# A description of all possible decay rates for solutions of some semilinear parabolic equations

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**Abstract**

We consider an abstract first order evolution equation in a Hilbert space in which the linear part is represented by a self-adjoint nonnegative operator $A$ with discrete spectrum, and the nonlinear term has order greater than one at the origin. We investigate the asymptotic behavior of solutions.

We prove that two different regimes coexist. Close to the kernel of $A$ the dynamic is governed by the nonlinear term, and solutions (when they decay to 0) decay as negative powers of $t$. Close to the range of $A$, the nonlinear term is negligible, and solutions behave as solutions of the linearized problem. This means that they decay exponentially to 0, with a rate and an asymptotic profile given by a simple mode, namely a one-frequency solution of the linearized equation.

The abstract results apply to semilinear parabolic equations.

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**Key words:** semilinear parabolic equation, decay rates, slow solutions, exponentially decaying solutions, Dirichlet quotients.

# 1 Introduction

In this paper we study the asymptotic behavior of decaying solutions to the first order evolution equation

$$u'(t) + Au(t) = f(u(t)) \qquad \forall t \geq 0, \tag{1.1}$$

where $A$ is a self-adjoint linear operator on a Hilbert space $H$ and $f$ is a nonlinear term.

We assume that $A$ is non-negative, but not necessarily strictly positive, and that its spectrum is a finite set or an increasing sequence of eigenvalues. We also assume that the nonlinear term has order greater than one in the origin, in the sense that it satisfies inequalities such as

$$|f(u)| \leq K_0 \left(|u|^{1+p} + |A^{1/2}u|^{1+q}\right) \tag{1.2}$$

for some positive exponents $p$ and $q$.

As a model example, we have in mind semilinear parabolic equations such as

$$u_t - \Delta u + |u|^p u = 0$$

with Neumann boundary conditions in a bounded domain $\Omega \subseteq \mathbb{R}^n$, or

$$u_t - \Delta u - \lambda_1(\Omega)u + |u|^p u = 0$$

with Dirichlet boundary conditions in a bounded domain $\Omega \subseteq \mathbb{R}^n$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet Laplacian. We point out that in both cases the operator associated to the linear part has a nontrivial kernel.

These two model examples have been investigated in the last decade in a series of papers by the third author and collaborators. Let us outline their achievements and the main problems which remained open in order to motivate our analysis.

In the Neumann case a simple application of the maximum principle gives that all solutions decay at least as $t^{-1/p}$, even in the $L^\infty$ norm. Moreover, it is easy to exhibit examples both of solutions decaying exactly as $t^{-1/p}$, and of solutions decaying exponentially. To this end, in the first case it is enough to consider spatially homogeneous solutions $u(t,x) := v(t)$, where necessarily $v(t)$ satisfies $v'(t) + |v(t)|^p v(t) = 0$, and observe that all nontrivial solutions of this ordinary differential equation decay to 0 exactly as $t^{-1/p}$. In the second case it is enough to consider a symmetric domain $\Omega$, for example the interval $(-1,1)$ in one dimension, and restrict ourselves to odd functions. The Neumann Laplacian, when restricted to odd functions, becomes a coercive operator, hence it is not difficult to see that all non-trivial solutions with odd initial datum decay to 0 exponentially.

This means that in the Neumann case there is coexistence between *slow solutions*, decaying as negative powers of $t$, and *fast solutions* decaying exponentially. It was later shown in [2] that actually one has the so called *slow-fast alternative*, meaning that all non-zero solutions decay to 0 either exponentially, or exactly as $t^{-1/p}$. Finally, in [4] it was shown that all initial data which are small enough and close enough to constant functions in the norm of $L^\infty(\Omega)$ give rise to slow solutions. Despite of these achievements, several problems remained open, for example describing all possible exponential decay rates, or even proving the existence of fast solutions when $\Omega$ has no special symmetry, or providing an explicit description of an open set of initial data in the norm of the phase space $L^2(\Omega)$ originating slow solutions.

The Dirichlet case proved to be more difficult to tackle. It is still true, but more delicate to establish, that all solutions decay to 0 at least as $t^{-1/p}$. Moreover, a comparison argument with suitable sub-solutions was enough to prove that all solutions with non-negative initial data are actually slow solutions. Both results have been proved in [12] (see also [13]). Later on, a weak form of slow-fast alternative was established in [3], meaning that all non-zero solutions decay either as $t^{-1/p}$, or faster than all negative powers of $t$. In the same paper also the existence of exponentially fast solutions was shown, but assuming the domain $\Omega$ to be symmetric, and the existence of intermediate decay rates was not excluded. Several problems remained open, among them existence of an open set of slow solutions, description of fast solutions, existence of fast solutions in domains without symmetries, and the true slow-fast alternative.

We stress that all these results were proved exploiting special symmetries, or the existence of $A$-invariant subspaces of $H$ which are invariant also for the nonlinear term, or comparison arguments.

