint nonnegative operator on a separable Hilbert space H,*
- *the coefficient $c : [0,+\infty) \to \mathbb{R}$ satisfies the strict hyperbolicity assumption (1.4) and the ω-continuity assumption (2.7) for some continuity modulus $\omega(x)$,*
- *$\sigma \geq 0$ and $\delta \geq 0$ are two real numbers,*

- *initial conditions satisfy*

$$(u_0, u_1) \in \mathcal{G}_{\varphi,r_0,1/2}(A) \times \mathcal{G}_{\varphi,r_0,0}(A)$$

for some $r_0 > 0$ *and some function* $\varphi : (0, +\infty) \to (0, +\infty)$ *such that*

$$\limsup_{x \to +\infty} \frac{x}{\varphi(x)}\, \omega\left(\frac{1}{x}\right) < +\infty. \tag{2.10}$$

Let u *be the unique solution to the problem provided by Theorem A.*
Then there exist $T > 0$ *and* $r > 0$ *such that* $rT < r_0$ *and*

$$u \in C^0\left([0, T], \mathcal{G}_{\varphi,r_0-rt,1/2}(A)\right) \cap C^1\left([0, T], \mathcal{G}_{\varphi,r_0-rt,0}(A)\right). \tag{2.11}$$

Remark 2.4. The key assumptions of Theorem B and Theorem C are (2.9) and (2.10), respectively, representing the exact compensation between space-regularity of initial data and time-regularity of the coefficient $c(t)$ required in order to obtain well-posedness.

These conditions do not appear explicitly in [6], where they are replaced by suitable specific choices of ω, φ, ψ, which of course satisfy the same relations. To our knowledge, those conditions were stated for the first time in [8], thus unifying several papers that in the last 30 years had been devoted to special cases (see for example [5] and the references quoted therein).

Remark 2.5. The standard example of application of Theorem B and Theorem C is the following. Let us assume that $c(t)$ is α-Hölder continuous for some $\alpha \in (0, 1)$, namely $\omega(x) = Mx^\alpha$ for a suitable constant M. Then (2.9) and (2.10) hold true with $\psi(x) = \varphi(x) := x^{1-\alpha}$. This leads to well-posedness both in the large space of Gevrey ultradistributions of order $(1 - \alpha)^{-1}$, and in the small space of Gevrey functions of the same order.

Remark 2.6. The choice of ultradistributions in Theorem B is not motivated by the search for generality, but it is in some sense the only possible one because of the (DGCS)-phenomenon exhibited in [6], at least in the non-dissipative case. When $\delta = 0$, if initial data are taken in Sobolev spaces or in any space larger than the Gevrey spaces of Theorem C, then it may happen that for all positive times the solution lies in the space of ultradistributions specified in Theorem B, and nothing more. In other words, for $\delta = 0$ there is no well-posedness result in between the Gevrey spaces of Theorem C and the Gevrey ultradistributions of Theorem B, and conditions (2.9) and (2.10) are optimal.

The aim of this paper is to provide an optimal picture for the case $\delta > 0$.

3 Main results

In this section we state our main regularity results for solutions to (1.1)–(1.2). To this end, we need some further notation. Given any $\nu \geq 0$, we write H as an orthogonal direct sum

$$H := H_{\nu,-} \oplus H_{\nu,+}, \tag{3.1}$$

where $H_{\nu,-}$ is the closure of the subspace generated by all eigenvectors of A relative to eigenvalues $\lambda_k < \nu$, and $H_{\nu,+}$ is the closure of the subspace generated by all eigenvectors of A relative to eigenvalues $\lambda_k \geq \nu$. For every vector $u \in H$, we write $u_{\nu,-}$ and $u_{\nu,+}$ to denote its components with respect to the decomposition (3.1). We point out that $H_{\nu,-}$ and $H_{\nu,+}$ are A-invariant subspaces of H, and that A is a bounded operator when restricted to $H_{\nu,-}$, and a coercive operator when restricted to $H_{\nu,+}$ if $\nu > 0$.

In the following statements we provide separate estimates for low-frequency components $u_{\nu,-}(t)$ and high-frequency components $u_{\nu,+}(t)$ of solutions to (1.1). This is due to the fact that the energy of $u_{\nu,-}(t)$ can be unbounded as $t \to +\infty$, while in many cases we are able to prove that the energy of $u_{\nu,+}(t)$ is bounded in time.

3.1 Existence results in Sobolev spaces

The first result concerns the supercritical regime $\sigma \geq 1/2$, in which case the dissipation always dominates the time-dependent coefficient.

Theorem 3.1 (Supercritical dissipation)**.** *Let us consider problem (1.1)–(1.2) under the following assumptions:*

- *A is a self-adjoint nonnegative operator on a separable Hilbert space H,*
- *the coefficient $c : [0, +\infty) \to \mathbb{R}$ belongs also to $L^\infty((0, +\infty))$ and satisfies the degenerate hyperbolicity assumption (1.8),*
- *σ and δ are two positive real numbers such that either $\sigma > 1/2$, or $\sigma = 1/2$ and $4\delta^2 \geq \mu_2$,*
- *$(u_0, u_1) \in D(A^\alpha) \times D(A^\beta)$ for some real numbers α and β satisfying (1.7).*

Let u be the unique solution to the problem provided by Theorem A.
Then u actually satisfies

$$(u, u') \in C^0\left([0, +\infty), D(A^\alpha) \times D(A^\beta)\right). \tag{3.2}$$

Moreover, for every $\nu \geq 1$ such that $4\delta^2\nu^{4\sigma-2} \geq \mu_2$, it turns out that

$$|A^\beta u'_{\nu,+}(t)|^2 + |A^\alpha u_{\nu,+}(t)|^2 \leq \left(2 + \frac{2}{\delta^2} + \frac{\mu_2^2}{\delta^4}\right) |A^\beta u_{1,\nu,+}|^2 + 3\left(1 + \frac{\mu_2^2}{2\delta^2}\right) |A^\alpha u_{0,\nu,+}|^2 \tag{3.3}$$

for every $t \geq 0$.

Our second result concerns the subcritical regime $\sigma \in [0, 1/2]$, in which case the time-regularity of $c(t)$ competes with the exponent σ.

Theorem 3.2 (Subcritical dissipation)**.** *Let us consider problem (1.1)–(1.2) under the following assumptions:*

- *A is a self-adjoint nonnegative operator on a separable Hilbert space H,*
- *the coefficient $c : [0, +\infty) \to \mathbb{R}$ satisfies the strict hyperbolicity assumption (1.4) and the ω-continuity assumption (2.7) for some continuity modulus $\omega(x)$,*
- *$\sigma \in [0, 1/2]$ and $\delta > 0$ are two real numbers such that*

$$4\delta^2\mu_1 > \Lambda_\infty^2 + 2\delta\Lambda_\infty, \tag{3.4}$$

  where we set

$$\Lambda_\infty := \limsup_{\varepsilon\to 0^+} \frac{\omega(\varepsilon)}{\varepsilon^{1-2\sigma}}, \tag{3.5}$$

- *$(u_0, u_1) \in D(A^{1/2}) \times H$.*

Let u be the unique solution to the problem provided by Theorem A.
Then u actually satisfies

$$u \in C^0\left([0, +\infty), D(A^{1/2})\right) \cap C^1\left([0, +\infty), H\right).$$

Moreover, for every $\nu \geq 1$ such that

$$4\delta^2\mu_1 \geq \left[\lambda^{1-2\sigma}\omega\left(\frac{1}{\lambda}\right)\right]^2 + 2\delta\left[\lambda^{1-2\sigma}\omega\left(\frac{1}{\lambda}\right)\right] \tag{3.6}$$

for every $\lambda \geq \nu$, it turns out that

$$|u'_{\nu,+}(t)|^2 + 2\mu_1|A^{1/2}u_{\nu,+}(t)|^2 \leq 4|u_{1,\nu,+}|^2 + 2(3\delta^2 + \mu_2)|A^{1/2}u_{0,\nu,+}|^2 \tag{3.7}$$

for every $t \geq 0$.

Let us make a few comments on the first two statements.

Remark 3.3. In both results we proved that a suitable high-frequency component of the solution can be uniformly bounded in terms of initial data. Low-frequency components might in general diverge as $t \to +\infty$. Nevertheless, they can always be estimated as follows.

Let us just assume that $c \in L^1((0, T))$ for every $T > 0$. Then for every $\nu \geq 0$ the component $u_{\nu,-}(t)$ satisfies

$$|u'_{\nu,-}(t)|^2 + |A^{1/2}u_{\nu,-}(t)|^2 \leq \left(|u_{1,\nu,-}|^2 + |A^{1/2}u_{0,\nu,-}|^2\right)\exp\left(\nu t + \nu\int_0^t |c(s)|\,ds\right) \tag{3.8}$$

for every $t \geq 0$. Indeed, let $F(t)$ denote the left-hand side of (3.8). Then

$$\begin{aligned} F'(t) &= -4\delta|A^{\sigma/2}u'_{\nu,-}(t)|^2 + 2(1 - c(t))\langle u'_{\nu,-}(t), Au_{\nu,-}(t)\rangle \\ &\leq 2(1 + |c(t)|)\cdot|u'_{\nu,-}(t)|\cdot\nu|A^{1/2}u_{\nu,-}(t)| \\ &\leq \nu(1 + |c(t)|)F(t) \end{aligned}$$

for almost every $t \geq 0$, so that (3.8) follows by integrating this differential inequality.

Remark 3.4. The phase spaces involved in Theorem 3.1 and Theorem 3.2 are exactly the same which are known to be optimal when $c(t)$ is constant (see [10]). In particular, the only possible choice in the subcritical regime is the classic energy space $D(A^{1/2}) \times H$, or more generally $D(A^{\alpha+1/2}) \times D(A^{\alpha})$. This "gap 1/2" between the powers of A involved in the phase space is typical of hyperbolic problems, and it is the same which appears in the classic results of section 2.

On the contrary, in the supercritical regime there is an interval of possible gaps, described by (1.7). This interval is always centered in 1/2, but also different values are allowed, including negative ones when $\sigma > 1$.

Remark 3.5. The classic example of application of Theorem 3.2 is the following. Let us assume that $c(t)$ is α-Hölder continuous for some $\alpha \in (0,1)$, namely $\omega(x) = Mx^{\alpha}$ for some constant M. Then problem (1.1)–(1.2) is well-posed in the energy space provided that either $\alpha > 1-2\sigma$, or $\alpha = 1-2\sigma$ and M is small enough. Indeed, for $\alpha > 1-2\sigma$ we get $\Lambda_\infty = 0$, and hence (3.4) is automatically satisfied. For $\alpha = 1-2\sigma$ we get $\Lambda_\infty = M$, so that (3.4) is satisfied provided that M is small enough.

In all other cases, namely when either $\alpha < 1-2\sigma$, or $\alpha = 1-2\sigma$ and M is large enough, only Theorem B applies to initial data in Sobolev spaces, providing global existence just in the sense of Gevrey ultradistributions of order $(1-\alpha)^{-1}$.

Remark 3.6. Let us examine the limit case $\sigma = 0$, which falls in the subcritical regime.

