SPHERICAL NILPOTENT ORBITS AND ABELIAN SUBALGEBRAS IN ISOTROPY REPRESENTATIONS

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Abstract. Let G be a simply connected semisimple algebraic group with Lie algebra $\mathfrak{g}$, let $G_0 \subset G$ be the symmetric subgroup defined by an algebraic involution σ and let $\mathfrak{g}_1 \subset \mathfrak{g}$ be the isotropy representation of G_0. Given an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_1$ and stable under the action of some Borel subgroup $B_0 \subset G_0$, we classify the B_0-orbits in $\mathfrak{a}$ and we characterize the sphericity of $G_0\mathfrak{a}$. Our main tool is the combinatorics of σ-minuscule elements in the affine Weyl group of $\mathfrak{g}$ and that of strongly orthogonal roots in Hermitian symmetric spaces.

1. Introduction

Let G be a connected simply connected semisimple complex algebraic group with Lie algebra $\mathfrak{g}$. Let B be a Borel subgroup, and set $\mathfrak{b} = \mathrm{Lie}B$. Recall that a G-variety X is called *G-spherical* if it possesses an open B-orbit. The relationships between spherical nilpotent orbits and abelian ideals of $\mathfrak{b}$ have been first investigated in [22]. There it is shown that if $\mathfrak{a}$ is an abelian ideal of $\mathfrak{b}$, then any nilpotent orbit meeting $\mathfrak{a}$ is a G-spherical variety and $G\mathfrak{a}$ is the closure a spherical nilpotent orbit. In particular, B acts on $\mathfrak{a}$ with finitely many orbits.

Subsequently, Panyushev [20] dealt with similar questions in the $\mathbb{Z}_2$-graded case. Let σ be an involution of G and $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be the corresponding eigenspace decomposition at the Lie algebra level. Let G_0 be the connected subgroup of G corresponding to $\mathfrak{g}_0$ and $B_0 \subset G_0$ a Borel subgroup of G_0. The "graded" analog of the set of abelian ideals of $\mathfrak{b}$ is the set $\mathcal{I}_{ab}^{\sigma}$ of (abelian) B_0-stable subalgebras of $\mathfrak{g}_1$.

Definition 1.1. We say that $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$ is *G-spherical* (resp. *G_0-spherical*) if all orbits Gx, $x \in \mathfrak{a}$ are G-spherical (resp. if all orbits G_0x, $x \in \mathfrak{a}$ are G_0-spherical).

Panyushev [18] started the classification of the spherical nilpotent G_0-orbits in $\mathfrak{g}_1$. The classification of the spherical nilpotent G_0-orbits in $\mathfrak{g}_1$ was then completed by King [12] (see also [2], where the classification is reviewed and a missing case is pointed out). Shortly afterwards, Panyushev [20] noticed the emergence of non-spherical subalgebras $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$, and classified the involutions σ for which these subalgebras exist. After explicit verifications, he also noticed that an element $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$ is G-spherical if and only if it is G_0-spherical, but no verification was given, and no conceptual proof was known.

The purpose of the present paper is to deepen and expand the results quoted above in the following directions. Let $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$.

i) We clarify the connections between G_0-orbits of nilpotent elements in $\mathfrak{g}_1$, spherical G-orbits of nilpotent elements in $\mathfrak{g}_1$ and G_0-orbits of abelian subalgebras in $\mathfrak{g}_1$ which are stable under some Borel subalgebra of $\mathfrak{g}_0$.

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ii) We prove that B_0 acts on $\mathfrak{a}$ with finitely many orbits, independently of its sphericity. Moreover, we parametrize orbits via orthogonal set of weights of $\mathfrak{a}$.
iii) Assume that there exist non-spherical subalgebras. We give a construction of a canonical non-spherical subalgebra $\mathfrak{a}_p$.
iv) We give a simple criterion to decide whether $\mathfrak{a}$ is spherical or not: in Theorem 6.7 we show that there exists $\overline{\mathfrak{a}} \in \mathcal{I}^\sigma_{ab}$ such that $\mathfrak{a}$ is non-spherical if and only if $\mathfrak{a} \supset \overline{\mathfrak{a}}$.

One important feature of our approach lies in the methods used. The theory of abelian ideals and its graded version rely on a strict relationship with the geometry of alcoves of the affine Weyl group $\widehat{W}$ of $\mathfrak{g}$ [13], [4], and, for the graded case, with Kac's classification of finite order automorphisms of semisimple Lie algebras [11], [6], [7].

The main link is that a B_0-stable subalgebra $\mathfrak{a}$ can be encoded by an element $w_\mathfrak{a} \in \widehat{W}$ defined through its set of inversions $N(w_\mathfrak{a})$ (cf. (2.1)). The elements so obtained, called σ-minuscule (Definition 2.4), pave a convex polytope in the dual space of a Cartan subalgebra of $\mathfrak{g}$ and have remarkable properties: see Section 2.3 for a recollection of these facts. It has been explicitly asked (e.g., in [20]) to use the above connections as a tool for dealing with problems about sphericity. This is what we do here.

We start discussing items i)-iv) by making the content of i) more precise. Define the *height* of a nilpotent element $x \in \mathfrak{g}$ as

$$\mathrm{ht}(x) = \max\{n \in \mathbb{N} \mid \mathrm{ad}(x)^n \neq 0\}.$$

In the adjoint case, Panyushev [18] completely characterized the spherical nilpotent G-orbits in $\mathfrak{g}$ by showing that, for $x \in \mathfrak{g}$, the orbit Gx is spherical if and only if $\mathrm{ht}(x) \leqslant 3$. Subsequently, Panyushev and Röhrle [22] proved that, if $\mathfrak{a} \subset \mathfrak{b}$ is an abelian ideal, then the saturation $G\mathfrak{a}$ is spherical. On the other hand, if Gx is spherical, by chosing $\mathfrak{b}$ properly it is always possible to construct an abelian ideal $\mathfrak{a} \subset \mathfrak{b}$ such that $G\mathfrak{a} = \overline{Gx}$. Therefore we may regard both these properties as consequences of the small height of the nilpotent element x.

For $i = 0, 1$ define the i-height of a nilpotent element $x \in \mathfrak{g}_1$ as

$$\mathrm{ht}_i(x) = \max\{n \in \mathbb{N} \mid \mathrm{ad}(x)^n_{|\mathfrak{g}_i} \neq 0\}.$$

In [18] Panyushev showed that, for $x \in \mathfrak{g}_1$, the following implications hold

$$\mathrm{ht}(x) \leqslant 3 \Longrightarrow G_0 x \text{ spherical} \Longrightarrow \mathrm{ht}_0(x) \leqslant 4, \mathrm{ht}_1(x) \leqslant 3.$$

In Corollary 6.3, we show the following result.

Theorem. *If* $\mathfrak{a} \in \mathcal{I}^\sigma_{ab}$ *and* $x \in \mathfrak{a}$, *then* $\mathrm{ht}_0(x) \leqslant 3$ *and* $\mathrm{ht}_1(x) \leqslant 4$.

These properties completely characterize the elements of abelian subalgebras of $\mathfrak{g}_1$ which are stable under some Borel subgroup of G_0 (see Section 5.1). As a corollary, using Panyushev's criterion for the G_0-sphericity of a nilpotent element in $\mathfrak{g}_1$, it follows that $\mathfrak{a} \in \mathcal{I}^\sigma_{ab}$ is G_0-spherical if and only if it is G-spherical.

Regarding ii), a well known result independently due to Brion [3] and Vinberg [26], states that every spherical G-variety contains finitely many B-orbits. In particular, every abelian ideal $\mathfrak{a}$ of $\mathfrak{b}$ contains finitely many B-orbits. In [21], the same result has been proved avoiding the use of the sphericity of $G\mathfrak{a}$. In Section 3, we prove, along the same lines, the finiteness theorem quoted in ii), in the more general context of finite order automorphisms of G (see Theorem 3.1).

To streamline our approach to iii), recall that involutions of $\mathfrak{g}$ are encoded by the datum of one or two simple roots (with suitable features) in the extended Dynkin diagram $\widehat{\Pi}$ of $\mathfrak{g}$. The main result of [20] has been rephrased by Panyushev as follows: there exists a non spherical $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$ if and only if if σ is defined by a single simple root α_p, which is long and non-complex (see Definition 2.3). (As usual, if $\widehat{\Pi}$ is simply laced, every root is regarded as long). However this claim was obtained as a by-product of direct considerations on various classes of involutions and by constructing case-by-case a non-spherical element $\mathfrak{a} \in \mathcal{I}_{ab}^{\sigma}$ for all involution satisfying the previous condition.

In this paper we observe that, precisely when α_p is long and non-complex, there exists a *special* element $\mathfrak{a}_p \in \mathcal{I}_{ab}^{\sigma}$, which plays a role in the classification of maximal elements in $\mathcal{I}_{ab}^{\sigma}$ performed in [7]. In Section 5 we study the properties of $\mathfrak{a}_p$. In particular, using the combinatorics of $N(w_{\mathfrak{a}_p})$, we prove that $\mathfrak{a}_p$ is not G_0-spherical. The method is combinatorial: we associate to any orthogonal set of maximal cardinality in $N(w_{\mathfrak{a}_p}) \setminus \{\alpha_p\}$ a generalized Cartan matrix of affine type, whose type is severely restricted (see Proposition 5.3). The information we obtain from this Cartan matrix allows us to build up a generic element $x \in \mathfrak{a}_p$ with $\mathrm{ht}_1(x) = 4$, proving that $\mathfrak{a}_p$ is not G_0-spherical.

The same strategy is applied in a wider context in Section 6, and it enables us to classify the spherical elements of $\mathcal{I}_{ab}^{\sigma}$, as outlined in iv). The construction of the minimal non-spherical subalgebra $\overline{\mathfrak{a}}$ is based on the combinatorics of strongly orthogonal roots in Hermitian symmetric spaces. Many related technical results might be of independent interest, and they are displayed in Section 4. To construct $\overline{\mathfrak{a}}$, decompose $\widehat{\Pi} \setminus \{\alpha_p\}$ into a disjoint union of connected components Σ. Then in each Σ there exists a unique simple root α_Σ non-orthogonal to α_p, and it turns out that α_Σ determines an Hermitian involution of tube type of the Lie algebra $\mathfrak{g}_\Sigma$ having Σ as set of simple roots (see Subsection 4.4 and Proposition 5.7). If $\Phi(\Sigma)_1^+$ denotes the set of positive roots of $\mathfrak{g}_\Sigma$ having α_Σ in their support, in Lemma 4.12 we prove that there exists a unique subset $\mathcal{A}_\Sigma$ which is an antichain in $\Phi(\Sigma)_1^+$ w.r.t. the dominance order $\leqslant_\Sigma$ defined by Σ and which is a maximal orthogonal subset of $\Phi(\Sigma)_1^+$. Next, we prove that $\bigcup_\Sigma \bigcup\limits_{\eta \in \mathcal{A}_\Sigma} \{\xi + \alpha_p \mid \xi \leqslant_\Sigma \eta\}$ is the set of inversions of a σ-minuscule element, hence it determines an element $\overline{\mathfrak{a}} \in \mathcal{I}_{ab}^{\sigma}$, which turns out to have the property described in iv).

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2. Setup

Let G be a semisimple, connected and simply connected complex algebraic group with Lie algebra $\mathfrak{g}$, and let $B \subset G$ be a Borel subgroup with Lie algebra $\mathfrak{b}$. Throughout the paper, $\sigma : G \longrightarrow G$ will be an indecomposable automorphism of finite order m. Then σ induces an automorphism of $\mathfrak{g}$ as well, still denoted by σ. Fix a primitive m^{th}-root of unity ζ and consider the corresponding $\mathbb{Z}_m$-grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}_m} \mathfrak{g}_i,$$

where $\mathfrak{g}_i$ denotes the eigenspace of σ of weight ζ^i. Then $\mathfrak{g}_0$ is a reductive subalgebra of $\mathfrak{g}$ (see [11, Lemma 8.1]), and the connected reductive subgroup $G_0 \subset G$ defined by $\mathfrak{g}_0$ coincides with the set of fixed points of σ. Fix a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$, which is abelian since $\mathfrak{g}_0$ is reductive. If $\mathfrak{a} \subset \mathfrak{g}$ is a $\mathfrak{h}_0$-stable subspace, we let $\Psi(\mathfrak{a})$ denote its set of $\mathfrak{h}_0$-weights and, for $\lambda \in \Psi(\mathfrak{a})$, we let $\mathfrak{a}^\lambda$ be the corresponding weight space.

Every eigenspace $\mathfrak{g}_i$ is a G_0-module under the restriction of the adjoint action. If $i \in \mathbb{Z}_m$, we denote by Φ_i the set of the non-zero $\mathfrak{h}_0$-weights in $\mathfrak{g}_i$. Denote finally $\Phi = \cup_i \Phi_i$ the set of non-zero weights. We say $\mu, \nu \in \Phi$ are *strongly orthogonal* if $(\mu, \nu) = 0$ and $\mu \pm \nu \notin \Phi$. (This definition agrees with the usual notion of strongly orthogonal roots in a semisimple Lie algebra).

Observe that Φ_0 is the set of $\mathfrak{h}_0$-roots for $\mathfrak{g}_0$. As shown in [11, Chapter 8], $\mathfrak{h}_0$ contains a regular element h_{reg} of $\mathfrak{g}$. In particular the centralizer $\mathrm{Cent}(\mathfrak{h}_0)$ of $\mathfrak{h}_0$ in $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{g}$ and h_{reg} defines a set of positive roots in the set of roots of $(\mathfrak{g}, \mathrm{Cent}(\mathfrak{h}_0))$ and a set Φ_0^+ of positive roots in Φ_0. We let Π_0 be the corresponding set of simple roots, $\mathfrak{b}_0$ the corresponding Borel subalgebra, and $B_0 \subset G_0$ the corresponding Borel subgroup.

2.1. **The grading associated to a nilpotent element $x \in \mathfrak{g}_1$.** We fix in this subsection notation concerning the grading of $\mathfrak{g}$ associated to nilpotent elements in $\mathfrak{g}_1$ and the correspoding notion of height. The main references for this subsection are [25] and [18]. By [25], an element $x \in \mathfrak{g}_1$ is semisimple if and only if G_0x is closed, whereas it is nilpotent if and only if $0 \in \overline{G_0x}$.

Let $x \in \mathfrak{g}_1$ be a nilpotent element. By a modification of the Jacobson-Morozov theorem, there exists a $\mathfrak{sl}(2)$-triple (x, h, y) with $y \in \mathfrak{g}_1$ and $h \in \mathfrak{g}_0$. Such triples are usually called *normal triples*, or *adapted triples*. Let

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$$

be the $\mathbb{Z}$-grading defined by h; then we get a bigrading of $\mathfrak{g}$ by setting

$$\mathfrak{g}_j(i) = \mathfrak{g}_j \cap \mathfrak{g}(i).$$

Since all normal triples containing x are conjugated by the stabilizer of x in G_0 (see [25, Theorem 1]), it follows that the structure of this bigrading does not depend on the choice of the normal triple.

Following Panyushev [18], define the *height* of x as

$$\mathrm{ht}(x) = \max\{n \in \mathbb{N} \mid \mathfrak{g}(n) \neq 0\}.$$

Since $[x, \mathfrak{g}(i)] = \mathfrak{g}(i+2)$, this notion agrees with the height defined in the Introduction, namely the maximum n such that $\mathrm{ad}(x)^n \neq 0$.

2.2. **Twisted loop algebra and finite order automorphisms.** Since σ fixes h_{reg}, we see that the action of σ on the positive roots defines, once Chevalley generators are fixed, a diagram automorphism η of $\mathfrak{g}$ that, clearly, fixes $\mathfrak{h}_0$. Set, using the notation of [11], $\widehat{\mathfrak{h}} = \mathfrak{h}_0 \oplus \mathbb{C}K \oplus \mathbb{C}d$. Recall that d is the element of

$$\widehat{L}(\mathfrak{g}, \sigma) = (\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}d$$

acting on $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$ as $t\frac{d}{dt}$, while K is a central element. Define $\delta' \in \widehat{\mathfrak{h}}^*$ by setting $\delta'(d) = 1$ and $\delta'(\mathfrak{h}_0) = \delta'(K) = 0$ and let $\lambda \mapsto \overline{\lambda}$ be the restriction map $\widehat{\mathfrak{h}} \to \mathfrak{h}_0$. There is a unique extension, still denoted by $(\cdot, \cdot)$, of the Killing form of $\mathfrak{g}$ to a nondegenerate symmetric bilinear invariant form on $\widehat{L}(\mathfrak{g}, \sigma)$. Let $\nu : \widehat{\mathfrak{h}} \to \widehat{\mathfrak{h}}^*$ be the isomorphism induced by the form $(\cdot, \cdot)$, and denote again by $(\cdot, \cdot)$ the form induced on $\widehat{\mathfrak{h}}^*$. One has $(\delta', \delta') = (\delta', \mathfrak{h}_0^*) = 0$.

We let $\widehat{\Phi}$ be the set of $\widehat{\mathfrak{h}}$-roots of $\widehat{L}(\mathfrak{g}, \sigma)$. We can choose as set of positive roots $\widehat{\Phi}^+ = \Phi_0^+ \cup \{\alpha \in \widehat{\Phi} \mid \alpha(d) > 0\}$. We let $\widehat{\Pi} = \{\alpha_0, \dots, \alpha_n\}$ be the corresponding set of simple roots. It is known that n is the rank of $\mathfrak{g}_0$. Recall that any $\widehat{L}(\mathfrak{g}, \sigma)$ is a Kac-Moody Lie algebra $\mathfrak{g}(A)$ defined by generators and relations starting from a generalized Cartan matrix A of affine type. These matrices are classified by means of Dynkin diagrams listed in [11].

Let $\widehat{W}$ be the Weyl group of $\widehat{L}(\mathfrak{g}, \sigma)$ and let $\widehat{\Phi}_{re} = \widehat{W}\widehat{\Pi}$ be the set of real roots of $\widehat{L}(\mathfrak{g}, \sigma)$. Recall that if $\beta = w(\alpha)$, $\alpha \in \widehat{\Pi}$, one defines $\beta^\vee = w(\alpha^\vee)$.

If $\gamma \in \mathfrak{h}_0^*$, we set

$$h_\gamma = \nu^{-1}(\gamma), \qquad \gamma^\vee = \frac{2h_\gamma}{(\gamma, \gamma)} \quad (\gamma \neq 0).$$

By [11, 5.1] if $\lambda\in\mathbb{C}\delta'+\mathfrak{h}_0^*$ and $\beta\in\widehat{\Phi}_{re}$, then

$$\lambda(\beta^\vee)=2\frac{(\lambda,\beta)}{(\beta,\beta)}=2\frac{(\lambda,\overline{\beta})}{(\overline{\beta},\overline{\beta})}=\lambda(\overline{\beta}^\vee)=2\frac{(\overline{\lambda},\overline{\beta})}{(\overline{\beta},\overline{\beta})}=\overline{\lambda}(\overline{\beta}^\vee).$$

If $\lambda,\mu\in\mathbb{C}\delta'+\mathfrak{h}_0^*$ and $(\mu,\mu)\neq 0$, we set

$$\langle\lambda,\mu^\vee\rangle=2\frac{(\lambda,\mu)}{(\mu,\mu)}.$$

In particular, if $\alpha\in\widehat{\Phi}$ and $\beta\in\widehat{\Phi}_{re}$,

$$\langle\alpha,\beta^\vee\rangle=\alpha(\beta^\vee)=\langle\overline{\alpha},\overline{\beta}^\vee\rangle=\overline{\alpha}(\overline{\beta}^\vee).$$

We will use these equalities many times without comment.

Following [11, Chapter 8], we can assume that σ is the automorphism of type $(\eta;s_0,\dots,s_n)$, where η is the diagram automorphism defined above. Recall that, if $a_0,\dots,a_n$ are the labels of the Dynkin diagram of $\widehat{L}(\mathfrak{g},\sigma)$ and k is the order of η, then $k(\sum_{i=0}^n s_ia_i)=m$. Recall also that $s_0,\dots,s_n$ are relatively prime so, in the case of involutions ($m=2$), we must have that $s_i\in\{0,1\}$ and $s_i=0$ for all but at most two indices.

Since σ is the automorphism of type $(\eta;s_0,\dots,s_n)$, we can write $\alpha_i=s_i\delta'+\overline{\alpha_i}$ and it turns out that $\Pi_0=\{\alpha_i\mid s_i=0\}$. Set also $\Pi_1=\widehat{\Pi}\setminus\Pi_0$. Introduce $\delta=\sum_{i=0}^n a_i\alpha_i$ and note that $\delta=(\sum_{i=0}^n a_is_i)\delta'=\frac{m}{k}\delta'$.