What was missing is a unifying abstract theory. Filling this gap is the aim of this paper. So we consider the abstract evolution equation (1.1) and we address the following issues.

(1) Existence of an open set of slow solutions.

(2) Existence of fast solutions.

(3) Slow-fast alternative.

(4) Classification of all possible decay rates.

(5) Description of the set of solutions with a prescribed decay rate.

In Theorem 2.1 we focus on the slow-fast alternative. Instead of (1.1) and (1.2), we consider more generally an evolution inequality of the form

$$|u'(t) + Au(t)| \leq K_0 \left(|u(t)|^{1+p} + |A^{1/2}u(t)|^{1+q}\right) \qquad \forall t \geq 0, \tag{1.3}$$

and we prove that all its non-zero solutions, when they decay to 0, decay either exponentially or at most as $t^{-1/p}$. We prove also that slow solutions move closer and closer to the kernel of $A$, in the sense that $|A^{1/2}u(t)|$ decays faster than $|u(t)|$. This is clearly impossible if $A$ is strictly positive, in which case slow solutions cannot exist. Finally, we prove that fast solutions have an asymptotic profile of the form $u(t) \sim v_0 e^{-\lambda t}$, where $\lambda$ is an eigenvalue of $A$ and $v_0 \neq 0$ is a corresponding eigenvector. This settles (3) and (4).

In Theorem 2.6 we prove that slow solutions of (1.1) exist whenever $\ker(A)$ is non-trivial and the nonlinear term satisfies (1.2) and an additional sign condition allowing the existence of global solutions. What we actually show is the existence of an open set of initial data generating slow solutions, and this open set is characterized by simple explicit inequalities such as (3.48). This settles (1).

Finally, in Theorem 2.9 we address the existence of fast solutions. We prove that for every eigenvalue $\lambda$ of $A$, and every corresponding eigenvector $v_0 \neq 0$ which is small enough, there exists a nonempty set of initial data originating solutions whose asymptotic profile is exactly $v_0 e^{-\lambda t}$. This nonempty set is parametrized by an open set in the subspace of $H$ generated by all eigenvectors of $A$ greater than $\lambda$, in analogy with solutions of the linearized equation (see also Remark 2.11). This settles (2) and (5).

Our results are apparently new, at least in the sense that they are not explicitly stated elsewhere, even in the special situation where $A$ is strictly positive and $H$ is a finite dimensional space, in which case (1.1) reduces to a system of ordinary differential equations. That case is usually handled by means of Lyapunov functions or linearization theorems. Lyapunov functions lead to a simple proof that all solutions decay exponentially. Classical linearization theorems (see [14, 15, 16]) state that in a neighborhood of the origin the dynamic induced by the nonlinear system is homeomorphic to the dynamic induced by the linearized one. Nevertheless, both methods do not lead to a classification of decay rates, the first one because it only provides an estimate from above, the second one because the homeomorphisms given by the linearization theorems are just Hölder continuous, hence they do not preserve decay rates.

When $H$ is infinite dimensional and $A$ is strictly positive, our results seem to be new as well, at least in full generality. The only related literature we are aware of is [6], where an analogous classification of exponential decay rates has been provided for solutions of the Navier-Stokes equation in a bounded domain, in which case the operator $A$ is coercive and the nonlinear term is quadratic at the origin. As in [6], our proof requires a careful analysis of the asymptotic behavior of the Dirichlet quotient (3.16). Apart from this, our approach is quite different, especially in the construction of fast solutions with a prescribed asymptotic profile. In [6] the set of such solutions, called nonlinear spectral manifold, is characterized as a level set of a suitable function. That approach seems to assume that global solutions already exist and generate a semigroup with some regularity, assumptions which are well suited for the Navier-Stokes equation but not for our general framework. Therefore, what we do is proving a stand-alone existence result through a Banach fixed point argument. Since fast solutions with a prescribed profile are non-unique, it might seem impossible to obtain them by a contraction argument. Nevertheless, the trick is to produce them one by one, by looking for them carefully in suitable classes of functions where one and only one such solution is supposed to be.

In any case, the strength of our results lies mainly in dealing with the case where the kernel of $A$ is nontrivial, which causes the coexistence of slow and fast solutions. This case was completely open, even in finite dimension, apart from the Neumann and Dirichlet examples quoted before. When we apply our abstract results to those examples, we obtain new proofs of the previous known results, and we solve all open problems, at least in dimension one and two, or under a smallness condition on $p$ depending on the dimension (this restriction comes from the need of Sobolev embeddings in order to verify the assumptions of our abstract results). Actually some results persist for any $p$, for instance the slow-fast alternative for the Dirichlet case. Moreover applications are not limited to the model examples, but larger classes of semilinear parabolic equations fit in our general framework.

From the technical point of view, both the existence of slow solutions and the slow-fast alternative require now a careful asymptotic analysis of what we call generalized Dirichlet quotients, defined in (3.17). Similar quotients have been used also in [9, 10, 11] in different contexts (quasilinear and semilinear dissipative hyperbolic equations), but always with the aim of estimating decay rates from below.

