

NORMALITY AND SMOOTHNESS OF SIMPLE LINEAR GROUP COMPACTIFICATIONS

JACOPO GANDINI AND ALESSANDRO RUZZI

ABSTRACT. Given a semisimple algebraic group G , we characterize the normality and the smoothness of its simple linear compactifications, namely those equivariant $G \times G$ -compactifications possessing a unique closed orbit which arise in a projective space of the shape $\mathbb{P}(\text{End}(V))$, where V is a finite dimensional rational G -module. Both the characterizations are purely combinatorial and are expressed in terms of the highest weights of V . In particular, we show that $\text{Sp}(2r)$ (with $r \geq 1$) is the unique non-adjoint simple group which admits a simple smooth compactification.

INTRODUCTION.

Let G be a connected semisimple algebraic group defined over an algebraically closed field \mathbb{k} of characteristic zero. Fix a maximal torus $T \subset G$ and a Borel subgroup $B \supset T$, denote $\mathcal{X}(T)$ the character lattice of T and denote $\mathcal{X}(T)^+$ the semigroup of dominant characters. If $\Pi \subset \mathcal{X}(T)^+$ is a finite set of dominant weights and if $V(\mu)$ denotes the simple G -module of highest weight μ , set $V_\Pi = \bigoplus_{\mu \in \Pi} V(\mu)$ and consider the $G \times G$ -variety

$$X_\Pi = \overline{(G \times G)[\text{Id}]} \subset \mathbb{P}(\text{End}(V_\Pi)),$$

which is a compactification of a quotient of G . Suppose moreover that X_Π is a simple $G \times G$ -variety, i.e. that it possesses a unique closed orbit: the aim of this paper is to characterize the normality and the smoothness of X_Π by giving explicit combinatorial conditions on the set of weights Π .

In [19], Timashev studied the general situation of a connected reductive group without any assumption on the number of closed orbits, giving necessary and sufficient conditions for the normality and for the smoothness of these compactifications, however the conditions of normality in particular are not completely explicit. On the other hand, in [3] the authors together with P. Bravi and A. Maffei studied the case of a simple compactification of a connected semisimple adjoint group G : in that case the conditions of normality and smoothness were considerably simplified and this has been the starting point of the paper.

To explain our results we need some further notation. Let Φ be the root system associated to T and let $\Delta \subset \Phi$ be the set of simple roots associated to B , which we identify with the set of vertices of the Dynkin diagram of Φ . Recall the *dominance order* and the *rational dominance order* on $\mathcal{X}(T)$, respectively defined by $\nu \leq \mu$ if and only if $\mu - \nu \in \mathbb{N}[\Delta]$ and by $\nu \leq_{\mathbb{Q}} \mu$ if and only if $\mu - \nu \in \mathbb{Q}^+[\Delta]$.

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If $\lambda \in \mathcal{X}(T)^+$, denote $\Pi(\lambda) \subset \mathcal{X}(T)$ the set of weights occurring in $V(\lambda)$ and denote $\mathcal{P}(\lambda)$ the convex hull of $\Pi(\lambda)$ in $\mathcal{X}(T) \otimes \mathbb{Q}$. Denote

$$\begin{aligned}\Pi^+(\lambda) &= \Pi(\lambda) \cap \mathcal{X}(T)^+ = \{\mu \in \mathcal{X}(T)^+ : \mu \leq \lambda\}, \\ \Pi_G^+(\lambda) &= \mathcal{P}(\lambda) \cap \mathcal{X}(T)^+ = \{\mu \in \mathcal{X}(T)^+ : \mu \leq_{\mathbb{Q}} \lambda\}.\end{aligned}$$

Define moreover the *support* of λ as the set $\text{Supp}(\lambda) = \{\alpha \in \Delta : \langle \lambda, \alpha^\vee \rangle \neq 0\}$.

If $\Pi \subset \mathcal{X}(T)^+$, then X_Π is a simple compactification of a quotient of G if and only if Π possesses a unique maximal element w.r.t. $\leq_{\mathbb{Q}}$, whereas X_Π is a simple compactification of a quotient of the adjoint group G_{ad} if and only if Π possesses a unique maximal element w.r.t. \leq . We will say that a subset $\Pi \subset \mathcal{X}(T)^+$ is *simple* if it possesses a unique maximal element w.r.t. $\leq_{\mathbb{Q}}$, and we say that Π is *adjoint* if it possesses a unique maximal element w.r.t. \leq . For instance, the subsets $\Pi^+(\lambda)$ and $\Pi_G^+(\lambda)$ are both simple and $\Pi^+(\lambda)$ is also adjoint.

Suppose that Π is a simple adjoint subset with maximal element λ . The normality of X_Π was characterized in [3] by introducing a subset $\text{Lb}(\lambda, G_{\text{ad}}) \subset \Pi^+(\lambda)$, called the set of *adjoint little brothers* of λ , with the following property: X_Π is a normal compactification of G_{ad} if and only if $\Pi \supset \text{Lb}(\lambda, G_{\text{ad}})$. Every adjoint little brother of λ is a weight covered by λ (i.e. it is maximal in $\Pi^+(\lambda) \setminus \{\lambda\}$ w.r.t. \leq) which arises correspondingly to a non-simply-laced connected component $\Delta' \subset \Delta$ and is defined in a purely combinatorial way by $\text{Supp}(\lambda) \cap \Delta'$ (see Definition 1.7).

In order to explain how previous characterization of the normality extends to the non-adjoint case, we need to introduce a partial order on $\mathcal{X}(T)$ which is less fine than the dominance order. If $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha \in \mathbb{Q}[\Delta]$, define its *support over Δ* as $\text{Supp}_\Delta(\beta) = \{\alpha \in \Delta : n_\alpha \neq 0\}$. If $\lambda \in \mathcal{X}(T)^+$, consider the set of positive roots which are non-orthogonal to λ

$$\Phi^+(\lambda) = \{\beta \in \Phi^+ : \text{Supp}_\Delta(\beta) \cap \text{Supp}(\lambda) \neq \emptyset\}$$

and consider the associated partial orders on $\mathcal{X}(T)$:

$$\begin{aligned}\nu \leq^\lambda \mu &\quad \text{if and only if} \quad \mu - \nu \in \mathbb{N}[\Phi^+(\lambda)], \\ \nu \leq_{\mathbb{Q}}^\lambda \mu &\quad \text{if and only if} \quad \mu - \nu \in \mathbb{Q}^+[\Phi^+(\lambda)].\end{aligned}$$

These orderings arise naturally in the representation theory of G : for instance if $\mu \in \Pi(\lambda)$ then $\mu \leq^\lambda \lambda$, while if $\mu \in \mathcal{P}(\lambda)$ then $\mu \leq_{\mathbb{Q}}^\lambda \lambda$ (see Proposition 2.1). Define the set of *little brothers* of λ w.r.t. G as follows:

$$\text{Lb}(\lambda, G) = \{\mu \in (\Pi_G^+(\lambda) \setminus \Pi^+(\lambda)) \cup \text{Lb}(\lambda, G_{\text{ad}}) : \mu \text{ is maximal w.r.t. } \leq_{\mathbb{Q}}^\lambda\}.$$

Then the normality of the variety X_Π is characterized as follows, where for $\alpha \in \Delta$ we denote by r_α the number of vertices of the connected component of the Dynkin diagram of G containing α and where for $I \subset \Delta$ we denote by $I^\circ = \{\alpha \in I : \langle \alpha, \beta^\vee \rangle = 0 \ \forall \beta \in \Delta \setminus I\}$ the *inner part* of I .

Theorem 1 (see Theorem 3.10). *Let $\Pi \subset \mathcal{X}(T)^+$ be a simple subset with maximal element λ and suppose that $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ$. Then the variety X_Π is a normal compactification of G if and only if $\Pi \supset \text{Lb}(\lambda, G)$.*

The assumption on the coefficients of λ in previous theorem involves no loss of generality. If indeed we define $\Pi' = \{\mu + \lambda : \mu \in \Pi\}$, then the varieties X_Π and $X_{\Pi'}$ are equivariantly isomorphic: rather than Π , the variety X_Π depends only on the set of simple roots $\text{Supp}(\lambda)$ and on the set of differences $\{\mu - \lambda : \mu \in \Pi\}$ (see Proposition 1.5).

Roughly speaking, if $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ$, then the set of differences $\{\mu - \lambda : \mu \in \text{Lb}(\lambda, G)\}$ is constant and depends only on $\text{Supp}(\lambda)$, while otherwise it may happen that $\Pi_G^+(\lambda)$ “misses” some maximal element of the partial order $\leq_{\mathbb{Q}}^\lambda$. This fact is linked to the problem of the surjectivity of the multiplication map between sections of globally generated line bundles on the canonical compactification of G : while the multiplication is always surjective in the case of an adjoint group (this was shown by S. S. Kannan in [11]), the surjectivity of the multiplication may fail in the general case (see Example 3.1 and Proposition 3.3).

In the case of an odd orthogonal group, a combinatorial classification of its simple linear compactifications was given by the first author in [10] by means of a partial order tightly related to \leq^λ , which makes use only of the positive long roots which are non-orthogonal to the dominant weight λ . A similar classification should be expectable in the case of any (non simply-laced) semisimple group by using similar partial orders.

While adjoint groups possess many simple smooth compactifications, $\text{Sp}(2r)$ is essentially the unique non-adjoint group possessing such compactifications. This is the content of the following theorem: together with the characterization of the smoothness given in [3] in the adjoint case (see Theorem 4.4), it gives a combinatorial characterization of the smoothness of a simple linear group compactification in the general case.

Theorem 2 (see Proposition 2.15, Lemma 4.3 and Theorem 4.7). *Let $\Pi \subset \mathcal{X}(T)^+$ be simple with maximal element λ . If $G_i \subset G$ is a simple normal subgroup and if $T_i = T \cap G_i$, set $\Pi_i = \{\mu|_{T_i} : \mu \in \Pi\}$.*

- i) *If X_Π is smooth, then $G = G_1 \times \dots \times G_n$ is a direct product of simple groups and $X_\Pi = X_{\Pi_1} \times \dots \times X_{\Pi_n}$, where X_{Π_i} is a smooth compactification of G_i . Moreover, every G_i is either adjoint or isomorphic to $\text{Sp}(2r_i)$ for some $r_i \geq 1$.*
- ii) *Suppose that $G = \text{Sp}(2r)$ with $r \geq 1$. Then X_Π is a smooth compactification of G if and only if $\text{Supp}(\lambda)$ is connected, $\alpha_r \in \text{Supp}(\lambda)$ and $\lambda + \omega_{r-1} - \omega_r \in \Pi$ (where we set $\omega_0 = 0$).*

The paper is organized as follows. In Section 1 we introduce the variety X_Π and we give some preliminary results: almost all of these results come from [3] and [10] and although some of them are claimed in a more general form than the original ones, the proofs are substantially the same. In Section 2 we introduce the partial orders \leq^λ and $\leq_{\mathbb{Q}}^\lambda$. After showing how these orderings arise naturally in the representation theory of G , we show some properties of the elements in $\Pi_G^+(\lambda)$ which are maximal w.r.t. \leq^λ : these results will be fundamental in the proofs of the main results of the paper. Finally, in Section 3 we characterize the normality of the variety X_Π , while in Section 4 we characterize its smoothness.

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1. PRELIMINARIES.

Let G be a connected semisimple algebraic group with Lie algebra \mathfrak{g} and with adjoint group G_{ad} . Let $B \subset G$ be a Borel subgroup and $T \subset B$ a maximal torus, B^- denotes the opposite Borel subgroup of B w.r.t. T . Denote by Λ the weight lattice of G and $\Lambda^+ \subset \Lambda$ the semigroup of

dominant weights. Denote Φ the root system associated to T and $\Delta \subset \Phi$ the basis associated to B , denote W the Weyl group of Φ . Denote $\mathcal{X}(T)$ the character lattice of T and set $\Lambda_{\text{rad}} = \mathbb{Z}[\Delta]$ the root lattice, set $\mathcal{X}(T)^+ = \mathcal{X}(T) \cap \Lambda^+$ and $\mathcal{X}(T)_{\mathbb{Q}} = \mathcal{X}(T) \otimes \mathbb{Q}$. Denote Φ^\vee the set of coroots and $\Delta^\vee \subset \Phi^\vee$ the basis of Φ^\vee associated to B . Denote $\mathcal{X}(T)^\vee$ the cocharacter lattice of T , denote Λ^\vee the coweight lattice of G and $\Lambda_{\text{rad}}^\vee = \mathbb{Z}[\Delta^\vee]$ the coroot lattice of G . If V is a G -module and if $H \subset G$ is a closed subgroup, we denote by $V^{(H)} \subset V$ the subset of H -eigenvectors.

We consider the following partial orders on Λ : the *dominance order*, defined by $\nu \leq \mu$ if and only if $\mu - \nu \in \mathbb{N}[\Delta]$, and the *rational dominance order*, defined by $\nu \leq_{\mathbb{Q}} \mu$ if and only if $\mu - \nu \in \mathbb{Q}^+[\Delta]$. Notice that, if $\nu, \mu \in \Lambda^+$, then $\nu \leq_{\mathbb{Q}} \mu$ if and only if $\det(C)\nu \leq \det(C)\mu$, where C is the Cartan matrix of Φ : indeed $\det(C)\Lambda \subset \mathbb{Z}[\Delta]$, hence $\mu - \nu \in \mathbb{Q}^+[\Delta]$ if and only if $\det(C)(\mu - \nu) \in \mathbb{N}[\Delta]$.

If $I \subset \Delta$, define its *border* ∂I and its *interior* I° as follows:

$$\partial I = \{\alpha \in \Delta \setminus I : \exists \beta \in I \text{ such that } \langle \beta, \alpha^\vee \rangle \neq 0\},$$

$$I^\circ = I \setminus \partial(\Delta \setminus I).$$

We say that $\alpha \in I$ is an *extremal root* of I if it is connected to at most one other element of I in the Dynkin diagram of G .

