ON THE STABILITY OF STANDING WAVES OF KLEIN-GORDON EQUATIONS IN A SEMICLASSICAL REGIME

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ABSTRACT. We investigate the orbital stability and instability of standing waves for two classes of Klein-Gordon equations in the semi-classical regime.

1. Introduction and results

The nonlinear Klein-Gordon equation

$$\varepsilon^2 u_{tt} - \varepsilon^2 \Delta u + mu - |u|^{p-1}u = 0 \qquad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.1}$$

where $\varepsilon, m > 0$, $p > 1$ for $N = 1, 2$ and $1 < p < (N+2)/(N-2)$ for $N \geq 3$, is one of the simplest nonlinear partial differential equations invariant for the Poincaré group. We are interested in the study of the nonlinear Klein Gordon equation in presence of a potential depending on the space variable. Two different choices are viable. We can simply add a potential term $W(x)u$ to equation (1.1). This case has been studied, for the linear wave equation, for example, by Beals and Strauss in [3]. Otherwise, if we want to fully preserve the invariance for the Poincaré group of (1.1), we have to change the temporal derivative $\varepsilon^2\partial_{tt}$ with a covariant derivative, depending on the potential D^2_{tt}, where $D_t = \varepsilon\partial_t + \mathrm{i}V(x)$. This second approach is classical, in quantum electrodynamics, when considering electromagnetic waves. The first approach leads us to consider the equation

$$\varepsilon^2 u_{tt} - \varepsilon^2 \Delta u + mu - Wu - |u|^{p-1}u = 0, \qquad \text{in } \mathbb{R}^N, \tag{1.2}$$

while the second one to investigate the equation

$$\varepsilon^2 u_{tt} + 2\mathrm{i}\varepsilon V u_t - \varepsilon^2 \Delta u + mu - V^2 u - |u|^{p-1}u = 0, \qquad \text{in } \mathbb{R}^N. \tag{1.3}$$

In this paper, we treat simultaneously the two previous Klein-Gordon equations by studying

$$\varepsilon^2 u_{tt} + 2\mathrm{i}\varepsilon V u_t - \varepsilon^2 \Delta u + mu - Wu - |u|^{p-1}u = 0, \qquad \text{in } \mathbb{R}^N, \tag{1.4}$$

where $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, $\varepsilon > 0$ and V, W are real valued potential functions. Equation (1.4) formally yields to (1.3) for the choice $W = V^2$ as well as to (1.2) when $V = 0$. We shall study the stability of standing waves of this equation in the semiclassical regime $\varepsilon \to 0$. It admits standing waves, namely solutions of the form $u(x,t) = e^{\mathrm{i}\omega t/\varepsilon}\varphi_\omega(x/\varepsilon)$, where $\omega \in \mathbb{R}$ and φ_ω satisfies

$$-\Delta\varphi_\omega + \big(m - \omega^2 - 2\omega V(\varepsilon y) - W(\varepsilon y)\big)\varphi_\omega - |\varphi_\omega|^{p-1}\varphi_\omega = 0, \qquad \text{in } \mathbb{R}^N. \tag{1.5}$$

Date: November 8, 2018.

2000 *Mathematics Subject Classification.* 35D99, 35J62, 58E05, 35J70.

Key words and phrases. Klein-Gordon equations, stability, semi-classical limit.

Research supported in part by 2009 MIUR project: *"Variational and Topological Methods in the Study of Nonlinear Phenomena"*, ANR project ESONSE and GNAMPA project *"Dinamica e proprietà di soluzioni concentrate in teorie di campo nonlineari"* .