We are quite optimistic about the possibility to extend our techniques to cases where our results, as stated here, do not apply immediately. In particular we have in mind both the model examples in any dimension with any $p$ (for the complete result), and hyperbolic equations with damping terms under growth conditions on the non-linearity. These are likely to be the directions of future investigations.

This paper is organized as follows. In Section 2 we clarify the functional setting, we recall two classical local existence theorems, and we state our main abstract results, which we prove in Section 3. In Section 4 we apply the abstract theory to semilinear parabolic problems.

# 2 Statements

## 2.1 Notation and classical existence results

Throughout this paper $H$ denotes a Hilbert space, $|x|$ denotes the norm of an element $x \in H$, and $\langle x, y\rangle$ denotes the scalar product of two elements $x$ and $y$ in $H$. We consider a self-adjoint linear operator $A$ on $H$ with dense domain $D(A)$. We assume that $A$ is nonnegative, namely $\langle Au, u\rangle \geq 0$ for every $u \in D(A)$, so that for every $\alpha \geq 0$ the power $A^\alpha u$ is defined provided that $u$ lies in a suitable domain $D(A^\alpha)$, which is itself a Hilbert space with norm

$$|u|_{D(A^\alpha)} := \left(|u|^2 + |A^\alpha u|^2\right)^{1/2}.$$

Before stating our results, let us spend a few words on the notion of solution. Let us start with the linear equation

$$u'(t) + Au(t) = g(t), \tag{2.1}$$

with initial condition

$$u(0) = u_0. \tag{2.2}$$

For our purposes we can limit ourselves to consider *strong solutions* of (2.1), namely functions $u$ defined in some time-interval $[0, T]$ and such that for almost every $t \in (0, T)$ one has that $u'(t)$ exists, $u(t) \in D(A)$, and (2.1) is satisfied.

We recall the following classical result (we refer for example Theorem 3.6 in [5] where the same regularity is obtained in a more general nonlinear setting).

**Theorem A** (Linear equation – Existence)**.** *Let $H$ be a Hilbert space, and let $A$ be a self-adjoint nonnegative operator on $H$ with dense domain $D(A)$. Let us assume that $T > 0$, $g \in L^2((0,T), H)$ and $u_0 \in D(A^{1/2})$.*

*Then problem (2.1)–(2.2) has a unique solution with the following regularity*

$$u \in C^0\left([0,T], D(A^{1/2})\right), \tag{2.3}$$

$$u \in W^{1,2}\left((0,T), H\right)) \cap L^2\left((0,T), D(A)\right), \tag{2.4}$$

$$\textit{the function } t \to |A^{1/2}u(t)|^2 \textit{ is absolutely continuous in } [0,T]. \tag{2.5}$$

We point out that the solution is defined as long as the forcing term $g(t)$ is defined. If $g \in L^2\left((0,T), H\right)$ for every $T > 0$, then the solution is defined for every $t \geq 0$.

We also mention that the solution provided by Theorem A can be represented by the well-known integral formula

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}g(s)\, ds.$$

Now we consider a semilinear equation of the form

$$u'(t) + Au(t) = f(u(t)), \tag{2.6}$$

for which we have the following local existence result.

**Theorem B** (Semi-linear equation – Local existence)**.** *Let $H$ be a Hilbert space, and let $A$ be a self-adjoint nonnegative operator on $H$ with dense domain $D(A)$. Let $R > 0$, let $B_R := \left\{u \in D(A^{1/2}) : |u|_{D(A^{1/2})} < R\right\}$, and let $f : B_R \to H$ be a function.*

*Let us assume that there exists a constant $L$ such that*

$$|f(u) - f(v)| \leq L|u - v|_{D(A^{1/2})} \qquad \forall (u,v) \in [B_R]^2. \tag{2.7}$$

*Then for every $u_0 \in B_R$ there exist $T > 0$, and a unique local solution $u$ to problem (2.6)–(2.2) satisfying (2.3) through (2.5). This solution can be continued to a solution defined in a maximal interval $[0, T_*)$, where either $T_* = +\infty$ or*

$$\lim_{t \to T_*^-} |u(t)|_{D(A^{1/2})} = R. \tag{2.8}$$

The proof of Theorem B is completely standard. It is enough to consider the map that associates to every $v \in C^0([0,T], D(A^{1/2}))$ the solution of the linear problem (2.1)–(2.2) with $g(t) := f(v(t))$. If $T > 0$ is small enough, this map turns out to be a contraction, and the unique fixed point is the required (strong) local solution.

## 2.2 Main results

Our first result provides a classification of all possible decay rates for decaying solutions to differential inequalities of the form (1.3).