When $\sigma = 0$, assumption (3.4) is satisfied if and only if $c(t)$ is Lipschitz continuous and its Lipschitz constant is small enough. On the other hand, in the Lipschitz case it is a classic result that problem (1.1)–(1.2) is well-posed in the energy space, regardless of the Lipschitz constant. Therefore, the result stated in Theorem 3.2 is non-optimal when $\sigma = 0$ and $c(t)$ is Lipschitz continuous.

A simple refinement of our argument would lead to the full result also in this case, but unfortunately it would be useless in all other limit cases in which $c(t)$ is α-Hölder continuous with $\alpha = 1-2\sigma$ and $\sigma \in (0, 1/2]$. We refer to section 4 for further details.

Remark 3.7. Let us examine the limit case $\sigma = 1/2$, which falls both in the subcritical and in the supercritical regime, so that the conclusions of Theorem 3.1 and Theorem 3.2 coexist. Both of them provide well-posedness in the energy space, but with different assumptions.

Theorem 3.1 needs less assumptions on $c(t)$, which is only required to be measurable and to satisfy the degenerate hyperbolicity assumption (1.8), but it requires δ to be large enough so that $4\delta^2 \geq \mu_2$.

On the contrary, Theorem 3.2 needs less assumptions on δ, which is only required to be positive, but it requires $c(t)$ to be continuous and to satisfy the strict hyperbolicity assumption (1.4). Indeed, inequality (3.4) is automatically satisfied in the case $\sigma = 1/2$ because $\Lambda_\infty = 0$.

The existence of two different sets of assumptions leading to the same conclusion suggests the existence of a unifying statement, which could probably deserve further investigation.

3.2 Gevrey regularity for positive times

A strong dissipation in the range $\sigma \in (0,1)$ has a regularizing effect on initial data, provided that the solution exists in Sobolev spaces. In the following two statements we quantify this effect in terms of scales of Gevrey spaces.

Both results can be summed up by saying that the solution lies, for positive times, in Gevrey spaces of order $(2\min\{\sigma, 1-\sigma\})^{-1}$. It is not difficult to show that this order is optimal, even in the case where $c(t)$ is constant.

Theorem 3.8 (Supercritical dissipation)**.** *Let us consider problem (1.1)–(1.2) under the same assumptions of Theorem 3.1, and let u be the unique solution to the problem provided by Theorem A.*

Let us assume in addition that either $\sigma \in (1/2, 1)$, or $\sigma = 1/2$ and $4\delta^2 > \mu_2$. Let us set $\varphi(x) := x^{2(1-\sigma)}$, and

$$C(t) := \int_0^t c(s)\,ds. \tag{3.9}$$

Then there exists $r > 0$ such that

$$(u, u') \in C^0\left((0, +\infty), \mathcal{G}_{\varphi, rC(t), \alpha}(A) \times \mathcal{G}_{\varphi, rC(t), \beta}(A)\right), \tag{3.10}$$

and there exist $\nu \geq 1$ and $K > 0$ such that

$$\|u'_{\nu,+}(t)\|^2_{\varphi, rC(t), \beta} + \|u_{\nu,+}(t)\|^2_{\varphi, rC(t), \alpha} \leq K\left(|A^\beta u_{1,\nu,+}|^2 + |A^\alpha u_{0,\nu,+}|^2\right) \tag{3.11}$$

for every $t > 0$. The constants r, ν, and K depend only on δ, μ_2, and σ.

Of course, (3.10) and (3.11) are nontrivial only if $C(t) > 0$, which is equivalent to saying that the coefficient $c(t)$ is not identically 0 in $[0, t]$. On the other hand, this weak form of hyperbolicity is necessary, since no regularizing effect on $u(t)$ can be expected as long as $c(t)$ vanishes.

Theorem 3.9 (Subcritical dissipation)**.** *Let us consider problem (1.1)–(1.2) under the same assumptions of Theorem 3.2, and let u be the unique solution to the problem provided by Theorem A.*

Let us assume in addition that $\sigma \in (0, 1/2]$ (instead of $\sigma \in [0, 1/2]$), and let us set $\varphi(x) := x^{2\sigma}$.

Then there exists $r > 0$ such that

$$u \in C^0\left((0, +\infty), \mathcal{G}_{\varphi, rt, 1/2}(A)\right) \cap C^1\left((0, +\infty), \mathcal{G}_{\varphi, rt, 0}(A)\right),$$

and there exist $\nu \geq 1$ and $K > 0$ such that

$$\|u'_{\nu,+}(t)\|^2_{\varphi, rt, 0} + \|u_{\nu,+}(t)\|^2_{\varphi, rt, 1/2} \leq K\left(|u_{1,\nu,+}|^2 + |A^{1/2} u_{0,\nu,+}|^2\right) \tag{3.12}$$

for every $t > 0$. The constants r, ν, and K depend only on δ, μ_1, μ_2, σ and ω.

The estimates which provide Gevrey regularity of high-frequency components provide also the decay of the same components as $t \to +\infty$. We refer to Lemma 5.1 and Lemma 5.2 for further details.

3.3 Counterexamples

The following result shows that even strongly dissipative hyperbolic equations can exhibit the (DGCS)-phenomenon, provided that we are in the subcritical regime.

Theorem 3.10 ((DGCS)-phenomenon)**.** *Let A be a linear operator on a Hilbert space H. Let us assume that there exists a countable (not necessarily complete) orthonormal system $\{e_k\}$ in H, and an unbounded sequence $\{\lambda_k\}$ of positive real numbers such that $Ae_k = \lambda_k^2 e_k$ for every $k \in \mathbb{N}$. Let $\sigma \in [0, 1/2)$ and $\delta > 0$ be real numbers.*

Let $\omega : [0, +\infty) \to [0, +\infty)$ be a continuity modulus such that

$$\lim_{\varepsilon \to 0^+} \frac{\omega(\varepsilon)}{\varepsilon^{1-2\sigma}} = +\infty. \tag{3.13}$$

Let $\varphi : (0, +\infty) \to (0, +\infty)$ and $\psi : (0, +\infty) \to (0, +\infty)$ be two functions such that

$$\lim_{x \to +\infty} \frac{x}{\varphi(x)}\, \omega\left(\frac{1}{x}\right) = \lim_{x \to +\infty} \frac{x}{\psi(x)}\, \omega\left(\frac{1}{x}\right) = +\infty. \tag{3.14}$$

Then there exist a function $c : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{1}{2} \le c(t) \le \frac{3}{2} \qquad \forall t \in \mathbb{R}, \tag{3.15}$$

$$|c(t) - c(s)| \le \omega(|t - s|) \qquad \forall (t, s) \in \mathbb{R}^2, \tag{3.16}$$

and a solution $u(t)$ to equation (1.1) such that

$$(u(0), u'(0)) \in \mathcal{G}_{\varphi,r,1/2}(A) \times \mathcal{G}_{\varphi,r,0}(A) \qquad \forall r > 0, \tag{3.17}$$

$$(u(t), u'(t)) \notin \mathcal{G}_{-\psi,R,1/2}(A) \times \mathcal{G}_{-\psi,R,0}(A) \qquad \forall R > 0,\ \forall t > 0. \tag{3.18}$$

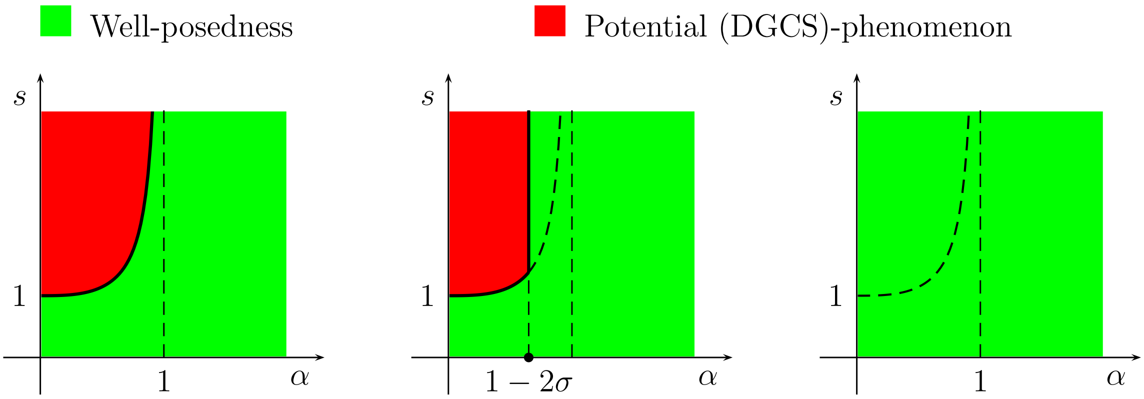
Remark 3.11. Due to (3.15), (3.16), and (3.17), the function $u(t)$ provided by Theorem 3.10 is a solution to (1.1) in the sense of Theorem A with $\psi(x) := x$, or even better in the sense of Theorem B with $\psi(x) := x\omega(1/x)$.

Remark 3.12. Assumption (3.13) is equivalent to saying that Λ_∞ defined by (3.5) is equal to $+\infty$, so that (3.4) can not be satisfied. In other words, Theorem 3.2 gives well-posedness in the energy space if Λ_∞ is 0 or small, while Theorem 3.10 provides the (DGCS)-phenomenon if $\Lambda_\infty = +\infty$. The case where Λ_∞ is finite but large remains open. We suspect that the (DGCS)-phenomenon is still possible, but our construction does not work. We comment on this issue in the first part of section 6.

Finally, Theorem 3.10 shows that assumptions (2.9) and (2.10) of Theorems B and C are optimal also in the subcritical dissipative case with $\Lambda_\infty = +\infty$. If initial data are in the Gevrey space with $\varphi(x) = x\omega(1/x)$, solutions remain in the same space. If initial are in a Gevrey space corresponding to some $\varphi(x) \ll x\omega(1/x)$, then it may happen that for positive times the solution lies in the space of ultradistributions with $\psi(x) := x\omega(1/x)$, but not in the space of ultradistributions corresponding to any given $\psi(x) \ll x\omega(1/x)$.

4 Heuristics

The following pictures summarize roughly the results of this paper. In the horizontal axis we represent the time-regularity of $c(t)$. With some abuse of notation, values $\alpha \in (0,1)$ mean that $c(t)$ is α-Hölder continuous, $\alpha = 1$ means that it is Lipschitz continuous, $\alpha > 1$ means even more regular. In the vertical axis we represent the space-regularity of initial data, where the value s stands for the Gevrey space of order s (so that higher values of s mean lower regularity). The curve is $s = (1-\alpha)^{-1}$.



$\delta = 0$ $\qquad$ $\delta > 0, \quad 0 < \sigma < 1/2$ $\qquad$ $\delta > 0, \quad \sigma > 1/2$

For $\delta = 0$ we have the situation described in Remark 2.5 and Remark 2.6, namely well-posedness provided that either $c(t)$ is Lipschitz continuous or $c(t)$ is α-Hölder continuous and initial data are in Gevrey spaces of order less than or equal to $(1-\alpha)^{-1}$, and (DGCS)-phenomenon otherwise. The same picture applies if $\delta > 0$ and $\sigma = 0$.