If $\lambda \in \Lambda$, define its *support* as follows

$$\text{Supp}(\lambda) = \{\alpha \in \Delta : \langle \lambda, \alpha^\vee \rangle \neq 0\}.$$

If $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha \in \mathbb{Q}[\Delta]$, define its *support over* Δ as follows:

$$\text{Supp}_\Delta(\beta) = \{\alpha \in \Delta : n_\alpha \neq 0\}.$$

If $\lambda \in \mathcal{X}(T)^+$, denote $V(\lambda)$ the simple G -module of highest weight λ . Denote $\Pi(\lambda)$ the set of weights occurring in $V(\lambda)$: then $\Pi(\lambda) = W\Pi^+(\lambda)$, where

$$\Pi^+(\lambda) = \Pi(\lambda) \cap \mathcal{X}(T)^+ = \{\mu \in \mathcal{X}(T)^+ : \mu \leq \lambda\}.$$

Similarly, denote $\mathcal{P}(\lambda) \subset \mathcal{X}(T)_{\mathbb{Q}}$ the convex hull of $\Pi(\lambda)$ and denote $\Pi_G(\lambda) = \mathcal{P}(\lambda) \cap \mathcal{X}(T)$: then $\Pi_G(\lambda) = W\Pi_G^+(\lambda)$, where

$$\Pi_G^+(\lambda) = \mathcal{P}(\lambda) \cap \mathcal{X}(T)^+ = \{\mu \in \mathcal{X}(T)^+ : \mu \leq_{\mathbb{Q}} \lambda\}.$$

If $\Pi \subset \mathcal{X}(T)^+$, denote $V_\Pi = \bigoplus_{\mu \in \Pi} V(\mu)$ and consider the following variety

$$X_\Pi = \overline{(G \times G)[\text{Id}]} \subset \mathbb{P}(\text{End}(V_\Pi)),$$

which is a compactification of a quotient of $G \simeq G \times G / \text{diag}(G \times G)$. Notice that, if $\text{Id}_\mu \in \text{End}(V(\mu))$ is the identity, then we may define X_Π as well as follows

$$X_\Pi = \overline{(G \times G)[(\text{Id}_\mu)_{\mu \in \Pi}]} \subset \mathbb{P}\left(\bigoplus_{\mu \in \Pi} \text{End}(V(\mu))\right):$$

indeed $(V(\lambda) \otimes V(\mu)^*)^{\text{diag}(G \times G)}$ is non-zero if and only if $\lambda = \mu$, in which case it is generated by Id_μ . If $\Pi = \{\lambda_1, \dots, \lambda_m\}$, for simplicity sometimes we will denote X_Π also by $X_{\lambda_1, \dots, \lambda_m}$.

Definition 1.1. Let $\Pi \subset \mathcal{X}(T)^+$ be a set of dominant characters. We say that X_Π is *simple* if it contains a unique closed orbit. We say that X_Π is *faithful* (resp. *almost faithful*, *adjoint*) if its open $G \times G$ -orbit is isomorphic to G (resp. to a quotient of G by a finite group, to a quotient of G_{ad}). If X_Π is simple (resp. faithful, almost faithful, adjoint), then we will say also that Π

is *simple* (resp. *faithful, almost faithful, adjoint*). We say that a weight $\lambda \in \mathcal{X}(T)^+$ is *almost faithful* if $\{\lambda\}$ is almost faithful, namely if X_λ is a compactification of G_{ad} .

Proposition 1.2 (see [19, §8]). *Let $\Pi \subset \mathcal{X}(T)^+$ be a finite subset and denote $\mathcal{P}(\Pi) \subset \mathcal{X}(T)_{\mathbb{Q}}$ the polytope generated by the T -weights occurring in V_Π .*

- i) Π is faithful if and only if the set $\{\mu - \nu\}_{\mu, \nu \in \Pi}$ generates $\mathcal{X}(T)/\mathbb{Z}[\Delta]$, whereas it is adjoint if and only if $\{\mu - \nu\}_{\mu, \nu \in \Pi} \subset \mathbb{Z}[\Delta]$.
- ii) If $\mu \in \Pi$, then X_Π contains the closed orbit of $\mathbb{P}(\text{End}(V(\mu)))$ if and only if μ is a vertex of $\mathcal{P}(\Pi)$. In particular, Π is simple if and only if it contains a unique maximal element w.r.t. $\leq_{\mathbb{Q}}$.

In particular it follows that a simple set Π with maximal element λ is almost faithful if and only if $\text{Supp}(\lambda) \cap \Delta' \neq \emptyset$ for every connected component $\Delta' \subset \Delta$. Notice also that $\Pi \subset \mathcal{X}(T)^+$ is simple and adjoint if and only if it possesses a unique maximal element w.r.t. \leq . For instance, if $\lambda \in \mathcal{X}(T)^+$, then the set $\Pi_G^+(\lambda)$ is simple, whereas $\Pi^+(\lambda)$ is simple and adjoint. If moreover λ is almost faithful, then $\Pi_G^+(\lambda)$ is also faithful provided that the non-zero coefficients of λ are big enough (see Theorem 3.5).

We now do some recalls from the theory of the embeddings of a semisimple algebraic group, which we regard as a special case of the general theory of spherical embeddings developed by Luna and Vust (see [13] and [19]). Given a semisimple group G , recall that a normal G -variety is called *spherical* if it possesses an open orbit for some Borel subgroup of G . We are here interested in the case of the group G itself regarded as a $G \times G$ -homogeneous space: the open cell BB^- is then an orbit for the Borel subgroup $B \times B^-$, so that we may regard G as a spherical $G \times G$ -variety.

Let X be a simple and complete normal embedding of G . Denote $\mathcal{D}(G)$ the set of $B \times B^-$ -stable prime divisors of G and $\mathcal{D}(X) \subset \mathcal{D}(G)$ the set of divisors whose closure in X contains the closed $G \times G$ -orbit: then the Picard group $\text{Pic}(X)$ is free and it is generated by the classes of divisors in $\mathcal{D}(G) \setminus \mathcal{D}(X)$ (see [4, Proposition 2.2]), while the class group $\text{Cl}(X)$ is generated by the classes of $B \times B^-$ -stable prime divisors of X (see [4, section 2.2]). Denote $\mathcal{N}(X)$ the set of $G \times G$ -stable prime divisors of X , so that the set of $B \times B^-$ -stable prime divisors of X is identified with $\mathcal{D}(G) \cup \mathcal{N}(X)$.

Notice that $\mathbb{k}(G)^{(B \times B^-)}/\mathbb{k}^* \simeq \mathcal{X}(T)$ and consider the natural map $\rho : \mathcal{D}(G) \cup \mathcal{N}(X) \rightarrow \mathcal{X}(T)^\vee$ defined by associating to a $B \times B^-$ -stable prime divisor D the cocharacter associated to the rational discrete valuation induced by D . If $D \in \mathcal{N}(X)$, then $\rho(D)$ is a negative multiple of a fundamental coweight, while if $D \in \mathcal{D}(G)$, then $\rho(D)$ is a simple coroot; moreover ρ is injective and $\rho(\mathcal{D}(G)) = \Delta^\vee$ (see [19, § 7]).

Denote $\mathcal{C}(X) \subset \mathcal{X}(T)_{\mathbb{Q}}^\vee$ the convex cone generated by $\rho(\mathcal{D}(X) \cup \mathcal{N}(X))$; by the general theory of spherical embeddings we have that $\mathcal{C}(X)$ is generated by $\rho(\mathcal{D}(X))$ together with the negative Weyl chamber of Φ (see [19, Theorem 3 and Corollary of Proposition 4]). By definition, the *colored cone* of X is the couple $(\mathcal{C}(X), \mathcal{D}(X))$: up to equivariant isomorphisms, it uniquely determines X as a $G \times G$ -compactification of G (see [19, Theorem 2]).

In the case of our interest, let $\lambda \in \mathcal{X}(T)^+$ and denote $\tilde{X}_\lambda \rightarrow X_\lambda$ the normalization of X_λ in $\mathbb{k}(G)$: this variety depends only on the set of simple roots $\text{Supp}(\lambda)$ (see [3, Proposition 1.2]) and we have $\rho(\mathcal{D}(\tilde{X}_\lambda)) = \Delta^\vee \setminus \text{Supp}(\lambda)^\vee$ (see [19, Theorem 7]). When λ is regular \tilde{X}_λ is called the *canonical compactification* of G and we will denote it by M . In case G is adjoint, then $M = M_{\text{ad}}$ is the *wonderful compactification* of G_{ad} considered by De Concini and Procesi in [8]

in the more general context of adjoint symmetric varieties. By [8] we have that M_{ad} coincides with the variety X_λ whenever λ is a regular dominant weight, so that M is the normalization of M_{ad} in $\mathbb{k}(G)$. Following the general theory of spherical varieties (see [13, Theorem 5.1] and [19, Proposition 1]), M dominates every simple linear compactification of a quotient of G .

The closed orbit of M is isomorphic to $G/B \times G/B$ and the restriction of line bundles determines an embedding of $\text{Pic}(M)$ into $\text{Pic}(G/B \times G/B)$, that we identify with $\Lambda \times \Lambda$, which identifies $\text{Pic}(M)$ with the set of weights of the form (λ, λ^*) . Therefore $\text{Pic}(M)$ is identified with Λ and we denote by $\mathcal{M}_\lambda \in \text{Pic}(M)$ the line bundle whose restriction to $G/B \times G/B$ is isomorphic to $\mathcal{L}_\lambda \boxtimes \mathcal{L}_{\lambda^*}$.

Since M possesses an open $B \times B^-$ -orbit, it follows that the $G \times G$ -module $\Gamma(M, \mathcal{M}_\lambda)$ is multiplicity free. Let $\Pi \subset \mathcal{X}(T)^+$ be a simple set with maximal element λ . Then the map $G \rightarrow \mathbb{P}(\text{End}(V_\Pi))$ extends to a map $M \rightarrow \mathbb{P}(\text{End}(V_\Pi))$ whose image is X_Π . Moreover \mathcal{M}_λ is the pullback of $\mathcal{O}(1)$, and if we pull back the homogeneous coordinates of $\mathbb{P}(\text{End}(V_\Pi))$ to M we get a submodule of $\Gamma(M, \mathcal{M}_\lambda)$ which is isomorphic to $\bigoplus_{\mu \in \Pi} \text{End}(V(\mu))^*$. In particular, if $\mu \in \Pi_G^+(\lambda)$, then $\Gamma(M, \mathcal{M}_\lambda)$ possesses a unique submodule isomorphic to $\text{End}(V(\mu))^*$, and by abuse of notation we will still denote it by $\text{End}(V(\mu))^*$. Conversely, for $\lambda \in \mathcal{X}(T)^+$, if we consider the projective map associated to the complete linear system of \mathcal{M}_λ we get a linear compactification of a quotient of G , say X_Π , which is simple since M is so, and by Proposition 1.2 it follows that Π is simple with maximal element λ . Therefore, as $G \times G$ -module, the space of sections of \mathcal{M}_λ decomposes as follows:

$$\Gamma(M, \mathcal{M}_\lambda) \simeq \bigoplus_{\mu \in \Pi_G^+(\lambda)} \text{End}(V(\mu))^*.$$

Consider the graded algebra $\tilde{A}(\lambda) = \bigoplus_{n=0}^{\infty} \tilde{A}_n(\lambda)$, where $\tilde{A}_n(\lambda) = \Gamma(M, \mathcal{M}_{n\lambda})$. By identifying sections of line bundles on M with functions on G , it is possible to describe the multiplication in $\tilde{A}(\lambda)$ in the following way.

Lemma 1.3 ([11, Lemma 3.1] or [7, Lemma 3.4]). *Let $\lambda, \mu \in \mathcal{X}(T)^+$ and let $\lambda' \in \Pi_G^+(\lambda)$ and $\mu' \in \Pi_G^+(\mu)$. Then the image of $\text{End}(V(\lambda'))^* \times \text{End}(V(\mu'))^* \subset \Gamma(M, \mathcal{M}_\lambda) \times \Gamma(M, \mathcal{M}_\mu)$ in $\Gamma(M, \mathcal{M}_{\lambda+\mu})$ via the multiplication map is*

$$\bigoplus_{V(\nu) \subset V(\lambda') \otimes V(\mu')} \text{End}(V(\nu))^*.$$

Denote $A(\Pi)$ the homogeneous coordinate ring of $X_\Pi \subset \mathbb{P}(\text{End}(V_\Pi))$. As a subalgebra of $\tilde{A}(\lambda)$, the algebra $A(\Pi)$ is generated by $\bigoplus_{\mu \in \Pi} \text{End}(V(\mu))^* \subset \Gamma(M, \mathcal{M}_\lambda)$ and it inherits a grading defined by $A_n(\Pi) = A(\Pi) \cap A_n(\lambda)$. Denote $\phi_\lambda \in \text{End}(V_\lambda)$ a highest weight vector and consider the $B \times B^-$ -stable affine open subsets $X_\lambda^\circ \subset X_\lambda$ and $X_\Pi^\circ \subset X_\Pi$ defined by the non-vanishing of ϕ_λ . The coordinate rings of previous affine sets are described as follows:

$$\mathbb{k}[X_\lambda^\circ] = \left\{ \frac{\phi}{\phi_\lambda^n} : n \in \mathbb{N}, \phi \in A_n(\lambda) \right\}, \quad \mathbb{k}[X_\Pi^\circ] = \left\{ \frac{\phi}{\phi_\lambda^n} : n \in \mathbb{N}, \phi \in A_n(\Pi) \right\}.$$

and the natural morphism $X_\Pi \rightarrow X_\lambda$ induces a natural inclusion $\mathbb{k}[X_\Pi^\circ] \subset \mathbb{k}[X_\lambda^\circ]$. Previous rings are not $G \times G$ -modules, however they are $\mathfrak{g} \oplus \mathfrak{g}$ -modules.

Following lemma and proposition were given in [10] in the case of a simple linear adjoint compactification, however their proofs generalize straightforwardly to the non-adjoint case.