**Theorem 2.1** (Classification of decay rates)**.** *Let $H$ be a Hilbert space, and let $A$ be a self-adjoint nonnegative operator on $H$ with dense domain $D(A)$.*

*Let $g \in L^2((0,T),H)$ for every $T > 0$, and let $u \in C^0\left([0,+\infty), D(A^{1/2})\right)$ be a global solution of (2.1) in the sense of Theorem A. Let us assume that*

*(i) the spectrum of $A$ is a finite set or an increasing sequence of eigenvalues,*

*(ii) $u$ is a decaying solution in the sense that*

$$\lim_{t\to+\infty} |u(t)|_{D(A^{1/2})} = 0, \tag{2.9}$$

*(iii) there exist $p > 0$, $q > 0$, and $K_0 \geq 0$ such that*

$$|g(t)| \leq K_0 \left(|u(t)|^{1+p} + |A^{1/2}u(t)|^{1+q}\right) \qquad \forall t \geq 0. \tag{2.10}$$

*Then one and only one of the following statements apply.*

*(1)* (Null solution) *The solution is the zero-solution $u(t) \equiv 0$ for every $t \geq 0$.*

*(2)* (Slow solutions) *There exist positive constants $M_1$ and $M_2$ such that*

$$|u(t)| \geq \frac{M_1}{(1+t)^{1/p}} \qquad \forall t \geq 0, \tag{2.11}$$

$$|A^{1/2}u(t)| \leq M_2 |u(t)|^{1+p} \qquad \forall t \geq 0. \tag{2.12}$$

*(3)* (Spectral fast solutions) *There exist an eigenvalue $\lambda > 0$ of $A$, and a corresponding eigenvector $v_0 \neq 0$, such that*

$$\lim_{t\to+\infty} \left|u(t) - v_0 e^{-\lambda t}\right|_{D(A^{1/2})} e^{\gamma t} = 0 \tag{2.13}$$

*for every*

$$\gamma < \min\left\{\beta, (1+p)\lambda, (1+q)\lambda\right\}, \tag{2.14}$$

*where $\beta = +\infty$ if the spectrum of $A$ is finite and $\lambda$ is its maximum, and $\beta$ is the smallest eigenvalue of $A$ larger than $\lambda$ otherwise.*

**Remark 2.2.** When the kernel of $A$ is non-trivial, a differential inequality such as (1.3) does not guarantee that all its solutions in a neighborhood of the origin tend to 0 (just think to the ordinary differential equation $u' = u^3$). This is the reason why we need assumption (2.9).

In other words, there might be coexistence of solutions that decay to 0 and solutions that do not decay, or even do not globally exist. When this is the case, our result classifies all possible decay rates of decaying solutions, regardless of non-decaying ones.

**Remark 2.3.** Concerning the null solution, Theorem 2.1 extends the well-known backward uniqueness results of the seminal papers [1, 8]. Assuming forward uniqueness, which holds true for large classes of equations, classical backward uniqueness results read as follows. If $u(t) = 0$ for some $t \geq 0$ (hence also for all subsequent times), then $u(t) = 0$ for all $t \geq 0$. Our result extends the classical one by showing that, if $u(t)$ decays at infinity faster than $e^{-ct}$ for all $c > 0$, then $u(t) = 0$ for all $t \geq 0$.

**Remark 2.4.** Concerning slow solutions, we point out that only the exponent $p$ in (2.10) appears in the decay rate, while $q$ is irrelevant provided it is positive. Roughly speaking, this happens because slow solutions move closer and closer to the kernel of $A$, as suggested by the otherwise unnatural estimate (2.12) in which $|A^{1/2}u(t)|$ is controlled with a higher power of $|u(t)|$. Close to the kernel of $A$, the term $|A^{1/2}u(t)|$ can be neglected, and this justifies the disappearance of $q$ in the final decay rate.

Let us write $u(t)$ as the sum of its projection $P_K u(t)$ into $\ker(A)$, and its "range component" $u(t) - P_K u(t)$ orthogonal to $\ker(A)$. Since the operator $A$ is coercive when restricted to the range of $A$, estimate (2.12) implies that there exists a constant $c$ such that

$$|u(t) - P_K u(t)| \leq c|A^{1/2}u(t)| \leq cM_2|u(t)|^{1+p}.$$

In other words, when $u(t)$ decays to 0, its range component always decays faster, so that the slowness of $u(t)$ is due uniquely to its component $P_K u(t)$ with respect to the kernel. This is consistent with previous results (see [13]). This shows also that slow solutions cannot exist when the operator $A$ is coercive.

The exponent $(1 + p)$ in (2.12) is optimal. This can be seen by considering the case where $H = \mathbb{R}^2$, $p = 2$, and the evolution problem reduces to the following system of ordinary differential equations

$$\begin{cases} x'(t) = -x^3(t), \\ y'(t) + y(t) = x^3(t). \end{cases}$$

A solution of the first equation is $x(t) = (1 + 2t)^{-1/2}$. At this point it is possible to prove that all solutions of the second equation decay as the forcing term, hence as $(1+2t)^{-3/2}$. Therefore, in this example we have that $|u(t)| \sim |x(t)| \sim (1+2t)^{-1/2}$, while $|A^{1/2}u(t)| = |y(t)| \sim (1 + 2t)^{-3/2} = |u(t)|^{1+p}$.