When $\delta > 0$ and $0 < \sigma < 1/2$, the full strip with $\alpha > 1 - 2\sigma$ falls in the well-posedness region, as stated in Theorem 3.2. The region with $\alpha < 1 - 2\sigma$ is divided as in the non-dissipative case. Indeed, Theorem C still provides well-posedness below the curve and on the curve, while Theorem 3.10 provides the (DGCS)-phenomenon above the curve. What happens on the vertical half-line which separates the two regions is less clear (it is the region where Λ_∞ is positive and finite, see Remark 3.12).

Finally, when $\delta > 0$ and $\sigma > 1/2$ well-posedness dominates because of Theorem 3.1, even in the degenerate hyperbolic case.

Now we present a rough justification of this threshold effect. As already observed, existence results for problem (1.1)–(1.2) are related to estimates for solutions to the family of ordinary differential equations (1.9).

Let us consider the simplest energy function $\mathcal{E}(t) := |u_\lambda'(t)|^2 + \lambda^2 |u_\lambda(t)|^2$, whose time-derivative is

$$\begin{aligned} \mathcal{E}'(t) &= -4\delta\lambda^{2\sigma}|u_\lambda'(t)|^2 + 2\lambda^2(1-c(t))u_\lambda(t)u_\lambda'(t) \\ &\leq -4\delta\lambda^{2\sigma}|u_\lambda'(t)|^2 + \lambda(1+|c(t)|)\mathcal{E}(t). \end{aligned} \tag{4.1}$$

Since $\delta \geq 0$, a simple integration gives that

$$\mathcal{E}(t) \leq \mathcal{E}(0) \exp\left(\lambda t + \lambda \int_0^t |c(s)|\, ds\right), \tag{4.2}$$

which is almost enough to establish Theorem A.

If in addition $c(t)$ is ω-continuous and satisfies the strict hyperbolicity condition (1.4), then (4.2) can be improved to

$$\mathcal{E}(t) \leq M_1 \mathcal{E}(0) \exp\left(M_2 \lambda \omega(1/\lambda) t\right) \tag{4.3}$$

for suitable constants M_1 and M_2. Estimates of this kind are the key point in the proof of both Theorem B and Theorem C. Moreover, the (DGCS)-phenomenon is equivalent to saying that the term $\lambda\omega(1/\lambda)$ in (4.3) is optimal.

Let us assume now that $\delta > 0$. If $\sigma > 1/2$, or $\sigma = 1/2$ and δ is large enough, then it is reasonable to expect that the first (negative) term in the right-hand side of (4.1) dominates the second one, and hence $\mathcal{E}(t) \leq \mathcal{E}(0)$, which is enough to establish well-posedness in Sobolev spaces. Theorem 3.1 confirms this intuition.

If $\sigma \leq 1/2$ and $c(t)$ is constant, then (1.9) can be explicitly integrated, obtaining that

$$\mathcal{E}(t) \leq \mathcal{E}(0) \exp\left(-2\delta\lambda^{2\sigma} t\right). \tag{4.4}$$

If $c(t)$ is ω-continuous and satisfies the strict hyperbolicity assumption (1.4), then we expect a superposition of the effects of the coefficient, represented by (4.3), and the effects of the damping, represented by (4.4). We end up with

$$\mathcal{E}(t) \leq M_1 \mathcal{E}(0) \exp\left([M_2 \lambda\omega(1/\lambda) - 2\delta\lambda^{2\sigma}]t\right). \tag{4.5}$$

Therefore, it is reasonable to expect that $\mathcal{E}(t)$ satisfies an estimate independent of λ, which guarantees well-posedness in Sobolev spaces, provided that $\lambda\omega(1/\lambda) \ll \lambda^{2\sigma}$, or $\lambda\omega(1/\lambda) \sim \lambda^{2\sigma}$ and δ is large enough. Theorem 3.2 confirms this intuition. The same argument applies if $\sigma = 0$ and $\omega(x) = Lx$, independently of L (see Remark 3.6).

On the contrary, in all other cases the right-hand side of (4.5) diverges as $\lambda \to +\infty$, opening the door to the (DGCS)-phenomenon. We are able to show that it does happen provided that $\lambda\omega(1/\lambda) \gg \lambda^{2\sigma}$. We refer to the first part of section 6 for further comments.

5 Proofs of well-posedness and regularity results

All proofs of our main results concerning well-posedness and regularity rely on suitable estimates for solutions to the ordinary differential equation (1.9) with initial data

$$u_\lambda(0) = u_0, \qquad u_\lambda'(0) = u_1. \tag{5.1}$$

For the sake of simplicity in the sequel we write $u(t)$ instead of $u_\lambda(t)$.

5.1 Supercritical dissipation

Let us consider the case $\sigma \geq 1/2$. The key tool is the following.

Lemma 5.1. *Let us consider problem (1.9)–(5.1) under the following assumptions:*

- *the coefficient* $c : [0, +\infty) \to \mathbb{R}$ *is measurable and satisfies the degenerate hyperbolicity assumption (1.8),*
- δ*,* λ*,* σ *are positive real numbers such that*

$$4\delta^2 \lambda^{4\sigma-2} \geq \mu_2. \tag{5.2}$$

Then the solution $u(t)$ *satisfies the following estimates.*

(1) For every $t \geq 0$ *it turns out that*

$$|u(t)|^2 \leq \frac{2}{\delta^2 \lambda^{4\sigma}} u_1^2 + 3u_0^2, \tag{5.3}$$

$$|u'(t)|^2 \leq \left(2 + \frac{\mu_2^2}{\delta^4 \lambda^{8\sigma-4}}\right) u_1^2 + \frac{3\mu_2^2}{2\delta^2 \lambda^{4\sigma-4}} u_0^2. \tag{5.4}$$

(2) Let us assume in addition that $\lambda \geq 1$ *and* $\sigma \geq 1/2$*, and let* α *and* β *be two real numbers satisfying (1.7).*

Then for every $t \geq 0$ *it turns out that*

$$\lambda^{4\beta}|u'(t)|^2 + \lambda^{4\alpha}|u(t)|^2 \leq \left(2 + \frac{2}{\delta^2} + \frac{\mu_2^2}{\delta^4}\right) \lambda^{4\beta} u_1^2 + 3\left(1 + \frac{\mu_2^2}{2\delta^2}\right) \lambda^{4\alpha} u_0^2. \tag{5.5}$$

(3) In addition to the assumptions of the statement (2), let us assume also that there exists $r > 0$ *satisfying the following three inequalities:*

$$\delta \lambda^{4\sigma-2} > r\mu_2, \qquad 2\delta r \leq 1, \qquad 4\delta^2 \lambda^{4\sigma-2} \geq (1 + 2r\delta)\mu_2. \tag{5.6}$$

Then for every $t \geq 0$ *it turns out that*

$$\begin{aligned} \lambda^{4\beta}|u'(t)|^2 + \lambda^{4\alpha}|u(t)|^2 \ &\leq \ \left[2\left(1 + \frac{2\mu_2^2}{\delta^4} + \frac{1}{\delta^2}\right) \lambda^{4\beta} u_1^2 + 3\left(1 + \frac{2\mu_2^2}{\delta^2}\right) \lambda^{4\alpha} u_0^2\right] \times \\ &\quad \times \exp\left(-2r\lambda^{2(1-\sigma)} \int_0^t c(s)\, ds\right). \end{aligned} \tag{5.7}$$

Proof Let us consider the energy $E(t)$ defined in (1.10). Since

$$-\frac{3}{4}|u'(t)|^2 - \frac{4}{3}\delta^2 \lambda^{4\sigma}|u(t)|^2 \leq 2\delta\lambda^{2\sigma} u(t) u'(t) \leq |u'(t)|^2 + \delta^2 \lambda^{4\sigma}|u(t)|^2,$$

we easily deduce that

$$\frac{1}{4}|u'(t)|^2 + \frac{2}{3}\delta^2 \lambda^{4\sigma}|u(t)|^2 \leq E(t) \leq 2|u'(t)|^2 + 3\delta^2 \lambda^{4\sigma}|u(t)|^2 \qquad \forall t \geq 0. \tag{5.8}$$

Statement (1) The time-derivative of $E(t)$ is

$$E'(t) = -2\left(\delta\lambda^{2\sigma}|u'(t)|^2 + \delta\lambda^{2\sigma+2}c(t)|u(t)|^2 + \lambda^2 c(t)u(t)u'(t)\right). \tag{5.9}$$

The right-hand side is a quadratic form in $u(t)$ and $u'(t)$. The coefficient of $|u'(t)|^2$ is negative. Therefore, this quadratic form is less than or equal to 0 for all values of $u(t)$ and $u'(t)$ if and only if

$$4\delta^2\lambda^{4\sigma-2}c(t) \geq c^2(t),$$

and this is always true because of (1.8) and (5.2). It follows that $E'(t) \leq 0$ for (almost) every $t \geq 0$, and hence

$$\delta^2\lambda^{4\sigma}|u(t)|^2 \leq E(t) \leq E(0) \leq 2u_1^2 + 3\delta^2\lambda^{4\sigma}u_0^2, \tag{5.10}$$

which is equivalent to (5.3).

In order to estimate $u'(t)$, we rewrite (1.9) in the form

$$u''(t) + 2\delta\lambda^{2\sigma}u'(t) = -\lambda^2 c(t)u(t),$$

which we interpret as a first order linear equation with constant coefficients in the unknown $u'(t)$, with the right-hand side as a forcing term. Integrating this differential equation in $u'(t)$, we obtain that

$$u'(t) = u_1\exp\left(-2\delta\lambda^{2\sigma}t\right) - \int_0^t \lambda^2 c(s)u(s)\exp\left(-2\delta\lambda^{2\sigma}(t-s)\right)\,ds. \tag{5.11}$$

From (1.8) and (5.3) it follows that

$$\begin{aligned}
|u'(t)| &\leq |u_1| + \mu_2\lambda^2 \cdot \max_{t\in[0,T]}|u(t)| \cdot \int_0^t e^{-2\delta\lambda^{2\sigma}(t-s)}\,ds \\
&\leq |u_1| + \frac{\mu_2\lambda^2}{2\delta\lambda^{2\sigma}}\left(\frac{2}{\delta^2\lambda^{4\sigma}}u_1^2 + 3u_0^2\right)^{1/2},
\end{aligned}$$

and therefore

$$|u'(t)|^2 \leq 2|u_1|^2 + \frac{\mu_2^2\lambda^4}{2\delta^2\lambda^{4\sigma}}\left(\frac{2}{\delta^2\lambda^{4\sigma}}u_1^2 + 3u_0^2\right),$$

which is equivalent to (5.4).

Statement (2) Exploiting (5.3) and (5.4), with some simple algebra we obtain that

$$\begin{aligned}
\lambda^{4\beta}|u'(t)|^2 + \lambda^{4\alpha}|u(t)|^2 &\leq \left(2 + \frac{\mu_2^2}{\delta^4}\cdot\frac{1}{\lambda^{4(2\sigma-1)}} + \frac{2}{\delta^2}\cdot\frac{1}{\lambda^{4(\beta+\sigma-\alpha)}}\right)\lambda^{4\beta}u_1^2 \\
&\quad + 3\left(1 + \frac{\mu_2^2}{2\delta^2}\cdot\frac{1}{\lambda^{4(\alpha-\beta+\sigma-1)}}\right)\lambda^{4\alpha}u_0^2.
\end{aligned}$$

All exponents of λ's in denominators are nonnegative owing to (1.7). Therefore, since $\lambda \geq 1$, all those fractions can be estimated with 1. This leads to (5.5).