To ensure existence of solutions to (1.5) for ε close to 0, we assume the following. Let V and W be $\mathcal{C}^2$. For the function

$$Z(y) = m - \omega^2 - 2\omega V(y) - W(y), \quad y \in \mathbb{R}^N$$

there exists $x_0 \in \mathbb{R}^N$ such that

$$\nabla Z(x_0) = 0, \qquad \nabla^2 Z(x_0) \text{ is non-degenerate.} \tag{1.6}$$

Furthermore, we assume that

$$Z(x_0) = m - \omega^2 - 2\omega V(x_0) - W(x_0) > 0. \tag{1.7}$$

Under these hypotheses, it is well-know (see e.g. [1] or [2, Section 8.2]) that when ε is close to 0 the equation (1.5) admits a family of positive, exponentially decaying, solutions $\varphi_\omega \subset H^1(\mathbb{R}^N)$ (hiding the dependance upon ε). More precisely, there exists $\xi_\varepsilon \in \mathbb{R}^N$ and $\psi_\omega \in H^1(\mathbb{R}^N)$ such that $\varphi_\omega(\cdot) = \psi_\omega(\cdot - \xi_\varepsilon) + \mathcal{O}(\varepsilon^2)$ in $H^1(\mathbb{R}^N)$ as $\varepsilon \to 0$, where $\xi_\varepsilon = o(\varepsilon)$ and

$$-\Delta\psi_\omega + Z(x_0)\psi_\omega = |\psi_\omega|^{p-1}\psi_\omega, \qquad \text{in } \mathbb{R}^N.$$

We are interested in the (orbital) stability or instability of standing waves when ε goes to 0.

A standing wave of (1.4) is said to be *(orbitally) stable* if any solution of (1.4) starting close to the standing wave remains close for all time, up to the invariances of the equation. More precisely, we say that $e^{\frac{i\omega t}{\varepsilon}}\varphi_\omega\left(\frac{x}{\varepsilon}\right)$ is stable if for all $\eta > 0$ there exists $\delta > 0$ such that for all $u_0 \in H^1(\mathbb{R}^N)$ verifying $\|u_0 - \varphi_\omega\|_{H^1(\mathbb{R}^N)} < \delta$ the solution $u(x,t)$ of (1.4) with initial data u_0 satisfies

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|u - e^{i\theta}\varphi_\omega\|_{H^1(\mathbb{R}^N)} < \eta.$$

Since the pioneering works [4,6–8,24,25], the study of orbital stability for standing waves of dispersive PDE has attracted a lot of attention. Among many others, one can refer to [10,11,13]; see also the books and surveys [5,12,22,23] and the references therein. Relatively few works [9, 14, 16] are concerned with stability at the semi-classical limit for Schrödinger type equations. For Klein-Gordon equations, after the ground works [20, 21], there has been a recent interest for stability by blow-up [15, 17–19].

To study stability, we first rewrite (1.4) in Hamiltonian form

$$\varepsilon\frac{\partial U}{\partial t} = JE'(U), \tag{1.8}$$

where $U = \begin{pmatrix} u \\ v \end{pmatrix}$, $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and

$$E(U) = \frac{1}{2}\|v - iVu\|^2_{L^2(\mathbb{R}^N)} + \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + m\frac{1}{2}\|u\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{2}\int_{\mathbb{R}^N} W|u|^2 dx - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}.$$

It is easy to see that if u solves (1.4) and v is defined by $v := \varepsilon u_t + iVu$, then $U = \begin{pmatrix} u \\ v \end{pmatrix}$ solves (1.8). The charge $Q(U)$ is defined by

$$Q(U) = \Im \int_{\mathbb{R}^N} \bar{u}v dx.$$

In particular, for a standing wave $u = e^{i\omega t/\varepsilon}\varphi_\omega(x/\varepsilon)$, the charge is given by

$$Q(\varphi_\omega) := Q(U) = \varepsilon^N \left(\omega\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} V(\varepsilon y)|\varphi_\omega|^2\right). \tag{1.9}$$

According to the theory developed in [7,8], a standing wave $e^{\frac{i\omega t}{\varepsilon}}\varphi_\omega\left(\frac{x}{\varepsilon}\right)$ is stable if two conditions are satisfied.