**Remark 2.5.** Concerning fast solutions, the possible asymptotic profiles are described by (2.13), which also provides an estimate for the remainder. We point out that (2.14) is optimal. This can be seen by considering the case where $H = \mathbb{R}^2$ and the evolution problem reduces to the following system of ordinary differential equations

$$\begin{cases} x'(t) + \lambda x(t) = 0, \\ y'(t) + \beta y(t) = |x(t)|^{1+p} + |x(t)|^{1+q}. \end{cases}$$

A solution of the first equation is $x(t) = e^{-\lambda t}$. At this point, solutions of the second equation can decay as $e^{-\eta t}$, where $\eta$ is the right-hand side of (2.14), or even as $te^{-\eta t}$ in case of resonance.

We conclude this long discussion on Theorem 2.1 by mentioning the following.

**Open problem.** Is it possible to weaken assumption (2.9) by asking just that $|u(t)| \to 0$ as $t \to +\infty$, namely by requiring the limit in $H$ instead of $D(A^{1/2})$? Our proof requires the assumption as stated, but we have no counterexamples with the weaker requirement. Actually we have no examples at all of solutions that decay to 0 in $H$ but not in $D(A^{1/2})$.

The slow-fast alternative alone does not guarantee the existence of both slow solutions and fast solutions. Next result provides sufficient conditions for the existence of an open set of slow solutions.

**Theorem 2.6** (Existence of slow solutions)**.** *Let $H$ be a Hilbert space, and let $A$ be a self-adjoint nonnegative operator on $H$ with dense domain $D(A)$. Let $f : B_R \to H$ be a function, with $R > 0$ and $B_R$ as in Theorem B.*

*Let us assume that*

*(i)* $\ker(A) \neq \{0\}$, *and there exists a constant $\nu > 0$ such that*

$$|Au|^2 \geq \nu |A^{1/2}u|^2 \qquad \forall u \in D(A), \tag{2.15}$$

*(ii) there exists a constant $L$ such that (2.7) holds true,*

*(iii) there exist $p > 0$, $q > 0$, and $K_0 \geq 0$ such that*

$$|f(u)| \leq K_0 \left(|u|^{1+p} + |A^{1/2}u|^{1+q}\right) \qquad \forall u \in B_R, \tag{2.16}$$

*and in addition*

$$\langle u, f(u)\rangle \leq 0 \qquad \forall u \in B_R. \tag{2.17}$$

*Then there exists an open set $\mathcal{S} \subseteq B_R$ (open with respect to the norm of $D(A^{1/2})$) with the following property. For every $u_0 \in \mathcal{S}$, the unique solution $u$ of (2.6)–(2.2) provided by Theorem B is actually global, and slow in the sense that it satisfies (2.11).*

**Remark 2.7.** Let us briefly comment on the hypotheses of Theorem 2.6. Concerning the operator $A$, we already pointed out that $\ker(A) \neq \{0\}$ is a necessary condition for the existence of slow solutions, while (2.15) is automatic if the spectrum of $A$ is a finite set or an increasing sequence of eigenvalues.

Concerning the nonlinear term, assumption (2.7) comes from the local existence result, while (2.16) means that the nonlinear term has order higher than one at the origin, in accordance with (2.10).

Assumption (2.17) guarantees that the function $t \to |u(t)|$ is nonincreasing, and this is exploited in the proof in order to keep the solution inside $B_R$. This assumption can be weakened in several ways, for example by requiring only that $\langle u, f(u)\rangle \leq |A^{1/2}u|^2$ for every $u \in B_R$, but it can not be dropped completely. Indeed this is a sort of sign condition, and when it is violated one can not guarantee even the existence of global solutions, as in the case of the ordinary differential equation $u' = u^3$.

**Remark 2.8.** Concerning the conclusion of Theorem 2.6, we point out that we prove the existence of an open set of solutions decaying *at most* as $t^{-1/p}$. In general it is not true that solutions decay exactly as $t^{-1/p}$. As a matter of fact, they can even not to decay at all. For example, the assumptions are satisfied in the extreme case where both $A$ and the nonlinear term are identically 0, and in that case all solutions are stationary (which implies slow).

Of course, when we know that a solution is slow, we can always apply Theorem 2.1 and deduce that it satisfies (2.12) in addition to (2.11).

Our last result concerns the existence of families of fast solutions.