Statement (3) Let us define $C(t)$ as in (3.9). To begin with, we prove that in this case the function $E(t)$ satisfies the stronger differential inequality

$$E'(t) \leq -2r\lambda^{2(1-\sigma)}c(t)E(t), \tag{5.12}$$

and hence

$$E(t) \leq E(0)\exp\left(-2r\lambda^{2(1-\sigma)}C(t)\right) \qquad \forall t \geq 0. \tag{5.13}$$

Coming back to (5.9), inequality (5.12) is equivalent to

$$\lambda^{2\sigma}\left(\delta - r\lambda^{2-4\sigma}c(t)\right)|u'(t)|^2 + \delta\lambda^{2\sigma+2}(1-2r\delta)c(t)|u(t)|^2 + \lambda^2(1-2r\delta)c(t)u(t)u'(t) \geq 0.$$

As in the proof of statement (1), we consider the whole left-hand side as a quadratic form in $u(t)$ and $u'(t)$. Since $c(t) \leq \mu_2$, from the first inequality in (5.6) it follows that

$$\delta\lambda^{4\sigma-2} > r\mu_2 \geq rc(t),$$

which is equivalent to saying that the coefficient of $|u'(t)|^2$ is positive. Therefore, the quadratic form is nonnegative for all values of $u(t)$ and $u'(t)$ if and only if

$$4\lambda^{2\sigma}\left(\delta - r\lambda^{2-4\sigma}c(t)\right)\cdot\delta\lambda^{2\sigma+2}c(t)(1-2r\delta) \geq \lambda^4c^2(t)(1-2r\delta)^2,$$

hence if and only if

$$(1-2r\delta)c(t)\left[4\delta^2\lambda^{4\sigma-2} - (1+2r\delta)c(t)\right] \geq 0,$$

and this follows from (1.8) and from the last two inequalities in (5.6).

Now from (5.13) it follows that

$$\delta^2\lambda^{4\sigma}|u(t)|^2 \leq E(t) \leq E(0)\exp\left(-2r\lambda^{2(1-\sigma)}C(t)\right), \tag{5.14}$$

which provides an estimate for $|u(t)|$. In order to estimate $u'(t)$, we write it in the form (5.11), and we estimate the two terms separately. The third inequality in (5.6) implies that $2\delta\lambda^{4\sigma-2} \geq r\mu_2$. Since $C(t) \leq \mu_2 t$, it follows that

$$2\delta\lambda^{2\sigma}t \geq r\lambda^{2-2\sigma}\mu_2 t \geq r\lambda^{2-2\sigma}C(t),$$

and hence

$$\left|u_1\exp\left(-2\delta\lambda^{2\sigma}t\right)\right| \leq |u_1|\exp\left(-2\delta\lambda^{2\sigma}t\right) \leq |u_1|\exp\left(-r\lambda^{2(1-\sigma)}C(t)\right). \tag{5.15}$$

As for the second terms in (5.11), we exploit (5.14) and we obtain that

$$\left|\int_0^t \lambda^2 c(s)u(s)\exp\left(-2\delta\lambda^{2\sigma}(t-s)\right)\,ds\right| \leq \lambda^2\mu_2\int_0^t |u(s)|\exp\left(-2\delta\lambda^{2\sigma}(t-s)\right)\,ds$$

$$\leq \frac{\mu_2[E(0)]^{1/2}}{\delta\lambda^{2\sigma-2}}\exp\left(-2\delta\lambda^{2\sigma}t\right)\int_0^t \exp\left(-r\lambda^{2(1-\sigma)}C(s) + 2\delta\lambda^{2\sigma}s\right)ds.$$

From the first inequality in (5.6) it follows that

$$2\delta\lambda^{2\sigma} - r\lambda^{2(1-\sigma)}c(s) \geq 2\delta\lambda^{2\sigma} - r\lambda^{2(1-\sigma)}\mu_2 \geq \delta\lambda^{2\sigma},$$

hence

$$\begin{aligned}
&\int_0^t \exp\left(-r\lambda^{2(1-\sigma)}C(s) + 2\delta\lambda^{2\sigma}s\right) ds \\
&\qquad \leq \frac{1}{\delta\lambda^{2\sigma}} \int_0^t \left(2\delta\lambda^{2\sigma} - r\lambda^{2(1-\sigma)}c(s)\right) \exp\left(2\delta\lambda^{2\sigma}s - r\lambda^{2(1-\sigma)}C(s)\right) ds \\
&\qquad \leq \frac{1}{\delta\lambda^{2\sigma}} \exp\left(2\delta\lambda^{2\sigma}t - r\lambda^{2(1-\sigma)}C(t)\right),
\end{aligned}$$

and therefore

$$\left|\int_0^t \lambda^2 c(s)u(s)\exp\left(-2\delta\lambda^{2\sigma}(t-s)\right) ds\right| \leq \frac{\mu_2[E(0)]^{1/2}}{\delta^2\lambda^{4\sigma-2}} \exp\left(-r\lambda^{2(1-\sigma)}C(t)\right). \quad (5.16)$$

From (5.11), (5.15) and (5.16) we deduce that

$$|u'(t)| \leq \left(|u_1| + \frac{\mu_2[E(0)]^{1/2}}{\delta^2\lambda^{4\sigma-2}}\right) \exp\left(-r\lambda^{2(1-\sigma)}C(t)\right),$$

and hence

$$|u'(t)|^2 \leq \left(2|u_1|^2 + \frac{2\mu_2^2E(0)}{\delta^4\lambda^{8\sigma-4}}\right) \exp\left(-2r\lambda^{2(1-\sigma)}C(t)\right). \quad (5.17)$$

Finally, we estimate $E(0)$ as in (5.10). At this point, estimate (5.7) follows from (5.17) and (5.14) with some simple algebra (we need to exploit that $\lambda \geq 1$ and assumption (1.7) exactly as in the proof of statement (2)). □

5.1.1 Proof of Theorem 3.1

Let us fix a real number $\nu \geq 1$ such that $4\delta^2\nu^{4\sigma-2} \geq \mu_2$ (such a number exists because of our assumptions on δ and σ). Let us consider the components $u_k(t)$ of $u(t)$ corresponding to eigenvalues $\lambda_k \geq \nu$. Since $\lambda_k \geq 1$ and $4\delta^2\lambda_k^{4\sigma-2} \geq \mu_2$, we can apply statement (2) of Lemma 5.1 to these components. If u_{0k} and u_{1k} denote the corresponding components of initial data, estimate (5.5) read as

$$\lambda_k^{4\beta}|u_k'(t)|^2 + \lambda_k^{4\alpha}|u_k(t)|^2 \leq \left(2 + \frac{2}{\delta^2} + \frac{\mu_2^2}{\delta^4}\right)\lambda_k^{4\beta}|u_{1,k}|^2 + 3\left(1 + \frac{\mu_2^2}{2\delta^2}\right)\lambda_k^{4\alpha}|u_{0,k}|^2.$$

Summing over all $\lambda_k \geq \nu$ we obtain exactly (3.3).

This proves that $u_{\nu,+}(t)$ is bounded with values in $D(A^\alpha)$ and $u_{\nu,+}'(t)$ is bounded with values in $D(A^\beta)$. The same estimate guarantees the uniform convergence in the whole half-line $t \geq 0$ of the series defining $A^\alpha u_{\nu,+}(t)$ and $A^\beta u_{\nu,+}'(t)$. Since all summands are continuous, and the convergence is uniform, the sum is continuous as well. Since low-frequency components $u_{\nu,-}(t)$ and $u_{\nu,-}'(t)$ are continuous (see Remark 3.3), (3.2) is proved. □

5.1.2 Proof of Theorem 3.8

Let us fix a real number $\nu \geq 1$ such that $4\delta^2\nu^{4\sigma-2} > \mu_2$ (such a number exists because of our assumptions on δ and σ). Then there exists $r > 0$ such that the three inequalities in (5.6) hold true for every $\lambda \geq \nu$. Therefore, we can apply statement (3) of Lemma 5.1 to all components $u_k(t)$ of $u(t)$ corresponding to eigenvalues $\lambda_k \geq \nu$. If u_{0k} and u_{1k} denote the corresponding components of initial data, estimate (5.7) read as

$$\left(\lambda_k^{4\beta}|u_k'(t)|^2 + \lambda_k^{4\alpha}|u_k(t)|^2\right) \exp\left(2r\lambda_k^{2(1-\sigma)}\int_0^t c(s)\,ds\right) \leq K\left(\lambda_k^{4\beta}|u_{1k}|^2 + \lambda_k^{4\alpha}|u_{0k}|^2\right)$$

for every $t \geq 0$, where K is a suitable constant depending only on μ_2 and δ. Summing over all $\lambda_k \geq \nu$ we obtain exactly (3.11). The continuity of $u(t)$ and $u'(t)$ with values in the suitable spaces follows from the uniform convergence of series as in the proof of Theorem 3.1. $\square$

5.2 Subcritical dissipation

Let us consider the case $0 \leq \sigma \leq 1/2$. The key tool is the following.

Lemma 5.2. *Let us consider problem (1.9)–(5.1) under the following assumptions:*

- *the coefficient $c : [0, +\infty) \to \mathbb{R}$ satisfies the strict hyperbolicity assumption (1.4) and the ω-continuity assumption (2.7) for some continuity modulus $\omega(x)$,*
- *$\delta > 0$, $\lambda > 0$, and $\sigma \geq 0$ are real numbers satisfying (3.6).*

Then the solution $u(t)$ satisfies the following estimates.