(i) The *Slope Condition*: $\dfrac{\partial}{\partial\omega}Q(\varphi_\omega) < 0$.

(ii) The *Spectral Condition*: $L_\varepsilon := -\Delta + Z(\varepsilon y) - p|\varphi_\omega|^{p-1}$ has exactly one negative eigenvalue and is non degenerate.

On the other hand, denote by $n(L_\varepsilon)$ the number of negative eigenvalues of L_ε and set $p(\omega) = 0$ if $\dfrac{\partial}{\partial\omega}Q(\varphi_\omega) > 0$, $p(\omega) = 1$ if $\dfrac{\partial}{\partial\omega}Q(\varphi_\omega) < 0$. Then the standing wave is unstable if $n(L_\varepsilon) - p(\omega)$ is odd.

Then we have the following

Theorem 1.1. *Assume that conditions (1.6)-(1.7) hold. Then, we have the following facts.*

(1) *If $p < 1 + 4/N$, then the Slope Condition $\frac{\partial}{\partial\omega}Q(\varphi_\omega) < 0$ is fulfilled if*

$$Z(x_0) < (\omega + V(x_0))^2\Big(\frac{4}{p-1} - N\Big) \qquad \textit{(non-critical case)}$$

or if

$$\begin{cases} Z(x_0) = (\omega + V(x_0))^2\Big(\frac{4}{p-1} - N\Big), \\ \Big(\Delta Z(0) - \Delta V(0)\Big(1 + \frac{2(\omega+V(0))}{Z(0)}\Big)\Big) < 0, \end{cases} \qquad \textit{(critical case).}$$

(2) *If $p \geq 1 + 4/N$, then we always have $\frac{\partial}{\partial\omega}Q(\varphi_\omega) > 0$.*

(3) *We have the equality $n(L_\varepsilon) = n(\nabla^2 Z(x_0)) + 1$, where $n(\nabla^2 Z(x_0))$ is the number of negative eigenvalues of $\nabla^2 Z(x_0)$.*

In particular, the standing waves $e^{i\omega t}\varphi_\omega$ are stable if x_0 is non-degenerate local minimum of Z, $p < 1 + 4/N$ and

$$Z(x_0) < (\omega + V(x_0))^2\Big(\frac{4}{p-1} - N\Big).$$

Note that, conversely to what was happening in the case of Schrödinger equations studied in [14], the values of the potentials V and W at x_0 comes into play for the Slope Condition even in the noncritical case. Note also that only the local behavior of Z around x_0 influences the stability or instability.

Notations : Most of the objects we consider will depend both on ε and ω. We will emphasize the most important parameter by indicating it as a subscript, the dependence in the other parameter being understood.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We start be focusing on the Slope Condition and then we study the Spectral Condition. For the sake of simplicity in notations and without loss of generality, in the rest of this section we assume that $x_0 = 0$.

2.1. **The Slope Condition.** We start with the noncritical case.

2.1.1. *Noncritical case.* We assume that
$$Z(0) \neq (\omega + V(0))^2\Big(\frac{4}{p-1} - N\Big).$$

We first rewrite $Q(\varphi_\omega)$ by expanding $V(\varepsilon y)$ and using the exponential decay of φ_ω:
$$Q(\varphi_\omega) = \varepsilon^N\,(\omega + V(0))\,\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + \mathcal{O}(\varepsilon^{N+1}).$$
Therefore, since
$$\frac{\partial}{\partial\omega}Q(\varphi_\omega) = \varepsilon^N\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + \varepsilon^N(\omega + V(0))\frac{\partial}{\partial\omega}\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + \mathcal{O}(\varepsilon^{N+1}), \tag{2.1}$$
to evaluate the sign of the map $\omega \mapsto \frac{\partial}{\partial\omega}Q(\varphi_\omega)$ one should compute the quantity
$$\frac{\partial}{\partial\omega}\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} = 2\int_{\mathbb{R}^N} R_\omega\varphi_\omega, \tag{2.2}$$
where $R_\omega(x) := \frac{\partial\varphi_\omega}{\partial\omega}(x)$.