Lemma 1.4 ([10, Lemma 1.3]). *Let $\Pi \subset \mathcal{X}(T)^+$ be simple with maximal element λ . As a $\mathfrak{g} \oplus \mathfrak{g}$ -algebra, $\mathbb{k}[X_\Pi^\circ]$ is generated by $\mathbb{k}[X_\lambda^\circ]$ together with the set $\{\phi_\mu/\phi_\lambda\}_{\mu \in \Pi}$.*

Proposition 1.5 ([10, Prop. 1.6 and Cor. 1.10]). *Let $\Pi, \Pi' \subset \mathcal{X}(T)^+$ be simple subsets with maximal elements resp. λ and λ' and suppose that $\text{Supp}(\lambda) = \text{Supp}(\lambda')$.*

- i) *There exists a $G \times G$ -equivariant morphism $X_\Pi \rightarrow X_{\Pi'}$ if and only if, for every $\mu' \in \Pi'$, there exist $n \in \mathbb{N}$ and $\mu_1, \dots, \mu_n \in \Pi$ such that $V(\mu' - \lambda' + n\lambda) \subset V(\mu_1) \otimes \dots \otimes V(\mu_n)$.*
- ii) *If $\{\mu - \lambda\}_{\mu \in \Pi} = \{\mu' - \lambda'\}_{\mu' \in \Pi'}$, then $X_\Pi \simeq X_{\Pi'}$ as $G \times G$ -varieties.*

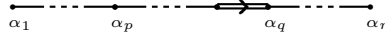
We now recall the criterion of normality for a simple adjoint compactification given in [3]. Let $\Pi \subset \mathcal{X}(T)^+$ be simple and adjoint with maximal element λ . Then $\Pi \subset \Pi^+(\lambda)$ and we have a natural projection $X_{\Pi^+(\lambda)} \rightarrow X_\Pi$. While Kannan shown in [11] that $X_{\Pi^+(\lambda)}$ is a projectively normal variety, De Concini proved in [7] that $X_{\Pi^+(\lambda)} \rightarrow X_\Pi$ is the normalization of X_Π . Hence we deduce by Lemma 1.5 the following characterization of the normality of X_Π .

Proposition 1.6 ([3, Prop. 2.3]). *Let $\Pi \subset \mathcal{X}(T)^+$ be simple and adjoint with maximal element λ . Then the variety X_Π is normal if and only if for every $\mu \in \Pi^+(\lambda)$ they exist $n \in \mathbb{N}$ and $\lambda_1, \dots, \lambda_n \in \Pi$ such that*

$$V(\mu + (n-1)\lambda) \subset V(\lambda_1) \otimes \dots \otimes V(\lambda_n).$$

The following definition allows to make effective previous characterization.

Definition 1.7. If $\Delta' \subset \Delta$ is a non-simply laced connected component, order the simple roots in $\Delta' = \{\alpha_1, \dots, \alpha_r\}$ starting from the extremal long root and denote α_q the first short root in Δ' . Let $\lambda \in \Lambda^+$ be such that $\alpha_q \notin \text{Supp}(\lambda)$ and such that $\text{Supp}(\lambda) \cap \Delta'$ contains a long root, denote α_p the last long root which occurs in $\text{Supp}(\lambda) \cap \Delta'$. For instance, if Δ' is not of type G_2 , then the numbering is as follows:



The *adjoint little brother* of λ with respect to Δ' is the dominant weight

$$\lambda_{\Delta'}^{\text{lb}} = \lambda - \sum_{i=p}^q \alpha_i = \begin{cases} \lambda - \omega_1 + \omega_2 & \text{if } G \text{ is of type } G_2 \\ \lambda + \omega_{p-1} - \omega_p + \omega_{q+1} & \text{otherwise} \end{cases}$$

where ω_i is the fundamental weight associated to α_i if $1 \leq i \leq r$, while $\omega_0 = \omega_{r+1} = 0$.

The set of adjoint little brothers of λ will be denoted by $\text{Lb}(\lambda, G_{\text{ad}})$, while we set $\overline{\text{Lb}}(\lambda, G_{\text{ad}}) = \text{Lb}(\lambda, G_{\text{ad}}) \cup \{\lambda\}$. Notice that if G is simply laced then $\text{Lb}(\lambda, G_{\text{ad}}) = \emptyset$ for every $\lambda \in \mathcal{X}(T)^+$.

Theorem 1.8 ([3, Thm. 2.10]). *Let $\Pi \subset \mathcal{X}(T)^+$ be simple and adjoint with maximal element λ . Then the variety X_Π is normal if and only if $\Pi \supset \text{Lb}(\lambda, G_{\text{ad}})$.*

In particular, it follows from the previous theorem that, if G is simply laced, then every simple adjoint compactification is normal.

If $n \in \mathbb{N}$ consider the set

$$\text{Tens}_n(G) = \{(\lambda_0, \dots, \lambda_n) \in \mathcal{X}(T)^+ \times \dots \times \mathcal{X}(T)^+ : V(\lambda_0) \subset V(\lambda_1) \otimes \dots \otimes V(\lambda_n)\}.$$

Following lemma has been proved in several references, usually in the case $n = 2$. For later use, we state it in a slightly more general form, which is easily reduced to the case $n = 2$ proceeding by induction on n .

Lemma 1.9 ([16, Lemma 3.9]). *The set $\text{Tens}_n(G)$ is a semigroup with respect to the addition.*

An easy consequence of previous lemma is the following.

Corollary 1.10. *Let $\lambda, \mu, \nu \in \mathcal{X}(T)^+$ be such that $V(\nu) \subset V(\lambda) \otimes V(\mu)$. Then, for any $\nu' \in \mathcal{X}(T)^+$, it also holds $V(\nu + \nu') \subset V(\lambda + \nu') \otimes V(\mu)$.*

When G is of type **A**, the following saturation property of $\text{Tens}_n(G)$ holds:

If $N > 0$ and $\lambda_0, \dots, \lambda_n \in \mathcal{X}(T)^+$ are such that $(N\lambda_0, \dots, N\lambda_n) \in \text{Tens}_n(G)$, then $(\lambda_0, \dots, \lambda_n) \in \text{Tens}_n(G)$ as well.

In case $n = 2$, previous property was proved by Knutson and Tao in [14], and then it has been conjectured for every simply laced group by Kapovich and Millson in [12]. As discussed in the survey of Fulton [9], when G is of type **A** the case of a general n can be deduced from the case $n = 2$.

Remark 1.11. We already noticed that Theorem 1.8 implies in particular that, if G is simply laced, then every simple adjoint compactification is normal. When G is of type **A**, a very easy proof of this fact follows by the saturation of $\text{Tens}_n(G)$ thanks to Proposition 1.6. Let indeed $\lambda \in \mathcal{X}(T)^+$ and $\mu \in \Pi^+(\lambda)$. Then there exists $n \in \mathbb{N}$ such that $V(n\mu) \subset V(\lambda)^{\otimes n}$ (see [1, Lemma 4.9] or [19, Lemma 1]). By Corollary 1.10, this implies that $V(n\mu + n(n-1)\lambda) \subset V(n\lambda)^{\otimes n}$, hence the saturation property implies that $V(\mu + (n-1)\lambda) \subset V(\lambda)^{\otimes n}$.

2. THE PARTIAL ORDERS \leq^λ AND $\leq_{\mathbb{Q}}^\lambda$.

Let $\lambda \in \Lambda^+$. This section is devoted to proving some properties of the partial orderings \leq^λ and $\leq_{\mathbb{Q}}^\lambda$ that we defined in the introduction. Notice that if $\lambda, \lambda' \in \Lambda^+$ are such that $\text{Supp}(\lambda) = \text{Supp}(\lambda')$, then λ and λ' induce the same partial orders on Λ .

Proposition 2.1. *If $\lambda \in \mathcal{X}(T)^+$ and $\pi \in \Pi(\lambda)$, then $\pi \leq^\lambda \lambda$.*

Proof. Let $v_\lambda, v_\lambda^- \in V(\lambda)$ be respectively a highest weight vector and a lowest weight vector and denote P, P^- their stabilizers in G . Denote \mathfrak{p} the Lie algebra of P and \mathfrak{n}^- the Lie algebra of the unipotent radical of P^- and let $\mathfrak{U}(\mathfrak{g})$ (resp. $\mathfrak{U}(\mathfrak{n}^-)$) be the universal enveloping algebra of \mathfrak{g} (resp. of \mathfrak{n}^-). Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{p}$ and it follows $V(\lambda) = \mathfrak{U}(\mathfrak{g})v_\lambda = \mathfrak{U}(\mathfrak{n}^-)v_\lambda$. On the other hand, if $\mathfrak{g}_\alpha \subset \mathfrak{g}$ denotes the root space of $\alpha \in \Phi$, then $\mathfrak{n}^- = \bigoplus_{\alpha \in \Phi^-(\lambda)} \mathfrak{g}_\alpha$, hence by looking at T -weights it follows $\pi - \lambda \in \mathbb{N}[\Phi^-(\lambda)]$. \square

Corollary 2.2. *Let $\lambda \in \mathcal{X}(T)^+$ and let $\lambda_1, \dots, \lambda_n \in \Pi^+(\lambda)$. If $\mu, \nu \in \mathcal{X}(T)^+$ are such that*

$$V(\nu) \subset V(\mu) \otimes V(\lambda_1) \otimes \dots \otimes V(\lambda_n),$$

then $\nu \leq^\lambda \mu + n\lambda$.

Proof. We proceed by induction on n , suppose first that $n = 1$. Recall that every highest weight in $V(\mu) \otimes V(\lambda_1)$ is of the shape $\mu + \pi$ for some $\pi \in \Pi(\lambda_1)$ (see for instance [16, Proposition

3.2]). On the other hand $\Pi(\lambda_1) \subset \Pi(\lambda)$, therefore previous proposition implies that $\nu \leq^\lambda \mu + \lambda$. Suppose now $n > 1$ and let $\mu' \in \mathcal{X}(T)^+$ be such that

$$V(\mu') \subset V(\mu) \otimes V(\lambda_1) \otimes \dots \otimes V(\lambda_{n-1}) \quad \text{and} \quad V(\nu) \subset V(\mu') \otimes V(\lambda_n) :$$

Then by the inductive hypothesis we get $\mu' \leq^\lambda \mu + (n-1)\lambda$ and $\nu \leq^\lambda \mu' + \lambda$ and the claim follows. \square

Following proposition will be the core of the necessity part of Theorem 1.

Proposition 2.3. *Suppose that $\nu, \mu_1, \dots, \mu_n \in \Pi_G^+(\lambda)$ are such that*

$$V(\nu + (n-1)\lambda) \subset V(\mu_1) \otimes \dots \otimes V(\mu_n).$$

Then $\nu \leq_{\mathbb{Q}}^\lambda \mu_i$ for every i .

Proof. We only show the inequality $\nu \leq_{\mathbb{Q}}^\lambda \mu_1$, the others are analogous. Set $c = \det(C)$, where C is the Cartan matrix of Φ , and denote $\pi = \nu + (n-1)\lambda$: we will prove the claim by showing that

$$c\pi \leq^\lambda c\mu_1 + (n-1)c\lambda.$$

By Theorem 1.8 the variety $X_{\overline{\text{Lb}}(c\lambda, G_{\text{ad}})}$ is normal. Set $\overline{\text{Lb}}(c\lambda, G_{\text{ad}}) = \{\lambda_1, \dots, \lambda_s\}$ and let $i > 1$. Since $c\mu_i \leq c\lambda$, by Proposition 1.6 they exist $M_i^1, \dots, M_i^s \in \mathbb{N}$ such that

$$V(c\mu_i + (N_i - 1)c\lambda) \subset \bigotimes_{j=1}^s V(c\lambda_j)^{\otimes M_i^j}$$

where we set $N_i = \sum_{j=1}^s M_i^j$. On the other hand, the hypothesis together with Lemma 1.9 imply that

$$V(c\pi) \subset V(c\mu_1) \otimes \dots \otimes V(c\mu_n).$$

Combining previous inclusions together with Corollary 1.10 we get then

$$V(c\pi + \sum_{i=2}^n (N_i - 1)c\lambda) \subset V(c\mu_1) \otimes \bigotimes_{j=1}^s V(c\lambda_j)^{\otimes M_2^j + \dots + M_n^j}.$$

Since the partial orderings induced by λ and by $c\lambda$ are the same, Proposition 2.1 together with Corollary 2.2 show that

$$c\pi + \sum_{i=2}^n (N_i - 1)c\lambda \leq^\lambda c\mu_1 + \sum_{i=2}^n N_i c\lambda,$$

i.e. $c\pi \leq^\lambda c\mu_1 + (n-1)c\lambda$. \square

Denote $\mathcal{C}(\lambda) \subset \mathcal{X}(T)_{\mathbb{Q}}$ the cone generated by $\Phi^-(\lambda)$ and recall that $\mathcal{P}(\lambda)$ is the polytope generated in $\mathcal{X}(T)_{\mathbb{Q}}$ by the orbit $W\lambda$. It follows from the definition that, if $\nu, \mu \in \mathcal{X}(T)$, then $\nu \leq_{\mathbb{Q}}^\lambda \mu$ if and only if $\nu - \mu \in \mathcal{C}(\lambda)$.