(1) It turns out that

$$|u'(t)|^2 + 2\lambda^2\mu_1|u(t)|^2 \leq 4u_1^2 + 2\left(3\delta^2\lambda^{4\sigma} + \lambda^2\mu_2\right)u_0^2 \qquad \forall t \geq 0. \tag{5.18}$$

(2) Let us assume in addition that $\lambda \geq 1$, $\sigma \in [0, 1/2]$, and there exists a constant $r \in (0, \delta)$ such that

$$4(\delta - r)(\delta\mu_1 - r\mu_2) \geq \left[\lambda^{1-2\sigma}\omega\left(\frac{1}{\lambda}\right)\right]^2 + 2\delta(1+2r)\left[\lambda^{1-2\sigma}\omega\left(\frac{1}{\lambda}\right)\right] + 8r\delta^3. \tag{5.19}$$

Then for every $t \geq 0$ it turns out that

$$|u'(t)|^2 + 2\lambda^2\mu_1|u(t)|^2 \leq \left[4u_1^2 + 2\left(3\delta^2\lambda^{4\sigma} + \lambda^2\mu_2\right)u_0^2\right]\exp\left(-2r\lambda^{2\sigma}t\right). \tag{5.20}$$

Proof For every $\varepsilon > 0$ we introduce the regularized coefficient

$$c_\varepsilon(t) := \frac{1}{\varepsilon}\int_t^{t+\varepsilon} c(s)\,ds \qquad \forall t \geq 0.$$

It is easy to see that $c_\varepsilon \in C^1([0,+\infty),\mathbb{R})$ and satisfies the following estimates:

$$\mu_1 \leq c_\varepsilon(t) \leq \mu_2 \qquad \forall t \geq 0, \tag{5.21}$$

$$|c(t) - c_\varepsilon(t)| \leq \omega(\varepsilon) \qquad \forall t \geq 0, \tag{5.22}$$

$$|c_\varepsilon'(t)| \leq \frac{\omega(\varepsilon)}{\varepsilon} \qquad \forall t \geq 0. \tag{5.23}$$

Approximated energy For every $\varepsilon > 0$ we consider the approximated hyperbolic energy $E_\varepsilon(t)$ defined in (1.11). Since

$$-\frac{1}{2}|u'(t)|^2 - 2\delta^2\lambda^{4\sigma}|u(t)|^2 \leq 2\delta\lambda^{2\sigma}u(t)u'(t) \leq |u'(t)|^2 + \delta^2\lambda^{4\sigma}|u(t)|^2,$$

we deduce that

$$\frac{1}{2}|u'(t)|^2 + \mu_1\lambda^2|u(t)|^2 \leq E_\varepsilon(t) \leq 2|u'(t)|^2 + (3\delta^2\lambda^{4\sigma} + \lambda^2\mu_2)|u(t)|^2 \tag{5.24}$$

for every $\varepsilon > 0$ and $t \geq 0$. The time-derivative of $E_\varepsilon(t)$ is

$$\begin{aligned} E_\varepsilon'(t) &= -2\delta\lambda^{2\sigma}|u'(t)|^2 - 2\delta\lambda^{2\sigma+2}c(t)|u(t)|^2 \\ &\quad -2\lambda^2(c(t) - c_\varepsilon(t))u(t)u'(t) + \lambda^2 c_\varepsilon'(t)|u(t)|^2, \end{aligned} \tag{5.25}$$

hence

$$\begin{aligned} E_\varepsilon'(t) &\leq -2\delta\lambda^{2\sigma}|u'(t)|^2 - \left(2\delta\lambda^{2\sigma+2}c(t) - \lambda^2|c_\varepsilon'(t)|\right)|u(t)|^2 \\ &\quad +2\lambda^2|c(t) - c_\varepsilon(t)|\cdot|u(t)|\cdot|u'(t)|. \end{aligned} \tag{5.26}$$

Statement (1) We claim that, for a suitable choice of ε, it turns out that

$$E_\varepsilon'(t) \leq 0 \qquad \forall t \geq 0. \tag{5.27}$$

If we prove this claim, then we apply (5.24) with that particular value of ε and we obtain that

$$\frac{1}{2}|u'(t)|^2 + \mu_1\lambda^2|u(t)|^2 \leq E_\varepsilon(t) \leq E_\varepsilon(0) \leq 2u_1^2 + (3\delta^2\lambda^{4\sigma} + \lambda^2\mu_2)u_0^2,$$

which is equivalent to (5.18).

In order to prove (5.27), we consider the whole right-hand side of (5.26) as a quadratic form in $|u(t)|$ and $|u'(t)|$. Since the coefficient of $|u'(t)|^2$ is negative, this quadratic form is less than or equal to 0 for all values of $|u(t)|$ and $|u'(t)|$ if and only if

$$2\delta\lambda^{2\sigma}\cdot\left(2\delta\lambda^{2\sigma+2}c(t)-\lambda^2|c_\varepsilon'(t)|\right)-\lambda^4|c(t)-c_\varepsilon(t)|^2\geq 0,$$

hence if and only if

$$4\delta^2\lambda^{4\sigma-2}c(t)\geq|c(t)-c_\varepsilon(t)|^2+2\delta\lambda^{2\sigma-2}|c_\varepsilon'(t)|. \tag{5.28}$$

Now in the left-hand side we estimate $c(t)$ from below with μ_1, and we estimate from above the terms in the right-hand side as in (5.22) and (5.23). We obtain that (5.28) holds true whenever

$$4\delta^2\mu_1\geq\frac{\omega^2(\varepsilon)}{\lambda^{4\sigma-2}}+2\delta\frac{\omega(\varepsilon)}{\lambda^{2\sigma}\varepsilon}.$$

This condition is true when $\varepsilon:=1/\lambda$ thanks to assumption (3.6). This completes the proof of (5.18).

Statement (2) Let us assume now that $\lambda\geq 1$ and that (5.19) holds true for some $r\in(0,\delta)$. In this case we claim that, for a suitable choice of $\varepsilon>0$, the stronger estimate

$$E_\varepsilon'(t)\leq-2r\lambda^{2\sigma}E_\varepsilon(t)\qquad\forall t\geq 0 \tag{5.29}$$

holds true, hence

$$E_\varepsilon(t)\leq E_\varepsilon(0)\exp\left(-2r\lambda^{2\sigma}t\right)\qquad\forall t\geq 0.$$

Due to (5.24), this is enough to deduce (5.20). So it remains to prove (5.29).

Owing to (5.25), inequality (5.29) is equivalent to

$$2\lambda^{2\sigma}(\delta-r)|u'(t)|^2+\left[2\lambda^{2\sigma+2}(\delta c(t)-rc_\varepsilon(t))-\lambda^2c_\varepsilon'(t)-4r\delta^2\lambda^{6\sigma}\right]|u(t)|^2$$
$$+2\left[\lambda^2(c(t)-c_\varepsilon(t))-2r\delta\lambda^{4\sigma}\right]u(t)u'(t)\geq 0.$$

Keeping (1.4) and (5.21) into account, the last inequality is proved if we show that

$$2\lambda^{2\sigma}(\delta-r)|u'(t)|^2+\left[2\lambda^{2\sigma+2}(\delta\mu_1-r\mu_2)-\lambda^2|c_\varepsilon'(t)|-4r\delta^2\lambda^{6\sigma}\right]|u(t)|^2$$
$$-2\left[\lambda^2|c(t)-c_\varepsilon(t)|+2r\delta\lambda^{4\sigma}\right]|u(t)|\cdot|u'(t)|\geq 0.$$

As in the proof of the first statement, we consider the whole left-hand side as a quadratic form in $|u(t)|$ and $|u'(t)|$. The coefficient of $|u'(t)|$ is positive because $r<\delta$. Therefore, this quadratic form is nonnegative for all values of $|u(t)|$ and $|u'(t)|$ if and only if

$$2\lambda^{2\sigma}(\delta-r)\cdot\left[2\lambda^{2\sigma+2}(\delta\mu_1-r\mu_2)-\lambda^2|c_\varepsilon'(t)|-4r\delta^2\lambda^{6\sigma}\right]\geq\left[\lambda^2|c(t)-c_\varepsilon(t)|+2r\delta\lambda^{4\sigma}\right]^2.$$

Now we rearrange the terms, and we exploit (5.22) and (5.23). We obtain that the last inequality is proved if we show that

$$4(\delta-r)(\delta\mu_1-r\mu_2)\geq\lambda^{2-4\sigma}\omega^2(\varepsilon)+2\delta\left(1+2r\varepsilon\lambda^{2\sigma}\right)\frac{\omega(\varepsilon)}{\varepsilon\lambda^{2\sigma}}+\frac{8r\delta^3}{\lambda^{2-4\sigma}}. \tag{5.30}$$

Finally, we choose $\varepsilon := 1/\lambda$, so that (5.30) becomes

$$4(\delta-r)(\delta\mu_1-r\mu_2)\geq\left[\lambda^{1-2\sigma}\omega\left(\frac{1}{\lambda}\right)\right]^2+2\delta\left(1+\frac{2r}{\lambda^{1-2\sigma}}\right)\left[\lambda^{1-2\sigma}\omega\left(\frac{1}{\lambda}\right)\right]+\frac{8r\delta^3}{\lambda^{2-4\sigma}}.$$

Since $\lambda\geq 1$ and $\sigma\leq 1/2$, this inequality follows from assumption (5.19). □

5.2.1 Proof of Theorem 3.2

Let us rewrite (3.5) in the form

$$\Lambda_\infty=\limsup_{\lambda\to+\infty}\lambda^{1-2\sigma}\omega\left(\frac{1}{\lambda}\right). \tag{5.31}$$

Due to (3.4), there exists $\nu\geq 1$ such that (3.6) holds true for every $\lambda\geq\nu$. Therefore, we can apply statement (1) of Lemma 5.2 to the components $u_k(t)$ of $u(t)$ corresponding to eigenvalues $\lambda_k\geq\nu$. If u_{0k} and u_{1k} denote the corresponding components of initial data, estimate (5.18) read as

$$|u_k'(t)|^2+2\lambda_k^2\mu_1|u_k(t)|^2\leq 4|u_{1k}|^2+2\left(3\delta^2\lambda_k^{4\sigma}+\lambda_k^2\mu_2\right)|u_{0k}|^2.$$

Since $\sigma\leq 1/2$ and we chose $\nu\geq 1$, this implies that

$$|u_k'(t)|^2+2\lambda_k^2\mu_1|u_k(t)|^2\leq 4|u_{1k}|^2+2\left(3\delta^2+\mu_2\right)\lambda_k^2|u_{0k}|^2.$$

Summing over all $\lambda_k\geq\nu$ we obtain exactly (3.7).

This proves that $u_{\nu,+}(t)$ is bounded with values in $D(A^{1/2})$ and $u_{\nu,+}'(t)$ is bounded with values in H. The continuity of $u(t)$ and $u'(t)$ with values in the same spaces follows from the uniform convergence of series as in the proof of Theorem 3.1. □

5.2.2 Proof of Theorem 3.9

Let us rewrite (3.5) in the form (5.31). Due to (3.4), there exists $r>0$ and $\nu\geq 1$ such that (5.19) holds true for every $\lambda\geq\nu$. Therefore, we can apply statement (2) of Lemma 5.2 to the components $u_k(t)$ of $u(t)$ corresponding to eigenvalues $\lambda_k\geq\nu$. If u_{0k} and u_{1k} denote the corresponding components of initial data, estimate (5.20) reads as

$$\left(|u_k'(t)|^2+2\lambda_k^2\mu_1|u_k(t)|^2\right)\exp\left(2r\lambda_k^{2\sigma}t\right)\leq 4|u_{1k}|^2+2\left(3\delta^2\lambda_k^{4\sigma}+\lambda_k^2\mu_2\right)|u_{0k}|^2.$$

Since $\sigma\leq 1/2$ and we chose $\nu\geq 1$, this implies that

$$\left(|u_k'(t)|^2+2\lambda_k^2\mu_1|u_k(t)|^2\right)\exp\left(2r\lambda_k^{2\sigma}t\right)\leq 4|u_{1k}|^2+2\left(3\delta^2+\mu_2\right)\lambda_k^2|u_{0k}|^2$$

for every $t\geq 0$. Summing over all $\lambda_k\geq\nu$ we obtain (3.12) with a constant K depending only on μ_1, μ_2, and δ. The continuity of $u(t)$ and $u'(t)$ with values in the suitable spaces follows from the uniform convergence of series as in the proof of Theorem 3.1. □

6 The (DGCS)-phenomenon

In this section we prove Theorem 3.10. Let us describe the strategy before entering into details. Roughly speaking, what we need is a solution $u(t)$ whose components $u_k(t)$ are small at time $t = 0$ and huge at time $t > 0$. The starting point is given by the following functions

$$b(\varepsilon, \lambda, t) := (2\varepsilon\lambda - \delta\lambda^{2\sigma})t - \varepsilon \sin(2\lambda t),$$

$$w(\varepsilon, \lambda, t) := \sin(\lambda t) \exp(b(\varepsilon, \lambda, t)), \tag{6.1}$$

$$\gamma(\varepsilon, \lambda, t) := 1 + \frac{\delta^2}{\lambda^{2-4\sigma}} - 16\varepsilon^2 \sin^4(\lambda t) - 8\varepsilon \sin(2\lambda t). \tag{6.2}$$

With some computations it turns out that

$$w''(\varepsilon, \lambda, t) + 2\delta\lambda^{2\sigma} w'(\varepsilon, \lambda, t) + \lambda^2 \gamma(\varepsilon, \lambda, t) w(\varepsilon, \lambda, t) = 0 \qquad \forall t \in \mathbb{R},$$

where "primes" denote differentiation with respect to t. As a consequence, if we set $c(t) := \gamma(\varepsilon, \lambda, t)$ and $\varepsilon := \omega(1/\lambda)$, the function $u(t) := w(\varepsilon, \lambda, t)$ turns out to be a solution to (1.9) which grows as the right-hand side of (4.5). Unfortunately this is not enough, because we need to realize a similar growth for countably many components with the same coefficient $c(t)$.