We remark that differentiation of (1.5) with respect to ω easily yields
$$L_\varepsilon R_\omega = 2(\omega + V(\varepsilon y))\varphi_\omega. \tag{2.3}$$
If we now introduce the rescaling $\varphi_\omega(x) = \lambda^{\frac{1}{p-1}}\varphi_\lambda(\sqrt{\lambda}x)$, it follows that φ_λ satisfies
$$-\Delta\varphi_\lambda + \lambda^{-1}Z\left(\frac{\varepsilon y}{\sqrt{\lambda}}\right)\varphi_\lambda - |\varphi_\lambda|^{p-1}\varphi_\lambda = 0, \qquad \text{in } \mathbb{R}^N. \tag{2.4}$$
Now, differentiating equation (2.4) with respect to λ and denoting $T_\lambda = \frac{\partial\varphi_\lambda}{\partial\lambda}{}_{|\lambda=1}$ yields
$$L_\varepsilon T_\lambda - Z(\varepsilon y)\varphi_\omega - \frac{1}{2}\varepsilon y\cdot\nabla Z(\varepsilon y)\varphi_\omega = 0. \tag{2.5}$$

Since 0 is a critical point of Z, a Taylor expansion gives
$$Z(\varepsilon y) = Z(0) + \mathcal{O}(\varepsilon^2|y|^2), \tag{2.6}$$
$$\frac{1}{2}\varepsilon y\cdot\nabla Z(\varepsilon y) = \mathcal{O}(\varepsilon^2|y|^2). \tag{2.7}$$
Then, from (2.5), as $\varepsilon \to 0$ we have
$$L_\varepsilon T_\lambda = Z(0)\varphi_\omega + \mathcal{O}(\varepsilon^2|y^2|\varphi_\omega), \quad \text{in } \mathbb{R}^N. \tag{2.8}$$
Then, in turn, taking into account identity (2.3) we get
$$\begin{aligned}
Z(0)\int_{\mathbb{R}^N} R_\omega\varphi_\omega &= \int_{\mathbb{R}^N} R_\omega L_\varepsilon T_\lambda + \mathcal{O}(\varepsilon^2)\\
&= \int_{\mathbb{R}^N} L_\varepsilon R_\omega T_\lambda + \mathcal{O}(\varepsilon^2)\\
&= \int_{\mathbb{R}^N} 2(\omega + V(\varepsilon y))\varphi_\omega T_\lambda + \mathcal{O}(\varepsilon^2)\\
&= 2(\omega + V(0))\int_{\mathbb{R}^N} \varphi_\omega T_\lambda + \mathcal{O}(\varepsilon)\\
&= (\omega + V(0))\frac{\partial}{\partial\lambda}\|\varphi_\lambda\|^2_{L^2(\mathbb{R}^N)|\lambda=1} + \mathcal{O}(\varepsilon)\\
&= (\omega + V(0))\Big(\frac{N}{2} - \frac{2}{p-1}\Big)\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + \mathcal{O}(\varepsilon).
\end{aligned} \tag{2.9}$$

In conclusion, by combining (2.1), (2.2) and (2.8), we have

$$\frac{\partial}{\partial\omega}Q(\varphi_\omega) = \varepsilon^N\left(1 + \frac{(\omega+V(0))^2}{Z(0)}\Big(N - \frac{4}{p-1}\Big)\right)\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + \mathcal{O}(\varepsilon^{N+1}).$$