Proposition 2.4. *The cone $\mathcal{C}(\lambda)$ is generated by $\mathcal{P}(\lambda) - \lambda$. Moreover, the following equality holds:*

$$\mathcal{C}(\lambda) = \bigcup_{n \in \mathbb{N}} \mathcal{P}(n\lambda) - n\lambda.$$

Proof. If $\pi_1, \pi_2 \in \mathcal{X}(T)$, denote by $\overline{\pi_1, \pi_2} \subset \mathcal{X}(T)_{\mathbb{Q}}$ the line segment connecting π_1 and π_2 . Since $\mathcal{P}(\lambda)$ is a convex polytope, the cone generated by $\mathcal{P}(\lambda) - \lambda$ is generated by the set

$$\mathcal{G}(\lambda) = \{w\lambda - \lambda : \overline{w\lambda, \lambda} \text{ is an edge of } \mathcal{P}(\lambda)\}.$$

By the following lemma, every element in $\mathcal{G}(\lambda)$ is a positive multiple of an element in $\Phi^-(\lambda)$. Therefore $\mathcal{P}(\lambda) - \lambda$ generates $\mathcal{C}(\lambda)$ and the second claim is trivial. \square

Lemma 2.5. *Let $w \in W$ be such that $\overline{w\lambda, \lambda}$ is an edge of $\mathcal{P}(\lambda)$. Then $w = s_\tau$ is the reflection associated to a positive root $\tau \in \Phi^+(\lambda)$.*

Proof. Denote \mathcal{C} the cone generated by $\mathcal{P}(\lambda) - \lambda$. Notice that $\Phi^-(\lambda) \subset \mathcal{C}$: indeed if $\sigma \in \Phi^-(\lambda)$ then $s_\sigma \lambda \in \mathcal{P}(\lambda)$ and $s_\sigma \lambda - \lambda$ is a non-zero multiple of σ .

By hypothesis, the vector $w\lambda - \lambda$ generates an extremal ray of \mathcal{C} , while by Proposition 2.1 we may write $w\lambda - \lambda = \sum_{\sigma \in \Phi^-(\lambda)} a_\sigma \sigma$ with $a_\sigma \geq 0$ for all σ . Since $\Phi^-(\lambda) \subset \mathcal{C}$, it follows that there exists only one $\tau \in \Phi^-(\lambda)$ with $a_\tau \neq 0$. In particular, $w\lambda - \lambda$ is a multiple of τ , hence $s_\tau \lambda$ coincides with $w\lambda$ because they are both vertices of $\mathcal{P}(\lambda)$. \square

If $\alpha \in \Delta$, denote $\Delta(\alpha) \subset \Delta$ the connected component containing α and denote r_α the rank of $\Delta(\alpha)$. If $\beta \in \Delta(\alpha)$, denote $d(\alpha, \beta) \in \mathbb{N}$ the distance between α and β in the Dynkin diagram of $\Delta(\alpha)$. Denote by $I(\alpha, \beta) \subset \Delta$ the minimal connected subset containing both α and β , so that $|I(\alpha, \beta)| = d(\alpha, \beta) + 1$. We now describe a property of the maximal elements in $\Pi_G^+(\lambda)$ w.r.t. \leq^λ which will be one of the main tools in the characterization of the normality of a simple linear compactification of G . First we need an auxiliary lemma.

Lemma 2.6. *Let $\lambda \in \mathcal{X}(T)^+$ and let $\mu \in \Pi_G^+(\lambda)$. Let $\alpha \in \text{Supp}(\lambda)$ be such that $\langle \lambda - \mu, \alpha^\vee \rangle > r_\alpha$ and denote $S(\alpha) \subset (\Delta(\alpha) \setminus \text{Supp}(\mu)) \cup \{\alpha\}$ the connected component of α . Fix β an extremal root of $S(\alpha)$, set $I(\alpha, \beta) = \{\gamma_0, \gamma_1, \dots, \gamma_k\}$ (where $k = d(\alpha, \beta)$ and where the numbering is defined by $\gamma_0 = \alpha$ and $\langle \gamma_i, \gamma_{i-1}^\vee \rangle \neq 0$ for every $i = 1, \dots, k$) and denote $(\lambda - \mu)|_{I(\alpha, \beta)} = \sum_{i=0}^k c_i \gamma_i$.*

i) *Set $c_{k+1} = 0$. For every $i \leq k$, it holds*

$$c_i \geq \frac{(i+1)c_{i+1} + r_\alpha + 1}{i+2}$$

ii) *For every $i \leq k$ it holds*

$$c_i \geq k - i + 1 + \frac{r_\alpha - k - 1}{i+2}$$

Proof. i) We prove the claim by induction on i . The case $i = 0$ follows immediately by the inequality

$$r_\alpha + 1 \leq \langle \lambda - \mu, \alpha^\vee \rangle \leq 2c_0 - c_1,$$

so we assume $i > 0$. Notice that

$$\langle \lambda, \gamma_i^\vee \rangle + c_{i-1} - 2c_i + c_{i+1} \leq \langle \mu, \gamma_i^\vee \rangle = 0:$$

hence by the inductive hypothesis we get

$$2c_i \geq c_{i-1} + c_{i+1} \geq \frac{ic_i + r_\alpha + 1}{i+1} + c_{i+1}$$

and the inequality follows.

type of Φ	highest short root
A_r	$\alpha_1 + \cdots + \alpha_r$
B_r	$\alpha_1 + \cdots + \alpha_r$
C_r	$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{r-1}) + \alpha_r$
D_r	$\alpha_1 + 2(\alpha_2 + \cdots + \alpha_{r-2}) + \alpha_{r-1} + \alpha_r$
E_6	$\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$
E_7	$2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$
E_8	$2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$
F_4	$\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$
G_2	$2\alpha_1 + \alpha_2$

TABLE 1. Highest short roots

ii) We prove the claim by induction on $k - i$. If $i = k$, then by part i) we get

$$c_k \geq \frac{r_\alpha + 1}{k + 2} = 1 + \frac{r_\alpha - k - 1}{k + 2}.$$

Assume $i < k$. Then by part i) together with the hypotheses we get

$$c_i \geq \frac{(i + 1)c_{i+1} + r_\alpha + 1}{i + 2} \geq \frac{(i + 1)(k - i) + r_\alpha + 1}{i + 2} = k - i + 1 + \frac{r_\alpha - k - 1}{i + 2}. \quad \square$$

Proposition 2.7. *Let $\lambda \in \mathcal{X}(T)^+$ and let $\mu \in \Pi_G^+(\lambda)$ be maximal w.r.t. \leq^λ . Then $\langle \lambda - \mu, \alpha^\vee \rangle \leq r_\alpha$ for every $\alpha \in \Delta$.*

Proof. We will prove the proposition by showing that, if $\alpha \in \text{Supp}(\lambda)$ is such that $\langle \lambda - \mu, \alpha^\vee \rangle > r_\alpha$, then there exists a weight $\mu' \in \Pi_G^+(\lambda)$ such that $\mu' - \mu \in \Phi^+(\lambda)$, contradicting the maximality of μ . Fix such a root α , denote $S(\alpha) \subset (\Delta(\alpha) \setminus \text{Supp}(\mu)) \cup \{\alpha\}$ its connected component and denote $\theta_\alpha \in \mathbb{N}[S(\alpha)]$ the highest short root of the root subsystem associated to $S(\alpha)$.

Suppose that the root lattice $\mathbb{Z}[\Delta(\alpha)]$ equals the weight lattice of $\Delta(\alpha)$. Then by Proposition 2.1 it follows $\mu|_{\Delta(\alpha)} \leq^\lambda \lambda|_{\Delta(\alpha)}$, so that the maximality of μ implies the equality $\lambda|_{\Delta(\alpha)} = \mu|_{\Delta(\alpha)}$ and no root such as α can exist. In particular, this excludes that $\Delta(\alpha)$ is of type E_8, F_4 or G_2 .

Set $r = r_\alpha$ and denote $\Delta(\alpha) = \{\alpha_1, \dots, \alpha_r\}$, where the numbering is that of [2], and set

$$(\lambda - \mu)|_{\Delta(\alpha)} = \sum_{i=1}^r a_i \alpha_i.$$

We claim that $\theta_\alpha \leq_{\mathbb{Q}} \lambda - \mu$. Let indeed $\alpha_i \in S(\alpha)$ and denote

$$d(\alpha_i) = \min\{d(\alpha_i, \beta) : \beta \text{ is an extremal root of } S(\alpha)\}.$$

If β is an extremal root of $S(\alpha)$ such that $\alpha_i \in I(\alpha, \beta)$, then Lemma 2.6 ii) implies that

$$a_i \geq d(\alpha_i, \beta) + 1 + \frac{r - d(\alpha, \beta) - 1}{d(\alpha, \alpha_i) + 1}. \quad (1)$$

In particular we get $a_i \geq d(\alpha_i) + 1$, which implies that $a_i \geq 1$ for every $\alpha_i \in S(\alpha)$ and $a_i \geq 2$ for every $\alpha_i \in S(\alpha)$ which is not extremal in $S(\alpha)$. Therefore by Table 1 it follows that $\theta_\alpha \leq_{\mathbb{Q}} \lambda - \mu$ whenever $S(\alpha)$ is of type A,B,C or D.

Suppose that $S(\alpha)$ is of type E. Then $d(\alpha, \beta) + 1 < r$ for every extremal root $\beta \in S(\alpha)$, therefore inequality (1) implies that

$$a_i > d(\alpha_i, \beta) + 1 \quad (2)$$

for every $\alpha_i \in S(\alpha)$. Suppose that $S(\alpha)$ is of type E_6 : then by the description of the fundamental weights we get that $a_2, a_4 \in \mathbb{Z}$, hence inequality (2) implies $a_2 \geq 2$ and $a_4 \geq 3$, and by Table 1 it follows $\theta_\alpha \leq_{\mathbb{Q}} \lambda - \mu$.

Suppose finally that $S(\alpha)$ is of type E_7 . Then by the description of the fundamental weights we get that $a_1, a_3, a_4 \in \mathbb{Z}$ and $a_2 \in \frac{1}{2}\mathbb{Z}$, hence inequality (2) implies $a_1 \geq 2$, $a_2 \geq 3/2$, $a_3 \geq 3$, $a_4 \geq 3$. Therefore it follows from Table 1 that $\theta_\alpha \leq_{\mathbb{Q}} \lambda - \mu$ unless $a_2 = 3/2$ or $a_4 = 3$. To show that $a_4 > 3$, notice that we may choose the extremal root $\beta \in S(\alpha)$ in such a way that $\alpha_4 \in I(\alpha, \beta)$ and $d(\alpha_4, \beta) \geq 2$: therefore inequality (2) implies $a_4 > 3$. To show that $a_2 > 3/2$, suppose first that $\alpha \neq \alpha_2$: since $\text{Supp}(\mu) \cap S(\alpha) \subset \{\alpha\}$, it follows that

$$\langle \lambda, \alpha_2^\vee \rangle - 2a_2 + a_4 = \langle \mu, \alpha_2^\vee \rangle = 0,$$

whence $a_2 \geq \frac{1}{2}a_4 \geq 2$. Suppose now that $\alpha = \alpha_2$: then we may choose as extremal root $\beta = \alpha_7$ and inequality (2) implies that $a_2 > d(\alpha_2, \alpha_7) + 1 = 5$.

We proved so far that $\theta_\alpha \leq_{\mathbb{Q}} \lambda - \mu$. Since μ is maximal w.r.t. \leq^λ , the weight $\mu + \theta_\alpha$ must be non-dominant and an easy case-by-case consideration shows that $\Delta(\alpha)$ is of type B_r , that $S(\alpha) = \{\alpha_p, \dots, \alpha_{r-1}\}$ for some $p < r$ and that $\langle \mu, \alpha_r^\vee \rangle = 1$, and in particular it follows

$$\langle \lambda, \alpha_r^\vee \rangle + 2a_{r-1} - 2a_r = \langle \mu, \alpha_r^\vee \rangle = 1.$$

On the other hand inequality (1) implies $a_{r-1} > 1$, and being $a_{r-1} \in \frac{1}{2}\mathbb{Z}$ it follows that $a_{r-1} \geq 3/2$. Thus we get $2a_r \geq 2a_{r-1} - 1 \geq 2$ and it follows that $\theta_\alpha + \alpha_r \leq_{\mathbb{Q}} \lambda - \mu$: hence $\mu + \theta_\alpha + \alpha_r \in \Pi_G^+(\lambda)$, contradicting the maximality of μ . \square

Corollary 2.8. *Let $\lambda \in \mathcal{X}(T)^+$ be such that $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ$ and let $\mu \in \Pi_G^+(\lambda)$ be maximal w.r.t. \leq^λ . If $\alpha \in \text{Supp}(\lambda)^\circ$, then $\langle \lambda - \mu, \alpha^\vee \rangle \leq 1$.*

Proof. Suppose that $\mu \in \Pi_G^+(\lambda)$ is maximal w.r.t. \leq^λ and denote $\lambda - \mu = \sum_{\alpha \in \Delta} c_\alpha \alpha$. Let $\beta \in \text{Supp}(\lambda)^\circ$ be such that $\langle \lambda - \mu, \beta^\vee \rangle \geq 2$: since $\langle \lambda - \mu, \beta^\vee \rangle \leq 2c_\beta$, it follows then $c_\beta \geq 1$. Since μ is maximal, the weight $\mu + \beta$ is not dominant, on the other hand since $\beta \in \text{Supp}(\lambda)^\circ$ there exists $n > 1$ such that the weight $\mu + (n-1)\lambda + \beta$ is dominant. Denote $\lambda' = n\lambda$ and $\mu' = \mu + (n-1)\lambda$ and let $\nu' \in \Pi_G^+(\lambda)$ be maximal w.r.t. \leq^λ such that $\mu' <^\lambda \nu' \leq_{\mathbb{Q}} \lambda'$. Since $\lambda' - \nu' < \lambda - \mu$, proceeding inductively we may assume that, taking n big enough, the inequality $\langle \lambda' - \nu', \alpha^\vee \rangle \leq 1$ holds for every $\alpha \in \text{Supp}(\lambda)^\circ$. Since $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ$, by Proposition 2.7 it follows then that the weight $\nu = \lambda + \nu' - \lambda'$ is dominant, hence $\mu <^\lambda \nu \leq_{\mathbb{Q}}^\lambda \lambda$ and we get a contradiction since μ was supposed to be maximal. \square

Definition 2.9. Define the following sets:

$$\Pi_G^m(\lambda) = \{ \mu \in \Pi_G^+(\lambda) : \mu \text{ is maximal w.r.t. } \leq^\lambda \} \cup \text{Lb}(\lambda, G_{\text{ad}}),$$

$$\text{Lb}(\lambda, G) = \{ \mu \in (\Pi_G^+(\lambda) \setminus \Pi^+(\lambda)) \cup \text{Lb}(\lambda, G_{\text{ad}}) : \mu \text{ is maximal w.r.t. } \leq_{\mathbb{Q}}^\lambda \}.$$

We call $\text{Lb}(\lambda, G)$ the set of *little brothers* of λ w.r.t. G and we set $\overline{\text{Lb}}(\lambda, G) = \text{Lb}(\lambda, G) \cup \{\lambda\}$.