Given $\lambda\in\widehat{\Phi}$ we denote by $\widehat{L}(\mathfrak{g},\sigma)_\lambda$ the corresponding root space in $\widehat{L}(\mathfrak{g},\sigma)$. Recall the following properties (see [11, Exercise 8.2]).

Proposition 2.1. *Let $\lambda\in\widehat{\Phi}_{re}$, then the following holds:*

- i) $\dim\widehat{L}(\mathfrak{g},\sigma)_\lambda=1$.
- ii) *If $\mu\in\widehat{\Phi}$, then the set of $\mu+i\lambda\in\widehat{\Phi}\cup\{0\}$ is a string $\mu-p\lambda\dots,\mu+q\lambda$, where p,q are non-negative integers such that $p-q=\langle\mu,\lambda^\vee\rangle$.*
- iii) *If $\mu\in\widehat{\Phi}$ and $\mu+\lambda\in\widehat{\Phi}$, then $[\widehat{L}(\mathfrak{g},\sigma)_\lambda,\widehat{L}(\mathfrak{g},\sigma)_\mu]\neq 0$.*

Notice that, if $\lambda=i\delta'+\alpha\in\widehat{\Phi}_{re}$, then

$$\widehat{L}(\mathfrak{g},\sigma)_\lambda=t^i\otimes\mathfrak{g}_i^\alpha.$$

This implies that we can rephrase the previous proposition in terms of $\mathfrak{h}_0$-weights in $\mathfrak{g}$ as follows.

Corollary 2.2. *Let $\alpha\in\Phi_i$ and $\beta\in\Phi_j$. If $i\equiv j$ mod m, assume also that $\alpha\neq\beta$.*

- i) $\dim\mathfrak{g}_i^\alpha=1$*, and if* $-\alpha\in\Phi_i$ *then* $[\mathfrak{g}_i^\alpha,\mathfrak{g}_i^{-\alpha}]\neq 0$.
- ii) *If $(\alpha,\beta)<0$ then $\alpha+\beta\in\Phi_{i+j}$, and if $(\alpha,\beta)>0$ then $\alpha-\beta\in\Phi_{i-j}$.*
- iii) *If $\alpha+\beta\in\Phi_{i+j}$, then $[\mathfrak{g}_i^\alpha,\mathfrak{g}_j^\beta]\neq 0$.*

In general, Π_0 is disconnected and we write $\Sigma|\Pi_0$ to mean that Σ is a connected component of Π_0. Clearly, the Weyl group W_0 of $\mathfrak{g}_0$ is the direct product of the $W(\Sigma)$, $\Sigma|\Pi_0$. If θ_Σ is the highest root of $\Phi(\Sigma)$, set

$$\begin{aligned}\widehat{\Phi}_0&=\{\alpha+\mathbb{Z}k\delta\mid\alpha\in\Phi_0\}\cup\pm\mathbb{N}k\delta,\\ \widehat{\Pi}_0&=\Pi_0\cup\{k\delta-\theta_\Sigma\mid\Sigma|\Pi_0\},\\ \widehat{\Phi}_0^+&=\Phi_0^+\cup\{\alpha\in\widehat{\Phi}_0\mid\alpha(d)>0\}.\end{aligned}$$

Denote by $\widehat{W_0}$ the Weyl group of $\widehat{\Phi}_0$. If $\alpha\in\widehat{\Phi}$, let $[\alpha:\alpha_i]$ be the coefficient of α_i in the expansion of α in terms of $\widehat{\Pi}$. Set

$$\mathrm{ht}_\sigma(\alpha)=\sum_{i=0}^n s_i[\alpha:\alpha_i]$$

and, for $i \in \mathbb{Z}$,

$$\widehat{\Phi}_i = \{\alpha \in \widehat{\Phi} \mid \mathrm{ht}_\sigma(\alpha) = i\}.$$

Note that if $\alpha \in \widehat{\Phi}_i$, then $\overline{\alpha}$ is a weight of $\mathfrak{g}_i$.

2.3. **B_0-stable subalgebras in $\mathfrak{g}_1$ and σ-minuscule elements.** In this subsection we assume that σ is an (indecomposable) involution. With this assumption, Π_1 has at most two elements. If $\widehat{\Pi}$ is simply laced, the real roots of $\widehat{\Phi}$ are regarded as long.

Definition 2.3. We say that $\eta \in \widehat{\Phi}$ is complex if $\overline{\eta} \in \Phi_0 \cap \Phi_1$.

It is clear that complex roots can occurr only if rk $\mathfrak{g}_0 <$ rk $\mathfrak{g}$. Moreover, if $\mathfrak{g}$ is simple and rk $\mathfrak{g}_0 <$ rk $\mathfrak{g}$, then $\eta \in \widehat{\Phi}$ is complex if and only if it belongs to $\widehat{\Phi}_{re}$ and it is not long (see [7]). The case of $\mathfrak{g}$ semisimple and not simple corresponds to $\mathfrak{g}$ equal to the sum $\mathfrak{k} \oplus \mathfrak{k}$ of two isomorphic simple ideals, σ the flip involution, $\mathfrak{g}_0 = \mathfrak{k}$, and $\mathfrak{g}_1 \simeq \mathfrak{k}$ with $\mathfrak{k}$ acting on itself via the adjoint representation.

For $w \in \widehat{W}$, define its set of inversions

$$N(w) = \{\alpha \in \widehat{\Phi}^+ \mid w^{-1}(\alpha) \in -\widehat{\Phi}^+\}. \tag{2.1}$$

Recall that a finite subset A of positive roots of an affine root system is of the form $N(w)$ for some $w \in \widehat{W}$ if and only if both A and $\widehat{\Phi}^+ \setminus A$ are closed under root addition (see e.g. [5]). We will refer to this property as *biconvexity*.

If α is a real root in $\widehat{\Phi}^+$, we let s_α denote the reflection in α. If α_i is a simple root we set $s_i = s_{\alpha_i}$.

Recall from [6] the following

Definition 2.4. An element $w \in \widehat{W}$ is called σ-minuscule if $N(w) \subset \widehat{\Phi}_1$.

We denote by $\mathcal{W}_\sigma^{ab}$ the set of σ-minuscule elements of $\widehat{W}$, and we regard it as a poset under the weak Bruhat order.

We let $\mathcal{I}_{ab}^\sigma$ be the set of abelian subalgebras of $\mathfrak{g}$ contained in $\mathfrak{g}_1$ that are stable under the action of the Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$ corresponding to Φ_0^+, or equivalently under the action of the Borel subgroup $B_0 \subset G_0$ with Lie algebra $\mathfrak{b}_0$. Inclusion turns $\mathcal{I}_{ab}^\sigma$ into a poset.

Proposition 2.5. [6, Theorem 3.2] *Let $w \in \mathcal{W}_\sigma^{ab}$. Suppose $N(w) = \{\beta_1, \dots, \beta_k\}$. The map $\Theta : \mathcal{W}_\sigma^{ab} \to \mathcal{I}_{ab}^\sigma$ defined by*

$$w \mapsto \bigoplus_{i=1}^{k} \mathfrak{g}_1^{-\overline{\beta}_i}$$

is a poset isomophism.

Assume that $\mathfrak{g}_0$ is semisimple; then there is an index p such that $\Pi_0 = \widehat{\Pi} \setminus \{\alpha_p\}$. Assume furthermore that α_p is non-complex (in particular, $\mathfrak{g}$ is simple). Set $\Pi_{0,\alpha_p} = \Pi_0 \cap \alpha_p^\perp$, $W_{0,\alpha_p} = W(\Pi_{0,\alpha_p})$, and denote by w_{0,α_p} the longest element of W_{0,α_p}. Let w_0 be the longest element of W_0. Set

$$w_p = s_p w_{0,\alpha_p} w_0$$

In [7] it is proved that $w_p \in \mathcal{W}_\sigma^{ab}$ if and only if α_p is long; in such a case, the abelian B_0-stable subalgebra $\mathfrak{a}_p$ corresponding to w_p is maximal.

3. B_0-orbits in B_0-stable subalgebras contained in $\mathfrak{g}_1$

Throughout this section, we will assume that $\sigma : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an (indecomposable) automorphism of order m, and that $\mathfrak{a}$ is a B_0-stable subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_1$. By [19, Proposition 4.9], $\mathfrak{a}$ contains no semisimple element. In particular, $\mathfrak{a} \cap \mathfrak{g}_1^0 = 0$ and $\mathfrak{a}$ is completely determined by its set of weights $\Psi(\mathfrak{a}) \subset \Phi_1$:

$$\mathfrak{a} = \bigoplus_{\alpha \in \Psi(\mathfrak{a})} \mathfrak{g}_1^\alpha.$$

Since $\mathfrak{a}$ is B_0-stable it follows that $G_0\mathfrak{a}$ is closed; since it contains no semisimple element it follows that every element in $\mathfrak{a}$ is nilpotent. Since there are only finitely many nilpotent G_0-orbits in $\mathfrak{g}_1$, it follows that $G_0\mathfrak{a}$ is the closure of such an orbit.

For all $\alpha \in \Phi_i$, fix a non-zero element $x_i^\alpha \in \mathfrak{g}_i^\alpha$. If $v \in \mathfrak{a}$ and $v = \sum_\alpha c_\alpha x_1^\alpha$, then we set $\mathrm{supp}(v) = \{\alpha \in \Psi(\mathfrak{a}) \mid c_\alpha \neq 0\}$. If $\mathcal{S} \subset \Psi(\mathfrak{a})$ we set

$$x_{\mathcal{S}} = \sum_{\alpha \in \mathcal{S}} x_1^\alpha.$$

Theorem 3.1. *Let $\mathfrak{a}$ be a B_0-stable abelian subalgebra in $\mathfrak{g}_1$. For all $x \in \mathfrak{a}$, there is a unique orthogonal subset $\mathcal{S}$ of $\Psi(\mathfrak{a})$ such that $B_0x = B_0x_{\mathcal{S}}$. In particular, B_0 acts on $\mathfrak{a}$ with finitely many orbits, which are parametrized by the orthogonal subsets of $\Psi(\mathfrak{a})$.*

In the special case of the involution $\sigma : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$, $(x, y) \mapsto (y, x)$, $\mathcal{I}_{ab}^\sigma$ is the set of abelian ideals of $\mathfrak{b}$ and, for such an ideal $\mathfrak{a}$, $G\mathfrak{a}$ is always the closure of a spherical nilpotent G-orbit in $\mathfrak{g}$ [22]. Moreover, the closure of a spherical nilpotent orbit in $\mathfrak{g}$ can be realized as $G\mathfrak{a}$ for some abelian ideal $\mathfrak{a}$ (up to choosing the Borel subalgebra $\mathfrak{b}$ in a compatible way). In this case, Panyushev [21] has recently given a new proof of the finiteness of the B-orbits, by giving an explicit parametrization of the B-orbits in $\mathfrak{a}$. Our proof of Theorem 3.1 will follow closely the proof of [21, Theorem 2.2].

By Corollary 2.2, the following properties for a B_0-stable abelian subalgebra $\mathfrak{a} \subset \mathfrak{g}_1$ hold.

- (A1) If $\alpha \in \Psi(\mathfrak{a})$, then $-\alpha \notin \Psi(\mathfrak{a})$.
- (A2) Let $\alpha, \beta \in \Psi(\mathfrak{a})$, then $\alpha + \beta \notin \Phi_2$.
- (A3) Let $\alpha \in \Psi(\mathfrak{a})$ and $\gamma \in \Phi_0^+$ be such that $\alpha + \gamma \in \Phi_1$, then $\alpha + \gamma \in \Psi(\mathfrak{a})$.

Lemma 3.2. *For $\alpha, \beta \in \Psi(\mathfrak{a})$, $\alpha \neq \beta$, the following statements are equivalent:*

- i) *α, β are orthogonal;*
- ii) *$\alpha - \beta \notin \Phi_0$;*
- iii) *$\delta' + \alpha$ and $\delta' + \beta$ are strongly orthogonal in $\widehat{\Phi}$.*

Proof. (1) $\Rightarrow$ (2) Suppose that $(\alpha, \beta) = 0$ and $\alpha - \beta \in \Phi_0$. Then $\delta' + \alpha, \delta' + \beta \in \widehat{\Phi}$ are orthogonal as well, and

$$s_{\delta'+\beta}(\alpha - \beta) = s_{\delta'+\beta}(\delta' + \alpha) - s_{\delta'+\beta}(\delta' + \beta) = 2\delta' + \alpha + \beta.$$

It follows that $2\delta' + \alpha + \beta \in \widehat{\Phi}$, that is $\alpha + \beta \in \Phi_2$, contradicting (A2).

(2) $\Rightarrow$ (1) Suppose that $(\alpha, \beta) \neq 0$. If $(\alpha, \beta) < 0$, then Corollary 2.2 implies $\alpha + \beta \in \Phi_2$, contradicting (A2). Therefore it must be $(\alpha, \beta) > 0$, and again by Corollary 2.2 we get $\alpha - \beta \in \Phi_0$.

Statement (3) is clearly equivalent to the others. $\square$

As in [21], the key step to prove Theorem 3.1 is the following combinatorial lemma, which generalizes [21, Lemma 1.2] to the graded setting.

Lemma 3.3. *Let $\alpha, \beta \in \Psi(\mathfrak{a})$ be orthogonal weights and let $\gamma \in \Phi_0$. If $\alpha + \gamma \in \Psi(\mathfrak{a})$, then $\beta + \gamma \notin \Psi(\mathfrak{a})$*

Proof. Assume that both $\alpha+\gamma$ and $\beta+\gamma$ belong to $\Psi(\mathfrak{a})$. Suppose that $(\alpha,\gamma)<0$: then $(\beta+\gamma,\alpha)<0$ as well, and Corollary 2.2 implies $\alpha+\beta+\gamma\in\Phi_2$, against (A2). Similarly it cannot be $(\beta,\gamma)<0$. Suppose that $(\alpha,\gamma)=(\beta,\gamma)=0$: then $(\alpha+\gamma,\beta+\gamma)>0$, hence Corollary 2.2 ii) implies $\alpha-\beta\in\Phi_0$, which contradicts the fact that α and β are orthogonal by Lemma 3.2. Therefore it must be $(\alpha,\gamma)\geqslant 0$ and $(\beta,\gamma)\geqslant 0$.

On the other hand, again by Lemma 3.2, we have that

$$(\alpha,\beta)=0 \iff \alpha-\beta\notin\Phi_0 \iff (\alpha+\gamma)-(\beta+\gamma)\notin\Phi_0 \iff (\alpha+\gamma,\beta+\gamma)=0.$$

In turn, by the orthogonality of α and β, the last equality implies that either $(\alpha,\gamma)<0$ or $(\beta,\gamma)<0$, which is a contradiction. □

We denote by $\leqslant_0$ the dominance order on $\mathfrak{h}_0^*$: $\lambda\leqslant_0\mu$ if $\mu-\lambda\in\mathbb{N}\Phi_0^+$. If $\mathcal{S}$ is subset of $\Psi(\mathfrak{a})$ we denote by $\min(\mathcal{S})$ the set of the minimal elements of $\mathcal{S}$ w.r.t. $\leqslant_0$, and define two subsets of $\Psi(\mathfrak{a})$ as follows

$$\begin{aligned}\Psi_{\mathcal{S}} &= \{\beta\in\Phi_1 \mid \text{there is } \alpha\in\mathcal{S} \text{ with } \beta-\alpha\in\Phi_0^+\},\\ \mathcal{S}^{\geqslant_0} &= \{\beta\in\Phi_1 \mid \text{there is } \alpha\in\mathcal{S} \text{ with } \alpha\leqslant_0\beta\}.\end{aligned}$$

We also denote by $\mathfrak{a}_{\mathcal{S}}$ the minimal B_0-stable subalgebra of $\mathfrak{a}$ containing the weight space $\mathfrak{g}_1^\alpha$ for all $\alpha\in\mathcal{S}$, namely

$$\mathfrak{a}_{\mathcal{S}}=\bigoplus_{\beta\in\mathcal{S}^{\geqslant_0}}\mathfrak{g}_1^\beta.$$

Lemma 3.4. *Let $\mathfrak{a}\in\mathcal{I}_{ab}^\sigma$. Let $\mathcal{S}$ be an orthogonal subset of $\Psi(\mathfrak{a})$. Let $x\in\mathfrak{a}$ be such that $\mathcal{S}\subset\operatorname{supp}(x)$ and $\mathcal{S}$ is a lower order ideal in $\operatorname{supp}(x)$. Then there is $y\in B_0x$ with the same property such that $\operatorname{supp}(y)\cap\Psi_{\mathcal{S}}=\varnothing$.*

Proof. Set $Z=\{x\in\mathfrak{a}\mid \mathcal{S}\text{ is a lower order ideal in }\operatorname{supp}(x)\}$. If $x\in Z$, we set $\mathcal{T}(x)=\operatorname{supp}(x)\smallsetminus\mathcal{S}$. We prove the claim by induction on $\dim\mathfrak{a}_{\mathcal{T}(x)}$. If $\dim\mathfrak{a}_{\mathcal{T}(x)}=0$, then $\operatorname{supp}(x)=\mathcal{S}$ and there is nothing to prove.

Assume $\dim\mathfrak{a}_{\mathcal{T}(x)}>0$. Set $\mathcal{S}'=\Psi_{\mathcal{S}}$, and for $v\in\mathfrak{a}$, let $\mathcal{S}'(v)=\mathcal{S}'\cap\operatorname{supp}(v)$. We can assume that $\mathcal{S}'(x)\neq\varnothing$, for, otherwise, we can take $y=x$. Then there are $\alpha\in\mathcal{S}$ and $\gamma\in\Phi_0^+$ such that $\alpha+\gamma\in\operatorname{supp}(x)$. By Lemma 3.3, it follows that $\varepsilon+\gamma\notin\Psi(\mathfrak{a})$ for all $\varepsilon\in\mathcal{S}\smallsetminus\{\alpha\}$. If $u_\gamma(\xi)\in B_0$ is the element defined by exponentiating ξx_0^γ ($\xi\in\mathbb{C}$), it follows that

(3.1)

$$u_\gamma(\xi)x_1^\beta=x_1^\beta \text{ if } \beta\in\mathcal{S}\smallsetminus\{\alpha\},\quad u_\gamma(\xi)x_1^\beta=x_1^\beta+\xi[x_0^\gamma,x_1^\beta]+\frac{\xi^2}{2}[x_0^\gamma,[x_0^\gamma,x_1^\beta]]+\dots \text{ otherwise.}$$

Let $\pi:\mathfrak{a}\longrightarrow\bigoplus_{\alpha\in\mathcal{S}}\mathfrak{g}_1^\alpha$ be the projection. We claim that $\pi(u_\gamma(\xi)x)=\pi(x)$ for all $\xi\in\mathbb{C}$. In fact, if $\varepsilon\in\mathcal{S}\smallsetminus\{\alpha\}$ then, by (3.1), $\pi(u_\gamma(\xi)x_1^\varepsilon)=x_1^\varepsilon$. Since $\mathcal{S}$ is strongly orthogonal, $\alpha+k\gamma\notin\mathcal{S}$ so $\pi(u_\gamma(\xi)x_1^\alpha)=x_1^\alpha$. Finally, if $\beta\notin\mathcal{S}$ and $\beta\in\operatorname{supp}(x)$, then, since $\mathcal{S}$ is a lower order ideal in $\operatorname{supp}(x)$, $\beta+k\gamma\notin\mathcal{S}$, so $\pi(u_\gamma(\xi)x_1^\beta)=\pi(x_1^\beta)=0$. Choose $\xi_0\in\mathbb{C}$ such that $\alpha+\gamma\notin\operatorname{supp}(u_\gamma(\xi_0)x)$ and set $x'=u_\gamma(\xi_0)x$. By construction $\mathcal{S}\subset\operatorname{supp}(x')$. We have to prove that $\mathcal{S}$ is a lower order ideal in $\operatorname{supp}(x')$. Take $\eta\in\operatorname{supp}(x')$ such that there exists $\beta\in\mathcal{S}$ with $\eta\leqslant\beta$. We know that $\eta=\zeta+r\gamma$, $\zeta\in\operatorname{supp}(x)$. Since $\zeta\leqslant_0\beta$, we have that $\zeta\in\mathcal{S}$. If $\zeta\neq\alpha$, then, by Lemma 3.3, $\zeta+\gamma$ is not in Φ_1, in particular $\eta\notin\operatorname{supp}(x')$. If $\zeta=\alpha$ then, by construction, either $r>1$ or $\eta=\zeta$. If $r>1$ then $\alpha+\gamma\leqslant_0\alpha+r\gamma\leqslant_0\beta\in\mathcal{S}$ and $\alpha+\gamma\in\operatorname{supp}(x)$, so $\alpha+\gamma\in\mathcal{S}$. We already observed that this is not possible, hence $\eta=\zeta\in\mathcal{S}$.