Then, taking into account the fact that $Z(0) > 0$ and that φ_ω converges to ψ_ω in $L^2(\mathbb{R}^N)$ as $\varepsilon \to 0$, the sign of $\frac{\partial}{\partial\omega}Q(\varphi_\omega)$ is the sign of

$$Z(0) - (\omega + V(0))^2\Big(\frac{4}{p-1} - N\Big).$$

2.1.2. *Critical case.* We assume now that

$$Z(0) = (\omega + V(0))^2\Big(\frac{4}{p-1} - N\Big). \tag{2.10}$$

In the critical case, the term of order ε^N in the expansion of $\frac{\partial}{\partial\omega}Q(\varphi_\omega)$ vanishes and we need to calculate the expansion at a higher order. We first refine (2.6)-(2.7).

$$Z(\varepsilon y) = Z(0) + \frac{\varepsilon^2}{2}\nabla^2 Z(0)(y,y) + \mathcal{O}(\varepsilon^3|y|^3)$$
$$\frac{1}{2}\varepsilon y\cdot\nabla Z(\varepsilon y) = \frac{\varepsilon^2}{2}\nabla^2 Z(0)(y,y) + \mathcal{O}(\varepsilon^3|y|^3).$$

Then (2.5) gives

$$L_\varepsilon T_\lambda = Z(0)\varphi_\omega + \varepsilon^2\nabla^2 Z(0)(y,y)\varphi_\omega + \mathcal{O}(\varepsilon^3|y^3|)\varphi_\omega.$$

Now, we have

$$Z(0)\int_{\mathbb{R}^N} R_\omega\varphi_\omega = \int_{\mathbb{R}^N} R_\omega L_\varepsilon T_\lambda - \varepsilon^2\int_{\mathbb{R}^N}\nabla^2 Z(0)(y,y)R_\omega\varphi_\omega + \mathcal{O}(\varepsilon^3). \tag{2.11}$$

From (2.3), we obtain

$$\int_{\mathbb{R}^N} R_\omega L_\varepsilon T_\lambda = \int_{\mathbb{R}^N} L_\varepsilon R_\omega T_\lambda = \int_{\mathbb{R}^N} 2(\omega + V(\varepsilon y))\varphi_\omega T_\lambda. \tag{2.12}$$

Expanding the potential V we get

$$\begin{aligned}\int_{\mathbb{R}^N} 2V(\varepsilon y)\varphi_\omega T_\lambda = \int_{\mathbb{R}^N} 2V(0)\varphi_\omega T_\lambda + 2\varepsilon\int_{\mathbb{R}^N} y\cdot\nabla V(0)\varphi_\omega T_\lambda \\ + \varepsilon^2\int_{\mathbb{R}^N}\nabla^2 V(0)(y,y)\varphi_\omega T_\lambda + \mathcal{O}(\varepsilon^3).\end{aligned} \tag{2.13}$$

Note that since $\varphi_\omega = \psi_\omega(\cdot - \xi_\varepsilon) + \mathcal{O}(\varepsilon^2)$ and $\xi_\varepsilon = o(\varepsilon)$, we have

$$2\varepsilon\int_{\mathbb{R}^N} y\cdot\nabla V(0)\varphi_\omega T_\lambda = 2\varepsilon\int_{\mathbb{R}^N} y\cdot\nabla V(0)\psi_\omega T_\lambda + o(\varepsilon^2) = o(\varepsilon^2) \tag{2.14}$$

where the last cancellation comes from the fact that ψ_ω is radial. Coming back to (2.12) and as in (2.9), we have

$$\begin{aligned}\int_{\mathbb{R}^N} R_\omega L_\varepsilon T_\lambda = (\omega + V(0))\Big(\frac{N}{2} - \frac{2}{p-1}\Big)\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} \\ + \varepsilon^2\int_{\mathbb{R}^N}\nabla^2 V(0)(y,y)\varphi_\omega T_\lambda + o(\varepsilon^2).\end{aligned} \tag{2.15}$$