To this end, we argue as in [6]. We introduce a suitable decreasing sequence $t_k \to 0^+$, and in the interval $[t_k, t_{k-1}]$ we design the coefficient $c(t)$ so that $u_k(t_k)$ is small and $u_k(t_{k-1})$ is huge. Then we check that the piecewise defined coefficient $c(t)$ has the required time-regularity, and that $u_k(t)$ remains small for $t \in [0, t_k]$ and remains huge for $t \geq t_{k-1}$. This completes the proof.

Roughly speaking, the coefficient $c(t)$ plays on infinitely many time-scales in order to "activate" countably many components, but these countably many actions take place one by one in disjoint time intervals. Of course this means that the lengths $t_{k-1} - t_k$ of the "activation intervals" tend to 0 as $k \to +\infty$. In order to obtain enough growth, despite of the vanishing length of activation intervals, we are forced to assume that $\lambda\omega(1/\lambda) \gg \lambda^{2\sigma}$ as $\lambda \to +\infty$. In addition, components do not grow exactly as $\exp(\lambda\omega(1/\lambda)t)$, but just more than $\exp(\varphi(\lambda)t)$ and $\exp(\psi(\lambda)t)$.

This is the reason why this strategy does not work when $\lambda\omega(1/\lambda) \sim \lambda^{2\sigma}$ and δ is small. In this case one would need components growing exactly as $\exp(\lambda\omega(1/\lambda)t)$, but this requires activation intervals of non-vanishing length, which are thus forced to overlap. In a certain sense, the coefficient $c(t)$ should work once again on infinitely many time-scales, but now the countably many actions should take place in the same time.

Definition of sequences From (3.13) and (3.14) it follows that

$$\lim_{x \to +\infty} x^{1-2\sigma} \omega\left(\frac{1}{x}\right) = +\infty, \tag{6.3}$$

$$\lim_{x \to +\infty} \frac{1}{x^{1-2\sigma}\omega(1/x)} + \frac{\varphi(x)}{x\omega(1/x)} + \frac{\psi(x)}{x\omega(1/x)} = 0, \tag{6.4}$$

and a fortiori

$$\lim_{x\to+\infty} x^{1+2\sigma}\omega\left(\frac{1}{x}\right) = +\infty, \tag{6.5}$$

$$\lim_{x\to+\infty} \frac{x^{2\sigma}+\varphi(x)+\psi(x)}{x}\,\omega\left(\frac{1}{x}\right) = 0. \tag{6.6}$$

Let us consider the sequence $\{\lambda_k\}$, which we assumed to be unbounded. Due to (6.5) and (6.4) we can assume, up to passing to a subsequence (not relabeled), that the following inequalities hold true for every $k \geq 1$:

$$\lambda_k > 4\lambda_{k-1}, \tag{6.7}$$

$$\lambda_k^{1+2\sigma}\omega\left(\frac{1}{\lambda_k}\right) \geq \frac{\delta^4}{2^{10}\pi^2}\frac{1}{\lambda_{k-1}^{2-8\sigma}} + \frac{4k^2}{\pi^2}\lambda_{k-1}^2, \tag{6.8}$$

$$\lambda_k^{1+2\sigma}\omega\left(\frac{1}{\lambda_k}\right) \geq \frac{4k^2}{\pi^2}\lambda_{k-1}^3\left(\lambda_{k-1}^{2\sigma}+\varphi(\lambda_{k-1})+\psi(\lambda_{k-1})\right)\omega\left(\frac{1}{\lambda_{k-1}}\right), \tag{6.9}$$

$$\lambda_k^{1+2\sigma}\omega\left(\frac{1}{\lambda_k}\right) \geq \lambda_{k-1}\left(\lambda_{k-1}^{2\sigma}+\varphi(\lambda_{k-1})+\psi(\lambda_{k-1})\right)\omega\left(\frac{1}{\lambda_{k-1}}\right), \tag{6.10}$$

$$\frac{1}{\lambda_k^{1-2\sigma}\omega(1/\lambda_k)} + \frac{\varphi(\lambda_k)}{\lambda_k\omega(1/\lambda_k)} + \frac{\psi(\lambda_k)}{\lambda_k\omega(1/\lambda_k)} \leq \frac{\pi^2}{4k^2}\frac{1}{\lambda_{k-1}^2}. \tag{6.11}$$

Now let us set

$$t_k := \frac{4\pi}{\lambda_k}, \qquad s_k := \frac{\pi}{\lambda_k}\left\lfloor 2\frac{\lambda_k}{\lambda_{k-1}}\right\rfloor, \tag{6.12}$$

where $\lfloor\alpha\rfloor$ denotes the largest integer less than or equal to α, and

$$\varepsilon_k := \left\{\frac{\lambda_k^{2\sigma}+\varphi(\lambda_k)+\psi(\lambda_k)}{\lambda_k}\,\omega\left(\frac{1}{\lambda_k}\right)\right\}^{1/2}.$$

Properties of the sequences We collect in this section of the proof all the properties of the sequences which are needed in the sequel. First of all, it is clear that $\lambda_k \to +\infty$, hence $t_k \to 0$ and $\varepsilon_k \to 0$ (because of (6.6)). Moreover it turns out that

$$\frac{t_{k-1}}{4} = \frac{\pi}{\lambda_{k-1}} \leq s_k \leq \frac{2\pi}{\lambda_{k-1}} = \frac{t_{k-1}}{2}. \tag{6.13}$$

Keeping (6.7) into account, it follows that

$$t_k < s_k < t_{k-1} \qquad \forall k \geq 1,$$

and in particular also $s_k \to 0$. In addition, it turns out that

$$\sin(\lambda_k t_k) = \sin(\lambda_k s_k) = 0 \tag{6.14}$$

and

$$|\cos(\lambda_k t_k)| = |\cos(\lambda_k s_k)| = 1. \tag{6.15}$$

Since $\sigma < 1/2$, $\lambda_k \to +\infty$, $\varepsilon_k \to 0$, $t_k \to 0$, keeping (6.3) and (6.4) into account, we deduce that the following seven inequalities are satisfied provided that k is large enough:

$$\frac{\delta^2}{\lambda_k^{2-4\sigma}} + 16\varepsilon_k^2 + 8\varepsilon_k \leq \frac{1}{2}, \tag{6.16}$$

$$\varepsilon_k \leq \frac{1}{4}, \tag{6.17}$$

$$16\pi\varepsilon_k + \frac{16\pi\delta}{\lambda_k^{1-2\sigma}} \leq 2\pi, \tag{6.18}$$

$$\frac{1}{\lambda_k^{1-2\sigma}\omega(1/\lambda_k)} + \frac{\varphi(\lambda_k)}{\lambda_k\omega(1/\lambda_k)} + \frac{\psi(\lambda_k)}{\lambda_k\omega(1/\lambda_k)} \leq \frac{1}{5^2 \cdot 2^{10} \cdot \pi^2}, \tag{6.19}$$

$$\frac{\delta^2}{(4\pi)^{2-4\sigma}}(2t_k)^{1-2\sigma} \sup\left\{\frac{x^{1-2\sigma}}{\omega(x)} : x \in (0, t_k)\right\} \leq \frac{1}{5}, \tag{6.20}$$

$$\lambda_k^{1-2\sigma}\omega\left(\frac{1}{\lambda_k}\right) \geq \delta^2, \tag{6.21}$$

$$\frac{2\delta^2}{\lambda_{k-1}^{2-4\sigma}\omega(1/\lambda_{k-1})} \leq \frac{1}{5}. \tag{6.22}$$

Let $k_0 \in \mathbb{N}$ be a positive integer such that (6.16) through (6.22) hold true for every $k \geq k_0$. From (6.21) it follows that

$$\varepsilon_k\lambda_k \geq \delta\lambda_k^{2\sigma} \qquad \forall k \geq k_0. \tag{6.23}$$

From (6.19) it follows that

$$32\pi\frac{\varepsilon_k}{\omega(1/\lambda_k)} \leq \frac{1}{5} \qquad \forall k \geq k_0. \tag{6.24}$$

Since $s_k \geq \pi/\lambda_{k-1}$ (see the estimate from below in (6.13)), from (6.8) it follows that

$$\varepsilon_k\lambda_k s_k \geq \frac{\delta^2}{32}\frac{1}{\lambda_{k-1}^{2-4\sigma}} \qquad \forall k \geq k_0, \tag{6.25}$$

$$\varepsilon_k\lambda_k s_k \geq 2k \qquad \forall k \geq k_0, \tag{6.26}$$

from (6.9) it follows that

$$\varepsilon_k\lambda_k s_k \geq 2k\varepsilon_{k-1}\lambda_{k-1} \qquad \forall k \geq k_0, \tag{6.27}$$

and from (6.11) it follows that

$$\varepsilon_k\lambda_k s_k \geq 2k\left(\lambda_k^{2\sigma} + \varphi(\lambda_k) + \psi(\lambda_k)\right) \qquad \forall k \geq k_0. \tag{6.28}$$

As a consequence of (6.26) through (6.28) it turns out that

$$2\varepsilon_k\lambda_k s_k \geq k\varepsilon_{k-1}\lambda_{k-1} + 2k\left(\lambda_k^{2\sigma} + \varphi(\lambda_k) + \psi(\lambda_k)\right) + k \qquad \forall k \geq k_0. \tag{6.29}$$

Finally, from (6.10) it follows that

$$\varepsilon_k\lambda_k \geq \varepsilon_{k-1}\lambda_{k-1} \qquad \forall k \geq k_0. \tag{6.30}$$

Definition of $c(t)$ and $u(t)$ For every $k \geq 1$, let $\ell_k : \mathbb{R} \to \mathbb{R}$ be defined by

$$\ell_k(t) := \frac{\delta^2}{t_{k-1} - s_k}\left(\frac{1}{\lambda_{k-1}^{2-4\sigma}} - \frac{1}{\lambda_k^{2-4\sigma}}\right)(t - s_k) + 1 + \frac{\delta^2}{\lambda_k^{2-4\sigma}} \qquad \forall t \in \mathbb{R}.$$