**Theorem 2.9** (Existence of fast solutions)**.** *Let $H$ be a Hilbert space, and let $A$ be a self-adjoint nonnegative operator on $H$ with dense domain $D(A)$. Let $f : B_R \to H$ be a function, with $R > 0$ and $B_R$ as in Theorem B.*

*Let us assume that*

*(i) the spectrum of $A$ is a finite set or an increasing sequence of eigenvalues,*

*(ii) there exist $p > 0$ and $L \geq 0$ such that*

$$|f(u) - f(v)| \leq L \left( |u|^p_{D(A^{1/2})} + |v|^p_{D(A^{1/2})} \right) |u - v|_{D(A^{1/2})} \tag{2.18}$$

*for every $u$ and $v$ in $B_R$, and in addition*

$$f(0) = 0. \tag{2.19}$$

*Let $\lambda > 0$ be an eigenvalue of $A$, and let $H = H_- \oplus H_+$ be the orthogonal decomposition of $H$ where $H_-$ is the closure of the subspace generated by all eigenvectors of $A$ relative to eigenvalues less than or equal to $\lambda$, and $H_+$ is the closure of the subspace generated by all eigenvectors of $A$ relative to eigenvalues greater than $\lambda$.*

*Then there exists $r_0 > 0$ with the following property. For every eigenvector $v_0$ relative to $\lambda$, and every $w_0 \in H_+ \cap D(A^{1/2})$ such that*

$$|v_0|_{D(A^{1/2})} + |w_0|_{D(A^{1/2})} \leq r_0, \tag{2.20}$$

*there exists $w_1 \in H_-$ such that the unique local solution, provided by Theorem B, to problem (2.6)–(2.2) with initial condition $u_0 := w_0 + w_1$ is actually global and*

$$\lim_{t \to +\infty} \left| e^{\lambda t} u(t) - v_0 \right|_{D(A^{1/2})} = 0. \tag{2.21}$$

**Remark 2.10.** Assumptions (2.18) and (2.19) imply both (2.7) and (2.16) with $p = q$. Moreover, (2.18) is stronger than (2.7) because it requires that the local Lipschitz constant of $f$ vanishes at the origin. We emphasize that we do not impose any sign condition on $f$, and therefore the assumptions of Theorem 2.9 are not enough to guarantee the existence of a global solution for every $u_0 \in B_R$. For this reason, also the global existence part of the statement is nontrivial.

We observe also that (2.18) could be stated with two different exponents as follows

$$|f(u) - f(v)| \leq L \left(|u|^p + |v|^p + |A^{1/2}u|^q + |A^{1/2}v|^q\right) |u - v|_{D(A^{1/2})},$$

or even with four exponents, but this would be useless because the exponents do not appear in the conclusion. What is relevant here is just that they are both positive, and thus there is no loss of generality in assuming that they are equal.

Of course, when we know that a solution is fast, we can always apply Theorem 2.1 and deduce that the remainder in (2.21) satisfies (2.13).

**Remark 2.11.** Let $\{\lambda_k\}$ denote the increasing sequence of eigenvalues of $A$, which only for simplicity we assume of multiplicity one, and let $\{e_k\}$ denote a corresponding orthonormal system. All solutions of the linear homogeneous equation $u'(t) + Au(t) = 0$ can be represented as

$$u(t) = \sum_{k=0}^{\infty} u_{0k} e^{-\lambda_k t} e_k,$$

where $\{u_{0k}\}$ are the components of the initial condition $u_0$ with respect to $\{e_k\}$.

If we fix an eigenvalue $\lambda_i$, then the solutions decaying as $e^{-\lambda_i t}$ (up to a multiplicative constant) are those of the form

$$u(t) = u_{0i} e^{-\lambda_i t} e_i + \sum_{k=i+1}^{\infty} u_{0k} e^{-\lambda_k t} e_k,$$

hence they are parametrized by the eigenvector $u_{0i} e_i$, which appears in the term which gives the asymptotic profile, and by the projection of the initial condition in the space $H_+$ generated by all eigenvectors relative to eigenvalues greater than $\lambda_i$.

Theorem 2.9 shows that the same parameters are involved in the nonlinear case, provided that we restrict to a neighborhood of the origin. Roughly speaking, what we prove is an existence result for solutions of (2.6) satisfying a mix of conditions at $t = 0$ and $t = +\infty$, namely

- a prescribed asymptotic profile (in a certain sense a condition at $t = +\infty$),
- a prescribed component of the initial datum $u_0$ with respect to the subspace $H_+$.

This suggests also that solutions with decay rate exactly $e^{-\lambda_i t}$ (up to multiplicative constants) are "generic" among solutions with decay rate at least $e^{-\lambda_i t}$ and, when the kernel of $A$ in non-trivial, slow solutions are "generic" among all decaying solutions. This rough idea that slower behaviors are always "generic", in a sense to be made precise, would probably deserve further investigation in the future.

# 3 Proofs

## 3.1 Estimates for linear equations

Let us start with two simple estimates for differential inequalities and integrals.

**Lemma 3.1.** *Let* $\varphi : [0, +\infty) \to [0, +\infty)$ *be a continuous function such that*

$$\lim_{t\to+\infty} \varphi(t) = 0. \tag{3.1}$$

*Let* $c > 0$*, and let* $z : [0, +\infty) \to [0, +\infty)$ *be an absolutely continuous function such that*

$$z'(t) \leq -cz(t) + \varphi(t) \tag{3.2}$$

*for almost every* $t > 0$.