It remains to compute the integrals involving the Hessians in (2.11) and (2.15). Since our problem is invariant by an orthonormal transformation, we can assume without loss of generality that $\nabla^2 V(0) = \mathrm{diag}(b_1, \dots, b_N)$. Hence $\nabla^2 V(0)(y,y) = \sum_{j=1}^N b_j y_j^2$. Recall also that T_λ can be computed explicitly to have

$$T_\lambda = -\frac{1}{p-1}\varphi_\omega - \frac{1}{2} y \cdot \nabla \varphi_\omega.$$

Therefore,

$$\int_{\mathbb{R}^N} b_j y_j^2 \varphi_\omega T_\lambda = -\frac{b_j}{p-1}\int_{\mathbb{R}^N} y_j^2 \varphi_\omega^2 - \frac{b_j}{2}\sum_{k=1}^N \int_{\mathbb{R}^N} y_j^2 y_k \varphi_\omega \frac{\partial \varphi_\omega}{\partial y_k}.$$

We have after integration by parts

$$2\sum_{k=1}^N \int_{\mathbb{R}^N} y_j^2 y_k \varphi_\omega \frac{\partial \varphi_\omega}{\partial y_k} = -\sum_{k=1}^N \int_{\mathbb{R}^N} (y_j^2 + 2\delta_{jk} y_j^2)\varphi_\omega^2 = -(N+2)\int_{\mathbb{R}^N} y_j^2 \varphi_\omega^2.$$

Thus

$$\int_{\mathbb{R}^N} \nabla^2 V(0)(y,y)\varphi_\omega T_\lambda = \sum_{j=1}^N \int_{\mathbb{R}^N} b_j y_j^2 \varphi_\omega T_\lambda = -\left(\frac{1}{p-1} - \frac{N+2}{4}\right)\sum_{j=1}^N b_j \int_{\mathbb{R}^N} y_j^2 \varphi_\omega^2.$$

Recall the following expansion in ε for R_ω and φ_ω.

$$\varphi_\omega = \psi_\omega + o(\varepsilon), \qquad R_\omega = \frac{\partial \psi_\omega}{\partial \omega} + o(\varepsilon).$$

Therefore, since ψ_ω is radial,

$$\int_{\mathbb{R}^N} y_j^2 \varphi_\omega^2 = \int_{\mathbb{R}^N} y_j^2 \psi_\omega^2 + o(\varepsilon) = \frac{1}{N}\| |y|\psi_\omega\|^2_{L^2(\mathbb{R}^N)} + o(\varepsilon),$$

and so

$$\int_{\mathbb{R}^N} \nabla^2 V(0)(y,y)\varphi_\omega T_\lambda = -\left(\frac{1}{p-1} - \frac{N+2}{4}\right)\frac{1}{N}\| |y|\psi_\omega\|^2_{L^2(\mathbb{R}^N)} \Delta V(0) + o(\varepsilon). \tag{2.16}$$

Similarly, we have

$$\int_{\mathbb{R}^N} \nabla^2 Z(0)(y,y) R_\omega \varphi_\omega = -\left(\frac{1}{p-1} - \frac{N+2}{4}\right)\frac{1}{N}\| |y|\psi_\omega\|^2_{L^2(\mathbb{R}^N)} \Delta Z(0) + o(\varepsilon). \tag{2.17}$$

Summarizing, using successively (2.11), (2.15), (2.16), (2.17) and (2.10) we have obtained

$$\int_{\mathbb{R}^N} R_\omega \varphi_\omega = -\frac{1}{2(\omega + V(0))}\|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + \varepsilon^2 \frac{(\Delta Z(0) - \Delta V(0))}{N Z(0)}\left(\frac{1}{p-1} - \frac{N+2}{4}\right)\| |y|\psi_\omega\|^2_{L^2(\mathbb{R}^N)} + o(\varepsilon^2). \tag{2.18}$$