We now prove that $\mathfrak{a}_{\mathcal{T}(x')}\subset\mathfrak{a}_{\mathcal{T}(x)}$. It suffices to prove that, if $\beta\in\mathcal{T}(x')$ then $\beta\in\Psi(\mathfrak{a}_{\mathcal{T}(x)})$. Write $\beta=\zeta+r\gamma$ with $r\geqslant 0$ and $\zeta\in\operatorname{supp}(x)$. If $\zeta\notin\mathcal{S}$ then we are already done. If $\zeta\in\mathcal{S}$ then, as shown above, we have $\zeta=\alpha$. Note that $r>0$, for, otherwise $\beta \in \mathcal{S}$. Now, if $r > 0$, as $\alpha + \gamma \in \mathcal{T}(x)$ and $\alpha + \gamma \leqslant_0 \beta$, we have that $\beta \in \Psi(\mathfrak{a}_{\mathcal{T}(x)})$.

Since $\alpha + \gamma \notin \mathrm{supp}(x')$, it follows that $\mathfrak{a}_{\mathcal{T}(x')} \subsetneq \mathfrak{a}_{\mathcal{T}(x)}$. By the induction hypothesis, it follows that $B_0 x'$ contains an element y such that $\mathcal{S}$ is a lower order ideal in $\mathrm{supp}(y)$ and $\mathrm{supp}(y) \cap \Psi_{\mathcal{S}} = \varnothing$. Since $x' \in B_0 x$, we have that $y \in B_0 x$. □

Thanks to previous lemmas, we can now reproduce the same argument given in [21, Theorem 2.2] to prove Theorem 3.1.

Proof of Theorem 3.1. We first show that every B_0-orbit in $\mathfrak{a}$ possesses a representative of the form $x_{\mathcal{S}}$, for some orthogonal subset $\mathcal{S} \subset \Psi(\mathfrak{a})$.

Let $v \in \mathfrak{a}$. Set $v_0 = v$ and $\mathcal{S}_0 = \min \mathrm{supp}(v)$. Notice that $\mathcal{S}_0$ is ortohogonal by Lemma 3.2: indeed for all $\alpha, \beta \in \mathcal{S}_0$ we have by construction that $\alpha - \beta \notin \Phi_0$. Therefore v_0 and $\mathcal{S}_0$ satisfy the assumptions of Lemma 3.4, and there is $v_1 \in B_0 v_0$ such that $\mathcal{S}_0 \subset \mathrm{supp}(v_1)$ and $\mathrm{supp}(v_1) \cap \Psi_{\mathcal{S}_0} = \varnothing$. Define $\mathcal{S}_1 = \mathcal{S}_0 \cup \min(\mathrm{supp}(v_1) \smallsetminus \mathcal{S}_0)$: then by construction we still have $\alpha - \beta \notin \Phi_0$ for all $\alpha, \beta \in \mathcal{S}_1$, so that $\mathcal{S}_1$ is again orthogonal by Lemma 3.2. It is clear that $\mathcal{S}_1$ is lower order ideal in $\mathrm{supp}(v_1)$.

More generally, let $i \geqslant 0$ and suppose that v_i and $\mathcal{S}_i$ are defined. Then there is $v_{i+1} \in B_0 v_i = B_0 v_0$ such that $\mathcal{S}_i$ is a lower order ideal in $\mathrm{supp}(v_{i+1})$, $\mathrm{supp}(v_{i+1}) \cap \Psi_{\mathcal{S}_i} = \varnothing$, and

$$\mathcal{S}_{i+1} = \mathcal{S}_i \cup \min(\mathrm{supp}(v_{i+1}) \smallsetminus \mathcal{S}_i)$$

is an orthogonal subset by Lemma 3.2 that is a lower order ideal in $\mathrm{supp}(v_{i+1})$.

Clearly, $\mathcal{S}_{i+1}$ is strictly bigger than $\mathcal{S}_i$, unless $\mathrm{supp}(v_{i+1}) = \mathcal{S}_i$. Therefore, proceeding inductively, we find an element $v_{k+1} \in B_0 v$ whose support equals $\mathcal{S}_k$, which is an orthogonal subset, hence $B_0 v = B_0 v_{k+1} = B_0 x_{\mathcal{S}_k}$.

We now show that every B_0-orbit contains a unique orthogonal representative $x_{\mathcal{S}}$. If $\mathcal{S} \subset \Psi(\mathfrak{a})$, notice that the vector space $\langle B_0 x_{\mathcal{S}} \rangle$ generated the orbit of $x_{\mathcal{S}}$ is B_0-stable, therefore it coincides with $\mathfrak{a}_{\mathcal{S}}$.

Let $\mathcal{S}, \mathcal{S}'$ be orthogonal subsets of $\Psi(\mathfrak{a})$ and suppose that $B_0 x_{\mathcal{S}} = B_0 x_{\mathcal{S}'}$. Then $\mathfrak{a}_{\mathcal{S}} = \mathfrak{a}_{\mathcal{S}'}$, and we set $\Gamma = \min \Psi(\mathfrak{a}_{\mathcal{S}})$. Notice that $\Gamma \subset \mathcal{S} \cap \mathcal{S}'$, therefore setting $\mathcal{R} = \mathcal{S} \smallsetminus \Gamma$ and $\mathcal{R}' = \mathcal{S}' \smallsetminus \Gamma$ we can decompose $x_{\mathcal{S}} = x_\Gamma + x_{\mathcal{R}}$ and $x_{\mathcal{S}'} = x_\Gamma + x_{\mathcal{R}'}$. Let $B_0 = T_0 U_0$ be the Levi decomposition of B_0. Let $b \in B_0$ be such that $b x_{\mathcal{S}} = x_{\mathcal{S}'}$ and write $b = t^{-1} u$ with $t \in T_0$ and $u \in U_0$. Then $u x_\Gamma + u x_{\mathcal{R}} = t x_\Gamma + t x_{\mathcal{R}'}$. It follows that $t x_\Gamma = u x_\Gamma = x_\Gamma$, hence $x_{\mathcal{R}'} \in B_0 x_{\mathcal{R}}$. Since $\mathfrak{a}_{\mathcal{R}} \subset \mathfrak{a}_{\mathcal{S}} \cap \mathfrak{a}_{\mathcal{S}'}$ is a smaller B_0-stable abelian subalgebra in $\mathfrak{g}_1$ and since $\mathcal{R}$ and $\mathcal{R}'$ are orthogonal subsets in $\Psi(\mathfrak{a}_{\mathcal{R}})$, the claim follows proceeding by downward induction. □

The following facts are also proved by adapting the same proofs of [21]. Denote by $\mathcal{S}_{\mathfrak{a}} \subset \Psi(\mathfrak{a})$ the subset constructed as follows: set $\mathcal{S}_1 = \min \Psi(\mathfrak{a})$, and for $i > 1$ define inductively

$$\mathcal{S}_i = \min \big(\Psi(\mathfrak{a}) \smallsetminus \bigcup_{j<i} (\mathcal{S}_j \cup \Psi_{\mathcal{S}_j})\big).$$

Define $\mathcal{S}_{\mathfrak{a}} = \bigcup_{i>0} \mathcal{S}_i$, which is an orthogonal subset thanks to Lemma 3.2.

Proposition 3.5. *Let $\mathcal{S} \subset \Psi(\mathfrak{a})$ be an orthogonal subset.*

i) *$B_0 x_{\mathcal{S}}$ is open in $\mathfrak{a}$ if and only if $\mathcal{S} = \mathcal{S}_{\mathfrak{a}}$.*
ii) *As a T_0-module, the tangent space $\mathrm{T}_{x_{\mathcal{S}}}(B_0 x_{\mathcal{S}})$ decomposes as follows:*

$$\mathrm{T}_{x_{\mathcal{S}}}(B_0 x_{\mathcal{S}}) = \bigoplus_{\alpha \in \mathcal{S} \cup \Psi_{\mathcal{S}}} \mathfrak{g}_1^\alpha.$$

In particular, $\dim B_0 x_{\mathcal{S}} = |\mathcal{S}| + |\Psi_{\mathcal{S}}|$.

4. Antichains of orthogonal roots in Hermitian symmetric spaces

Suppose that $\mathfrak{g}$ is a simple Lie algebra and let Π be a set of simple roots. Let θ be the corresponding highest root and let $\alpha_q \in \Pi$ be a simple root with $[\theta : \alpha_q] = 1$. Let $\mathfrak{p}^+ \subset \mathfrak{g}$ be the maximal parabolic subalgebra associated to the set of simple roots $\Pi \setminus \{\alpha_q\}$. Then its nilradical $\mathfrak{p}_{\mathrm{u}}^+$ is abelian; conversely any standard parabolic subalgebra with abelian nilradical arises in this way.

Let $\mathfrak{p}^+ = \mathfrak{l} \oplus \mathfrak{p}_{\mathrm{u}}^+$ be the Levi decomposition, and let $\mathfrak{p}^-$ be the opposite parabolic subalgebra of $\mathfrak{p}^+$. The decomposition into $\mathfrak{l}$-submodules $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}_{\mathrm{u}}^+ \oplus \mathfrak{p}_{\mathrm{u}}^-$ defines an involution σ of $\mathfrak{g}$ by setting $\sigma(x) = x$ if $x \in \mathfrak{l}$ and $\sigma(x) = -x$ if $x \in \mathfrak{p}_{\mathrm{u}}^+ \oplus \mathfrak{p}_{\mathrm{u}}^-$. It is then clear that

$$\mathfrak{g}_0 = \mathfrak{l}, \quad \mathfrak{g}_1 = \mathfrak{p}_{\mathrm{u}}^+ \oplus \mathfrak{p}_{\mathrm{u}}^-.$$

Recalling the notation introduced in Section 2.2, the sets Φ_i attached to σ are $\Phi_1 = \Phi_1^+ \cup -\Phi_1^+$, where Φ_1^+ is the set of roots β such $[\beta : \alpha_q] = 1$, while Φ_0 is the set of roots β such that $[\beta : \alpha_q] = 0$. Observe that in this case $\Phi = \Phi_0 \cup \Phi_1$ is the set of roots of $\mathfrak{g}$. Since in this case $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}$, we can choose $h_{reg} \in \mathfrak{h}_0$ so that $\alpha(h_{reg}) > 0$ for all $\alpha \in \Pi$. With this choice, letting Φ^+ denote the set of positive roots of $\mathfrak{g}$ corresponding to the choice of Π, we have that

$$\begin{aligned} \Phi_0^+ &= \{\beta \in \Phi^+ \mid [\beta : \alpha_q] = 0\}, \\ \Phi_1^+ &= \{\beta \in \Phi^+ \mid [\beta : \alpha_q] > 0\}, \\ \Pi_0 &= \Pi \setminus \{\alpha_q\}. \end{aligned}$$

We set $\Phi_i^- = -\Phi_i^+$ $(i = 0, 1)$. Clearly, $\Phi_1^\pm$ is the set of weights of $\mathfrak{h}_0$ in $\mathfrak{p}_{\mathrm{u}}^\pm$. We let $\mathfrak{b}_0^\pm$ be the Borel subalgebra of $\mathfrak{g}_0$ corresponding to $\Phi_0^\pm$. Recall that W_0 denotes the Weyl group of $\mathfrak{g}_0$.

Let $\mathrm{Ort}(\Phi_1^+)$ be the collection of the orthogonal subsets of Φ_1^+, and let $\mathrm{Ort}_{\max}(\Phi_1^+)$ be the collection of the orthogonal subsets of Φ_1^+ which are maximal with respect to inclusion. Regard Φ_1^+ as a partial ordered set via $\leqslant_0$. Since $\mathfrak{p}_{\mathrm{u}}^+$ is an abelian subalgebra of $\mathfrak{g}$, by Lemma 3.2 two elements $\alpha, \beta \in \Phi_1^+$ are orthogonal if and only if they are strongly orthogonal, if and only if $\alpha - \beta \notin \Phi_0$. In particular, every antichain $\mathcal{A} \subset \Phi_1^+$ is an orthogonal subset.

Given $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$, let $\mathfrak{a}_{\mathcal{B}} \subset \mathfrak{p}_{\mathrm{u}}^+$ be the B_0-stable subalgebra generated by $\mathcal{B}$. We define a preorder $\vdash$ on $\mathrm{Ort}(\Phi_1^+)$ as follows: if $\mathcal{B}_1, \mathcal{B}_2 \in \mathrm{Ort}(\Phi_1^+)$, then $\mathcal{B}_1 \vdash \mathcal{B}_2$ if $\mathfrak{a}_{\mathcal{B}_1} \subset \mathfrak{a}_{\mathcal{B}_2}$. Equivalently, $\mathcal{B}_1 \vdash \mathcal{B}_2$ if and only if $\mathcal{B}_1 \subset \mathcal{B}_2^{\geqslant_0}$, where, if $\mathcal{B} \subset \Phi_1^+$, we set

$$\mathcal{B}^{\geqslant_0} = \{\alpha \in \Phi_1^+ \mid \text{there is } \beta \in \mathcal{B} \text{ such that } \beta \leqslant_0 \alpha\}.$$

Given $m \leqslant r$, we will denote by $\mathrm{Ort}_m(\Phi_1^+)$ the set of the orthogonal subsets of cardinality m. If Φ is not simply laced, we will say $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$ *is of type* (h, k) if it contains exactly h short roots and k long roots, and we denote by $\mathrm{Ort}_{(h,k)}(\Phi_1^+)$ the set of the orthogonal subsets of type (h, k). To unify some notations, in the simply laced case we will regard every root as a long root. Therefore if Φ is simply laced we have $\mathrm{Ort}_m(\Phi_1^+) = \mathrm{Ort}_{(0,m)}(\Phi_1^+)$.

In this section we will study the antichains of Φ_1^+. We will show that for every orthogonal subset $\mathcal{B} \subset \Phi_1^+$ there is always an antichain $\mathcal{A} \subset \Phi_1^+$ such that $\mathcal{A} \vdash \mathcal{B}$. We first discuss the simply laced case uniformly; the two remaining cases (B_n, α_1) and (C_n, α_n) will be treated separately. We summarize our results in the following theorem.

Theorem 4.1.

i) *Suppose that Φ is simply laced. For all $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$, there is an antichain $\mathcal{A} \subset \Phi_1^+$ such that $|\mathcal{A}| = |\mathcal{B}|$ and $\mathcal{A} \vdash \mathcal{B}$.*

ii) *Suppose that Φ is not simply laced. For all $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$ of type (h,k), there is an antichain $\mathcal{A} \subset \Phi_1^+$ of type $(h+\lfloor k/2 \rfloor, k-2\lfloor k/2 \rfloor)$ such that $\mathcal{A} \vdash \mathcal{B}$.*

Proof. The claim follows combining Proposition 4.6, Proposition 4.7, Proposition 4.10. □

Let $P \subset G$ be the parabolic subgroup corresponding to $\mathfrak{p}^+$. It is well known that G/P is an irreducible simply connected Hermitian symmetric space of compact type, whose corresponding involution of G is σ, and every such a symmetric space arises in this way (see e.g. [23, Section 5.5]). Therefore we will refer to the pair (Π, α_q) as a *Hermitian pair*, and we will say that σ is an *involution of Hermitian type*, or simply a *Hermitian involution.* Correspondingly, we get also a symmetric variety G/G_0, where $G_0 = G^\sigma$ is the Levi factor of P.

Since $\mathfrak{p}_\mathrm{u}^+ \subset \mathfrak{g}_1$ is a B_0-stable abelian subalgebra of $\mathfrak{g}$, by Theorem 3.1 it possesses finitely many B_0-orbits, which are classified by $\mathrm{Ort}(\Phi_1^+)$. In this situation, the description of the B_0-orbits already follows by [21]: since $\mathfrak{p}_\mathrm{u}^+$ is abelian, the unipotent radical P_u acts trivially on its Lie algebra $\mathfrak{p}_\mathrm{u}^+$, therefore every B-orbit is actually a B_0-orbit. The G_0-orbits in $\mathfrak{p}_\mathrm{u}^+$ were studied by Muller, Rubenthaler, and Schiffmann [16] and by Richardson, Röhrle and Steinberg [23]. In the latter reference it is shown that they are parametrized by the W_0-orbits in $\mathrm{Ort}(\Phi_{1,\ell}^+)$, where $\Phi_{1,\ell}^+ \subset \Phi_1^+$ denotes the subset of the long roots.

Let $\mathcal{S}_{\Pi,\alpha_q} \subset \Phi_1^+$ be the orthogonal subset corresponding to the open B_0-orbit of $\mathfrak{p}_\mathrm{u}^+$, constructed recursively as in Proposition 3.5. In this case $\mathcal{S}_{\Pi,\alpha_q}$ is well known, and it coincides with the set of Harish-Chandra strongly orthogonal roots (see [9], [15]). Denote by $r = |\mathcal{S}_{\Pi,\alpha_q}|$ the rank of the symmetric variety G/G_0. By [15, Theorem 2] we have $\mathcal{S}_{\Pi,\alpha_q} \subset \Phi_{1,\ell}^+$, in particular $\mathcal{S}_{\Pi,\alpha_q}$ is a maximal orthogonal subset of Φ_1^+ consisting of long roots, and by [16, Theorem 2.12 and Proposition 2.13] it follows that $\mathcal{S}_{\Pi,\alpha_q}$ is an orthogonal subset of maximal cardinality in Φ_1^+. In particular, $\mathcal{S}_{\Pi,\alpha_q} \in \mathrm{Ort}_{\max}(\Phi_1^+)$, and $|\mathcal{B}| \leqslant r$ for all $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$.

We report in Table 1 the classification of the Hermitian pairs, together with the rank of the corresponding symmetric varieties G/G_0 (where $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is enumerated as in [1]).

(Π, α_q)	$\mathrm{rk}(G/G_0)$
(A_n, α_q) $(1 \leqslant q \leqslant n)$	$\min\{q, n+1-q\}$
(B_n, α_1)	2
(C_n, α_n)	n
(D_n, α_1)	2
(D_n, α_{n-1}), (D_n, α_n)	$\lfloor \frac{n}{2} \rfloor$
(E_6, α_1), (E_6, α_6)	2
(E_7, α_7)	3

Table 1 – Hermitian pairs and ranks of the corresponding symmetric varieties.

Remark 4.2. By [23, Proposition 2.8 and Remark], the Weyl group W_0 acts transitively on $\mathrm{Ort}_{(h,k)}(\Phi_1^+)$ for all h, k. In particular, we see that if Φ is simply laced then $\mathrm{Ort}_{\max}(\Phi_1^+) = \mathrm{Ort}_r(\Phi_1^+)$ coincides with the collection of the orthogonal subsets of maximal cardinality. On the other hand in the non-simply laced cases, corresponding to the Hermitian pairs (B_n, α_1) and (C_n, α_n), we will easily see that if $\mathcal{B}$ is an orthogonal subset of type (h,k), then $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$ if and only if $2h + k = r$. In particular, it follows that, if $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$ has maximal cardinality, then every root in $\mathcal{B}$ is long. Hence the orthogonal subsets of maximal cardinality coincide with the elements of $\mathrm{Ort}_{(0,r)}(\Phi_1^+)$, and W_0 acts transitively on these subsets. As well, it follows that, by choosing properly the Borel subgroup $B_0 \subset G_0$, every subset of orthogonal roots of maximal cardinality can be made into a set of strongly orthogonal Harish-Chandra roots for Φ_1^+.

4.1. **The simply laced case.** We start by recording a well known fact that holds for any root system. See e.g. [24, Lemma 3.2].

Lemma 4.3. *Let $\beta, \beta' \in \Phi^+$ and suppose that $\beta' - \beta$ is a sum of positive roots. Then there are $\gamma_1, \ldots, \gamma_m \in \Phi^+$ such that $\beta' - \beta = \gamma_1 + \ldots + \gamma_m$ and $\beta + \gamma_1 + \ldots + \gamma_i \in \Phi^+$ for all $i \leqslant m$.*

In the simply laced case, Lemma 4.3 can be improved as follows:

Proposition 4.4. *Suppose that Φ is simply laced. Let $\beta, \beta' \in \Phi^+$ and suppose that $\beta' - \beta$ is a sum of positive roots. Then $\beta' - \beta$ is a sum of positive pairwise orthogonal roots.*

Proof. By Lemma 4.3, there are $\gamma_1, \ldots, \gamma_m \in \Phi^+$ such that $\beta' - \beta = \gamma_1 + \ldots + \gamma_m$ and $\beta + \gamma_1 + \ldots + \gamma_i$ is a positive root for all $i \leqslant m$. Let m be minimal with the previous property, fix $\gamma_1, \ldots, \gamma_m \in \Phi^+$ as above and, if $0 \leqslant i \leqslant m$, denote $\beta_i = \beta + \gamma_1 + \ldots + \gamma_i$. We claim that $\gamma_1, \ldots, \gamma_m$ are pairwise orthogonal.