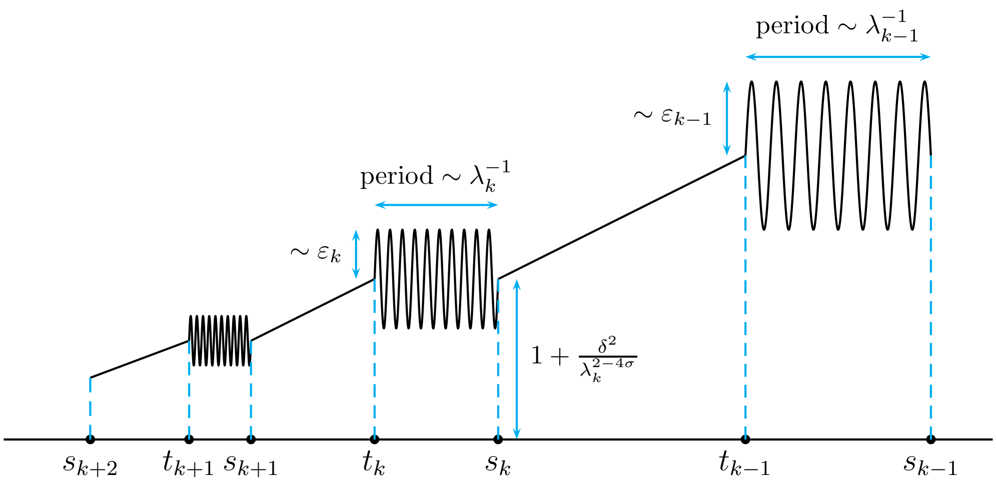
Thanks to (6.14), $\ell_k(t)$ is the affine function such that

$$\ell_k(s_k) = \gamma(\varepsilon_k, \lambda_k, s_k) \qquad \text{and} \qquad \ell_k(t_{k-1}) = \gamma(\varepsilon_{k-1}, \lambda_{k-1}, t_{k-1}).$$

Let $k_0 \in \mathbb{N}$ be such that (6.16) through (6.22) hold true for every $k \geq k_0$. Let us set

$$c(t) := \begin{cases} 1 & \text{if } t \leq 0, \\ \gamma(\varepsilon_k, \lambda_k, t) & \text{if } t \in [t_k, s_k] \text{ for some } k \geq k_0, \\ \ell_k(t) & \text{if } t \in [s_k, t_{k-1}] \text{ for some } k \geq k_0 + 1, \\ \gamma(\varepsilon_{k_0}, \lambda_{k_0}, s_{k_0}) & \text{if } t \geq s_{k_0}. \end{cases}$$

The following picture describes this definition. The coefficient $c(t)$ is constant for $t \leq 0$ and for $t \geq s_{k_0}$. In the intervals $[t_k, s_k]$ it coincides with $\gamma(\varepsilon_k, \lambda_k, t)$, hence it oscillates, with period of order λ_k^{-1} and amplitude of order ε_k, around a value which tends to 1. In the intervals $[s_k, t_{k-1}]$ it is just the affine interpolation of the values at the endpoints.



For every $k \geq k_0$, let us consider the solution $u_k(t)$ to the Cauchy problem

$$u_k''(t) + 2\delta\lambda_k^{2\sigma} u_k'(t) + \lambda_k^2 c(t) u_k(t) = 0,$$

with "initial" data

$$u_k(t_k) = 0, \qquad u_k'(t_k) = \lambda_k \exp\left((2\varepsilon_k\lambda_k - \delta\lambda_k^{2\sigma})t_k\right). \tag{6.31}$$

Then we set

$$a_k := \frac{1}{k\lambda_k}\exp(-k\varphi(\lambda_k)), \tag{6.32}$$

and finally

$$u(t) := \sum_{k=k_0}^{\infty} a_k u_k(t) e_k.$$

We claim that $c(t)$ satisfies (3.15) and (3.16), and that $u(t)$ satisfies (3.17) and (3.18). The rest of the proof is a verification of these claims.

Estimate and continuity of $c(t)$ We prove that

$$|c(t) - 1| \leq \frac{1}{2} \qquad \forall t \geq 0, \tag{6.33}$$

which is equivalent to (3.15), and that $c(t)$ is continuous on the whole real line.

To this end, it is enough to check (6.33) in the intervals $[t_k, s_k]$, because in the intervals $[s_k, t_{k-1}]$ the function $c(t)$ is just an interpolation of the values at the endpoints, and it is constant for $t \leq 0$ and for $t \geq s_{k_0}$.

In the intervals $[t_k, s_k]$ the function $c(t)$ coincides with $\gamma(\varepsilon_k, \lambda_k, t)$, hence from (6.2) it turns out that

$$|c(t) - 1| = |\gamma(\varepsilon_k, \lambda_k, t) - 1| \leq \frac{\delta^2}{\lambda_k^{2-4\sigma}} + 16\varepsilon_k^2 + 8\varepsilon_k, \tag{6.34}$$

so that (6.33) follows immediately from (6.16).

Since the right-hand side of (6.34) tends to 0 as $k \to +\infty$, the same estimate shows also that $c(t) \to 1$ as $t \to 0^+$, which proves the continuity of $c(t)$ in $t = 0$, the only point in which continuity was nontrivial.

Estimate on $c'(t)$ We prove that

$$|c'(t)| \leq 32\varepsilon_k\lambda_k \qquad \forall t \in (t_k, s_k),\ \forall k \geq k_0, \tag{6.35}$$

$$|c'(t)| \leq 32\varepsilon_k\lambda_k \qquad \forall t \in (s_k, t_{k-1}),\ \forall k \geq k_0 + 1. \tag{6.36}$$

Indeed in the interval (t_k, s_k) it turns out that

$$|c'(t)| = |\gamma'(\varepsilon_k, \lambda_k, t)| = \left|-64\varepsilon_k^2\lambda_k \sin^3(\lambda_k t)\cos(\lambda_k t) - 16\varepsilon_k\lambda_k\cos(2\lambda_k t)\right|$$

$$\leq 64\varepsilon_k^2\lambda_k + 16\varepsilon_k\lambda_k = 16\varepsilon_k\lambda_k(4\varepsilon_k+1),$$

so that (6.35) follows from (6.17).

In the interval (s_k, t_{k-1}) it turns out that

$$|c'(t)| = \frac{\delta^2}{t_{k-1}-s_k}\left(\frac{1}{\lambda_{k-1}^{2-4\sigma}} - \frac{1}{\lambda_k^{2-4\sigma}}\right) \leq \frac{\delta^2}{t_{k-1}-s_k}\cdot\frac{1}{\lambda_{k-1}^{2-4\sigma}} \leq \frac{\delta^2}{s_k}\cdot\frac{1}{\lambda_{k-1}^{2-4\sigma}},$$

where the last inequality follows from the estimate from above in (6.13). At this point (6.36) is equivalent to (6.25).

Modulus of continuity of $c(t)$ Let us prove that $c(t)$ satisfies (3.16). Since $c(t)$ is continuous, and constant for $t \leq 0$ and $t \geq s_{k_0}$, it is enough to verify its ω-continuity in $(0, s_{k_0}]$. In turn, the ω-continuity in $(0, s_{k_0}]$ is proved if we show that

$$|c(t_i) - c(t_j)| \leq \frac{1}{5}\,\omega(|t_i - t_j|) \qquad \forall i \geq k_0,\ \forall j \geq k_0, \tag{6.37}$$

$$|c(a) - c(b)| \leq \frac{1}{5}\,\omega(|a-b|) \qquad \forall (a,b) \in [t_k, s_k]^2,\ \forall k \geq k_0, \tag{6.38}$$

$$|c(a) - c(b)| \leq \frac{1}{5}\,\omega(|a-b|) \qquad \forall (a,b) \in [s_k, t_{k-1}]^2,\ \forall k \geq k_0+1. \tag{6.39}$$

Indeed, any interval $[s,t] \subseteq (0, s_{k_0}]$ can be decomposed as the union of at most 5 intervals whose endpoints fit in one of the 3 possibilities above.

Let us prove (6.37). From (6.14) it turns out that

$$|c(t_i) - c(t_j)| = \delta^2\left|\frac{1}{\lambda_i^{2-4\sigma}} - \frac{1}{\lambda_j^{2-4\sigma}}\right| \leq \delta^2\left|\frac{1}{\lambda_i^2} - \frac{1}{\lambda_j^2}\right|^{1-2\sigma},$$

where the inequality follows from the fact that the function $x \to x^{1-2\sigma}$ is $(1-2\sigma)$-Hölder continuous with constant equal to 1. Now from (6.12) it follows that

$$\delta^2\left|\frac{1}{\lambda_i^2} - \frac{1}{\lambda_j^2}\right|^{1-2\sigma} = \frac{\delta^2}{(4\pi)^{2-4\sigma}}|t_i^2 - t_j^2|^{1-2\sigma} = \frac{\delta^2}{(4\pi)^{2-4\sigma}}|t_i+t_j|^{1-2\sigma}\frac{|t_i-t_j|^{1-2\sigma}}{\omega(|t_i-t_j|)}\omega(|t_i-t_j|).$$

Since $|t_i + t_j| \leq 2t_{k_0}$ and $|t_i - t_j| \leq t_{k_0}$, we conclude that

$$|c(t_i) - c(t_j)| \leq \frac{\delta^2}{(4\pi)^{2-4\sigma}}(2t_{k_0})^{1-2\sigma}\sup\left\{\frac{x^{1-2\sigma}}{\omega(x)} : x \in (0, t_{k_0})\right\}\omega(|t_i - t_j|),$$

so that (6.37) follows from (6.20).

Let us prove (6.38). Since $c(t)$ is π/λ_k periodic in $[t_k, s_k]$, for every $(a,b) \in [t_k, s_k]^2$ there exists $(a_1, b_1) \in [t_k, s_k]^2$ such that $c(a) = c(a_1)$, $c(b) = c(b_1)$, and $|a_1 - b_1| \leq \pi/\lambda_k$. Thus from (6.35) it follows that

$$|c(a) - c(b)| = |c(a_1) - c(b_1)| \leq 32\varepsilon_k\lambda_k|a_1 - b_1| = 32\varepsilon_k\lambda_k\frac{|a_1-b_1|}{\omega(|a_1-b_1|)}\omega(|a_1-b_1|),$$

so that we are left to prove that

$$32\varepsilon_k\lambda_k\frac{|a_1-b_1|}{\omega(|a_1-b_1|)}\leq\frac{1}{5}. \tag{6.40}$$

Due to (2.6), (2.5), and the fact that $|a_1-b_1|\leq\pi/\lambda_k$, it turns out that

$$\frac{|a_1-b_1|}{\omega(|a_1-b_1|)}\leq\frac{\pi/\lambda_k}{\omega(\pi/\lambda_k)}\leq\frac{\pi}{\lambda_k\omega(1/\lambda_k)},$$

so that now (6.40) follows from (6.24).