*Then we have that*

$$\lim_{t\to+\infty} z(t) = 0. \tag{3.3}$$

*Proof* Integrating the differential inequality (3.2) it follows that

$$z(t) \leq z(0)e^{-ct} + e^{-ct}\int_0^t e^{cs}\varphi(s)\, ds.$$

The first term in the right-hand side tends to 0 as $t \to +\infty$. For the second term we can reason as follows. First $\varphi(t)$, being continuous and convergent at infinity, is bounded by some $M > 0$. Then for any $\varepsilon > 0$ there exists $T(\varepsilon) \geq 0$ such that

$$\varphi(t) \leq \varepsilon \qquad \forall t \geq T(\varepsilon).$$

By splitting the integral on the two sub-intervals $[0, T(\varepsilon)]$ and $[T(\varepsilon), t]$ we find

$$e^{-ct}\int_0^t e^{cs}\varphi(s)\, ds \leq \frac{M}{c}e^{c(T(\varepsilon)-t)} + \frac{\varepsilon}{c} \qquad \forall t \geq T(\varepsilon),$$

so that (3.3) follows by letting first $t \to +\infty$ and then $\varepsilon \to 0^+$. □

**Lemma 3.2.** *Let* $\delta > 0$*, and let* $\varphi : [0, +\infty) \to [0, +\infty)$ *be a continuous function such that*

$$\lim_{t\to+\infty} \varphi(t)e^{\gamma t} = 0 \qquad \forall \gamma < \delta. \tag{3.4}$$

*Then for every* $\alpha < \delta$ *we have that the integral*

$$\int_0^{+\infty} e^{\alpha s}\varphi(s)\, ds \tag{3.5}$$

*converges, and*

$$\lim_{t\to+\infty} e^{\gamma t}\int_t^{+\infty} e^{\alpha(s-t)}\varphi(s)\, ds = 0 \qquad \forall \gamma < \delta. \tag{3.6}$$

*Proof* Let us choose $\eta \in (\alpha, \delta)$. Due to (3.4), there exists a constant $c_\eta$ such that $\varphi(s) \leq c_\eta e^{-\eta s}$ for every $s \geq 0$, which easily implies the convergence of (3.5). At this point (3.6) is obvious if $\gamma \leq \alpha$. On the other hand, for $\gamma \in (\alpha, \delta)$ we can write

$$e^{\gamma t} \int_t^{+\infty} e^{\alpha(s-t)} \varphi(s)\, ds = \int_t^{+\infty} e^{(\gamma-\alpha)(t-s)} e^{\gamma s} \varphi(s)\, ds \leq \frac{1}{\gamma - \alpha} \sup_{s \geq t} e^{\gamma s} \varphi(s)$$

for every $t \geq 0$, so that in this case (3.6) follows from assumption (3.4). $\Box$

Now we prove estimates for solutions to the non-homogeneous linear equation

$$w'(t) + Aw(t) = \psi(t) \qquad \forall t \geq 0. \tag{3.7}$$

Here we assume that $X$ is a Hilbert space, $A$ is a self-adjoint nonnegative linear operator on $X$ with dense domain $D(A)$, the forcing term $\psi$ is in $L^2((0, +\infty), X)$, and $w \in C^0([0, +\infty), D(A^{1/2}))$ is a solution of (3.7) in the sense of Theorem A. When we apply these estimates in the proof of the main results, $X$ is a suitable subspace of $H$, different from case to case, and (3.7) is the projection of (2.1) onto $X$.

**Lemma 3.3** (Supercritical frequences). *Let us assume that there exist $\beta > 0$ and $\delta > 0$ such that*

$$|A^{1/2}x|^2 \geq \beta |x|^2 \qquad \forall x \in D(A^{1/2}), \tag{3.8}$$

$$\lim_{t \to +\infty} |\psi(t)| e^{\gamma t} = 0 \qquad \forall \gamma < \delta. \tag{3.9}$$

*Then we have that*

$$\lim_{t \to +\infty} |w(t)|_{D(A^{1/2})} e^{\gamma t} = 0 \qquad \forall \gamma < \min\{\beta, \delta\}. \tag{3.10}$$

*Proof* Let us consider the function $F_\gamma(t) := e^{2\gamma t} |A^{1/2} w(t)|^2$. Due to (3.8), the norm of $w(t)$ in $D(A^{1/2})$ is equivalent to $|A^{1/2} w(t)|$, hence (3.10) is equivalent to proving that $F_\gamma(t) \to 0$ as $t \to +\infty$ for every $\gamma < \min\{\beta, \delta\}$.