If $m = 1$ there is nothing to prove. Assume $m > 1$, and suppose that $\gamma_1, \ldots, \gamma_m$ are not orthogonal. Let $i_0 \leqslant m$ be the minimum such that $(\gamma_i, \gamma_j) = 0$ for all $i, j < i_0$ with $i \neq j$, and let $j_0 < i_0$ be such that $(\gamma_{j_0}, \gamma_{i_0}) \neq 0$. Since $\langle \beta, \gamma_{i_0}^\vee \rangle \geqslant -1$ and $\langle \gamma_i, \gamma_{i_0}^\vee \rangle \geqslant -1$, we can assume that $\langle \gamma_{j_0}, \gamma_{i_0}^\vee \rangle = -1$, thus $\gamma_{j_0} + \gamma_{i_0}$ is a root. To reach a contradiction, we show that the $m - 1$ positive roots

$$\gamma_1, \ldots, \gamma_{j_0 - 1}, \gamma_{j_0+1}, \ldots, \gamma_{i_0-1}, \gamma_{j_0} + \gamma_{i_0}, \gamma_{i_0+1}, \ldots, \gamma_m$$

also satisfy the assumptions of $\gamma_1, \ldots, \gamma_m$, contradicting the minimality of m. That is, we show that $\beta_i - \gamma_{j_0} \in \Phi^+$ whenever $j_0 < i < i_0$.

Indeed, if $i < i_0$, then $\beta_{i-1} + \gamma_i = \beta_i \in \Phi^+$, therefore $\langle \beta_{i-1}, \gamma_i^\vee \rangle = -1$, and being $(\gamma_i, \gamma_j) = 0$ for all $j < i$ it follows that $\langle \beta, \gamma_i^\vee \rangle = -1$. Therefore, if $j_0 < i < i_0$, then it follows $\langle \beta_i, \gamma_{j_0}^\vee \rangle = \langle \beta, \gamma_{j_0}^\vee \rangle + \langle \gamma_{j_0}, \gamma_{j_0}^\vee \rangle = 1$, and the claim follows. □

Lemma 4.5. *Suppose that Φ is simply laced. Let $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$ and suppose that it is not an antichain, then there exists $\mathcal{B}' \in \mathrm{Ort}(\Phi_1^+)$ with $|\mathcal{B}'| = |\mathcal{B}|$ such that $\mathcal{B}' \vdash \mathcal{B}$, and $\dim \mathfrak{a}_{\mathcal{B}'} < \dim \mathfrak{a}_{\mathcal{B}}$.*

Proof. Notice that W_0 acts on $\mathrm{Ort}(\Phi_1^+)$, we will find $\mathcal{B}'$ in the W_0-orbit of $\mathcal{B}$. Let $\beta \in \mathcal{B}$ be a minimal element and let $\beta' \in \mathcal{B}$ with $\beta < \beta'$. Write $\beta' - \beta = \gamma_1 + \ldots + \gamma_m$ for some pairwise orthogonal roots $\gamma_1, \ldots, \gamma_m \in \Phi^+$ as in Proposition 4.4, and notice that $\gamma_i \in \Phi_0^+$ for all i. Set $\gamma = \gamma_1$. Then $\langle \beta' - \beta, \gamma^\vee \rangle = 2$, and since Φ is simply laced it follows that $s_\gamma(\beta) = \beta + \gamma$ and $s_\gamma(\beta') = \beta' - \gamma$. On the other hand by Lemma 3.3 γ is orthogonal to every root in $\mathcal{B} \smallsetminus \{\beta, \beta'\}$, therefore

$$s_\gamma(\mathcal{B}) = (\mathcal{B} \smallsetminus \{\beta, \beta'\}) \cup \{s_\gamma(\beta), s_\gamma(\beta')\}.$$

Being $\gamma < \beta' - \beta$, we have $\beta < s_\gamma(\beta)$ and $\beta < s_\gamma(\beta')$. Hence $s_\gamma(\mathcal{B}) \vdash \mathcal{B}$, and since β is minimal in $\mathcal{B}$ we get $\beta \notin \Psi(\mathfrak{a}_{\mathcal{B}'})$. □

Proposition 4.6. *Suppose that Φ is simply laced. Let $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$, then there is an antichain $\mathcal{A} \in \mathrm{Ort}(\Phi_1^+)$ with $|\mathcal{A}| = |\mathcal{B}|$ such that $\mathcal{A} \vdash \mathcal{B}$.*

Proof. Notice that W_0 acts on the orthogonal subsets of cardinality $m = |\mathcal{B}|$. Suppose that $\mathcal{B}$ is not an antichain, then by Lemma 4.5 there is $\mathcal{B}_1 \in \mathrm{Ort}_{(m,0)}(\Phi_1^+)$ such that $\mathcal{B}_1 \vdash \mathcal{B}$ and $\dim \mathfrak{a}_{\mathcal{B}_1} < \dim \mathfrak{a}_{\mathcal{B}}$. Let $i \geqslant 1$ and suppose that $\mathcal{B}_1, \ldots, \mathcal{B}_i \in \mathrm{Ort}_{(m,0)}(\Phi_1^+)$ are such that $\mathcal{B}_i \vdash \ldots \vdash \mathcal{B}_1 \vdash \mathcal{B}$ and $\dim \mathfrak{a}_{\mathcal{B}_i} < \dim \mathfrak{a}_{\mathcal{B}_{i-1}} < \ldots <$

$\dim \mathfrak{a}_{\mathcal{B}}$. If $\mathcal{B}_i$ is not an antichain, then we can apply Lemma 4.5 again, and we find $\mathcal{B}_{i+1} \in \mathrm{Ort}_{(m,0)}(\Phi_1^+)$ such that $\mathcal{B}_{i+1} \vdash \mathcal{B}_i$ and $\dim \mathfrak{a}_{\mathcal{B}_{i+1}} < \dim \mathfrak{a}_{\mathcal{B}_i}$. Since $\mathcal{B}_{i+1}$ is not empty it must be $\dim \mathfrak{a}_{\mathcal{B}_{i+1}} > 0$, therefore the process must stop for some k, and $\mathcal{B}_k$ is an antichain. □

4.2. **The odd orthogonal case.** Consider the Hermitian pair (B_n, α_1). We enumerate the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ as in [1]. Given i, j such that $1 \leqslant i \leqslant n$ and $1 \leqslant j < n$ we set

$$\beta_i = \alpha_1 + \ldots + \alpha_i,$$
$$\beta'_j = \alpha_1 + \ldots + \alpha_j + 2\alpha_{j+1} + \ldots + 2\alpha_n.$$

Then $\Phi_1^+ = \{\beta_i \,|\, 1 \leqslant i \leqslant n\} \cup \{\beta'_j \,|\, 1 \leqslant j < n\}$. Notice that Φ_1^+ contains a unique short root, namely β_n.

In this case $\mathrm{Ort}_{\max}(\Phi_1^+) = \{\mathcal{B}_1, \dots, \mathcal{B}_n\}$, where we set $\mathcal{B}_i = \{\beta_i, \beta'_i\}$ for all $i < n$, and $\mathcal{B}_n = \{\beta_n\}$. In particular, the only possible types for an orthogonal subset are $(0,2)$, $(1,0)$ and $(0,1)$. Moreover $\mathcal{B}_n \vdash \mathcal{B}_{n-1} \vdash \ldots \vdash \mathcal{B}_1$, and $\mathcal{B}_n$ is the unique antichain in $\mathrm{Ort}_{\max}(\Phi_1^+)$. In particular, the following proposition trivially holds.

Proposition 4.7. *Consider the Hermitian pair (B_n, α_1), and let $\Phi^+ = \Phi_0^+ \cup \Phi_1^+$ be the corresponding decomposition.*

i) *Let $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$ of type (h,k), then there is an antichain $\mathcal{A} \subset \Phi_1^+$ of type $(h + \lfloor \frac{k}{2} \rfloor, k - 2\lfloor \frac{k}{2} \rfloor)$ such that $\mathcal{A} \vdash \mathcal{B}$.*

ii) *There exists a unique antichain $\mathcal{A}_* \in \mathrm{Ort}_{\max}(\Phi_1^+)$, and $\mathcal{A}_* \vdash \mathcal{B}$ for all $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$.*

4.3. **The symplectic case.** Consider the Hermitian pair (C_n, α_n). We enumerate the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_n\}$ as in [1], and we embed Φ into the euclidean vector space $\mathbb{R}^n$ with orthonormal basis $\varepsilon_1, \dots, \varepsilon_n$ by setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for all $i < n$ and $\alpha_n = 2\varepsilon_n$. Then

$$\Phi = \{\pm(\varepsilon_i \pm \varepsilon_j) \,|\, 1 \leqslant i, j \leqslant n\} \smallsetminus \{0\},$$

and $\Phi_1^+ = \{\varepsilon_i + \varepsilon_j \,|\, 1 \leqslant i \leqslant j \leqslant n\}$. Notice that, for $1 \leqslant i \leqslant j \leqslant n$, we have

$$\varepsilon_i + \varepsilon_j = \alpha_i + \ldots + \alpha_{j-1} + 2\alpha_j + \ldots + 2\alpha_{n-1} + \alpha_n.$$

In particular $\varepsilon_i + \varepsilon_j \leqslant \varepsilon_h + \varepsilon_k$ if and only if $h \leqslant i$ and $k \leqslant j$. Notice that $\mathcal{S}_{C_n,\alpha_1} = \{2\varepsilon_1, \dots, 2\varepsilon_n\}$, so that $r = n$.

Let $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$, and write $\mathcal{B} = \{\varepsilon_{i_1} + \varepsilon_{j_1}, \dots, \varepsilon_{i_m} + \varepsilon_{j_m}\}$ for some indices $i_1 \leqslant j_1, \dots, i_m \leqslant j_m$. Correspondingly, we have a disjoint union $\bigcup_{k=1}^m \{i_k, j_k\}$, and $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$ if and only if $\{1, \dots, n\} = \bigcup_{k=1}^m \{i_k, j_k\}$, and it immediately follows that, if $\mathcal{B}$ has type (h,k), then $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$ if and only if $2h + k = n$. Notice moreover that $\mathcal{B}$ is an antichain if and only if, up to some permutation of $\{1, \dots, m\}$, we have

$$i_1 < i_2 < \ldots < i_{m-1} < i_m \leqslant j_m < j_{m-1} < \ldots < j_2 < j_1.$$

It follows that there is a unique $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$ which satisfies the previous inequalities, therefore there is a unique antichain $\mathcal{A}_* \in \mathrm{Ort}_{\max}(\Phi_1^+)$.

The following lemma is an easy consequence of previous description of Φ_1^+.

Lemma 4.8. *Let $\beta, \beta' \in \Phi_1^+$ be orthogonal roots.*

i) *Suppose that $\{\beta, \beta'\}$ is of type $(1,1)$ and suppose that $\beta < \beta'$. Then $\beta' - \beta = 2\alpha + \alpha'$ for some short roots $\alpha, \alpha' \in \Phi_0^+$ with $\langle \alpha', \alpha^\vee \rangle = -1$.*

ii) *Suppose that $\{\beta, \beta'\}$ is of type $(2,0)$, and suppose that $\beta < \beta'$. Then $\beta' - \beta = \alpha + \alpha'$ for some orthogonal short roots $\alpha, \alpha' \in \Phi_0^+$.*

iii) *Suppose that $\{\beta, \beta'\}$ is of type $(0,2)$, then $\beta - \beta' = 2\alpha$ for some short root $\alpha \in \Phi_0$.*

Proof. Assume $\beta = \varepsilon_i + \varepsilon_j$ and $\beta' = \varepsilon_h + \varepsilon_k$, for some $i \leqslant j$ and $h \leqslant k$. The orthogonality implies that $i \neq h$ and $j \neq k$.

i) We have in this case $h \leqslant k < i \leqslant j$, and since $\{\beta, \beta'\}$ contains exactly one long root, either $h = k < i < j$ or $h < k < i = j$. Therefore the claim follows by setting $\alpha = \varepsilon_k - \varepsilon_i$ and $\alpha' = \varepsilon_h - \varepsilon_k + \varepsilon_i - \varepsilon_j$.

ii) We have in this case $i < j$ and $h < k$, and the claim follows by setting $\alpha = \varepsilon_i - \varepsilon_j = \alpha_k + \ldots + \alpha_{j-1}$ and $\alpha' = \varepsilon_h - \varepsilon_k = \alpha_h + \ldots + \alpha_{k-1}$.

iii) We have in this case $i = j$ and $h = k$, and the claim follows by setting $\alpha = \varepsilon_h - \varepsilon_i$. □

Lemma 4.9. *Let $\mathcal{B} \subset \Phi_1^+$ be an orthogonal subset of type (h, k), set $k' = \lfloor \frac{k}{2} \rfloor$ and suppose that $\mathcal{B}$ is not an antichain. Then there exists an orthogonal subset $\mathcal{B}'$ of type $(h + k', k - 2k')$ such that $\mathcal{B}' \vdash \mathcal{B}$, and $\dim \mathfrak{a}_{\mathcal{B}'} < \dim \mathfrak{a}_{\mathcal{B}}$.*

Proof. Let $\beta \in \mathcal{B}$ be a minimal element, and suppose that $\beta < \beta'$ for some $\beta' \in \mathcal{B}$. We construct an orthogonal subset $\mathcal{B}'$ such that $\mathcal{B}' \vdash \mathcal{B}$ and $\dim \mathfrak{a}_{\mathcal{B}'} < \dim \mathfrak{a}_{\mathcal{B}}$, whose type is $(h+1, k-2)$ if β, β' are both long, and (h, k) otherwise. Since two positive long roots in a root system of tyoe C_n are always comparable, the claim will follow repeating the argument until $\mathcal{B}'$ contains at most a single long root.

If β, β' are both long, then by Lemma 4.8 $\beta' - \beta = 2\alpha$ for some short root $\alpha \in \Phi_0^+$. Denote $\mathcal{B}' = (\mathcal{B} \smallsetminus \{\beta, \beta'\}) \cup \{\alpha + \beta\}$. Since $\alpha + \beta = \beta' - \alpha \in \Phi_1^+$, Lemma 3.3 implies that $(\alpha, \beta) = 0$ for all $\beta \in \mathcal{B} \smallsetminus \{\beta, \beta'\}$. Therefore $\mathcal{B}'$ is orthogonal, and it is of type $(h + 1, k - 2)$ since $\alpha + \beta$ is a short root. Moreover $\mathcal{B}' \vdash \mathcal{B}$, and since β is minimal in $\mathcal{B}$ we get $\dim \mathfrak{a}_{\mathcal{B}'} < \dim \mathfrak{a}_{\mathcal{B}}$ as well.

Suppose that β, β' are both short roots. Following Lemma 4.8, write $\beta' - \beta = \alpha + \alpha'$ with $\alpha, \alpha' \in \Phi_0^+$ short orthogonal roots. In particular, it must be $\langle \beta', \alpha^\vee \rangle = -\langle \beta, \alpha^\vee \rangle = 1$, and by Lemma 3.3 it follows $(\alpha, \beta'') = 0$ for all $\beta'' \in \mathcal{B} \smallsetminus \{\beta, \beta'\}$. Therefore

$$s_\alpha(\mathcal{B}) = (\mathcal{B} \smallsetminus \{\beta, \beta'\}) \cup \{\alpha + \beta, \beta' - \alpha\}.$$

On the other hand, being $\beta' - \beta = \alpha + \alpha'$, we get $\beta < s_\alpha(\beta)$ and $\beta < s_\alpha(\beta')$. Therefore $s_\alpha(\mathcal{B}) \vdash \mathcal{B}$, and since β is minimal in $\mathcal{B}$ it follows $\dim \mathfrak{a}_{s_\alpha(\mathcal{B})} < \dim \mathfrak{a}_{\mathcal{B}}$.

Suppose finally that $||\beta|| \neq ||\beta'||$. Following Lemma 4.8, we can write $\beta' - \beta = 2\alpha + \alpha'$ where $\alpha, \alpha' \in \Phi_0^+$ are short roots with $\langle \alpha', \alpha^\vee \rangle = -1$. In particular we get $\alpha + \alpha' \in \Phi_0^+$, hence $\beta + \alpha, \beta' - \alpha \in \Phi_1^+$, and by Lemma 3.3 it follows $(\alpha, \beta'') = 0$ for all $\beta'' \in \mathcal{B} \smallsetminus \{\beta, \beta'\}$. Therefore

$$s_\alpha(\mathcal{B}) = (\mathcal{B} \smallsetminus \{\beta, \beta'\}) \cup \{s_\alpha(\beta), s_\alpha(\beta')\}.$$

On the other hand, being $\beta' - \beta = 2\alpha + \alpha'$, we get $\beta < s_\alpha(\beta)$ and $\beta < s_\alpha(\beta')$. Therefore $s_\alpha(\mathcal{B}) \vdash \mathcal{B}$, and since β is minimal in $\mathcal{B}$ we get $\dim \mathfrak{a}_{s_\alpha(\mathcal{B})} < \dim \mathfrak{a}_{\mathcal{B}}$ as well. □

Proposition 4.10. *Consider the Hermitian pair (C_n, α_1).*

i) *Let $\mathcal{B} \subset \Phi_1^+$ be an orthogonal subset of type (h, k). Then there is an antichain $\mathcal{A} \subset \Phi_1^+$ of type $(h + \lfloor \frac{k}{2} \rfloor, k - 2\lfloor \frac{k}{2} \rfloor)$ such that $\mathcal{A} \vdash \mathcal{B}$.*

ii) *There exists a unique antichain $\mathcal{A}_* \in \mathrm{Ort}_{\max}(\Phi_1^+)$, and $\mathcal{A}_* \vdash \mathcal{B}$ for all $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$.*

Proof. i) Suppose that $\mathcal{B}$ is not an antichain, by Lemma 4.9 there is an orthogonal subset $\mathcal{B}_1 \subset \Phi_1^+$ of type $(h + \lfloor \frac{k}{2} \rfloor, k - 2\lfloor \frac{k}{2} \rfloor)$ such that $\mathcal{B}_1 \vdash \mathcal{B}$ and $\dim \mathfrak{a}_{\mathcal{B}_1} < \dim \mathfrak{a}_{\mathcal{B}}$. Suppose that $\mathcal{B}_i$ is defined, and suppose that $\mathcal{B}_i$ is not an antichain. Then we can apply Lemma 4.9 again, and we find an orthogonal subset $\mathcal{B}_{i+1} \subset \Phi_1^+$ such that $\mathcal{B}_{i+1} \vdash \mathcal{B}_i$ and $\dim \mathfrak{a}_{\mathcal{B}_{i+1}} < \dim \mathfrak{a}_{\mathcal{B}_i}$. Since $\mathcal{B}_{i+1}$ is not empty, $\mathfrak{a}_{\mathcal{B}_{i+1}}$ cannot be zero, therefore the process must stop for some k, and $\mathcal{B}_k$ is an antichain.

ii) As we already noticed, if $\mathcal{B} \in \mathrm{Ort}(\Phi_1^+)$ has type (h,k), then $\mathcal{B}$ is maximal if and only if $2h+k=n$. Therefore by i) for all $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$ there exists an antichain $\mathcal{A} \in \mathrm{Ort}_{\max}(\Phi_1^+)$ such that $\mathcal{A} \vdash \mathcal{B}$. We also already noticed that there is a unique antichain $\mathcal{A}_* \in \mathrm{Ort}_{\max}(\Phi_1^+)$, therefore $\mathcal{A}_* \vdash \mathcal{B}$ for all $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$. □

4.4. **Hermitian symmetric spaces of tube type.** Let (Π, α_q) be a Hermitian pair and let $\mathfrak{p}^+$ be the corresponding standard parabolic subalgebra of $\mathfrak{g}$. Let $\mathcal{S}_{\Pi,\alpha_q} = \{\gamma_1, \ldots, \gamma_r\}$ be the set of Harish-Chandra strongly orthogonal roots, and set $\mathfrak{h}^- = \mathrm{span}(\gamma_i^\vee \mid i = 1, \ldots, r)$. By [9], [15], a root $\alpha \in \Phi$ is in Φ_1^+ if and only if either $\alpha_{|\mathfrak{h}^-} = \frac{1}{2}(\gamma_i + \gamma_j)$, for some $i \leqslant j$, or $\alpha_{|\mathfrak{h}^-} = \frac{1}{2}\gamma_i$ for some i. A root $\alpha \in \Phi$ is in Φ_0^+ if and only if either $\alpha_{|\mathfrak{h}^-} = \frac{1}{2}(\gamma_i - \gamma_j)$, for some $i \leqslant j$, or $\alpha_{|\mathfrak{h}^-} = \pm\frac{1}{2}\gamma_i$, for some i. Recall that the Hermitian symmetric space G/P is called of *tube type* if it is holomorphically equivalent to the tube over a self dual cone. It is known (cf. [14]) that Hermitian symmetric spaces of tube type correspond to Hermitian involutions such that $\alpha \in \Phi$ is in Φ_1^+ if and only if $\alpha_{|\mathfrak{h}^-} = \frac{1}{2}(\gamma_i + \gamma_j)$ for some $i \leqslant j$, and a root $\alpha \in \Phi$ is in Φ_0^+ if and only if $\alpha_{|\mathfrak{h}^-} = \frac{1}{2}(\gamma_i - \gamma_j)$, for some $i \leqslant j$. We will call such involutions *Hermitian involutions of tube type.* Observe that a Hermitian involution is of tube type if and only if

$$(\sum_{i=1}^{r} \gamma_i, \alpha) = (\alpha_q, \alpha_q)\ i, \text{ for all } \alpha \in \Phi_i,\ i = 0, 1. \tag{4.1}$$

Hermitian symmetric spaces of tube type are classified by the Hermitian pairs (Π, α_q) such that $w_0(\alpha_q) = -\alpha_q$, in which case we say that (Π, α_q) is a *Hermitian pair of tube type* (see e.g. [10, Ch. X, D.4 pg. 528]). In particular, we have the following possibilities:

i) (A_{2q-1}, α_q);
ii) (B_n, α_1);
iii) (C_n, α_n);
iv) (D_n, α_{n-1}) with n even; (D_n, α_n) with n even; (D_n, α_1) for all n;
v) (E_7, α_7).