Define the following sets:

$$\mathcal{H}_G^m(\lambda) = \{ \mu - \lambda : \mu \in \Pi_G^m(\lambda) \}, \quad \mathcal{H}_G(\lambda) = \{ \mu - \lambda : \mu \in \text{Lb}(\lambda, G) \}.$$

Corollary 2.10. *If $\lambda \in \mathcal{X}(T)^+$, set $\bar{\lambda} = \sum_{\alpha \in \text{Supp}(\lambda)} n_\alpha \omega_\alpha$, where*

$$n_\alpha = \begin{cases} 1 & \text{if } \alpha \in \text{Supp}(\lambda)^\circ \\ r_\alpha & \text{if } \alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ \end{cases}$$

If $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ$, then $\mathcal{H}_G^m(\lambda) = \mathcal{H}_G^m(\bar{\lambda})$ and $\mathcal{H}_G(\lambda) = \mathcal{H}_G(\bar{\lambda})$.

Proof. Let $\lambda \in \mathcal{X}(T)^+$ be such that $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ$. If $\mu \in \Pi_G^+(\lambda)$ is maximal w.r.t. \leq^λ , then by Proposition 2.7 together with Corollary 2.8 it follows that $\bar{\lambda} + \mu - \lambda \in \Pi_G^+(\bar{\lambda})$, and its maximality w.r.t. \leq^λ follows by the maximality of μ . Similarly, if $\bar{\mu} \in \Pi_G^+(\bar{\lambda})$ is maximal w.r.t. \leq^λ , then $\mu = \bar{\mu} + \lambda - \bar{\lambda} \in \Pi_G^+(\lambda)$ is also maximal w.r.t. \leq^λ . Hence we get $\mathcal{H}_G^m(\lambda) = \mathcal{H}_G^m(\bar{\lambda})$, and the equality $\mathcal{H}_G(\lambda) = \mathcal{H}_G(\bar{\lambda})$ follows straightforwardly. \square

Example 2.11. Let $G = \text{SL}(r+1)$ and set $\lambda = n\omega_1$. Then $\Pi_G^+(\lambda)$ is faithful for every $n \geq 1$ and the following descriptions hold:

$$\begin{aligned} \Pi_G^m(\lambda) &= \{(n-i)\omega_1 : 0 \leq i \leq \min\{n, r\}\}, \\ \text{Lb}(\lambda, G) &= \{(n-1)\omega_1\}. \end{aligned}$$

Since $\min\{\langle \mu, \alpha_1^\vee \rangle : \mu \in \Pi_G^m(r\omega_1)\} = 0$ and $\min\{\langle \mu, \alpha_1^\vee \rangle : \mu \in \text{Lb}(r\omega_1, G)\} = r-1$, by Corollary 2.10 we get $\mathcal{H}_G^m(\lambda) = \mathcal{H}_G^m(r\omega_1)$ if and only if $n \geq r$, while $\mathcal{H}_G(\lambda) = \mathcal{H}_G(r\omega_1)$ for every $n \geq 1$.

Example 2.12. Let $G = \text{Spin}(2r+1)$ and set $\lambda = n\omega_1$. Denote $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ the floor and the ceiling functions. Then $\Pi_G^+(\lambda)$ is faithful for every $n \geq \lceil r/2 \rceil$ and the following descriptions hold:

$$\begin{aligned} \Pi_G^m(\lambda) &= \{\lfloor n - r/2 \rfloor \omega_1 + \omega_r, (n-1)\omega_1, n\omega_1\}, \\ \text{Lb}(\lambda, G) &= \{\lfloor n - r/2 \rfloor \omega_1 + \omega_r, (n-1)\omega_1\}. \end{aligned}$$

Hence by Corollary 2.10 we get $\mathcal{H}_G^m(\lambda) = \mathcal{H}_G^m(r\omega_1)$ if and only if $\mathcal{H}_G(\lambda) = \mathcal{H}_G(r\omega_1)$ if and only if $n \geq \lceil r/2 \rceil$.

Example 2.13. Let $G = \text{SL}(r+1)$ and set $\lambda = n\omega_2$. Then $\Pi_G^+(\lambda)$ is faithful if and only if $n \geq 2$ if r is odd, and for every $n \geq 1$ if r is even. Moreover

$$\text{Lb}(\lambda, G) = \{\mu_i : 0 \leq i < \min\{n, r\}\},$$

where we set

$$\mu_i = \begin{cases} (n-2)\omega_2 + \omega_3 & \text{if } i = 0 \\ (i-1)\omega_1 + (n-i)\omega_2 & \text{if } 1 \leq i < \min\{n, r\} \end{cases}$$

Since $\min\{\langle \mu, \alpha_2^\vee \rangle : \mu \in \text{Lb}(r\omega_2, G)\} = 1$, Corollary 2.10 implies that $\mathcal{H}_G(\lambda) = \mathcal{H}_G(r\omega_2)$ if and only if $n \geq r-1$.

Every little brother $\mu \in \text{Lb}(\lambda, G)$ is maximal in $\Pi_G^+(\lambda) \setminus \{\lambda\}$ w.r.t. \leq^λ , however it may not be maximal w.r.t. \leq .

Example 2.14. Let $G = \text{SL}(6)$ and set $\lambda = n\omega_3$. Then $\Pi_G^+(\lambda)$ is faithful if and only if $n \geq 2$ and $\text{Lb}(\lambda, G) = \{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \mu_6, \mu_7\}$, where we denote

- $\mu_1 = (n-1)\omega_3$,
- $\mu_2 = \omega_2 + (n-2)\omega_3$,
- $\mu_3 = (n-2)\omega_3 + \omega_4$,
- $\mu_4 = (n-2)\omega_3 + \omega_5$,
- $\mu_5 = \omega_1 + (n-3)\omega_3$,

- $\mu_6 = 2\omega_2 + (n-3)\omega_3 = \mu_4 + \alpha_2$,
- $\mu_7 = (n-3)\omega_3 + 2\omega_4 = \mu_5 + \alpha_4$.

Since $\min\{\langle \mu, \alpha_3^\vee \rangle : \mu \in \text{Lb}(5\omega_3, G)\} = 2$, Corollary 2.10 shows that $\mathcal{H}_G(\lambda) = \mathcal{H}_G(5\omega_3)$ if and only if $n \geq 3$.

Suppose that G is simply connected of type C_r with $r \geq 1$ (where we set $C_2 = B_2$ and $C_1 = A_1$). Then the description of the set $\text{Lb}(\lambda, G)$ is particularly simple.

Proposition 2.15. *Suppose $G = \text{Sp}(2r)$ with $r \geq 1$ and let $\lambda \in \Lambda^+$. Denote $q = \max\{i \leq r : \alpha_i \in \text{Supp}(\lambda)\}$ and set*

$$\lambda_G^{\text{lb}} = \lambda - \sum_{i=q}^{r-1} \alpha_i - \frac{1}{2}\alpha_r = \lambda + \omega_{q-1} - \omega_q :$$

then λ_G^{lb} is the unique little brother of λ w.r.t. G .

Proof. Suppose first that $\text{Lb}(\lambda, G_{\text{ad}}) \neq \emptyset$ and denote $\lambda_{\text{ad}}^{\text{lb}}$ the adjoint little brother of λ . Then $q = r \geq 2$ and $\lambda - \lambda_{\text{ad}}^{\text{lb}} = \alpha_{r-1} + \alpha_r$, hence

$$\lambda_G^{\text{lb}} - \lambda_{\text{ad}}^{\text{lb}} = \alpha_{r-1} + \frac{1}{2}\alpha_r = \frac{1}{2}(2\alpha_{r-1} + \alpha_r) \in \mathbb{Q}^+[\Phi^+(\lambda)]$$

and we get $\lambda_{\text{ad}}^{\text{lb}} <_{\mathbb{Q}}^{\lambda} \lambda_G^{\text{lb}}$.

Suppose now that $\mu \in \Pi_G^+(\lambda) \setminus \Pi^+(\lambda)$ and denote $\lambda - \mu = \sum_{i=1}^r a_i \alpha_i$. By the description of the fundamental weights of G , it follows that $a_i \in \mathbb{N}$ for every $i < r$, while $a_r \in \frac{1}{2}\mathbb{N}$: in particular we must have $a_r = \frac{n}{2}$ for some odd integer $n > 0$, since otherwise we would have $\mu < \lambda$. Notice that for $i > q$ we have

$$\langle \mu, \alpha_i^\vee \rangle = \begin{cases} a_{i-1} - 2a_i + a_{i+1} & \text{if } i < r-1 \\ a_{r-2} - 2a_{r-1} + 2a_r & \text{if } i = r-1 \\ a_{r-1} - 2a_r & \text{if } i = r \end{cases}$$

Since μ is dominant, it follows that $a_q \geq a_{q+1} \geq \dots \geq a_{r-1} \geq 2a_r$. Since $a_r \geq 1/2$, we get then $a_i \geq 1$ for every i such that $q \leq i < r$, and it follows that $\mu \leq \lambda_G^{\text{lb}}$.

We now show that every $\mu \in \Pi^+(\lambda_G^{\text{lb}})$ satisfies the inequality $\mu \leq^{\lambda} \lambda_G^{\text{lb}}$. Let $\mu \in \Pi^+(\lambda_G^{\text{lb}})$, set $\mu' = \mu - \omega_{q-1} + \omega_q$ and denote $w(\mu')$ its W -conjugate in the dominant Weyl chamber. To conclude it is enough to show that $w(\mu') \in \Pi^+(\lambda)$: indeed then $\mu' \in \Pi(\lambda)$ and by Proposition 2.1 it follows $\mu \leq^{\lambda} \lambda_G^{\text{lb}}$. The claim is clear if μ' is dominant, so we may assume that $\langle \mu, \alpha_{q-1}^\vee \rangle = 0$. Define $p = \max\{i < q : \langle \mu, \alpha_i^\vee \rangle \neq 0\}$ or set $p = 0$ in case $\text{Supp}(\mu) \cap \{\alpha_1, \dots, \alpha_{q-1}\} = \emptyset$: then w is the reflection associated to the root $\beta = \alpha_{p+1} + \dots + \alpha_{q-1}$ and $w(\mu') = \mu' + \beta$. On the other hand, since $\alpha_{q-1} \in \text{Supp}(\lambda_G^{\text{lb}}) \setminus \text{Supp}(\mu)$, by the definition of p it follows that $\{\alpha_{p+1}, \dots, \alpha_{q-1}\} \subset \text{Supp}_S(\lambda_G^{\text{lb}} - \mu)$. \square

If $\Delta' \subset \Delta$ is a connected component, denote by $\Lambda_{\Delta'} \subset \Lambda$ the associated weight lattice. If $\lambda_{\Delta'}^{\text{lb}} \in \text{Lb}(\lambda, G_{\text{ad}})$ is an adjoint little brother of λ which arises correspondingly to a non-simply laced connected component $\Delta' \subset \Delta$ such that $\mathcal{X}(T) \cap \Lambda_{\Delta'} = \mathbb{Z}[\Delta']$, then it follows from the definition that $\lambda_{\Delta'}^{\text{lb}} \in \text{Lb}(\lambda, G)$. For instance this happens whenever Δ' is of type FG. On the other hand, if Δ' is of type C_r with $r \geq 2$, then previous proposition shows that we may have $\lambda_{\Delta'}^{\text{lb}} \notin \text{Lb}(\lambda, G)$. This never happens if Δ' is of type B_r with $r \geq 3$.

Proposition 2.16. *Suppose that $G = \text{Spin}(2r+1)$ with $r \geq 3$, let $\lambda \in \Lambda^+$ be such that $\alpha_r \notin \text{Supp}(\lambda)$ and let $\lambda_{\text{ad}}^{\text{lb}} \in \text{Lb}(\lambda, G_{\text{ad}})$ be the adjoint little brother of λ . Then $\lambda_{\text{ad}}^{\text{lb}} \in \text{Lb}(\lambda, G)$.*

Proof. Denote $p < r$ the maximum such that $\alpha_p \in \text{Supp}(\lambda)$ and recall that $\lambda_{\text{ad}}^{\text{lb}} = \lambda - (\alpha_p + \dots + \alpha_r)$. Suppose that $\mu \in \Pi_G^+(\lambda)$ is such that $\lambda_{\text{ad}}^{\text{lb}} <_{\mathbb{Q}}^{\lambda} \mu <_{\mathbb{Q}} \lambda$ and denote $\lambda - \mu = \sum_{i=1}^r a_i \alpha_i$. Denote $t \geq p$ the maximum such that $a_t \neq 0$: then we must have $t = r$, since otherwise $\langle \mu, \alpha_{t+1} \rangle < 0$. Hence we get $\text{Supp}_S(\lambda - \mu) = \text{Supp}_S(\lambda - \lambda_{\text{ad}}^{\text{lb}})$ and in particular it follows that $\mu \in \Pi_G^+(\lambda) \setminus \Pi^+(\lambda)$.

By the description of the fundamental weights of G , it follows that $a_i \in \mathbb{Z}$ for every even i , while $a_i \in \frac{1}{2}\mathbb{Z}$ for every odd i . If $1 < i < r$, notice that

$$(a_{i+1} - a_i) - (a_i - a_{i-1}) = \langle \lambda - \mu, \alpha_i^{\vee} \rangle \in \mathbb{Z}.$$

Since $\mu \notin \Pi^+(\lambda)$, it follows then $a_i \notin \mathbb{Z}$ for every odd i : therefore we must have $p = 1$ and $a_i = 1/2$ for every odd i . On the other hand, since μ is dominant and since $p = 1$, it is easy to show that $a_1 \geq a_2 \geq \dots \geq a_r$: hence we get $a_2 = 1/2$, which is absurd. \square

3. NORMALITY.

Recall from Section 1 that M is the canonical compactification of G and, if $\lambda \in \mathcal{X}(T)^+$, recall the graded algebra

$$\tilde{A}(\lambda) = \bigoplus_{n=0}^{\infty} \Gamma(M, \mathcal{M}_{n\lambda}).$$

If G is adjoint, Kannan proved in [11] that $\tilde{A}(\lambda)$ is generated in degree one, however this is false for general semisimple groups. A counterexample is the following.