Now, we compute $\frac{\partial Q(\varphi_\omega)}{\partial \omega}$. First, recall that, coming back to the definition (1.9) of Q, we have

$$\varepsilon^{-N}\frac{\partial Q(\varphi_\omega)}{\partial \omega} = \|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + 2\omega \int_{\mathbb{R}^N} R_\omega \varphi_\omega + 2\int_{\mathbb{R}^N} V(\varepsilon y) R_\omega \varphi_\omega.$$

As in (2.13), (2.14), and (2.16) we can expand in ε and get

$$2\int_{\mathbb{R}^N} V(\varepsilon y)R_\omega\varphi_\omega = 2V(0)\int_{\mathbb{R}^N} R_\omega\varphi_\omega - \varepsilon^2\left(\frac{1}{p-1}-\frac{N+2}{4}\right)\frac{1}{N}\||y|\psi_\omega\|^2_{L^2(\mathbb{R}^N)}\Delta V(0)+o(\varepsilon^2).$$

Therefore,

$$\varepsilon^{-N}\frac{\partial Q(\varphi_\omega)}{\partial\omega} = \|\varphi_\omega\|^2_{L^2(\mathbb{R}^N)} + 2(\omega+V(0))\int_{\mathbb{R}^N} R_\omega\varphi_\omega - \varepsilon^2\left(\frac{1}{p-1}-\frac{N+2}{4}\right)\frac{1}{N}\||y|\psi_\omega\|^2_{L^2(\mathbb{R}^N)}\Delta V(0)+o(\varepsilon^2).$$

Using (2.18), we finally get

$$\varepsilon^{-N}\frac{\partial Q(\varphi_\omega)}{\partial\omega} = \varepsilon^2\left(\frac{1}{p-1}-\frac{N+2}{4}\right)\frac{1}{N}\||y|\psi_\omega\|^2_{L^2(\mathbb{R}^N)}\times \times\left(\Delta Z(0)-\Delta V(0)\left(1+\frac{2(\omega+V(0))}{Z(0)}\right)\right)+o(\varepsilon^2).$$

2.2. **The Spectral Condition.** We define the operator $L_0 := -\Delta + Z(0) - p\psi_\omega^{p-1}$. It is well known (see e.g. [2]) that the spectrum of L_0 consists of one negative eigenvalue, a N-dimensional kernel (generated by $\frac{\partial\psi_\omega}{\partial y_j}$ for $j = 1,\dots,N$) and the rest of the spectrum is bounded away from 0. When ε is close to 0, the spectrum of L_ε will be close to the spectrum of L_0. In particular, the 0 eigenvalue, of multiplicity N, will transform into N possibly different eigenvalues close to 0 but shifted either to the positive or to the negative side of the real axis, depending on the sign of the eigenvalues of the Hessian of Z at 0. More precisely, the following proposition was proved in [14] (see [9] for a detailed justification).

Proposition 2.1. *The spectrum of L_ε consists of positive spectrum away from 0 and a set of $N+1$ simple eigenvalues $\{\lambda_0,\lambda_1,\dots,\lambda_N\}$ such that*

$$\lambda_0 < \lambda_1 \le \cdots \le \lambda_N.$$

As $\varepsilon \to 0$, we have $\lambda_0 < 0$ and the following asymptotic expansion holds for the other eigenvalues:

$$\lambda_j = c_j\varepsilon^2 + o(\varepsilon^2), \qquad j = 1,\dots,N,$$

where $c_j = \frac{1}{2}\frac{\|\psi_\omega\|^2_{L^2(\mathbb{R}^N)}}{\|\frac{\partial\psi_\omega}{\partial x_j}\|^2_{L^2(\mathbb{R}^N)}}a_j$ and $\{a_1,\dots,a_N\}$ are the eigenvalues of the Hessian matrix $\nabla^2 Z(0)$.

Therefore, (3) in Theorem 1.1 is a direct consequence of Proposition 2.1. In particular, the spectral condition for stability will be satisfied if and only if 0 is a non-degenerate local minimum of Z.

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