Notice that being of tube type is equivalent to the fact that $\mathfrak{p}_\mathrm{u}^+$ is a regular prehomogeneous space under the action of G_0, namely the boundary of the open G_0-orbit has codimension 1 (see [16] and the references therein).

If (Π, α_q) is a Hermitian pair of tube type and Π is not simply laced, then the short roots in Φ_1 admit a nice description:

Lemma 4.11. *Suppose that Φ is not simply laced and let σ be a Hermitian involution of tube type. Let $\mathcal{S}$ be an orthogonal subset of Φ_1^+ of maximal cardinality and let $\beta \in \Phi_1^+$ be a short root. Then $\beta = \frac{1}{2}(\gamma + \gamma')$ for some distinct elements $\gamma, \gamma' \in \mathcal{S}$.*

Proof. By Remark 4.2, every $\gamma \in \mathcal{S}$ is a long root of Φ, and we can choose a set of positive roots in $\Phi_0^+ \subset \Phi_0$ so that $\mathcal{S}$ is the corresponding set of Harish-Chandra strongly orthogonal roots. Since σ is of tube type, we have that $\beta = \frac{1}{2}(\gamma + \gamma') + \lambda$, for some $\gamma, \gamma' \in \mathcal{S}$ with $(\gamma, \gamma') = 0$ and some λ with $(\lambda, \gamma'') = 0$ for all $\gamma'' \in \mathcal{S}$, therefore $\|\beta\|^2 = \frac{1}{4}(\|\gamma\|^2 + \|\gamma'\|^2) + \|\lambda\|^2 = \frac{1}{2}\|\gamma\|^2 + \|\lambda\|^2$. On the other hand β is a short root, therefore $||\beta||^2 = \frac{1}{2}||\gamma||^2$ and it follows $\lambda = 0$. □

If σ is the Hermitian involution of tube type associated to the Hermitian pair (B_n, α_1) or (C_n, α_n), we proved in Proposition 4.7 and Proposition 4.10 that there exists a unique antichain $\mathcal{A}_* \in \mathrm{Ort}_{\max}(\Phi_1^+)$, and that $\mathcal{A}_* \vdash \mathcal{B}$ for all $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$. We now show that this property holds whenever σ is a Hermitian involution of tube type.

Proposition 4.12. *Suppose that σ is a Hermitian involution of tube type.*

Then there exists a unique antichain $\mathcal{A}_* \in \mathrm{Ort}_{\max}(\Phi_1^+)$, and $\mathcal{A}_* \vdash \mathcal{B}$ for all $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$.

Proof. As we noticed, the claim has already been proved if Φ is not simply laced. Therefore we will assume that Φ is simply laced, so that $\mathrm{Ort}_{\max}(\Phi_1^+)$ coincides with the collection of the orthogonal subsets of maximal cardinality r. By Proposition 4.6, for all $\mathcal{B} \in \mathrm{Ort}_{\max}(\Phi_1^+)$, there is an antichain $\mathcal{A} \in \mathrm{Ort}_{\max}(\Phi_1^+)$ such that $\mathcal{A} \vdash \mathcal{B}$. Therefore we only need to show the uniqueness of the antichain in $\mathrm{Ort}_{\max}(\Phi_1^+)$.

Suppose $\mathcal{A}, \mathcal{A}' \in \mathrm{Ort}_{\max}(\Phi_1^+)$ are both antichains. Since they are both of maximal cardinality, by Remark 4.2, they are both sets of Harish-Chandra roots for some choice of positive sets of roots in Φ_0. Since the pair (Π, α_q) is of tube type we have, by (4.1),

$$(\sum_{\gamma \in \mathcal{A}} \gamma, \alpha_q) = (\alpha_q, \alpha_q) = (\sum_{\gamma' \in \mathcal{A}'} \gamma', \alpha_q)$$

and, if $\alpha \in \Pi_0$,

$$(\sum_{\gamma \in \mathcal{A}} \gamma, \alpha) = 0 = (\sum_{\gamma' \in \mathcal{A}'} \gamma', \alpha).$$

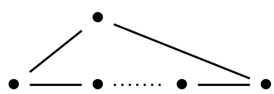
It follows that

$$\sum_{\gamma \in \mathcal{A}} \gamma = \sum_{\gamma' \in \mathcal{A}'} \gamma'. \tag{4.2}$$

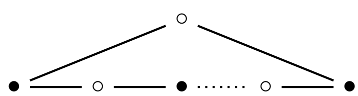
Let C be the matrix $((\gamma, \gamma'))_{\gamma \in \mathcal{A}, \gamma' \in \mathcal{A}'}$. Consider the matrix C' obtained by replacing the nonzero entries of C with 1. This is the incidence matrix of a relation. Let $\mathcal{G}$ be its incidence graph. Write $\mathcal{G} = \cup_i \mathcal{G}_i$, where $\mathcal{G}_i$ are the connected components of $\mathcal{G}$. Assume first that $\mathcal{G}_i$ has more than one node. If $\gamma \in \mathcal{G}_i \cap \mathcal{A} \cap \mathcal{A}'$ then $(\gamma, \gamma') = 0$ for all $\gamma' \in \mathcal{A}' \smallsetminus \{\gamma\}$, so γ is connected in $\mathcal{G}$ only to itself. Since $\mathcal{G}_i$ is connected this is not possible, hence $\mathcal{G}_i \cap \mathcal{A} \cap \mathcal{A}' = \emptyset$. If $\mathcal{G}_i \cap \mathcal{A} = \emptyset$ then any $\gamma' \in \mathcal{G}_i \cap \mathcal{A}'$ is orthogonal to $\mathcal{A}$, contradicting the fact that $\mathcal{A}$ is in $\mathrm{Ort}_{\max}(\Phi_1^+)$. Symmetrically we have that also $\mathcal{G}_i \cap \mathcal{A}'$ is not empty. Since $\mathcal{G}_i$ is connected, if $\gamma_0 \in \mathcal{G}_i \cap \mathcal{A}$, there must be $\gamma_0' \in \mathcal{G}_i \cap \mathcal{A}'$ such that $\gamma_0 \neq \gamma_0'$ and $(\gamma_0, \gamma_0') \neq 0$. Since

$$\langle \sum_{\gamma \in \mathcal{A}} \gamma, \gamma_0^\vee \rangle = 2 = \langle \sum_{\gamma' \in \mathcal{A}'} \gamma', \gamma_0^\vee \rangle = 1 + \langle \sum_{\gamma' \neq \gamma_0'} \gamma', \gamma_0^\vee \rangle,$$

we see that there are exactly two nodes in $\mathcal{G}_i$ to which γ_0 is connected. Symmetrically, the same property holds for all $\gamma' \in \mathcal{G}_i \cap \mathcal{A}'$. Thus every node has degree exactly 2 in $\mathcal{G}_i$. It follows that $\mathcal{G}_i$ is a cycle



Since $\mathcal{G}_i \cap \mathcal{A} \cap \mathcal{A}' = \emptyset$ and the nodes in $\mathcal{A}$ connect only to nodes in $\mathcal{A}'$ we have only this possibility (letting $\circ$ be the nodes in $\mathcal{A}$ and $\bullet$ the nodes in $\mathcal{A}'$):



If $(\gamma, \gamma') \neq 0$ then $(\gamma, \gamma') > 0$ so $\gamma - \gamma'$ is a root. It follows that either $\gamma > \gamma'$ or $\gamma' > \gamma$. We give an orientation to the graph $\mathcal{G}$ by orienting the edges so that they point from the larger to the smaller root. Since both $\mathcal{A}$ and $\mathcal{A}'$ are antichains, we cannot have consecutive arrows $\circ \to \bullet \to \circ$, $\bullet \to \circ \to \bullet$. It follows that the nodes $\circ$ are either all sources or all sinks. By eventually exchanging $\mathcal{A}$ and $\mathcal{A}'$, we can assume that all $\circ$ are sources and all $\bullet$ are sinks. This means that we can enumerate $\mathcal{A}\cap\mathcal{G}_i=\{\gamma_1,\dots,\gamma_s\}$ and $\mathcal{A}'\cap\mathcal{G}_i=\{\gamma_1',\dots,\gamma_s'\}$ so that $\gamma_i'<_0\gamma_i$.

It follows that $\lambda=\sum\gamma_i-\sum\gamma_i'>_0 0$. On the other hand

$$\begin{aligned}\|\lambda\|^2 &= \|\sum\gamma_i\|^2+\|\sum\gamma_i'\|^2-(\sum\gamma_i,\sum\gamma_i')-(\sum\gamma_i',\sum\gamma_i)\\ &= \|\sum\gamma_i\|^2+\|\sum\gamma_i'\|^2-(\sum\gamma_i,\sum_{\gamma'\in\mathcal{A}'}\gamma')-(\sum\gamma_i',\sum_{\gamma\in\mathcal{A}}\gamma).\end{aligned}$$

By (4.2),

$$\begin{aligned}\|\lambda\|^2 &= \|\sum\gamma_i\|^2+\|\sum\gamma_i'\|^2-(\sum\gamma_i,\sum_{\gamma\in\mathcal{A}}\gamma)-(\sum\gamma_i',\sum_{\gamma'\in\mathcal{A}'}\gamma')\\ &= \|\sum\gamma_i\|^2+\|\sum\gamma_i'\|^2-(\sum\gamma_i,\sum\gamma_i)-(\sum\gamma_i',\sum\gamma_i')=0.\end{aligned}$$

It follows that $\mathcal{G}_i$ has only one node for all i. As observed earlier $\mathcal{G}_i\cap\mathcal{A}\neq\emptyset$ and $\mathcal{G}_i\cap\mathcal{A}'\neq\emptyset$, thus, if $\mathcal{G}_i=\{\gamma\}$, then $\gamma\in\mathcal{A}\cap\mathcal{A}'$. Therefore $\mathcal{A}\cup\mathcal{A}'=\cup_i\mathcal{G}_i\subset\mathcal{A}\cap\mathcal{A}'$. Thus $\mathcal{A}=\mathcal{A}'$. □

Remark 4.13. The proof of uniqueness of the antichain we have given when Φ is simply laced can be extended (with some complications) to a uniform proof for any Φ. Since the two non simply laced cases are easily dealt individually, we preferred to omit this more complicated approach.

Given $\mathcal{B}\subset\Phi_1^+$, set

$$\mathcal{B}^{\leqslant_0}=\{\alpha\in\Phi_1^+\mid \text{ there is }\beta\in\mathcal{B}\text{ such that }\alpha\leqslant_0\beta\}.$$

Notice that $\mathcal{B}^{\leqslant_0}=\Psi(\mathfrak{a}_{\mathcal{B}}^-)$, where $\mathfrak{a}_{\mathcal{B}}^-\subset\mathfrak{p}_{\mathrm{u}}^+$ is the B_0^--stable subalgebra generated by $\mathcal{B}$ and $B_0^-\subset G_0$ is the opposite Borel subgroup of B.

If $\mathcal{A}_*\in\mathrm{Ort}_{\max}(\Phi_1^+)$ is the unique antichain, it follows by Proposition 4.12, that $\mathcal{A}_*\subset\mathcal{B}^{\geqslant_0}$ for all $\mathcal{B}\in\mathrm{Ort}_{\max}(\Phi_1^+)$. The following corollary shows that $\mathcal{A}_*\subset\mathcal{B}^{\leqslant_0}$ as well.

Corollary 4.14. *Let $\mathcal{A}_*\in\mathrm{Ort}_{\max}(\Phi_1^+)$ be the unique antichain. Then $\mathcal{A}_*\subset\mathcal{B}^{\leqslant_0}$ for all $\mathcal{B}\in\mathrm{Ort}_{\max}(\Phi_1^+)$.*

Proof. Let $\leqslant_0'$ be the partial order on Φ_1^+ defined by Φ_0^-. Then $\leqslant_0'$ is the reverse partial order of $\leqslant_0$, therefore a subset $\mathcal{A}\subset\Phi_1^+$ is an antichain w.r.t. $\leqslant_0$ if and only if it is an antichain w.r.t. $\leqslant_0'$. Therefore, if $\vdash'$ is the preorder on $\mathrm{Ort}(\Phi_1^+)$ defined by $\leqslant_0'$, it follows by Proposition 4.12 that $\mathcal{A}_*\vdash'\mathcal{B}$, namely $\mathcal{A}_*\subset\mathcal{B}^{\leqslant_0}$. □

5. The special B_0-stable abelian subalgebra

For the rest of the paper we will assume that $\sigma:G\longrightarrow G$ is an (indecomposable) involution. Moreover, throughout this section, we assume that $\mathfrak{g}_0$ is semisimple and that the simple root $\alpha_p\in\widehat{\Pi}$ corresponding to σ is long and non-complex.

Recall from Section 2.3 the element w_p of $\mathcal{W}_\sigma^{ab}$ and the corresponding subalgebra $\mathfrak{a}_p\in\mathcal{I}_{ab}^\sigma$. We call $\mathfrak{a}_p$ the *special B_0-stable abelian subalgebra*.

For each component Σ of Π_0, we let $\Phi(\Sigma)$ be the root subsystem of Φ_0 generated by Σ. As shown in [6, Lemma 5.7], there is a unique simple root $\alpha_\Sigma\in\Sigma$ which is connected to α_p. If moreover θ_Σ is the highest root in $\Phi(\Sigma)$, then $[\theta_\Sigma:\alpha_\Sigma]=1$, therefore (Σ,α_Σ) is a Hermitian pair. It is then clear that $\gamma\in\Phi(\Sigma)$ is orthogonal to α_p if and only if $[\gamma:\alpha_\Sigma]=0$. Following Section 4, we denote by $\Phi(\Sigma)_1^+$ the set of roots in $\Phi(\Sigma)^+$ that have α_Σ in their support.

Set $\mathcal{C}_\sigma=\{\alpha\in\widehat{\Phi}^+\mid\alpha_p+k\delta-\alpha\in\widehat{\Phi}^+\}$ and define

$$\mathcal{C}_\sigma^1=\mathcal{C}_\sigma\cap\widehat{\Phi}_1=\{\alpha\in\mathcal{C}_\sigma\mid[\alpha:\alpha_p]=1\}.$$

Notice that $\mathcal{C}^1_\sigma \subset \widehat{\Phi}^+_{re}$.

Lemma 5.1.

i) $N(w_p) = \bigcup_\Sigma \{\gamma + \alpha_p \mid \gamma \in \Phi(\Sigma)^+_1\} \cup \{\alpha_p\}$, *where* Σ *ranges among the components of* Π_0*;*

ii) $N(w_p) = \mathcal{C}^1_\sigma$*;*

iii) *The map* $\Upsilon : \mathcal{C}^1_\sigma \to \bigcup_\Sigma \Phi(\Sigma)^+_1 \cup \{0\}$ *mapping* η *to* $\eta - \alpha_p$ *is an order preserving bijection, where* Σ *varies among the components of* Π_0*;*

iv) *If* $\gamma, \eta \in \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$*, then* $(\Upsilon(\gamma), \Upsilon(\eta)) = (\gamma, \eta)$*.*

v) *If* $\eta \in \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$*, then* $(\alpha_p, \Upsilon(\eta)) = -(\alpha_p, \eta)$*.*

Proof. i) It is well known that $N(w_{0,\alpha_p} w_0) = \bigcup_\Sigma \Phi(\Sigma)^+_1$. If $\gamma \in \Phi(\Sigma)^+_1$, then $\langle \gamma, \alpha_p^\vee \rangle = \langle \alpha_\Sigma, \alpha_p^\vee \rangle$, and since α_p is long. we have $\langle \alpha_\Sigma, \alpha_p^\vee \rangle = -1$. It follows that $s_p(\gamma) = \gamma + \alpha_p \in \widehat{\Phi}^+$, hence

$$N(w_p) = N(s_p w_{0,\alpha_p} w_0) = \{\alpha_p\} \cup s_p(N(w_{0,\alpha_p} w_0)),$$

and the claim follows.

ii) Clearly $\alpha_p \in \mathcal{C}^1_\sigma$. Moreover, if $\gamma \in \Phi(\Sigma)^+_1$, then $k\delta + \alpha_p - (\gamma + \alpha_p) = k\delta - \gamma \in \widehat{\Phi}^+$. It follows that $N(w_p) \subseteq \mathcal{C}^1_\sigma$. Since $w_p(\alpha_p) = k\delta + \alpha_p$, it follows that $\ell(w_p s_p) = \ell(w_p) + 1$. This implies that $N(w_p s_p) = N(w_p) \cup \{k\delta + \alpha_p\}$. Let $\eta \neq \alpha_p$ be in $\mathcal{C}^1_\sigma$ so that there is β such that $\eta + \beta = k\delta + \alpha_p$. By biconvexity of $N(w_p s_p)$, exactly one between η and β is in $N(w_p)$. Since $[\eta : \alpha_p] = 1$ and $[\beta : \alpha_p] = 2$, it follows from the fact that $w_p \in \mathcal{W}^{ab}_\sigma$ that $\eta \in N(w_p)$. Thus $\mathcal{C}^1_\sigma \subseteq N(w_p)$, and the claim follows.

iii) Follows by i) and ii).

iv) Since α_p is long, if $\alpha \in \Phi(\Sigma)^+_1$ then $(\alpha, \alpha_p) = -\frac{(\alpha_p, \alpha_p)}{2}$. Therefore, if $\gamma, \eta \in \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$, we have

$$(\gamma, \eta) = (\gamma - \alpha_p, \eta - \alpha_p) + (\gamma - \alpha_p, \alpha_p) + (\eta - \alpha_p, \alpha_p) + (\alpha_p, \alpha_p) = (\gamma - \alpha_p, \eta - \alpha_p).$$

The second claim follows as well, since $(\Upsilon(\eta), \Upsilon(\eta)) = (\eta, \eta)$.

v) Since α_p is long and $\Upsilon(\eta) \in \Phi(\Sigma)^+_1$, it must be $\langle \Upsilon(\eta), \alpha_p^\vee \rangle = -1$. Therefore $\langle \eta, \alpha_p^\vee \rangle = 1$, and it follows $(\alpha_p, \Upsilon(\eta)) = -(\alpha_p, \eta)$. □

Lemma 5.2. *Let* $\{\eta_1, \dots, \eta_t\}$ *be an orthogonal set of real roots, let* η_{t+1} *be a real root such that* $(\eta_i, \eta_{t+1}) < 0$ *for all* $i \leqslant t$*, and set* $A = (\langle \eta_j, \eta_i^\vee \rangle)_{i,j=1,\dots,t+1}$*. Then* A *is a generalized Cartan matrix of finite or affine type.*

Proof. The fact that A is a generalized Cartan matrix [11, Section 1.1] is clear. It is also symmetrizable: setting $D = \mathrm{diag}(||\eta_1||^2, \dots, ||\eta_{t+1}||^2)$ we have that $DA = 2((\eta_i, \eta_j))$, which is symmetric. Since $(\eta_i, \eta_{t+1}) \neq 0$ for all $i \leqslant t$, DA is an indecomposable matrix. Therefore it is enough to check that DA is of finite or affine type. By [11, Lemma 4.5], we need to check that DA is positive semi-definite of corank less than or equal to 1. Since $\eta_1, \dots, \eta_t$ are orthogonal DA has rank at least t. It is clear that, given a positive semi-definite symmetric bilinear form $(\cdot, \cdot)$ on a vector space V and a set of vectors $R = \{v_1, \dots, v_k\}$ in V, the matrix $((v_i, v_j))$ is positive semi-definite. On the other hand, since $\widehat{\Phi}$ is an affine system, the invariant form $(\cdot, \cdot)$ is positive semi-definite, and being $DA = 2((\eta_i, \eta_j))$ it follows that DA is positive semi-definite. □

Let $\mathcal{S} \subset \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ be an orthogonal subset of maximal cardinality. Consider the set of roots

$$\Pi_\mathcal{S} = \mathcal{S} \cup \{-\alpha_p\}$$

and the matrix $A_\mathcal{S} = (\langle \eta', \eta^\vee \rangle)_{\eta, \eta' \in \Pi_\mathcal{S}}$. We will show that $G_0 \mathfrak{a}_p$ is not spherical. The following is the key result in this direction.