Example 3.1. Suppose $G = \text{SL}(5)$ and set $\lambda = \omega_1 + \omega_4$, $\nu = \omega_1 + \omega_2$. Then $\nu \leq_{\mathbb{Q}} 2\lambda$, hence $\text{End}(V(\nu))^* \subset \Gamma(M, \mathcal{M}_{2\lambda})$. However they do not exist dominant weights $\mu_1 \leq_{\mathbb{Q}} \lambda$ and $\mu_2 \leq_{\mathbb{Q}} \lambda$ such that $V(\nu) \subset V(\mu_1) \otimes V(\mu_2)$. In particular, the multiplication map

$$\Gamma(M, \mathcal{M}_{\lambda}) \times \Gamma(M, \mathcal{M}_{\lambda}) \longrightarrow \Gamma(M, \mathcal{M}_{2\lambda})$$

is not surjective and $\tilde{A}(\lambda)$ is not generated in degree one.

We now show that, if $\lambda \in \mathcal{X}(T)^+$ is sufficiently regular, then $\tilde{A}(\lambda)$ is generated in degree one. Before that, we recall a lemma from [11].

Lemma 3.2 ([11, Lemma 3.2]). *Let $\lambda, \mu \in \Lambda^+$ and let $\nu \in \Pi^+(\lambda + \mu)$. They exist $\lambda' \in \Pi^+(\lambda)$ and $\mu' \in \Pi^+(\mu)$ such that $V(\nu) \subset V(\lambda') \otimes V(\mu')$.*

Proposition 3.3. *Suppose that $\lambda \in \mathcal{X}(T)^+$ is such that $\langle \lambda, \alpha^{\vee} \rangle \geq r_{\alpha}$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^{\circ}$. Then the algebra $\tilde{A}(\lambda)$ is generated in degree one.*

Proof. By previous corollary it is enough to show that for every $\mu \in \Pi_G^+(n\lambda)$ which is maximal w.r.t. \leq the submodule $\text{End}(V(\mu))^* \subset \tilde{A}_n(\lambda)$ is contained in the power $\tilde{A}_1(\lambda)^n$. Since it is maximal w.r.t. \leq , the weight μ is maximal also w.r.t. \leq^{λ} , hence by Proposition 2.7 and Corollary 2.8 we get that

$$\langle n\lambda - \mu, \alpha^{\vee} \rangle \leq \begin{cases} 1 & \text{if } \alpha \in \text{Supp}(\lambda)^{\circ} \\ r_{\alpha} & \text{if } \alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^{\circ} \end{cases}$$

and by the assumption on the coefficients of λ it follows that $\langle n\lambda - \mu, \alpha^{\vee} \rangle \leq \langle \lambda, \alpha^{\vee} \rangle$. Therefore, if we set $\mu' = \mu - (n-1)\lambda$, then we get $\mu' \in \Pi_G^+(\lambda)$ and $V(\mu) \subset V(\mu') \otimes V(\lambda)^{\otimes n-1}$, hence $\text{End}(V(\mu))^* \subset \tilde{A}_1(\lambda)^n$. \square

Previous proposition allows to describe the normalization of a simple linear compactification of G as follows. If $\lambda \in \mathcal{X}(T)^+$ is an almost faithful weight, consider the adjoint compactification X_λ and denote $\tilde{X}_\lambda \rightarrow X_\lambda$ the normalization of X_λ in $\mathbb{k}(G)$. Before describing \tilde{X}_λ we recall a lemma from [6], which is given there in the context of symmetric varieties.

Lemma 3.4 ([6, Prop. 2.1]). *The algebra $\tilde{A}(\lambda)$ is integral over $A(\lambda)$.*

Theorem 3.5. *Let $\lambda \in \mathcal{X}(T)^+$ be an almost faithful weight with $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ$. Then $\tilde{X}_\lambda \simeq X_{\Pi_G^+(\lambda)}$.*

Proof. Since M is a normal variety, $\tilde{A}(\lambda)$ is an integrally closed algebra. On the other hand, Proposition 3.3 shows that $\tilde{A}(\lambda)$ is generated in degree one by

$$\Gamma(M, \mathcal{M}_\lambda) \simeq \bigoplus_{\mu \in \Pi_G^+(\lambda)} \text{End}(V(\mu)),$$

so that $X_{\Pi_G^+(\lambda)} \subset \mathbb{P}(\Gamma(M, \mathcal{M}_\lambda)^*)$ is a projectively normal variety.

Since λ is almost faithful it follows that $\mathcal{X}(T)^+ = \bigcup_{n \in \mathbb{N}} \Pi_G^+(n\lambda)$, hence there exists $n \in \mathbb{N}$ such that $\Pi_G^+(n\lambda)$ is faithful. On the other hand $\Pi_G^+(n\lambda)$ is faithful if and only if the set

$$\mathcal{H}_G^m(n\lambda) = \{\mu - n\lambda : \mu \in \Pi_G^+(n\lambda) \text{ is maximal w.r.t. } \leq^\lambda\}$$

generates $\mathcal{X}(T)/\mathbb{Z}[\Delta]$: therefore $\Pi_G^+(\lambda)$ as well is faithful, since Corollary 2.10 implies $\mathcal{H}_G^m(\lambda) = \mathcal{H}_G^m(n\lambda)$ for every $n > 0$.

Consider now the subalgebra $A(\lambda) \subset \tilde{A}(\lambda)$ generated by $\text{End}(V(\lambda))^* \subset \tilde{A}_1(\lambda)$, which is identified with the homogeneous coordinate ring of X_λ . Since $\tilde{A}(\lambda)$ is integral over $A(\lambda)$ by Lemma 3.4, it follows then that $X_{\Pi_G^+(\lambda)}$ is the normalization of X_λ in $\mathbb{k}(G)$. \square

As shown by following examples, the hypothesis on the coefficients of λ in previous theorem is necessary.

- Example 3.6.*
- i) Set $G = \text{SL}(r+1)$ and $\Pi = \Pi_G^+(n\omega_1)$. Then Π is faithful for every $n \geq 1$ and X_Π is normal for every $n \geq 1$ (see Example 2.11 and Theorem 3.10).
 - ii) Set $G = \text{Spin}(2r+1)$ and $\Pi = \Pi_G^+(n\omega_1)$. Then Π is faithful for every $n \geq \lceil r/2 \rceil$ and X_Π is normal for every $n \geq \lceil r/2 \rceil$ (see Example 2.12 and Theorem 3.10).
 - iii) Set $G = \text{SL}(r+1)$ and $\Pi = \Pi_G^+(n\omega_2)$. Then Π is faithful if and only if $n \geq 2$ if r is odd and for every $n \geq 1$ if r is even, whereas X_Π is normal if and only if $n \geq r-1$ (see Example 2.13 and Theorem 3.10).
 - iv) Suppose that G is a direct product of adjoint groups and of groups of type C_r with $r \geq 1$. Then $X_{\Pi_G^+(\lambda)}$ is a normal compactification of G for every almost faithful $\lambda \in \mathcal{X}(T)^+$ (see Proposition 2.15 and Theorem 3.10).

Combining previous theorem with Proposition 1.5 we get the following tensorial characterization of the normality of X_Π .

Proposition 3.7. *Let $\Pi \subset \mathcal{X}(T)^+$ be simple and faithful with maximal element λ and suppose that $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^\circ$. Then the variety X_Π is normal if and only if for every $\nu \in \Pi_G^+(\lambda)$ they exist $n \in \mathbb{N}$ and $\mu_1, \dots, \mu_n \in \Pi$ such that*

$$V(\nu + (n-1)\lambda) \subset V(\mu_1) \otimes \cdots \otimes V(\mu_n).$$

We may restate previous criterion in a more combinatorial way as follows. If $\Pi \subset \mathcal{X}(T)^+$ is simple with maximal element λ , denote

$$\Omega(\Pi) = \left\{ \nu - n\lambda : V(\nu) \subset \left(\bigoplus_{\mu \in \Pi} V(\mu) \right)^{\otimes n} \right\} :$$

it is a semigroup and by Lemma 1.3 it is the image of

$$\mathbb{k}[X_{\Pi}^{\circ}]^{(B \times B^{-})} / \mathbb{k}^* \subset \mathcal{X}(T) \times \mathcal{X}(T)$$

in $\mathcal{X}(T)$ via the projection on the first factor. If $\Pi = \{\mu_1, \dots, \mu_m\}$, for simplicity sometimes we will denote $\Omega(\Pi)$ also by $\Omega(\mu_1, \dots, \mu_m)$. Since X_{Π} possesses an open $B \times B^{-}$ -orbit, every $B \times B^{-}$ -semi-invariant function $\phi \in \mathbb{k}(X_{\Pi})^{(B \times B^{-})}$ is uniquely determined by its weight up to a non-zero scalar factor, therefore we may restate Proposition 1.5 and Proposition 3.7 as follows.

Proposition 3.8. *Let $\Pi \subset \mathcal{X}(T)^+$ be simple and faithful with maximal element λ .*

- i) *Let $\Pi' \subset \mathcal{X}(T)^+$ be simple with maximal element λ' and suppose that $\text{Supp}(\lambda) = \text{Supp}(\lambda')$. There exists an equivariant morphism $X_{\Pi} \rightarrow X_{\Pi'}$ if and only if $\Omega(\Pi') \subset \Omega(\Pi)$ if and only if $\mu' - \lambda' \in \Omega(\Pi)$ for every $\mu' \in \Pi'$.*
- ii) *Suppose that $\langle \lambda, \alpha^{\vee} \rangle \geq r_{\alpha}$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^{\circ}$. Then the variety X_{Π} is normal if and only if $\nu - \lambda \in \Omega(\Pi)$ for every $\nu \in \Pi_G^+(\lambda)$.*

Remark 3.9. Let $\lambda \in \mathcal{X}(T)^+$ be an almost faithful weight such that $\langle \lambda, \alpha^{\vee} \rangle \geq r_{\alpha}$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^{\circ}$ and denote $\tilde{\Omega}(\lambda)$ the image of the semigroup

$$\{B \times B^{-}\text{-weights in } \mathbb{k}[\tilde{X}_{\lambda}^{\circ}]\} \subset \mathcal{X}(T) \times \mathcal{X}(T)$$

in $\mathcal{X}(T)$ via the projection on the first factor: then $\tilde{\Omega}(\lambda) = \{\pi \in \mathcal{X}(T) : n\pi \in \Omega(\lambda) \exists n \in \mathbb{N}\}$ is the *saturation* of $\Omega(\lambda)$ in $\mathcal{X}(T)$. By the description of \tilde{X}_{λ} given in Theorem 3.5, we may describe $\tilde{\Omega}(\lambda)$ more explicitly as follows:

$$\tilde{\Omega}(\lambda) = \{\mu - n\lambda : \mu \in \Pi_G^+(n\lambda)\}.$$

Consider now the cone $\mathcal{C}(\lambda) \subset \mathcal{X}(T)_{\mathbb{Q}}$ generated by $\Phi^{-}(\lambda)$. Then Proposition 2.4 shows that

$$\mathcal{C}(\lambda) \cap \mathcal{X}(T) = \{\mu - n\lambda : \mu \in \Pi_G(n\lambda)\}$$

and by Theorem 3.5 it follows that $\mathcal{C}(\lambda) \cap \mathcal{X}(T)$ is the image of

$$\{T \times T\text{-weights in } \mathbb{k}[\tilde{X}_{\lambda}^{\circ}]\} \subset \mathcal{X}(T) \times \mathcal{X}(T)$$

in $\mathcal{X}(T)$ via the projection on the first factor.

Thanks to the description of \tilde{X}_{λ} given in Theorem 3.5, we are now able to characterize the normality of a simple linear compactification of G as follows.

Theorem 3.10. *Let $\Pi \subset \mathcal{X}(T)^+$ be a simple subset with maximal element λ and assume that $\langle \lambda, \alpha^{\vee} \rangle \geq r_{\alpha}$ for every $\alpha \in \text{Supp}(\lambda) \setminus \text{Supp}(\lambda)^{\circ}$. Then the variety X_{Π} is a normal compactification of G if and only if $\Pi \supset \text{Lb}(\lambda, G)$.*

Notice that the hypothesis on the coefficients of λ in the previous theorem is automatically fulfilled whenever λ is a regular weight. Notice also that it involves no loss of generality: if $\alpha \in \text{Supp}(\lambda)$ is such that $\langle \lambda, \alpha^{\vee} \rangle < r_{\alpha}$ then we may consider the simple set

$$\Pi' = \{\mu + (r_{\alpha} - \langle \lambda, \alpha^{\vee} \rangle)\omega_{\alpha} : \mu \in \Pi\}$$

and by Proposition 1.5 the varieties X_Π and $X_{\Pi'}$ are equivariantly isomorphic. However, it may also happen that the hypothesis on the coefficients of λ is not needed: for instance this is the case if $G = \mathrm{SL}(r+1)$ and $\mathrm{Supp}(\lambda) = \{\alpha_1\}$ (see Example 2.11) and when G is a direct product of adjoint groups and groups of type C_r with $r \geq 1$ (see Proposition 2.15).

For convenience we split the proof of previous theorem in the following two propositions, where we treat separately the necessity and the sufficiency of the condition.