Let us prove (6.39). Since $c(t)$ is affine in $[s_k,t_{k-1}]$, for every a and b in this interval it turns out that

$$|c(a)-c(b)|=\frac{\delta^2}{t_{k-1}-s_k}\left(\frac{1}{\lambda_{k-1}^{2-4\sigma}}-\frac{1}{\lambda_k^{2-4\sigma}}\right)|a-b|.$$

Since $s_k\leq t_{k-1}/2$, it follows that

$$|c(a)-c(b)|\leq\frac{2\delta^2}{t_{k-1}}\frac{1}{\lambda_{k-1}^{2-4\sigma}}\cdot|a-b|=\frac{2\delta^2}{t_{k-1}}\frac{1}{\lambda_{k-1}^{2-4\sigma}}\cdot\frac{|a-b|}{\omega(|a-b|)}\cdot\omega(|a-b|).$$

Due to (2.6), (2.5), and the fact that $|a-b|\leq t_{k-1}$, it turns out that

$$\frac{|a-b|}{\omega(|a-b|)}\leq\frac{t_{k-1}}{\omega(t_{k-1})}\leq\frac{t_{k-1}}{\omega(1/\lambda_{k-1})},$$

so that now (6.39) is a simple consequence of (6.22).

Energy functions Let us introduce the classic energy functions

$$E_k(t):=|u_k'(t)|^2+\lambda_k^2|u_k(t)|^2,$$

$$F_k(t):=|u_k'(t)|^2+\lambda_k^2c(t)|u_k(t)|^2.$$

Due to (3.15), they are equivalent in the sense that

$$\frac{1}{2}E_k(t)\leq F_k(t)\leq\frac{3}{2}E_k(t)\qquad\forall t\in\mathbb{R}.$$

Therefore, proving (3.17) is equivalent to showing that

$$\sum_{k=k_0}^{\infty}a_k^2E_k(0)\exp(2r\varphi(\lambda_k))<+\infty\qquad\forall r>0, \tag{6.41}$$

while proving (3.18) is equivalent to showing that

$$\sum_{k=k_0}^{\infty}a_k^2F_k(t)\exp(-2R\psi(\lambda_k))=+\infty\qquad\forall R>0,\ \forall t>0. \tag{6.42}$$

We are thus left to estimating $E_k(0)$ and $F_k(t)$.

Estimates in $[0, t_k]$ We prove that

$$E_k(0) \leq \lambda_k^2 \exp(4\pi) \qquad \forall k \geq k_0. \tag{6.43}$$

To this end, we begin by estimating $E_k(t_k)$. From (6.31) we obtain that $u_k(t_k) = 0$ and

$$|u_k'(t_k)| \leq \lambda_k \exp(2\varepsilon_k \lambda_k t_k) = \lambda_k \exp(8\pi\varepsilon_k),$$

so that

$$E_k(t_k) \leq \lambda_k^2 \exp(16\pi\varepsilon_k). \tag{6.44}$$

Now the time-derivative of $E_k(t)$ is

$$E_k'(t) = -4\delta\lambda_k^{2\sigma}|u_k'(t)|^2 - 2\lambda_k^2(c(t)-1)u_k'(t)u_k(t) \qquad \forall t \in \mathbb{R}.$$

Therefore, from (3.15) it follows that

$$E_k'(t) \geq -4\delta\lambda_k^{2\sigma} E_k(t) - \lambda_k|c(t)-1| \cdot 2|u_k'(t)| \cdot \lambda_k|u_k(t)| \geq -\left(4\delta\lambda_k^{2\sigma} + \frac{\lambda_k}{2}\right) E_k(t)$$

for every $t \in \mathbb{R}$. Integrating this differential inequality in $[0, t_k]$ we deduce that

$$E_k(0) \leq E_k(t_k) \exp\left[\left(4\delta\lambda_k^{2\sigma} + \frac{\lambda_k}{2}\right) t_k\right].$$

Keeping (6.44) and (6.12) into account, we conclude that

$$E_k(0) \leq \lambda_k^2 \exp\left(16\pi\varepsilon_k + \frac{16\pi\delta}{\lambda_k^{1-2\sigma}} + 2\pi\right),$$

so that (6.43) follows immediately from (6.18).

Estimates in $[t_k, s_k]$ In this interval it turns out that $u_k(t) := w(\varepsilon_k, \lambda_k, t)$, where $w(\varepsilon, \lambda, t)$ is the function defined in (6.1). Keeping (6.14) and (6.15) into account, we obtain that $u_k(s_k) = 0$ and

$$|u_k'(s_k)| = \lambda_k \exp(b(\varepsilon_k, \lambda_k, s_k)) = \lambda_k \exp\left((2\varepsilon_k\lambda_k - \delta\lambda_k^{2\sigma})s_k\right).$$

Therefore, from (6.23) it follows that

$$|u_k'(s_k)| \geq \lambda_k \exp(\varepsilon_k \lambda_k s_k),$$

and hence

$$F_k(s_k) = E_k(s_k) \geq \lambda_k^2 \exp(2\varepsilon_k \lambda_k s_k). \tag{6.45}$$

Estimates in $[s_k, t_{k-1}]$ We prove that

$$F_k(t_{k-1}) \geq \lambda_k^2 \exp(2\varepsilon_k\lambda_k s_k - 4\delta\lambda_k^{2\sigma} t_{k-1}). \tag{6.46}$$

Indeed the time-derivative of $F_k(t)$ is

$$F_k'(t) = -4\delta\lambda_k^{2\sigma}|u_k'(t)|^2 + \lambda_k^2 c'(t)|u_k(t)|^2 \qquad \forall t \in (s_k, t_{k-1}).$$

Since $c'(t) > 0$ in (s_k, t_{k-1}), it follows that

$$F_k'(t) \geq -4\delta\lambda_k^{2\sigma}|u_k'(t)|^2 \geq -4\delta\lambda_k^{2\sigma} F_k(t) \qquad \forall t \in (s_k, t_{k-1}),$$

and hence

$$F_k(t_{k-1}) \geq F_k(s_k)\exp\left(-4\delta\lambda_k^{2\sigma}(t_{k-1} - s_k)\right) \geq F_k(s_k)\exp\left(-4\delta\lambda_k^{2\sigma} t_{k-1}\right).$$

Keeping (6.45) into account, we have proved (6.46).

Estimates in $[t_{k-1}, +\infty)$ We prove that

$$F_k(t) \geq \lambda_k^2 \exp\left(2\varepsilon_k\lambda_k s_k - 8\delta\lambda_k^{2\sigma} t - 64\varepsilon_{k-1}\lambda_{k-1} t\right) \qquad \forall t \geq t_{k-1}. \tag{6.47}$$

To this end, let us set

$$I_k := [t_{k-1}, +\infty) \setminus \bigcup_{i=k_0}^{k-1} \{t_i, s_i\}.$$

First of all, we observe that

$$|c'(t)| \leq 32\varepsilon_{k-1}\lambda_{k-1} \qquad \forall t \in I_k \tag{6.48}$$

Indeed we know from (6.35) and (6.36) that

$$|c'(t)| \leq 32\varepsilon_i\lambda_i \qquad \forall t \in (t_i, s_i) \cup (s_i, t_{i-1}),$$

and of course $c'(t) = 0$ for every $t > s_{k_0}$. Now it is enough to observe that

$$I_k = (t_{k_0}, s_{k_0}) \cup (s_{k_0}, +\infty) \cup \bigcup_{i=k_0+1}^{k-1} [(t_i, s_i) \cup (s_i, t_{i-1})],$$

and that $\varepsilon_i\lambda_i$ is a nondecreasing sequence because of (6.30).

Now we observe that the function $t \to F_k(t)$ is continuous in $[t_{k-1}, +\infty)$ and differentiable in I_k, with

$$\begin{aligned} F_k'(t) &= -4\delta\lambda_k^{2\sigma}|u_k'(t)|^2 + \lambda_k^2 c'(t)|u_k(t)|^2 \\ &\geq -4\delta\lambda_k^{2\sigma}|u_k'(t)|^2 - \frac{|c'(t)|}{c(t)} \cdot \lambda_k^2 c(t)|u_k(t)|^2 \\ &\geq -\left(4\delta\lambda_k^{2\sigma} + \frac{|c'(t)|}{c(t)}\right) F_k(t). \end{aligned}$$

Therefore, from (6.48) and (3.15) it follows that

$$F_k'(t) \geq -\left(4\delta\lambda_k^{2\sigma} + 64\varepsilon_{k-1}\lambda_{k-1}\right) F_k(t) \qquad \forall t \in I_k,$$

and hence

$$\begin{aligned} F_k(t) &\geq F_k(t_{k-1}) \exp\left[-\left(4\delta\lambda_k^{2\sigma} + 64\varepsilon_{k-1}\lambda_{k-1}\right)(t - t_{k-1})\right] \\ &\geq F_k(t_{k-1}) \exp\left[-\left(4\delta\lambda_k^{2\sigma} + 64\varepsilon_{k-1}\lambda_{k-1}\right) t\right] \end{aligned}$$

for every $t \geq t_{k-1}$. Keeping (6.46) into account, we finally obtain that

$$F_k(t) \geq \lambda_k^2 \exp\left(2\varepsilon_k\lambda_k s_k - 4\delta\lambda_k^{2\sigma} t_{k-1} - 4\delta\lambda_k^{2\sigma} t - 64\varepsilon_{k-1}\lambda_{k-1} t\right),$$

from which (6.47) follows by simply remarking that $t \geq t_{k-1}$.

Conclusion We are now ready to verify (6.41) and (6.42). Indeed from (6.32) and (6.43) it turns out that

$$\begin{aligned} a_k^2 E_k(0) \exp(2r\varphi(\lambda_k)) &\leq \frac{1}{k^2\lambda_k^2} \exp(-2k\varphi(\lambda_k)) \cdot \lambda_k^2 \exp(4\pi) \cdot \exp(2r\varphi(\lambda_k)) \\ &= \frac{1}{k^2} \exp\left(4\pi + 2(r-k)\varphi(\lambda_k)\right). \end{aligned}$$

The argument of the exponential is less than 4π when k is large enough, and hence the series in (6.41) converges.

Let us consider now (6.42). For every $t > 0$ it turns out that $t \geq t_{k-1}$ when k is large enough. For every such k we can apply (6.47) and obtain that

$$\begin{aligned} & a_k^2 F_k(t) \exp(-2R\psi(\lambda_k)) \\ & \quad \geq \frac{1}{k^2} \exp\left(-2k\varphi(\lambda_k) - 2R\psi(\lambda_k) + 2\varepsilon_k\lambda_k s_k - 8\delta\lambda_k^{2\sigma} t - 64\varepsilon_{k-1}\lambda_{k-1} t\right). \end{aligned}$$

Keeping (6.29) into account, it follows that

$$\begin{aligned} & a_k^2 F_k(t) \exp(-2R\psi(\lambda_k)) \\ & \quad \geq \frac{1}{k^2} \exp\left((k - 64t)\varepsilon_{k-1}\lambda_{k-1} + 2(k-R)\psi(\lambda_k) + (2k - 8\delta t)\lambda_k^{2\sigma} + k\right) \\ & \quad \geq \frac{1}{k^2} \exp(k) \end{aligned}$$

when k is large enough. This proves that the series in (6.42) diverges. □

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