Proposition 5.3. *Let $\mathcal{S} \subset \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ be an orthogonal subset of maximal cardinality. Then $|\mathcal{S}| \leqslant 4$ and $A_{\mathcal{S}}$ is a Cartan matrix of affine type.*

Proof. Set $\mathcal{S} = \{\eta_1, \dots, \eta_t\}$. By Lemma 5.2, $A_{\mathcal{S}}$ has to be either of finite or of affine type. By a slight abuse of notation we denote by $\Pi_{\mathcal{S}}$ the corresponding Dynkin diagram. This diagram has $t+1$ nodes with t nodes connected only to the remaining node corresponding to α_p. This immediately implies that $t \leqslant 4$, and since α_p is long, the node connected to all other nodes corresponds to a long simple root.

If $t = 4$ then the diagram is of type $D_4^{(1)}$, hence it is affine. If $t = 3$ the only possibilities are D_4 or $B_3^{(1)}$. If $\Pi_{\mathcal{S}}$ is of type D_4, then $\eta = k\delta - (\eta_1 + \eta_2 + \eta_3 - 2\alpha_p)$ is a root in $\widehat{\Phi}$ with $[\eta : \alpha_p] = 1$. Set $\beta = \eta_1 + \eta_2 + \eta_3 - \alpha_p \in \widehat{\Phi}$. Then $\beta \in \widehat{\Phi}$, $[\beta : \alpha_p] = 2$, and $\eta + \beta = k\delta + \alpha_p$. It follows that $\eta \in \mathcal{C}^1_\sigma$. Being $(\eta, \eta_i) = 0$ for $i = 1, 2, 3$, we see that $\{\eta_1, \eta_2, \eta_3\}$ is not a set of maximal cardinality in $\mathcal{C}^1_\sigma \setminus \{\alpha_p\}$. This excludes the possibility that $\Pi_{\mathcal{S}}$ is of type D_4, so $\Pi_{\mathcal{S}}$ is of affine type.

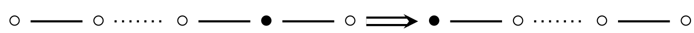
If $t = 2$ then $\Pi_{\mathcal{S}}$ can be only of type A_3, B_3, $G_2^{(1)}$, $D_3^{(2)}$. By Lemma 5.1 together with Remark 4.2, $\{\Upsilon(\eta_1), \Upsilon(\eta_2)\}$ is an orthogonal subset of maximal cardinality in $\bigcup_\Sigma \Phi(\Sigma)^+_1$, and both $\Upsilon(\eta_1)$ and $\Upsilon(\eta_2)$ are long roots in the respective components of Π_0. In particular, Π_0 contains at most two components.

Suppose that $t = 2$ and that $\Pi_0 = \Sigma_1 \cup \Sigma_2$ is the union of two components, then by Remark 4.2 the Hermitian symmetric spaces corresponding to $(\Sigma_1, \alpha_{\Sigma_1})$ and $(\Sigma_2, \alpha_{\Sigma_2})$ have both rank 1, hence Σ_1, Σ_2 are both of type A, and α_{Σ_i} is an extremal root in Σ_i (see Table 1). Since α_{Σ_i} and $\Upsilon(\eta_i)$ are both long in Σ_i, by Lemma 5.1 v) it follows that

$$\langle -\alpha_p, \eta_i^\vee \rangle = \langle \alpha_p, \Upsilon(\eta_i)^\vee \rangle = \langle \alpha_p, \alpha_{\Sigma_i}^\vee \rangle. \tag{5.1}$$

This means that α_p is connected to α_{Σ_i} in $\widehat{\Pi}$ with the same number of edges that connect $-\alpha_p$ to η_i in $\Pi_{\mathcal{S}}$. If $\Pi_{\mathcal{S}}$ is of type A_3, it follows that $\widehat{\Pi}$ is of type A, of the shape

(where the roots α_{Σ_i} are denoted by black nodes), which is absurd since $\widehat{\Pi}$ is not of finite type. If $\Pi_{\mathcal{S}}$ is of type B_3, then $\widehat{\Pi}$ is of the shape



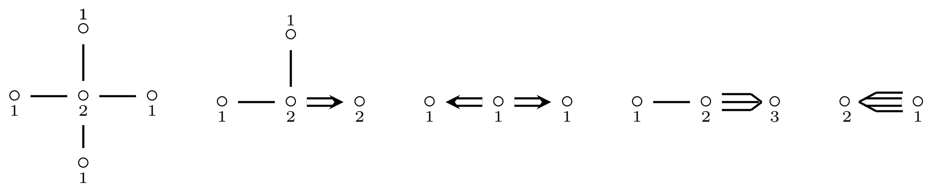
The only affine diagrams of this shape are $F_4^{(1)}$ and $E_6^{(2)}$. If $\widehat{\Pi}$ is of type $F_4^{(1)}$ then α_p has label 3, while if $\widehat{\Pi}$ is of type $E_6^{(2)}$, then α_p has label 2. Since in both cases the corresponding automorphism σ is not an involution, it follows that Π_0 must contain a unique component.

Suppose that $t = 2$ and that Π_0 is connected. By Lemma 5.1 together with Remark 4.2, it follows that $\{\Upsilon(\eta_1), \Upsilon(\eta_2)\}$ is an orthogonal subset of maximal cardinality in $\Phi(\Pi_0)^+_1$, and that $||\eta_1|| = ||\eta_2||$. Therefore, if $\Pi_{\mathcal{S}}$ is not affine, it must be of type A_3. Suppose that this is the case; then by (5.1) α_p is connected to α_{Π_0} by a single edge. On the other hand, by Remark 4.2 again, the symmetric space corresponding to the Hermitian pair (Π_0, α_{Π_0}) has rank 2. Up to an automorphism of Π_0, by Table 1 the possibilities for the pair (Π_0, α_{Π_0}) are the following: (A_n, α_2), (B_n, α_1), (D_n, α_1), (D_5, α_5), (E_6, α_1). But then by obvious considerations it follows that $\widehat{\Pi}$ also of finite type, a contradiction.

It remains to check the case when $t = 1$. If $\Pi_{\mathcal{S}}$ is of finite type, then it is of type A_2, C_2, or G_2. Since $\{\Upsilon(\eta_1)\}$ is a set of orthogonal roots of maximal cardinality in $\bigcup_\Sigma \Phi(\Sigma)^+_1$, it follows that Π_0 is connected, and the Hermitian symmetric variety corresponding to the pair (Π_0, α_{Π_0}) has rank one, therefore Π_0 is of type A and α_{Π_0}

is an extremal root in Π_0. By (5.1), α_p is connected to α_{Π_0} by the same number of edges that connect $-\alpha_p$ to η_1. Therefore, if $\Pi_{\mathcal{S}}$ is of finite type, then $\widehat{\Pi}$ would be of finite type as well, which is absurd. □

For the reader's convenience we list here all possible diagrams for the affine root system $\Pi_{\mathcal{S}}$, with the corresponding labels.



$D_4^{(1)}$ $B_3^{(1)}$ $D_3^{(2)}$ $G_2^{(1)}$ $A_2^{(2)}$

Table 2 – Affine root systems corresponding to non-spherical orbits in $G_0\mathfrak{a}_p$

Remark 5.4. Let Ξ be one of the affine diagrams of Table 2, let k_Ξ be the integer such that Ξ is of type $X_r^{(k_\Xi)}$ ($k_\Xi \in \{1,2\}$) and let $\alpha \in \Xi$ be the unique long simple root which is connected to all other simple roots. Let $\xi \in \Xi$ and let $a_{\Xi,\xi}$ be the corresponding label in Ξ, then for all $\xi \in \Xi$, we have $k_\Xi a_{\Xi,\xi} = |\langle \alpha, \xi^\vee \rangle|$. In particular, the equality

$$k_\Xi(\sum_{\xi\neq\alpha} a_{\Xi,\xi}) = 4$$

holds. If $\mathcal{S}$ is an orthogonal subset of $\mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ of maximal cardinality and $\xi \in \Pi_{\mathcal{S}}$, then we will denote $k_{\Pi_{\mathcal{S}}}$ and $a_{\Pi_{\mathcal{S}},\xi}$ simply by $k_{\mathcal{S}}$ and $a_{\mathcal{S},\xi}$.

As in Section 3, we fix a weight vector $x_1^\mu \in \mathfrak{g}_1^\mu$ for all $\mu \in \Phi_1^+$. Recall that $\overline{\alpha}_p$ is the lowest weight in Φ_1 and that $\Psi(\mathfrak{a}_p) = \{-\overline{\eta} \,|\, \eta \in \mathcal{C}^1_\sigma\}$. If $\mathcal{S} \subset \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ is an orthogonal subset, we set

$$x_{\mathcal{S}} = \sum_{\eta\in\mathcal{S}} x_1^{-\overline{\eta}} \in \mathfrak{a}_p.$$

For all $\eta \in \mathcal{S}$, we choose $y_1^{\overline{\eta}} \in \mathfrak{g}_1^{\overline{\eta}}$ so that $[x_1^{-\overline{\eta}}, y_1^{\overline{\eta}}] = -\overline{\eta}^\vee$. Since the weights $\overline{\eta}$ are strongly orthogonal, setting $y_{\mathcal{S}} = \sum_{\eta\in\mathcal{S}} y_1^{\overline{\eta}}$ and $h_{\mathcal{S}} = -\sum_{\eta\in\mathcal{S}} \overline{\eta}^\vee$, we have that $\{x_{\mathcal{S}}, h_{\mathcal{S}}, y_{\mathcal{S}}\}$ is a normal $\mathfrak{sl}(2)$-triple which contains $x_{\mathcal{S}}$ as a nilpositive element.

Theorem 5.5. *Suppose that $\mathfrak{g}_0$ is semisimple and α_p is long and non-complex, and let $\mathcal{S}$ be an orthogonal subset of $\mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ of maximal cardinality. Then the orbit $G_0 x_{\mathcal{S}} \subset \mathfrak{g}_1$ is not spherical. In particular, $G_0\mathfrak{a}_p$ is not spherical.*

Proof. Let $\mathcal{S} = \{\eta_1, \dots, \eta_t\}$ be an orthogonal subset of $\mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ of maximal cardinality. Since $\sum_{\alpha\in\Pi_{\mathcal{S}}} a_{\mathcal{S},\alpha}\alpha$ is an isotropic vector, it follows that $k_{\mathcal{S}}(\sum_{\alpha\in\Pi_{\mathcal{S}}} a_{\mathcal{S},\alpha}\alpha)$ is a multiple of δ. Since

$$[k_{\mathcal{S}}(\sum_{\alpha\in\Pi_{\mathcal{S}}} a_{\mathcal{S},\alpha}\alpha) : \alpha_p] = [k_{\mathcal{S}}(\sum_{i=1}^t a_{\mathcal{S},\eta_i}\eta_i) : \alpha_p] - k_{\mathcal{S}} a_{-\alpha_p,\mathcal{S}} = 4 - 2 = 2,$$

it follows that $k_{\mathcal{S}}(\sum_{\alpha\in\Pi_{\mathcal{S}}} a_{\mathcal{S},\alpha}\alpha) = k\delta$, hence

$$\sum_{i=1}^t k_{\mathcal{S}} a_{\mathcal{S},\eta_i}\eta_i - \alpha_p = k\delta + \alpha_p. \tag{5.2}$$

It follows moreover that $k_{\mathcal{S}} a_{\mathcal{S},\eta_i}(\eta_i,\eta_i) = 2(\alpha_p,\eta_i)$, namely

$$k_{\mathcal{S}} a_{\mathcal{S},\eta_i} = \langle \alpha_p, \eta_i^\vee \rangle. \tag{5.3}$$

The equalities (5.2) and (5.3) show that

$$\sum_{i=1}^t \langle \alpha_p, \eta_i^\vee \rangle \overline{\eta}_i = 2\overline{\alpha}_p,$$

hence $\sum_{i=1}^t \langle \alpha_p, \eta_i^\vee \rangle \langle \overline{\eta}_i, \alpha_p^\vee \rangle = 4$. Since α_p is long, we have that $1 = \langle \eta_i, \alpha_p^\vee \rangle = \langle \overline{\eta}_i, \alpha_p^\vee \rangle$, therefore considering the normal $\mathfrak{sl}(2)$-triple $\{x_{\mathcal{S}}, h_{\mathcal{S}}, y_{\mathcal{S}}\}$ we get

$$-\overline{\alpha}_p(h_{\mathcal{S}}) = \sum_{i=1}^t \langle \alpha_p, \overline{\eta}_i^\vee \rangle = \sum_{i=1}^t \langle \alpha_p, \eta_i^\vee \rangle = 4.$$

Considering the grading $\bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_1(i)$ induced by $h_{\mathcal{S}}$, we see that $\mathfrak{g}_1^{-\overline{\alpha}_p} \subset \mathfrak{g}_1(4)$, therefore $G_0 x_{\mathcal{S}}$ is not spherical by [18, Theorem 5.6]. □

Let $\Sigma \subset \Pi_0$ be a component, and let $e_\Sigma = -\langle \alpha_p, \alpha_\Sigma^\vee \rangle$ be the number of edges connecting α_Σ with α_p. As a corollary of the previous proof we get the following.

Corollary 5.6. *Let $\mathcal{S} = \{\eta_1, \dots, \eta_t\}$ be an orthogonal subset of maximal cardinality of $\mathcal{C}_\sigma^1 \smallsetminus \{\alpha_p\}$ and let $\Sigma \subset \Pi_0$ be a component.*

i) *If $\Upsilon(\eta_i) \in \Phi(\Sigma)$, then $k_{\mathcal{S}} a_{\mathcal{S},\eta_i} = \langle \alpha_p, \eta_i^\vee \rangle = e_\Sigma$.*
ii) *Let $(k\delta - 2\alpha_p)_\Sigma$ be the orthogonal projection of $k\delta - 2\alpha_p$ onto $\mathfrak{h}_\Sigma = \mathrm{span}(h_\alpha \mid \alpha \in \Sigma)$ and set $I_\Sigma = \{i \mid \Upsilon(\eta_i) \in \Phi(\Sigma)\}$. Then*

$$\sum_{i\in I_\Sigma} \Upsilon(\eta_i) = \frac{1}{e_\Sigma}(k\delta - 2\alpha_p)_\Sigma.$$

Proof. i) This follows immediately by formulas (5.1) and (5.3).

ii) Since $k_{\mathcal{S}}(\sum_{i=1}^t a_{\mathcal{S},\eta_i}) = 4$, formula (5.2) implies that

$$\sum_{i=1}^t k_{\mathcal{S}} a_{\mathcal{S},\eta_i} \Upsilon(\eta_i) = k\delta - 2\alpha_p.$$

On the other hand $(k\delta - 2\alpha_p)_\Sigma = \sum_{i\in I_\Sigma} k_{\mathcal{S}} a_{\mathcal{S},\eta_i} \Upsilon(\eta_i)$, therefore the claim follows by i). □

We already noticed at the beginning of the section that (Σ, α_Σ) is a Hermitian pair. More precisely, we have the following.

Proposition 5.7. *Let $\Sigma \subset \Pi_0$ be a component, then (Σ, α_Σ) is a Hermitian pair of tube type.*

Proof. For all $\alpha \in \Sigma \smallsetminus \{\alpha_\Sigma\}$, it clearly holds $(\alpha, k\delta - 2\alpha_p) = (\alpha, (k\delta - 2\alpha_p)_\Sigma) = 0$. By Corollary 5.6 we get then the equality

$$(\alpha, \sum_{i\in I_\Sigma} \Upsilon(\eta_i)) = i(\alpha_\Sigma, \alpha_\Sigma) \qquad \text{for all } \alpha \in \Phi(\Sigma)_i^+, \quad i = 0, 1.$$

By Lemma 5.1, $\{\Upsilon(\eta_i) \mid i \in I_\Sigma\}$ is an orthogonal set of maximal cardinality in $\Phi(\Sigma)_1^+$. By Remark 4.2 we can then choose a set of positive roots for $\Phi(\Sigma)_0$ in such a way that $\{\Upsilon(\eta_i) \mid i \in I_\Sigma\}$ is the corresponding set of strongly orthogonal roots in $\Phi(\Sigma)_1^+$. Therefore the claim follows immediately from (4.1). □

In Theorem 5.5 we constructed a non-spherical orbit $G_0 x_{\mathcal{S}} \subset G_0 \mathfrak{a}_p$ starting from an orthogonal subset $\mathcal{S} \subset \mathcal{C}_\sigma^1$ of maximal cardinality. We now show that this construction extends to any maximal orthogonal subset $\mathcal{S} \subset \mathcal{C}_\sigma^1$, and it always gives rise to the same G_0-orbit.

Theorem 5.8. *Let $\mathcal{S} \subset \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ be a maximal orthogonal subset.*

i) *For all $\alpha \in \Phi$ it holds $\alpha(h_\mathcal{S}) = -2\langle \alpha, \alpha_p^\vee \rangle$.*

ii) *The orbit $G_0 x_\mathcal{S} \subset \mathfrak{g}_1$ is not spherical, and $G_0 x_\mathcal{S} = G_0 x_\mathcal{T}$ for all maximal orthogonal subset $\mathcal{T} \subset \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$.*

Proof. If $\Sigma \subset \Pi_0$ is a component, set $\mathcal{S}_\Sigma = \Upsilon(\mathcal{S}) \cap \Phi(\Sigma)^+_1$, a maximal orthogonal subset in $\Phi(\Sigma)^+_1$. Write explicitly $\mathcal{S}_\Sigma = \{\gamma_1, \ldots, \gamma_s, \beta_1, \ldots, \beta_t\}$ with $\gamma_1, \ldots, \gamma_s$ long roots and $\beta_1, \ldots, \beta_t$ short roots. Let r_Σ be the rank of the Hermitian symmetric space corresponding to the pair (Σ, α_Σ) and let $\mathcal{S}'_\Sigma = \{\gamma_1, \ldots, \gamma_{r_\Sigma}\}$ be an orthogonal subset of maximal cardinality of $\Phi(\Sigma)^+_1$ containing $\{\gamma_1, \ldots, \gamma_s\}$. Then $\mathcal{S}' = \bigcup_\Sigma \Upsilon^{-1}(\mathcal{S}'_\Sigma)$ is an orthogonal subset of $\mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ of maximal cardinality.

By Proposition 5.7, the Hermitian pair (Σ, α_Σ) is of tube type, therefore by Lemma 4.11, upon relabelling $\gamma_{s+1}, \ldots, \gamma_{r_\Sigma}$, we may assume that $\beta_i = \frac{1}{2}(\gamma_{s+2i-1} + \gamma_{s+2i})$ for all $i = 1, \ldots, t$. By Corollary 5.6 it follows that

$$\sum_{i=1}^{s} \gamma_i + \sum_{j=1}^{t} 2\beta_j = \sum_{i=1}^{r_\Sigma} \gamma_i = \frac{1}{e_\Sigma}(k\delta - 2\alpha_p)_\Sigma.$$

Thus

$$e_\Sigma\Big(\sum_{i=1}^{s} \Upsilon^{-1}(\gamma_i) + \sum_{j=1}^{t} 2\Upsilon^{-1}(\beta_j)\Big) = (k\delta - 2\alpha_p)_\Sigma + e_\Sigma r_\Sigma \alpha_p.$$

Summing over all components $\Sigma \subset \Pi_0$, we get

$$\sum_{\eta \in \mathcal{S}} e_\eta \eta = k\delta - 2\alpha_p + \sum_\Sigma e_\Sigma r_\Sigma \alpha_p,$$

where for $\eta \in \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$ we define

$$e_\eta = -\langle \alpha_p, \Upsilon(\eta)^\vee \rangle = \begin{cases} e_\Sigma & \text{if } \Upsilon(\eta) \in \Phi(\Sigma)^+_1 \text{ is a long root,} \\ 2e_\Sigma & \text{if } \Upsilon(\eta) \in \Phi(\Sigma)^+_1 \text{ is a short root.} \end{cases}$$

Notice that by Lemma 5.1 and Corollary 5.6 i) we have $e_\eta = \langle \alpha_p, \eta^\vee \rangle = k_\mathcal{S} a_{\eta,\mathcal{S}}$ for all $\eta \in \mathcal{S}'$. Therefore by Corollary 5.6 ii) and Remark 5.4 we obtain

$$\sum_\Sigma e_\Sigma r_\Sigma = \sum_{\eta \in \mathcal{S}'} e_\eta = \sum_{\eta \in \mathcal{S}'} k_{\mathcal{S}'} a_{\eta,\mathcal{S}'} = 4.$$

On the other hand, by Lemma 5.1, we have $e_\eta = \langle \alpha_p, \eta^\vee \rangle$ for all $\eta \in \mathcal{S}$, therefore we get

$$\sum_{\eta \in \mathcal{S}} \langle \alpha_p, \eta^\vee \rangle \eta = \sum_{\eta \in \mathcal{S}} e_\eta \eta = k\delta + 2\alpha_p. \tag{5.4}$$

We now conclude the proof by showing that the orbit $G_0 x_\mathcal{S}$ does not depend on the maximal orthogonal subset $\mathcal{S} \subset \mathcal{C}^1_\sigma$, hence it is non-spherical by Theorem 5.5. We identify the orbit $G_0 x_\mathcal{S} \subset \mathfrak{g}_1$ by computing its weighted Dynkin diagram (see [8, Section 9.5]).