Proposition 3.11. *Let $\Pi \subset \mathcal{X}(T)^+$ be simple and faithful with maximal element λ and assume that $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\lambda)^\circ$. If X_Π is normal, then $\Pi \supset \mathrm{Lb}(\lambda, G)$.*

Proof. By Proposition 3.7 for every $\nu \in \Pi_G^+(\lambda)$ they exist $n \in \mathbb{N}$ and $\mu_1, \dots, \mu_n \in \Pi$ such that

$$V(\nu + (n-1)\lambda) \subset V(\mu_1) \otimes \cdots \otimes V(\mu_n).$$

Suppose that $\nu \in \Pi_G^+(\lambda) \setminus \Pi^+(\lambda)$ is maximal w.r.t. $\leq_{\mathbb{Q}}^\lambda$. Since $\sum_{i=1}^n \mu_i - \nu - (n-1)\lambda \in \mathbb{N}[\Delta]$, we must have $\mu_i \in \Pi_G^+(\lambda) \setminus \Pi^+(\lambda)$ for some i : hence by Proposition 2.3 we get $\nu \leq_{\mathbb{Q}}^\lambda \mu_i$ and the equality $\mu_i = \nu$ follows by maximality. Suppose now that $\nu \in \mathrm{Lb}(\lambda, G_{\mathrm{ad}})$ is maximal in $\Pi_G^+(\lambda) \setminus \{\lambda\}$ w.r.t. $\leq_{\mathbb{Q}}^\lambda$: then by Proposition 2.3 for every i we must have either $\mu_i = \lambda$ or $\mu_i = \nu$, while by Theorem 1.8 we cannot have $\mu_i = \lambda$ for every i . \square

Before to prove the sufficiency of the condition, we recall the former Parthasarathy-Ranga Rao-Varadarajan conjecture, which was proved independently by Kumar [15] and Mathieu [17].

Theorem 3.12 (PRV Conjecture). *Let $\lambda, \mu \in \mathcal{X}(T)^+$ be dominant weights and let $\nu \leq \lambda + \mu$ be a dominant weight of the shape $\nu = w\lambda + w'\mu$, with $w, w' \in W$. Then $V(\nu) \subset V(\lambda) \otimes V(\mu)$.*

Proposition 3.13. *Let $\Pi \subset \mathcal{X}(T)^+$ be a simple subset whose maximal element λ is such that $\langle \lambda, \alpha^\vee \rangle \geq r_\alpha$ for every $\alpha \in \mathrm{Supp}(\lambda) \setminus \mathrm{Supp}(\lambda)^\circ$. If $\Pi \supset \mathrm{Lb}(\lambda, G)$, then X_Π is a normal compactification of G .*

Proof. Suppose that $X_{\overline{\mathrm{Lb}}(\lambda, G)}$ is a normal compactification of G : then the claim follows by considering the natural projections

$$X_{\Pi_G^+(\lambda)} \longrightarrow X_\Pi \longrightarrow X_{\overline{\mathrm{Lb}}(\lambda, G)}.$$

Therefore we are reduced to the case $\Pi = \overline{\mathrm{Lb}}(\lambda, G)$. Since by Theorem 3.5 $X_{\Pi_G^+(\lambda)}$ is a normal compactification of G , by Proposition 3.7, it is enough to show that $\nu - \lambda \in \Omega(\overline{\mathrm{Lb}}(\lambda, G))$ for every $\nu \in \Pi_G^+(\lambda)$. Denote

$$\Pi_G^+(\lambda)^\circ = (\Pi_G^+(\lambda) \setminus \Pi^+(\lambda)) \cup \mathrm{Lb}(\lambda, G_{\mathrm{ad}}) :$$

by the normality of the adjoint compactification $X_{\overline{\mathrm{Lb}}(\lambda, G_{\mathrm{ad}})}$ (see Theorem 1.8), it is sufficient to show that $\nu - \lambda \in \Omega(\overline{\mathrm{Lb}}(\lambda, G))$ for any $\nu \in \Pi_G^+(\lambda)^\circ$.

Suppose that $\nu \in \Pi_G^+(\lambda)^\circ$ is not maximal w.r.t. $\leq_{\mathbb{Q}}^\lambda$ and let $\mu_1 \in \mathrm{Lb}(\lambda, G)$ be such that $\nu <_{\mathbb{Q}}^\lambda \mu_1$: then $\nu - \mu_1 \in \mathcal{C}(\lambda)$, hence by Proposition 2.4 there exists $m \in \mathbb{N}$ such that $m\lambda + \nu - \mu_1 \in \mathcal{P}(m\lambda)$. Denote

$$\nu' = \nu + (m-1)\lambda, \quad \mu'_1 = \mu_1 + (m-1)\lambda, \quad \lambda' = m\lambda.$$

and denote $\nu_2 = w(\lambda' + \nu' - \mu'_1) \in \mathcal{X}(T)^+$ the dominant weight which is conjugated to $\lambda' + \nu' - \mu'_1$ under the Weyl group W . Then $\nu' \leq_{\mathbb{Q}}^\lambda \mu'_1 <_{\mathbb{Q}} \lambda'$ and, since $\mathcal{P}(\lambda')$ is W -stable, it follows that $\nu_2 \leq_{\mathbb{Q}} \lambda'$. Moreover $\nu' + \lambda' = \mu'_1 + w^{-1}\nu_2$, hence the PRV conjecture implies

$$V(\nu' + \lambda') \subset V(\mu'_1) \otimes V(\nu_2).$$

Suppose that $\nu_2 \in \text{Lb}(\lambda', G)$: then by Corollary 2.10 there exists $\mu_2 \in \text{Lb}(\lambda, G)$ such that $\nu_2 = \mu_2 + (m-1)\lambda$. Hence $\lambda - \mu_1 = \lambda' - \mu'_1$ and $\lambda - \mu_2 = \lambda' - \nu_2$ and by Proposition 3.8 i) we get

$$\nu - \lambda = \nu' - \lambda' \in \Omega(\lambda', \mu'_1, \nu_2) = \Omega(\lambda, \mu_1, \mu_2) \subset \Omega(\overline{\text{Lb}}(\lambda, G)).$$

Suppose now that $\nu_2 \in \Pi_G^+(\lambda')^\circ$ is not maximal w.r.t. $\leq_{\mathbb{Q}}^\lambda$. Since $\nu' + \lambda' - \mu'_1 \leq \nu_2$, it follows that

$$\lambda' - \nu_2 \leq \lambda' - (\lambda' + \nu' - \mu'_1) = \mu_1 - \nu <_{\mathbb{Q}} \lambda - \nu :$$

hence we may proceed by decreasing induction on the rational dominance order and we may assume that they exist $n > 2$ and $\mu'_2, \dots, \mu'_n \in \text{Lb}(\lambda', G)$ such that

$$V(\nu_2 + (n-2)\lambda') \subset V(\mu'_2) \otimes \dots \otimes V(\mu'_n).$$

Combining previous tensorial inclusions and applying Corollary 1.10 we get then

$$V(\nu' + (n-1)\lambda') \subset V(\mu'_1) \otimes V(\mu'_2) \otimes \dots \otimes V(\mu'_n).$$

By Corollary 2.10 they exist $\mu_2, \dots, \mu_n \in \text{Lb}(\lambda, G)$ such that $\mu'_i = \mu_i + (m-1)\lambda$ for every $i = 2, \dots, n$. Hence $\lambda - \mu_i = \lambda' - \mu'_i$ for every i and by Proposition 3.8 i) we get

$$\nu - \lambda = \nu' - \lambda' \in \Omega(\lambda', \mu'_1, \dots, \mu'_n) = \Omega(\lambda, \mu_1, \dots, \mu_n) \subset \Omega(\overline{\text{Lb}}(\lambda, G)). \quad \square$$

4. SMOOTHNESS

If $\Pi \subset \mathcal{X}(T)^+$ is simple and faithful with maximal element λ , then the normalization of X_Π coincides with \tilde{X}_λ : hence by Theorem 3.10 it follows that X_Π is smooth if and only if \tilde{X}_λ is smooth and $\Pi \supset \text{Lb}(\lambda, G)$. Therefore the problem of studying the smoothness of X_Π reduces to the study of the smoothness of \tilde{X}_λ .

We now recall some results about the \mathbb{Q} -factoriality and the smoothness of \tilde{X}_λ . Recall that a normal variety X is called *locally factorial* if $\text{Pic}(X) = \text{Cl}(X)$, while it is called *\mathbb{Q} -factorial* if $\text{Pic}(X)_{\mathbb{Q}} = \text{Cl}(X)_{\mathbb{Q}}$. In the case of a simple spherical variety, these properties are nicely expressed by the combinatorial properties of the colored cone.

Proposition 4.1 ([4, Rem. 2.2.ii], [5, Prop. 4.2]). *Let $\lambda \in \mathcal{X}(T)^+$.*

- i) \tilde{X}_λ is \mathbb{Q} -factorial if and only if $\mathcal{C}(\tilde{X}_\lambda)$ is a simplicial cone (i.e. generated by linearly independent vectors).
- ii) \tilde{X}_λ is locally factorial if and only if $\mathcal{C}(\tilde{X}_\lambda)$ is generated by a basis of $\mathcal{X}(T)^\vee$.

More explicitly, the \mathbb{Q} -factoriality of \tilde{X}_λ can be characterized as follows, where we denote by Δ^e the set of the extremal roots of Δ .

Proposition 4.2 ([3, Prop. 3.4 and Cor. 3.5]). *Let $\lambda \in \mathcal{X}(T)^+$. The variety \tilde{X}_λ is \mathbb{Q} -factorial if and only if the following conditions are fulfilled:*

- i) For every connected component Δ' of Δ , $\text{Supp}(\lambda) \cap \Delta'$ is connected and, in case it contains a unique element, then this element is an extremal root of Δ' ;
- ii) $\text{Supp}(\lambda)$ contains every simple root which is adjacent to three other simple roots and at least two of the latter.

If this is the case, then the extremal rays of $\mathcal{C}(\tilde{X}_\lambda)$ are the half-lines generated by the elements in the set

$$\{\alpha^\vee : \alpha \in \Delta \setminus \text{Supp}(\lambda)\} \cup \{-\omega_\alpha^\vee : \alpha \in \text{Supp}(\lambda)^\circ \cup (\Delta^e \setminus \text{Supp}(\lambda))\}.$$

Previous result is stated in [3] in the case of a semisimple adjoint group, however it holds for any semisimple group since the cone $\mathcal{C}(\tilde{X}_\lambda)$ depends only on the set $\text{Supp}(\lambda)$. Previous proposition allows to reduce the study of the smoothness of the variety \tilde{X}_λ to the case of a simple group G as follows.

Lemma 4.3. *If $\lambda \in \mathcal{X}(T)^+$ is an almost faithful weight such that \tilde{X}_λ is locally factorial, then $G = G_1 \times \dots \times G_n$ is a direct product of simple groups. If moreover $T_i = T \cap G_i$ and $\lambda_i = \lambda|_{T_i}$, then*

$$\tilde{X}_\lambda = \tilde{X}_{\lambda_1} \times \dots \times \tilde{X}_{\lambda_n},$$

where \tilde{X}_{λ_i} denotes the normalization of X_{λ_i} in $\mathbb{k}(G_i)$.

Proof. Denote G_1, \dots, G_n the simple factors of G and denote Λ_{rad}^i the root lattice of G_i : then $\Lambda_{\text{rad}} = \Lambda_{\text{rad}}^1 \times \dots \times \Lambda_{\text{rad}}^n$ and G is the direct product of its simple factors if and only if $\mathcal{X}(T)^\vee$ is the direct product of the intersections $\mathcal{X}(T)^\vee \cap (\Lambda_{\text{rad}}^i)_{\mathbb{Q}}^\vee$. Hence the first claim follows by Proposition 4.1 ii) together with the description of the extremal rays of $\mathcal{C}(\tilde{X}_\lambda)$ given in Proposition 4.2. The second claim follows straightforwardly from the isomorphism $X_\lambda \simeq X_{\lambda_1} \times \dots \times X_{\lambda_n}$ (see [3, Lemma 1.1]). \square

In the case of an adjoint group, the smoothness of \tilde{X}_λ has been characterized in [3] as follows.

Theorem 4.4 ([3, Thm. 3.6]). *Suppose that G is simple and adjoint and let $\lambda \in \mathcal{X}(T)^+$. Then \tilde{X}_λ is smooth if and only if X_λ is smooth if and only if λ satisfies the following conditions:*

- i) *If $\text{Supp}(\lambda)$ contains a long root, then it contains also the unique short simple root which is non-orthogonal to a long simple root;*
- ii) *$\text{Supp}(\lambda)$ is connected and, in case it contains a unique element, then this element is an extremal root of Δ ;*
- iii) *$\text{Supp}(\lambda)$ contains every simple root which is adjacent to three other simple roots and at least two of the latter;*
- iv) *Every connected component of $\Delta \setminus \text{Supp}(\lambda)$ is of type A.*

Therefore we only need to consider the case of a simple non-adjoint group. A general criterion for the smoothness of a group compactification was given by Timashev in [19]; for convenience, we will use a generalization which can be found in [18] in the more general context of symmetric spaces. We recall it in the case of the compactification \tilde{X}_λ .

Theorem 4.5 (see [18, Thm. 2.2], [19, Thm. 9]). *Let $\lambda \in \mathcal{X}(T)^+$. Then \tilde{X}_λ is smooth if and only if the following conditions are fulfilled:*

- i) *All connected components of $\Delta \setminus \text{Supp}(\lambda)$ are of type A and there are no more than $|\text{Supp}(\lambda)|$ of them;*
- ii) *The cone $\mathcal{C}(\tilde{X}_\lambda)$ is simplicial and it is generated by a basis $\mathcal{B}(\lambda) \subset \mathcal{X}(T)^\vee$;*
- iii) *One can enumerate the simple roots in order of their positions at Dynkin diagrams of connected components $I_k = \{\alpha_1^k, \dots, \alpha_{n_k}^k\}$ of $\Delta \setminus \text{Supp}(\lambda)$, $k = 1, \dots, n$, and partition the dual basis $\mathcal{B}(\lambda)^* \subset \mathcal{X}(T)$ into subsets $J_k = \{\pi_1^k, \dots, \pi_{n_k+1}^k\}$, $k = 1, \dots, p$ (with $p \geq n$) in such a way that $\langle \pi_j^k, (\alpha_i^h)^\vee \rangle = \delta_{i,j} \delta_{h,k}$ and $\pi_j^k - \frac{j}{n_k+1} \pi_{n_k+1}^k$ is the j -th fundamental weight of the root system generated by I_k for all j, k .*

Before applying previous theorem in the case of our interest, we need an auxiliary lemma.