By Lemma 5.1 we have $\langle \eta, \alpha_p^\vee \rangle = 1$ for all $\eta \in \mathcal{C}^1_\sigma \setminus \{\alpha_p\}$. If $\alpha \in \Phi$, considering the normal $\mathfrak{sl}(2)$-triple $\{x_\mathcal{S}, h_\mathcal{S}, y_\mathcal{S}\}$, it follows that

$$\alpha(h_\mathcal{S}) = -\sum_{\eta \in \mathcal{S}} \langle \alpha, \eta^\vee \rangle \langle \eta, \alpha_p^\vee \rangle = -\frac{2}{||\alpha_p||^2}\Big(\alpha, \sum_{\eta \in \mathcal{S}} \langle \alpha_p, \eta^\vee \rangle \eta\Big).$$

By (5.4) we get then

$$\alpha(h_\mathcal{S}) = -\frac{2}{||\alpha_p||^2}(\alpha, k\delta + 2\alpha_p) = -2\langle \alpha, \alpha_p^\vee \rangle.$$

Since the right hand side of previous equality does not depend on $\mathcal{S}$, it follows that $G_0 x_\mathcal{S}$ does not depend on $\mathcal{S}$ either. $\square$

We conclude this Section by showing that $G_0\mathfrak{a}_p$ has complexity one. Recall that the complexity of an irreducible G_0-variety is defined as

$$c_{G_0}(X) = \min_{x\in X} \ \mathrm{codim} B_0 x.$$

In particular, the spherical G_0-varieties coincide with the G_0-varieties of complexity zero, and the complexity of a G_0-variety can be regarded as a measure of its non-sphericity. By Theorem 3.1 together with Theorem 5.8, if $\mathcal{S} \subset \mathcal{C}^1_\sigma$ is a maximal orthogonal subset, then $x_\mathcal{S}$ is in the open G_0-orbit of $G_0\mathfrak{a}_p$. Since the complexity of a G_0-variety coincides with that of its G_0-stable open subsets, it follows in particular that $c_{G_0}(G_0 x_\mathcal{S}) = 1$ for all maximal orthogonal subset $\mathcal{S} \subset \mathcal{C}^1_\sigma \smallsetminus \{\alpha_p\}$.

We start by recalling and proving some general results about the gradings associated to nilpotent elements of small height in $\mathbb{Z}_2$-graded Lie algebras.

5.1. **Remarks on the gradings associated to nilpotent elements of small height in $\mathfrak{g}_1$.** Let $x_1 \in \mathfrak{g}_1$ be a nilpotent element and let $\{x_1, h_0, y_1\}$ be a normal $\mathfrak{sl}(2)$-triple containing x_1. Denote $\mathfrak{p}_0 = \mathfrak{g}_0(\geqslant 0)$, $\mathfrak{u}_0 = \mathfrak{g}_0(\geqslant 1)$ and $\mathfrak{l}_0 = \mathfrak{g}_0(0)$, then $\mathfrak{p}_0$ is a parabolic subalgebra of $\mathfrak{g}_0$ with Levi factor $\mathfrak{l}_0$ and with nilradical $\mathfrak{u}_0$. Fix a Borel subalgebra $\mathfrak{b}_0 \subset \mathfrak{g}_0$ contained in $\mathfrak{p}_0$ and set $\mathfrak{b}_{00} = \mathfrak{b}_0 \cap \mathfrak{g}_0(0)$, a Borel subalgebra of $\mathfrak{l}_0$, so that $\mathfrak{b}_0 = \mathfrak{b}_{00} \oplus \mathfrak{u}_0$. Notice that $\mathfrak{g}_1(i)$ is $\mathfrak{l}_0$-stable for all $i \in \mathbb{Z}$.

Let P_0 be the parabolic subgroup of G_0 corresponding to $\mathfrak{p}_0$, $L_0 \subset P_0$ the Levi factor corresponding to $\mathfrak{l}_0$ and U_0 the unipotent radical of P_0, $B_0 \subset G_0$ the Borel subgroup corresponding to $\mathfrak{b}_0$, and $B_{00} \subset L_0$ be the Borel subgroup of L_0 corresponding to $\mathfrak{b}_{00}$. Recall that in this section the fixed points set $\mathfrak{g}_0$ of the involution is assumed to be semisimple and we denote by α_p the simple root in $\widehat{\Pi}$ corresponding to σ.

Proposition 5.9. *Suppose that $\mathfrak{g}_0$ is semisimple with corresponding simple root $\alpha_p \in \widehat{\Pi}$, and let $x_1 \in \mathfrak{g}_1$ be a nilpotent element with $\mathrm{ht}(x_1) = m$. Then $\mathfrak{g}_1(m)$ is an irreducible L_0-module. If moreover $\alpha(h_0) \geqslant 0$ for all $\alpha \in \Phi_0^+$, the highest weight vector of $\mathfrak{g}_1(m)$ is $x_1^{-\overline{\alpha}_p}$.*

Proof. Up to conjugating x_1 we may assume that $\alpha(h_0) \geqslant 0$ for all $\alpha \in \Phi_0^+$. Notice that $\mathfrak{u}_0$ acts trivially on $\mathfrak{g}_1(m)$, therefore every highest weight of $\mathfrak{g}_1(m)$ as a $\mathfrak{l}_0$-module is actually a highest weight for $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module. On the other hand, since $\mathfrak{g}_0$ is semisimple, $\mathfrak{g}_1$ is an irreducible $\mathfrak{g}_0$-module with highest weight vector $x_1^{-\overline{\alpha}_p}$, therefore $\mathfrak{g}_1(m)$ is an irreducible $\mathfrak{l}_0$-module as well, with highest weight vector $x_1^{-\overline{\alpha}_p}$. □

Assume furthermore that $\mathrm{ht}(x_1) \leqslant 4$ and $\mathfrak{g}_0(4) = 0$. For $i \geqslant 2$, set $\mathfrak{a}_i = \mathfrak{g}_1(\geqslant i)$. Notice that $\mathfrak{a}_i$ is a B_0-stable abelian subalgebra of $\mathfrak{g}$: indeed $\mathfrak{a}_i$ is $\mathfrak{p}_0$-stable, hence P_0-stable, and being $\mathrm{ht}(x_1) \leqslant 4$ and $\mathfrak{g}_0(4) = 0$, it follows that $[\mathfrak{a}_i, \mathfrak{a}_i] \subset \mathfrak{g}_0(4) = 0$. It follows then by Theorem 3.1 that $\mathfrak{a}_i$ possesses finitely many B_0-orbits, parametrized by the orthogonal subsets of $\Psi(\mathfrak{a}_i)$.

Proposition 5.10. *Suppose that $\mathrm{ht}(x_1) \leqslant 4$ and that $\mathfrak{g}_0(4) = 0$. Then $\mathfrak{g}_1(i)$ is a spherical L_0-module for all $i \geqslant 2$.*

Proof. Let $i \geqslant 2$ and consider the B_0-stable subalgebra $\mathfrak{a}_i$. By Theorem 3.1, there is $v_i \in \mathfrak{a}_i$ such that $\overline{B_0 v_i} = \mathfrak{a}_i$. Since $\mathfrak{a}_i = \mathfrak{g}_1(i) \oplus \mathfrak{a}_{i+1}$ and since $\mathfrak{a}_{i+1}$ is also B_0-stable, we may write $v_i = u_i + u_i'$, for some $u_i \in \mathfrak{g}_1(2)$ and $u_i' \in \mathfrak{a}_{i+1}$ with $u_i \neq 0$. Therefore

$$\mathfrak{g}_1(i) \oplus \mathfrak{a}_{i+1} = \mathfrak{a}_i = [\mathfrak{b}_0, v_i] \subset [\mathfrak{b}_{00}, u_i] \oplus ([\mathfrak{u}_0, u_i] + [\mathfrak{b}_0, u_i']).$$

Since $[\mathfrak{b}_{00}, u_i] \subset \mathfrak{g}_1(i)$ and $[\mathfrak{u}_0, u_i] + [\mathfrak{b}_0, u_i'] \subset \mathfrak{a}_{i+1}$, the equality $[\mathfrak{b}_{00}, u_i] = \mathfrak{g}_1(2)$ follows. Therefore $\mathfrak{g}_1(2) = \overline{B_{00} u_i}$ is a spherical L_0-module. □

5.2. **The complexity of $G_0\mathfrak{a}_p$.** We now apply the results of previous subsection to compute the complexity of $G_0\mathfrak{a}_p$. As we already noticed, it is enough to show that $c_{G_0}(G_0x_\mathcal{S})=1$ when $\mathcal{S}\subset\mathcal{C}^1_\sigma\smallsetminus\{\alpha_p\}$ is an orthogonal subset of maximal cardinality. In particular, under the bijection $\Upsilon:\mathcal{C}^1_\sigma\smallsetminus\{\alpha_p\}\to\bigcup_\Sigma\Phi(\Sigma)^+_1$ of Lemma 5.1, we may assume that $\alpha_\Sigma\in\Upsilon(\mathcal{S})$ for all components $\Sigma\subset\Pi_0$.
Let $\mathcal{S}\subset\mathcal{C}^1_\sigma\smallsetminus\{\alpha_p\}$ be an orthogonal subset of maximal cardinality, let $\{x_\mathcal{S},h_\mathcal{S},y_\mathcal{S}\}$ be the corresponding normal $\mathfrak{sl}(2)$-triple, let $\mathfrak{g}=\bigoplus\mathfrak{g}(i)$, and for $j=0,1$ set $\mathfrak{g}_j(i)=\mathfrak{g}_j\cap\mathfrak{g}(i)$. We keep the notation of previous subsection.

Let $K_0\subset L_0$ be the identity component of the stabilizer of $x_\mathcal{S}\in\mathfrak{g}_1(2)$. Then K_0 is reductive and $\mathfrak{g}_1(2)$ is a K_0-orthogonal module, therefore by a theorem of Luna there exists a reductive subgroup $M\subset K_0$ and a K_0-stable open subset $Z\subset\mathfrak{g}_1(2)$ such that every K_0-orbit in Z is isomorphic to K_0/M (see [18, Section 5]). Then by [18, Theorem 5.4] the following formula holds:

$$c_{G_0}(G_0x_\mathcal{S})=c_{L_0}(\mathfrak{g}_1(2))+c_M(\mathfrak{g}_1(\geqslant 3)).\tag{5.5}$$

Proposition 5.11. *We have $\mathfrak{g}_1(3)=\mathfrak{g}_0(4)=\{0\}$ and $\mathfrak{g}_1(4)=\mathfrak{g}_1^{-\overline{\alpha}_p}$ is the trivial one-dimensional representation of (L_0,L_0).*

Proof. By Theorem 5.5, we have $\alpha(h_\mathcal{S})=-2\langle\alpha,\alpha_p^\vee\rangle$ for all $\alpha\in\Phi$. In particular $\mathfrak{g}(3)=\{0\}$. If $\mathfrak{g}(4)^\alpha\neq\{0\}$, then $\langle\alpha,\alpha_p^\vee\rangle=-2$. Since α_p is long, it follows that $\alpha=-\overline{\alpha}_p$. Since α_p is non-complex, $\mathfrak{g}_0(4)=\{0\}$ and $\mathfrak{g}_1(4)=\mathfrak{g}_1^{-\overline{\alpha}_p}$.

By definition, L_0 is the Levi subgroup of G_0 whose set of simple roots is

$$\Pi_{00}=\bigcup_\Sigma\{\alpha\in\Sigma\mid\alpha(h_\mathcal{S})=0\}.$$

Let $\Sigma\subset\Pi_0$ be a component. Recall that $\alpha_\Sigma\in\Sigma$ is the unique simple root non-orthogonal to α_p. On the other hand by Theorem 5.8 we have $\alpha(h_\mathcal{S})=-2\langle\alpha,\alpha_p^\vee\rangle$, therefore $\Pi_{00}=\bigcup_\Sigma(\Sigma\smallsetminus\{\alpha_\Sigma\})$ and it follows that every simple root of L_0 is orthogonal to $-\overline{\alpha}_p$. This show that $\mathfrak{g}_1(4)=\mathfrak{g}_1^{-\overline{\alpha}_p}$ is the trivial one-dimensional representation of (L_0,L_0). □

Corollary 5.12. *Let $\mathcal{S}\subset\mathcal{C}^1_\sigma\smallsetminus\{\alpha_p\}$ be an orthogonal subset of maximal cardinality, then $c_{G_0}(G_0x_\mathcal{S})=1$. In particular, $c_{G_0}(G_0\mathfrak{a}_p)=1$.*

Proof. Proposition 5.10 implies that $c_{L_0}(\mathfrak{g}_1(2))=0$, whereas Proposition 5.11 shows that $\mathfrak{g}_1(\geqslant 3)=\mathfrak{g}_1(4)$ is one-dimensional, and by Theorem 5.8 together with (5.5) we get $1\leqslant c_{G_0}(G_0x_\mathcal{S})=c_M(\mathfrak{g}_1(4))\leqslant 1$. □

6. Classification of B_0-stable subalgebras of $\mathfrak{g}_1$

In this section $\mathfrak{g}$ is a semisimple Lie algebra and σ is an (indecomposable) involution of $\mathfrak{g}$. Theorem 3.1 prompts us to study the orbits $G_0x\subset\mathfrak{g}_0\oplus\mathfrak{g}_1$ with x of the form

$$x_\mathcal{S}=\sum_{\gamma\in\mathcal{S}_0}x_0^\gamma+\sum_{\gamma\in\mathcal{S}_1}x_1^\gamma,\tag{6.1}$$

where $\mathcal{S}_0\subset\Phi_0$, $\mathcal{S}_1\subset\Phi_1$, $\mathcal{S}_0\cap\mathcal{S}_1=\emptyset$, and $\mathcal{S}_0\cup\mathcal{S}_1$ is a set of strongly orthogonal weights in $\Phi=\Phi_0\cup\Phi_1$. We denote by $\mathcal{S}$ the disjoint union of $\mathcal{S}_0$ and $\mathcal{S}_1$. Notice that all $x_\mathcal{S}$ are nilpotent: indeed setting $y_\mathcal{S}=\sum_{\gamma\in\mathcal{S}_0}y_0^{-\gamma}+\sum_{\gamma\in\mathcal{S}_1}y_1^{-\gamma}$ and $h_\mathcal{S}=\sum_{\gamma\in\mathcal{S}_1}\gamma^\vee+\sum_{\gamma\in\mathcal{S}_2}\gamma^\vee$ we get a $\mathfrak{sl}(2)$-triple $\{x_\mathcal{S},h_\mathcal{S},y_\mathcal{S}\}$.

Let m be the height of $x_\mathcal{S}$. Since $h_\mathcal{S}\in\mathfrak{h}_0$, we can choose a weight $\alpha\in\Phi$ such that $\mathfrak{g}^\alpha\subset\mathfrak{g}(m)$, namely such that

$$\alpha(h_\mathcal{S})=\sum_{\gamma\in\mathcal{S}}\langle\alpha,\gamma^\vee\rangle=m.$$

Set $\mathcal{S}^+(\alpha) = \{\gamma \in \mathcal{S} \mid \langle \alpha, \gamma^\vee \rangle > 0\}$, so that

$$\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \alpha, \gamma^\vee \rangle \geqslant m. \tag{6.2}$$

Define $\hat{\alpha} = \alpha$ if $\alpha \in \Phi_0$, and $\hat{\alpha} = \delta' + \alpha$ if $\alpha \in \Phi_1 \smallsetminus \Phi_0$. Choose for each $\gamma \in \mathcal{S}^+(\alpha)$ a root $\hat{\gamma} \in \widehat{\Phi}$ such that $\overline{\hat{\gamma}} = \gamma$, and define $\widehat{\mathcal{S}^+(\alpha)} = \{\hat{\gamma} \mid \gamma \in \mathcal{S}^+(\alpha)\}$ and $\Pi_{\mathcal{S},\alpha} = \widehat{\mathcal{S}^+(\alpha)} \cup \{-\hat{\alpha}\}$. As $\mathcal{S}^+(\alpha) \cup \{-\alpha\} \subset \Phi$, we have that $\Pi_{\mathcal{S},\alpha} \subset \widehat{\Phi}^{re}$, so the matrix $A(\mathcal{S}, \alpha) = (\langle \beta, \xi^\vee \rangle)_{\beta, \xi \in \Pi_{\mathcal{S},\alpha}}$ is a generalized Cartan matrix, which is of finite or affine type by Lemma 5.2.

Lemma 6.1. *Let $\mathcal{S} \subset \Phi$ be a strongly orthogonal subset, and let $\alpha \in \Phi$ be such that $\alpha(h_\mathcal{S}) = \mathrm{ht}(x_\mathcal{S})$. Then the following statements hold.*

i) *$\mathrm{ht}(x_\mathcal{S})$ is less than or equal to the degree of $-\hat{\alpha}$ in $\Pi_{\mathcal{S},\alpha}$. In particular $\mathrm{ht}(x_\mathcal{S}) \leqslant 4$.*

ii) *If $Gx_\mathcal{S}$ is not spherical, then $\mathrm{ht}(x_\mathcal{S}) = 4$ and $\Pi_{\mathcal{S},\alpha}$ is of affine type, in which case its diagram is one of those listed in Table 2.*

Proof. i) If $\gamma \in \mathcal{S}^+(\alpha)$, notice that $\langle \alpha, \gamma^\vee \rangle = \langle \hat{\alpha}, \hat{\gamma}^\vee \rangle$, and that this number is less or equal to the number of edges connecting $-\hat{\alpha}$ with $\hat{\gamma}$. Therefore the claim follows by formula (6.2), by observing that the degree of any node in a finite or affine diagram is at most 4.

ii) By [17, Theorem 3.1], if $Gx_\mathcal{S}$ is not spherical then $\mathrm{ht}(x_\mathcal{S}) \geqslant 4$, hence $\mathrm{ht}(x_\mathcal{S}) = 4$ by i). Since in a Dynkin diagram of finite type any node has degree at most 3, it follows that $\Pi_{\mathcal{S},\alpha}$ is affine. Moreover, if $\langle \hat{\alpha}, \hat{\gamma}^\vee \rangle$ is less than the number of edges connecting $\hat{\alpha}$ and $\hat{\gamma}$ for some $\gamma \in \mathcal{S}^+(\alpha)$, then $\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \hat{\alpha}, \hat{\gamma}^\vee \rangle < 4$. Thus, for all $\gamma \in \mathcal{S}^+(\alpha)$, $\langle \hat{\alpha}, \hat{\gamma}^\vee \rangle$ equals the number of edges connecting $\hat{\alpha}$ and $\hat{\gamma}$, and $\hat{\alpha}$ is long in $\Pi_{\mathcal{S},\alpha}$. It follows that the diagram $\Pi_{\mathcal{S},\alpha}$ is one of the affine diagrams listed in Table 2. □

In the next result we use the main idea of Proposition 2.2 of [22].

Lemma 6.2. *Let $\mathcal{S} \subset \Phi$ be a strongly orthogonal subset and suppose that $\mathcal{S} \subset \Psi(\mathfrak{a})$ for some $\mathfrak{a} \in \mathcal{I}^\sigma_{ab}$. Let $\alpha \in \Phi$ be such that $\alpha(h_\mathcal{S}) = \mathrm{ht}(x_\mathcal{S})$, then $\alpha \in \Phi_1 \smallsetminus \Phi_0$ and*

$$\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \alpha, \gamma^\vee \rangle \hat{\gamma} = k\delta + 2\hat{\alpha}.$$

In particular, $\mathrm{ht}_0(x_\mathcal{S}) \leqslant 3$ and $\sum_{\gamma \in \mathcal{S}^+(\alpha)} \langle \beta, \gamma^\vee \rangle = 2\langle \beta, \alpha^\vee \rangle$ for all $\beta \in \Phi \cup \{0\}$.

Proof. Since $\mathcal{S}^+(\alpha) \subset \Phi_1$ we can assume that $\hat{\gamma} = \delta' + \gamma$ for all $\gamma \in \mathcal{S}^+$. As in the proof of Theorem 5.5, we find that, if $\Pi_{\mathcal{S},\alpha} = \Xi$, then $k_\Xi(\sum_{\xi \in \Xi} a_{\Xi,\xi} \xi)$ is an isotropic vector, hence it is a multiple of δ, say

$$k_\Xi(\sum_{\xi \in \Xi} a_{\Xi,\xi} \xi) = s\delta. \tag{6.3}$$

The coefficient s can be computed by counting the occurrences of roots in Φ_1^+ in the left hand side of (6.3). It follows that $s = 2k$ if $\alpha \in \Phi_0$, and $s = k$ if $\alpha \in \Phi_1 \smallsetminus \Phi_0$.

Define a multiset $\{\hat{\gamma}_i \mid i = 1, \cdots, 4\}$ by listing every $\hat{\gamma} \in \widehat{\mathcal{S}^+(\alpha)}$ with multiplicity $\langle \hat{\alpha}, \hat{\gamma}^\vee \rangle = k_\Xi a_{\Xi,\hat{\gamma}}$ (see Remark 5.4). Set $\beta = k\delta + \hat{\alpha}$. Since $\hat{\alpha} = \alpha \in \Phi_0$, it follows that β is a root. Notice that $\beta - \hat{\gamma}_i - \hat{\gamma}_j$ is a root for all $i \neq j$. If indeed $\hat{\gamma}_i = \hat{\gamma}_j$ for some $i \neq j$, then $\langle \hat{\alpha}, \hat{\gamma}_i^\vee \rangle \geqslant 2$, hence $\langle \beta, \hat{\gamma}_i^\vee \rangle \geqslant 2$ and $\beta - 2\hat{\gamma}_i$ is a root. If instead $\hat{\gamma}_i \neq \hat{\gamma}_j$, then $\langle \hat{\alpha}, \hat{\gamma}_i^\vee \rangle > 0$ and $\langle \hat{\alpha} - \hat{\gamma}_i, \hat{\gamma}_j^\vee \rangle > 0$, so $\beta - \hat{\gamma}_i - \hat{\gamma}_j$ is either a root or zero. On the other hand $\hat{\gamma}_i + \hat{\gamma}_j$ cannot be a root because $\mathfrak{a}$ is abelian, therefore $\beta - \hat{\gamma}_i - \hat{\gamma}_j$ is a root also in this case.

Since $\sum_{i=1}^4 \hat\gamma_i = \sum_{\gamma\in\mathcal{S}^+(\alpha)} k_\Xi a_{\Xi,\hat\gamma}\hat\gamma = 2k\delta + 2\hat\alpha = 2\beta$, we have

$$\sum_{i<j}(\beta - \hat\gamma_i - \hat\gamma_j) = 6\beta - 3\sum_{i=1}^4 \hat\gamma_i = 0,$$

thus $\beta - \hat\gamma_i - \hat\gamma_j$ is a positive root for some $i < j$. Since $\hat\gamma_i \in \widehat\Phi_1^+$ for each i, $\beta - \hat\gamma_i - \hat\gamma_j$ is a root in $\widehat\Phi_0^+$. Since $\langle\beta, \hat\gamma_i^\vee\rangle > 0$ and $\mathrm{ht}_\sigma(\beta - \hat\gamma_i) = 1$, $\beta - \hat\gamma_i \in \widehat\Phi_1$. Since $\beta - \hat\gamma_i = \hat\gamma_j + (\beta - \hat\gamma_i - \hat\gamma_j)$, we see that $\overline{\beta - \hat\gamma_i} \in \Psi(\mathfrak{a})$. Since $\overline{\beta - \hat\gamma_i}$ and γ_i are both in $\Psi(\mathfrak{a})$ and $\beta - \hat\gamma_i + \hat\gamma_i = \beta$, we reach a contradiction since $\mathfrak{a}$ is abelian.

To prove the last claim, notice that α is long in $\Pi_{\mathcal{S},\alpha}$, hence $\langle\gamma, \alpha^\vee\rangle = 1$. If $\beta \in \Phi_0 \cup \Phi_1 \cup \{0\}$, we get then the equality

$$\sum_{\gamma\in\mathcal{S}^+(\alpha)} \langle\beta, \gamma^\vee\rangle = \frac{2}{||\alpha||^2}\sum_{\gamma\in\mathcal{S}^+(\alpha)} \langle\alpha, \gamma^\vee\rangle(\beta, \hat\gamma) = 2\frac{(\beta, k\delta + 2\hat\alpha)}{||\alpha||^2} = 2\langle\beta, \alpha^\vee\rangle.$$

□

As a consequence of Lemma 6.2, we get the following result.

Corollary 6.3. *Let $\mathfrak{a} \in \mathcal{I}^\sigma_{ab}$ and let $x \in \mathfrak{a}$, then $\mathrm{ht}(x) \leqslant 4$ and $\mathrm{ht}_0(x) \leqslant 3$. In particular, Gx is spherical if and only if $\mathrm{ht}_1(x) \leqslant 3$, if and only if G_0x is spherical.*

Proof. By Theorem 3.1, acting with B_0 we may assume that $x = x_\mathcal{S}$ for some orthogonal subset $\mathcal{S} \subset \Psi(\mathfrak{a})$. Then by Lemma 6.1 we get $\mathrm{ht}(x) \leqslant 4$, and by Lemma 6.2 we get $\mathrm{ht}_0(x) \leqslant 3$. The last claim follows by [18, Theorem 5.6]. □

If in previous corollary we take x in the open B_0-orbit of $\mathfrak{a}$, then we get the following.

Corollary 6.4. *Let $\mathfrak{a} \in \mathcal{I}^\sigma_{ab}$, then $\mathfrak{a}$ is G-spherical if and only if it is G_0-spherical.*

Recall that Π_1 contains at most two elements, and that if $\widehat\Pi$ is simply laced, then the real roots of $\widehat\Phi$ are regarded as long. The next result has been proved in [20] as a consequence of a case-by-case inspection. We provide here a conceptual proof that follows from Lemma 6.2 and the results of Section 5.1. Note also that Theorem 6.5 includes Theorem 2.3 of [22].

Theorem 6.5. *There exists $\mathfrak{a} \in \mathcal{I}^\sigma_{ab}$ such that $G_0\mathfrak{a}$ is not spherical if and only if $\Pi_1 = \{\alpha_p\}$ and α_p is long and non-complex.*

Proof. If $\Pi_1 = \{\alpha_p\}$ with α_p long and non-complex then by Theorem 5.5 the special B_0-stable subalgebra α_p gives rise to a non-spherical variety $G_0\mathfrak{a}_p$.

Let now $\mathfrak{a} \in \mathcal{I}^\sigma_{ab}$ and suppose that $G_0\mathfrak{a}$ is not spherical. By Theorem 3.1 there is an orthogonal subset $\mathcal{S} \subset \Psi(\mathfrak{a})$ such that $G_0x_\mathcal{S}$ is not spherical, and by Lemma 6.1 we get $\mathrm{ht}(x_\mathcal{S}) = 4$. Fix $\alpha \in \Phi$ such that $\alpha(h_\mathcal{S}) = 4$, then $\alpha \in \Phi_1 \setminus \Phi_0$ by Lemma 6.2. Set $\hat\alpha = \delta' + \alpha \in \widehat\Phi_1$.

If $\Pi_1 = \{\alpha_i, \alpha_j\}$ consists of two distinct elements, then $k = 1$. Moreover, since $\hat\alpha \in \widehat\Phi_1^+$, we can assume $[\hat\alpha : \alpha_i] = 1$ and $[\hat\alpha : \alpha_j] = 0$. By Lemma 6.2 we have

$$\sum_{\gamma\in\mathcal{S}^+(\alpha)} \langle\alpha, \gamma^\vee\rangle\hat\gamma - 2\hat\alpha = \delta.$$

Being $(\hat\gamma, \hat\alpha) > 0$, for all $\gamma \in \mathcal{S}^+(\alpha)$ relation $\hat\gamma - \hat\alpha \in \widehat\Phi \cup \{0\}$ holds. Therefore $[\hat\gamma : \alpha_i] = 1$ for all $\gamma \in \mathcal{S}^+(\alpha)$, and we get $[\delta : \alpha_i] = 2$ which is absurd.

Thus $\Pi_1 = \{\alpha_p\}$ consists of a single element, and $\mathfrak{g}_0$ is semisimple. By Proposition 5.9, we can choose $\alpha = w(-\overline\alpha_p)$ with $w \in W_0$. Since $\alpha \in \Phi_1 \setminus \Phi_0$, we see that α_p cannot be complex. Suppose that α_p is short. Then also α is short, and there must be a component $\Sigma \subset \Pi_0$ such that θ_Σ is long. Since $(\theta_\Sigma, \alpha_p) = (\alpha_\Sigma, \alpha_p) < 0$, it follows that $\langle\theta_\Sigma, \alpha_p^\vee\rangle \leqslant -2$, and setting $\beta = w(\theta_\Sigma)$ we get $\langle\beta, \alpha^\vee\rangle \geqslant 2$. Since $\alpha(\sum_{\gamma\in\mathcal{S}^+(\alpha)}\gamma^\vee) = \mathrm{ht}(x_\mathcal{S}) = 4$, it follows that the element $x_{\mathcal{S}^+(\alpha)} = \sum_{\gamma\in\mathcal{S}^+(\alpha)} x_\gamma$ is still in $\mathfrak{a}$, and being $\mathrm{ht}(x_{\mathcal{S}^+(\alpha)}) \geqslant 4$ its G_0-orbit is still non-spherical by Corollary 6.3. Therefore we can assume that $\mathcal{S} = \mathcal{S}^+(\alpha)$. By Lemma 6.2 we get then

$$\beta(\sum_{\gamma\in\mathcal{S}}\gamma^\vee) = \beta(\sum_{\gamma\in\mathcal{S}^+(\alpha)}\gamma^\vee) = 2\langle\beta,\alpha^\vee\rangle \geqslant 4.$$

As $\beta \in \Phi_0$ and $\mathfrak{g}_0(i) = 0$ for $i > 3$, this is absurd. Therefore α_p must be long. □

Assume now that $\Pi_1 = \{\alpha_p\}$ with α_p long and non-complex. We give a classification of the subalgebras $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$ such that $G_0\mathfrak{a}$ is non-spherical. By Proposition 4.12, for any component $\Sigma \subset \Pi_0$ there is a unique maximally orthogonal antichain $\mathcal{A}_\Sigma$ in $\Phi(\Sigma)_1^+$. Regarding $\mathcal{C}_\sigma^1$ as a poset w.r.t. the dominance order, it is clear from Lemma 5.1 that $\mathcal{C}_\sigma^1 \smallsetminus \{\alpha_p\}$ contains a unique maximally orthogonal antichain $\mathcal{A}$, namely

$$\mathcal{A} = \bigcup_\Sigma \Upsilon^{-1}(\mathcal{A}_\Sigma).$$

For $\Gamma \subset \mathcal{C}_\sigma^1$, we set $\Gamma^{\leqslant_0} = \{\xi \in \mathcal{C}_\sigma^1 \mid \xi \leqslant_0 \eta \text{ for some } \eta \in \Gamma\}$.

Lemma 6.6. *There is $\overline{w} \in \mathcal{W}_\sigma^{ab}$ such that $N(\overline{w}) = \mathcal{A}^{\leqslant_0}$.*

Proof. We show that, if $\zeta, \xi \in \widehat{\Phi}^+$ are such that $\zeta + \xi \in \mathcal{A}^{\leqslant_0}$, then exactly one among ζ and ξ is in $\mathcal{A}^{\leqslant_0}$. Since $\mathcal{A}^{\leqslant_0} \subset \mathcal{C}_\sigma^1 = N(w_p)$, then exactly one among ζ and ξ (say ζ) is in $\mathcal{C}_\sigma^1$. Since $\zeta + \xi \in \mathcal{A}^{\leqslant_0}$, then $\zeta \in \mathcal{A}^{\leqslant_0}$. This implies that both $\mathcal{A}^{\leqslant_0}$ and its complement are closed under root addition. It follows that there is $\overline{w} \in \widehat{W}$ such that $N(\overline{w}) = \mathcal{A}^{\leqslant_0}$. Since $N(\overline{w}) \subset N(w_p)$, it is clear that $\overline{w} \in \mathcal{W}_\sigma^{ab}$. □

Let $\overline{\mathfrak{a}} = \Theta(\overline{w})$ (see Proposition 2.5).

Theorem 6.7. *Suppose that $\Pi_1 = \{\alpha_p\}$ with α_p long and non-complex, and let $\mathfrak{a} \in \mathcal{I}_{ab}^\sigma$. Then $G_0\mathfrak{a}$ is not spherical if and only if $\{-\overline{\eta} \mid \eta \in \mathcal{A}\} \subset \Psi(\mathfrak{a})$, if and only if $\overline{\mathfrak{a}} \subset \mathfrak{a}$.*

Proof. Suppose that $G_0\mathfrak{a}$ is non-spherical, and let $\mathcal{S} \subset \Psi(\mathfrak{a})$ be an orthogonal subset such that $G_0 x_\mathcal{S}$ is not spherical. Then by Lemma 6.1 we have $\mathrm{ht}(x_\mathcal{S}) = 4$, and by Corollary 6.3 there is $\alpha \in \Phi_1$ such that $\mathfrak{g}_1(4)^\alpha \neq 0$, namely $\alpha(h_\mathcal{S}) = \sum_{\gamma\in\mathcal{S}}\langle\alpha,\gamma^\vee\rangle = 4$. We may assume that α is maximal w.r.t. $\leqslant_0$ among the weights of $\mathfrak{g}_1(4)$. As in the proof of Theorem 6.5, we can assume that $\mathcal{S} = \mathcal{S}^+(\alpha)$.

Suppose that $\alpha = -\overline{\alpha}_p$. We have that $\hat\alpha = \delta' - \overline{\alpha}_p = -\alpha_p + 2\delta' = k\delta - \alpha_p$. If $\gamma \in \mathcal{S}$, since α_p is long we get

$$\langle\delta' - \gamma, (k\delta + \alpha_p)^\vee\rangle = \langle\gamma, \alpha^\vee\rangle = 1,$$

hence $\delta' - \gamma \in \mathcal{C}_\sigma^1$. Thus $\mathcal{O} = \{\delta' - \gamma \mid \gamma \in \mathcal{S}\}$ is an orthogonal subset of $\mathcal{C}_\sigma^1$. Notice moreover that $\mathcal{O}$ is a maximal orthogonal subset in $\mathcal{C}_\sigma^1$: if indeed $\eta \in \mathcal{O}^\perp \cap \mathcal{C}_\sigma^1$, then $\Pi_{\mathcal{S},\alpha} \cup \{\eta\}$ gives rise to a generalized Cartan matrix which is neither finite nor affine, contradicting Lemma 5.2. Therefore $\Upsilon(\mathcal{O})$ is a maximal orthogonal set in $\bigcup_\Sigma \Phi(\Sigma)_1^+$, and by Corollary 4.14 it follows that $\bigcup_\Sigma \mathcal{A}_\Sigma \subset \Upsilon(\mathcal{O})^{\leqslant_0}$. By Lemma 5.1, it follows that $\mathcal{A} \subset \mathcal{O}^{\leqslant_0} \subset N(\Theta^{-1}(\mathfrak{a}))$ which in turns means that $N(\overline{w}) \subset N(\Theta^{-1}(\mathfrak{a}))$, or, equivalently, that $\overline{\mathfrak{a}} \subset \mathfrak{a}$.

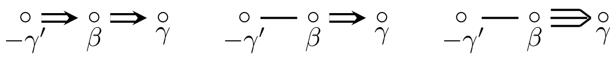
Suppose that $\alpha \neq -\overline{\alpha}_p$. Then there exists $\beta \in \Phi_0^+$ such that $\alpha+\beta \in \Phi_0\cup\Phi_1\cup\{0\}$, and the maximality of α among the weights of $\mathfrak{g}_1(4)$ implies that $\sum_{\gamma\in\mathcal{S}}\langle\alpha+\beta,\gamma^\vee\rangle < 4$. Therefore by Lemma 6.2 we get

$$2\langle\beta,\alpha^\vee\rangle = \sum_{\gamma\in\mathcal{S}}\langle\beta,\gamma^\vee\rangle < 0. \tag{6.4}$$

In particular there is γ such that $(\beta,\gamma)<0$ and, by Lemma 3.3 it follows $(\beta,\gamma')\geqslant 0$ for all $\gamma'\in\mathcal{S}\setminus\{\gamma\}$.

Suppose that $(\beta,\gamma')=0$ for all $\gamma'\in\mathcal{S}$. Then $s_\beta(\mathcal{S})$ is an orthogonal subset of $\Psi(\mathfrak{a})$ and $G_0x_{s_\beta(\mathcal{S})}=G_0x_{\mathcal{S}}$ is still not spherical, and $s_\beta(\alpha)$ is a maximal weight in Φ_1 w.r.t. $\leqslant_0$ such that $s_\beta(\alpha)(h_{s_\beta(\mathcal{S})})=\mathrm{ht}(x_{s_\beta(\mathcal{S})})=4$. On the other hand by (6.4) we have $\alpha\leqslant_0 s_\beta(\alpha)$, therefore we may proceed inductively by replacing $x_{\mathcal{S}}$ with $x_{s_\beta(\mathcal{S})}$ until either $\alpha=-\overline{\alpha}_p$ or we find a root $\gamma'\in\mathcal{S}$ such that $\langle\gamma',\beta^\vee\rangle>0$.

If $\alpha=-\overline{\alpha}_p$ then we are done, therefore we may assume that there are $\gamma,\gamma'\in\mathcal{S}$ such that $(\gamma,\beta)<0$ and $(\gamma',\beta)>0$. Consider the set $\Pi_\beta=\{\gamma,\beta,-\gamma'\}$, then $A_\beta=(\langle\nu,\xi^\vee\rangle)_{\nu,\xi\in\Pi_\beta}$ is a generalized Cartan matrix, and by Lemma 5.2 it is either of affine or of finite type. Identify Π_β with the corresponding Dynkin diagram; since $(\beta,\gamma')\geqslant 0$ for all $\gamma'\in\mathcal{S}\setminus\{\gamma\}$, by (6.4) we have that $\langle\beta,\gamma^\vee\rangle\leqslant -2$, so Π_β is not simply laced and γ is a short node. If moreover $\langle\beta,\gamma^\vee\rangle=-2$, then again by (6.4) it follows that $\langle\beta,\gamma'^\vee\rangle=1$. With these conditions at hand, we see that the only possibilities for the diagram of Π_β are the following:



The first case has to be discarded because, otherwise, the diagram of $\Pi_{\mathcal{S},\alpha}$ would have rank three with two nodes γ,γ' satisfying $\frac{\|\gamma'\|^2}{\|\gamma\|^2}=4$, and this never occurs for the diagrams of Table 2. The second case also has to be discarded, otherwise by Lemma 6.2 it would follow

$$2\langle\beta,\alpha^\vee\rangle=\sum_{\gamma\in\mathcal{S}}\langle\beta,\gamma^\vee\rangle=-1.$$

Therefore the diagram of Π_β is of type $G_2^{(1)}$. This is possible only if $\widehat{L}(\mathfrak{g},\sigma)$ is of type $G_2^{(1)}$. Let $\widehat{\Pi}=\{\alpha_0,\alpha_1,\alpha_2\}$ be as in [11, Table Aff 1] (in particular, α_2 is short). Since β is long and belongs to Φ_0^+, we have that $\beta=\alpha_0$. Moreover, $-\gamma'+2\beta+3\gamma$ is isotropic so $-\hat\gamma'+2\beta+3\hat\gamma=\delta$, hence $-\hat\gamma'+3\hat\gamma=3\alpha_0+2\alpha_1+3\alpha_2$. Since $\hat\gamma',\hat\gamma\in\widehat{\Phi}_1^+$, we have that $[\hat\gamma:\alpha_0]\leqslant 1$ and $[\gamma':\alpha_0]\leqslant 1$. It follows that $\hat\gamma'=\alpha_1+x\alpha_2$ and $\hat\gamma=\alpha_0+\alpha_1+y\alpha_2$. From $[-\hat\gamma'+3\hat\gamma:\alpha_2]=3$ and $(\hat\gamma',\hat\gamma)=0$ we obtain that either $\hat\gamma'=\alpha_1,\hat\gamma=\alpha_0+\alpha_1+\alpha_2$ or $\hat\gamma'=\alpha_1+3\alpha_2,\hat\gamma=\alpha_0+\alpha_1+2\alpha_2$. In both cases one easily verifies that $\overline{\hat\gamma}$ cannot belong to $\Psi(\mathfrak{a})$ with $\mathfrak{a}\in\mathcal{I}_{ab}^\sigma$. Hence we have obtained the desired contradiction. We conclude that $\alpha=-\overline{\alpha}_p$, and the proof is complete. □

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