Lemma 4.6. *Let $\lambda \in \mathcal{X}(T)^+$ be an almost faithful weight such that \tilde{X}_λ is smooth and denote $\mathcal{B}(\lambda) \subset \mathcal{X}(T)^\vee \cap \mathcal{C}(\tilde{X}_\lambda)$ the associated basis of $\mathcal{X}(T)^\vee$. If $\beta \in \Delta^e \setminus \text{Supp}(\lambda)$, then $-\omega_\beta^\vee \in \mathcal{B}(\lambda)$.*

Proof. Denote $\mathcal{B}(\lambda) = \{v_1, \dots, v_r\}$ and denote $\mathcal{B}(\lambda)^* = \{v_1^*, \dots, v_r^*\}$ the basis of $\mathcal{X}(T)$ which is dual to $\mathcal{B}(\lambda)$, defined by $\langle v_i^*, v_j \rangle = \delta_{ij}$. Denote $\Delta \setminus \text{Supp}(\lambda) = \bigcup_{k=1}^n I_k$ the decomposition in connected components and denote $\mathcal{B}(\lambda)^* = \bigcup_{k=1}^p J_k$ the partition which exists by Theorem 4.5 iii), where $p \geq n$.

Fix a connected component $I_k = \{\alpha_1^k, \dots, \alpha_{n_k}^k\} \subset \Delta \setminus \text{Supp}(\lambda)$ and consider the associated component $J_k = \{\pi_1^k, \dots, \pi_{n_k+1}^k\} \subset \mathcal{B}(\lambda)^*$, where the numberings in I_k and in J_k are those of Theorem 4.5 iii). Denote $L \subset G$ the Levi subgroup associated to I_k and denote by ω_α^L the fundamental weight of L associated to a simple root $\alpha \in I_k$.

By Proposition 4.2 it follows that I_k is of type A and it contains exactly an extremal root $\beta \in \Delta^e$. By the description of the extremal rays of $\mathcal{C}(\tilde{X}_\lambda)$ it follows that there is a unique element in $\mathcal{B}(\lambda) \setminus I_k^\vee$ which is non-orthogonal to I_k , namely the element $v_\beta \in \mathcal{B}(\lambda)$ which is proportional to ω_β^\vee . Therefore up to renumbering we must have $J_k = \{w_{\alpha_1^k}^*, \dots, w_{\alpha_{n_k}^k}^*, v_\beta^*\}$, where if $\alpha \in \Delta \setminus \text{Supp}(\lambda)$ we denote by $w_\alpha \in \mathcal{B}(\lambda)$ the unique element which is proportional with α^\vee .

Notice that $v_\beta^* = \pi_{n_k+1}^k$: indeed $\beta \in \text{Supp}_\Delta(\omega_\alpha^L)$ for every $\alpha \in \Delta \setminus \text{Supp}(\lambda)$, hence

$$\langle \pi_i^k, v_\beta \rangle - \frac{i}{n_k + 1} \langle \pi_{n_k+1}^k, v_\beta \rangle = \langle \omega_{\alpha_i^k}^L, v_\beta \rangle \neq 0.$$

Set now $v_\beta = -m\omega_\beta^\vee$, where $m > 0$, and set $\beta = \alpha_i^k$. Then by Theorem 4.5 iii) it follows that $\omega_\beta^L = \pi_i^k - \frac{i}{n_k+1}\pi_{n_k+1}^k$, hence

$$\langle \omega_\beta^L, v_\beta \rangle = -\frac{i}{n_k + 1} \langle v_\beta^*, v_\beta \rangle = -\frac{i}{n_k + 1}$$

for every $i \leq n_k$. On the other hand L is of type A_{n_k} , hence

$$\omega_\beta^L = \frac{1}{n_k + 1} \left(\sum_{j=1}^{i-1} j(n_k - i + 1) \alpha_j^k + \sum_{j=i}^{n_k} i(n_k - j + 1) \alpha_j^k \right)$$

and we get

$$\langle \omega_\beta^L, v_\beta \rangle = -\frac{im(n_k - i + 1)}{n_k + 1} \langle \beta, \omega_\beta^\vee \rangle = -\frac{im(n_k - i + 1)}{n_k + 1}.$$

Comparing the two equalities, it follows then $i = n_k$ and $m = 1$: therefore the numbering of I_k starts from the extremal root of I_k which is not extremal in Δ and $-\omega_\beta^\vee \in \mathcal{B}(\lambda)$. \square

Theorem 4.7. *Suppose that G is simple and non-adjoint and let $\lambda \in \mathcal{X}(T)^+$ be an almost faithful weight. Then \tilde{X}_λ is smooth if and only if $G = \text{Sp}(2r)$ with $r \geq 1$, $\text{Supp}(\lambda)$ is connected and $\alpha_r \in \text{Supp}(\lambda)$.*

Notice that $\text{Sp}(4)$ has type $C_2 = B_2$, while $\text{Sp}(2)$ has type $C_1 = A_1$.

Proof. Suppose that $G = \text{Sp}(2r)$ and assume that $\text{Supp}(\lambda)$ is connected and $\alpha_r \in \text{Supp}(\lambda)$: then Proposition 4.2 and Theorem 4.5 show that the variety \tilde{X}_λ is smooth.

Suppose conversely that \tilde{X}_λ is smooth. If λ is regular, then $\mathcal{C}(\tilde{X}_\lambda)$ is the negative Weyl chamber C^- , so it is simplicial and condition iii) of Theorem 4.5 is empty. Hence we only need to verify that C^- is generated by a basis of $\mathcal{X}(T)^\vee$, and it is easy to verify that this happens if and only if G is either adjoint or of type C_r with $r \geq 1$.

Therefore we may assume that λ is non-regular; by Proposition 4.2 it follows that $\text{Supp}(\lambda)$ is connected and that there exists a (unique) basis $\mathcal{B}(\lambda) \subset \mathcal{X}(T)^\vee$ which generates $\mathcal{C}(\tilde{X}_\lambda)$ as a cone. Since $\text{Supp}(\lambda)$ is connected and since λ is not regular, we must have $\Delta^e \setminus \text{Supp}(\lambda) \neq \emptyset$. Since otherwise G is necessarily adjoint, Δ cannot be of type E_8, F_4, G_2 .

Type A. If Δ is of type A_r with $r > 1$, then $\Lambda^\vee/\mathbb{Z}[\Delta^\vee]$ is generated both by ω_1^\vee and by ω_r^\vee . Since $\Delta^e \setminus \text{Supp}(\lambda) \neq \emptyset$, Lemma 4.6 implies then $\mathcal{X}(T)^\vee = \Lambda^\vee$, i.e. G is adjoint.

Type B. If Δ is of type B_r with $r > 2$, then Φ^\vee has type C_r and $\Lambda^\vee/\mathbb{Z}[\Delta^\vee]$ is generated by any fundamental coweight ω_i^\vee with i odd. If $\alpha_1 \notin \text{Supp}(\lambda)$, Lemma 4.6 implies that $\mathcal{X}(T)^\vee = \Lambda^\vee$, i.e. G is adjoint. Otherwise, by Theorem 4.5 i) it must be $\text{Supp}(\lambda) = \{\alpha_1, \dots, \alpha_{r-1}\}$, while by Propositions 4.1 and 4.2 there exists an element $v_1 \in \mathcal{B}(\lambda)$ which is a multiple of ω_1^\vee . Notice that $\alpha_1^\vee \notin \text{Supp}(v)$ for every $v \in \mathcal{B}(\lambda) \setminus \{v_1\}$: since $\langle \alpha_1, \alpha_2^\vee \rangle = -1$ and since $\alpha_2^\vee \in \mathcal{X}(T)^\vee$, it follows then $v_1 = -\omega_1^\vee$. Hence $\mathcal{X}(T)^\vee$ is the coweight lattice and G is adjoint.

Type C. If Δ has type C_r with $r \geq 1$, then Δ^\vee has type B_r and $\Lambda^\vee/\mathbb{Z}[\Delta^\vee]$ is generated by ω_r^\vee . Therefore Lemma 4.6 implies that, if $\alpha_r \notin \text{Supp}(\lambda)$, then $\mathcal{X}(T)^\vee$ is the coweight lattice and G is adjoint. Hence we get $\text{Supp}(\lambda) = \{\alpha_k, \dots, \alpha_r\}$.

Type D. If Δ is of type D_r , then $\Lambda^\vee/\mathbb{Z}[\Delta^\vee]$ is generated by any two fundamental coweights associated to some simple root in Δ^e . Therefore if $|\text{Supp}(\lambda) \cap \Delta^e| \leq 1$ Lemma 4.6 implies that G is adjoint. Suppose that $|\text{Supp}(\lambda) \cap \Delta^e| = 2$. If $\alpha_1 \notin \text{Supp}(\lambda)$, then by Proposition 4.2 we get that $\text{Supp}(\lambda) = \{\alpha_i, \dots, \alpha_r\}$ for some $i \leq r-2$ and by Propositions 4.1 and 4.2 they exist elements $v_{r-1}, v_r \in \mathcal{B}(\lambda)$ which are multiples resp. of ω_{r-1}^\vee and of ω_r^\vee . Notice that $\text{Supp}(v) \cap \{\alpha_{r-1}^\vee, \alpha_r^\vee\} = \emptyset$ for every $v \in \mathcal{B}(\lambda) \setminus \{v_{r-1}, v_r\}$: since $\alpha_{r-2}^\vee \in \mathcal{X}(T)^\vee$ and since $\langle \alpha_{r-1}, \alpha_{r-2}^\vee \rangle = \langle \alpha_r, \alpha_{r-2}^\vee \rangle = -1$, it follows then $v_{r-1} = -\omega_{r-1}^\vee$ and $v_r = -\omega_r^\vee$. Hence $\mathcal{X}(T)^\vee$ is the coweight lattice and G is adjoint.

Suppose now that $\alpha_1 \in \text{Supp}(\lambda)$: up to an automorphism of Φ , by Proposition 4.2 we may assume that $\text{Supp}(\lambda) = \{\alpha_1, \dots, \alpha_{r-1}\}$. By Propositions 4.1 and 4.2 there exist elements $v_1, v_{r-1} \in \mathcal{B}(\lambda)$ which are multiples resp. of ω_1^\vee and of ω_{r-1}^\vee . Notice that $\text{Supp}(v) \cap \{\alpha_1^\vee, \alpha_{r-1}^\vee\} = \emptyset$ for every $v \in \mathcal{B}(\lambda) \setminus \{v_1, v_{r-1}\}$: since $\{\alpha_2^\vee, \alpha_{r-2}^\vee\} \subset \mathcal{X}(T)^\vee$ and since $\langle \alpha_1, \alpha_2^\vee \rangle = \langle \alpha_{r-2}, \alpha_{r-1}^\vee \rangle = -1$, it follows then $v_1 = -\omega_1^\vee$ and $v_{r-1} = -\omega_{r-1}^\vee$. Hence $\mathcal{X}(T)^\vee$ is the coweight lattice and G is adjoint.

Type E₆. If Δ is of type E_6 , then Λ^\vee is generated both by ω_1^\vee and by ω_6^\vee : hence by Lemma 4.6 it follows that G is adjoint if $\text{Supp}(\lambda) \cap \{\alpha_1, \alpha_6\} \neq \emptyset$. If this is not the case, then by Theorem 4.5 i) it follows that $\text{Supp}(\lambda) = \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. By Propositions 4.1 and 4.2 there exists an element $v_1 \in \mathcal{B}(\lambda)$ which is a multiple of ω_1^\vee . Notice that $\alpha_1^\vee \notin \text{Supp}(v)$ for every $v \in \mathcal{B}(\lambda) \setminus \{v_1\}$: since $\langle \alpha_1, \alpha_3^\vee \rangle = -1$ and since $\alpha_3^\vee \in \mathcal{X}(T)^\vee$, it follows then $v_1 = -\omega_1^\vee$. Hence $\mathcal{X}(T)^\vee$ is the coweight lattice and G is adjoint.

Type E₇. If Δ has type E_7 , then Λ^\vee is generated both by $-\omega_2^\vee$ and by $-\omega_7^\vee$: hence by Lemma 4.6 it follows that G is adjoint if $\text{Supp}(\lambda) \cap \{\alpha_2, \alpha_7\} \neq \emptyset$. If this is not the case, then by Theorem 4.5 i) it follows that $\text{Supp}(\lambda) \supset \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$. By Propositions 4.1 and 4.2 there exists an element $v_2 \in \mathcal{B}(\lambda)$ which is a multiple of ω_2^\vee . Notice that $\alpha_2^\vee \notin \text{Supp}(v)$ for every $v \in \mathcal{B}(\lambda) \setminus \{v_2\}$: since $\langle \alpha_2, \alpha_4^\vee \rangle = -1$ and since $\alpha_4^\vee \in \mathcal{X}(T)^\vee$, it follows then $v_2 = -\omega_2^\vee$. Hence $\mathcal{X}(T)^\vee$ is the coweight lattice and G is adjoint. \square

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E-mail address: `gandini@mat.uniroma1.it`

Current address: DIPARTIMENTO DI MATEMATICA “GUIDO CASTELNUOVO”, “SAPIENZA” UNIVERSITÀ DI ROMA, PIAZZALE ALDO MORO 5, 00185 ROMA, ITALY

E-mail address: `Alessandro.Ruzzi@math.univ-bpclermont.fr`

Current address: UNIVERSITÉ BLAISE PASCAL - LABORATOIRE DE MATHÉMATIQUES, UMR 6620 - CNRS CAMPUS DES CÉZEAUX, 63171 AUBIÈRE CEDEX, FRANCE