To this end, we choose $\varepsilon \in (0, \beta - \gamma)$ and we estimate the time-derivative as follows

$$\begin{aligned} F_\gamma'(t) &= -2e^{2\gamma t} |Aw(t)|^2 + 2e^{2\gamma t} \langle Aw(t), \psi(t) \rangle + 2\gamma e^{2\gamma t} |A^{1/2} w(t)|^2 \\ &\leq -2e^{2\gamma t} |Aw(t)|^2 + \frac{2\varepsilon}{\beta} e^{2\gamma t} |Aw(t)|^2 + \frac{\beta}{2\varepsilon} e^{2\gamma t} |\psi(t)|^2 + 2\gamma e^{2\gamma t} |A^{1/2} w(t)|^2. \end{aligned}$$

From assumption (3.8) we have that $|Aw(t)|^2 \geq \beta |A^{1/2} w(t)|^2$, hence

$$F_\gamma'(t) \leq -2(\beta - \gamma - \varepsilon) F_\gamma(t) + \frac{\beta}{2\varepsilon} e^{2\gamma t} |\psi(t)|^2.$$

Now let us set

$$z(t) := F_\gamma(t), \qquad c := 2(\beta - \gamma - \varepsilon), \qquad \varphi(t) := \frac{\beta}{2\varepsilon} e^{2\gamma t} |\psi(t)|^2.$$

Since $\gamma < \delta$, assumption (3.9) implies (3.1). Therefore, we can apply Lemma 3.1 and deduce that $F_\gamma(t) \to 0$ as $t \to +\infty$, which completes the proof. $\Box$

**Lemma 3.4** (Subcritical frequences). *Let us assume that there exist $\delta > \alpha \geq 0$ such that*

$$|A^{1/2}x|^2 \leq \alpha|x|^2 \qquad \forall x \in X, \tag{3.11}$$

$$\lim_{t\to+\infty} |\psi(t)|e^{\gamma t} = 0 \qquad \forall \gamma < \delta. \tag{3.12}$$

*Then the following limit*

$$x_0 := \lim_{t\to+\infty} e^{tA}w(t) \tag{3.13}$$

*exists, and*

$$\lim_{t\to+\infty} \left|w(t) - e^{-tA}x_0\right|_{D(A^{1/2})} e^{\gamma t} = 0 \qquad \forall \gamma < \delta. \tag{3.14}$$

*Proof* Every solution of (3.7) is given by the explicit formula

$$w(t) = e^{-tA}\left(w(0) + \int_0^t e^{sA}\psi(s)\,ds\right) \qquad \forall t \geq 0. \tag{3.15}$$

We claim that the integral in the right-hand side has a finite limit when $t \to +\infty$. Indeed assumption (3.11) guarantees that $e^{sA}$ is a bounded operator on $X$ with norm less than or equal to $e^{\alpha s}$, hence it is enough to prove that the integral

$$\int_0^{+\infty} e^{\alpha s}|\psi(s)|\,ds$$

converges. Since $\alpha < \delta$, this follows from Lemma 3.2 applied with $\varphi(t) := |\psi(t)|$, and proves (3.13) with

$$x_0 := w(0) + \int_0^{+\infty} e^{sA}\psi(s)\,ds.$$

Now (3.15) can be rewritten as

$$w(t) = e^{-tA}\left(x_0 - \int_t^{+\infty} e^{sA}\psi(s)\,ds\right) = e^{-tA}x_0 - \int_t^{+\infty} e^{(s-t)A}\psi(s)\,ds.$$

Exploiting again (3.11), we have now that

$$\left|w(t) - e^{-tA}x_0\right| e^{\gamma t} \leq e^{\gamma t}\int_t^{+\infty} \left|e^{(s-t)A}\psi(s)\right|\,ds \leq e^{\gamma t}\int_t^{+\infty} e^{\alpha(s-t)}\left|\psi(s)\right|\,ds,$$

so that (3.14) follows from conclusion (3.6) of Lemma 3.2, applied once again with $\varphi(t) := |\psi(t)|$ (we remind that the norm in $H$ and $D(A^{1/2})$ are in this case equivalent owing to (3.11)). $\Box$

## 3.2 Generalized Dirichlet quotients

In this section we consider the classical *Dirichlet quotient*

$$Q(t) := \frac{|A^{1/2}u(t)|^2}{|u(t)|^2}. \tag{3.16}$$

We also consider the following *generalized Dirichlet quotient*

$$Q_d(t) := \frac{|A^{1/2}u(t)|^2}{|u(t)|^{2+d}}, \tag{3.17}$$

defined for every $d \geq 0$.

The aim of the next result is providing estimates for the time-derivative of $Q(t)$ and $Q_d(t)$ when $u$ is a solution of a linear equation such as (2.1).

**Lemma 3.5** (Time-derivatives of Dirichlet quotients)**.** *Let $H$ be a Hilbert space, and let $A$ be a self-adjoint nonnegative operator on $H$ with dense domain $D(A)$. Let $(a,b) \subseteq (0,+\infty)$ be an interval, let $g \in L^2((a,b),H)$, and let $u \in C^0((a,b),D(A^{1/2}))$ be a solution of (2.1) in $(a,b)$ in the sense of Theorem A.*

*Let us assume that $u(t) \neq 0$ for every $t \in (a,b)$.*

*Then we have the following conclusions.*

*(i) The Dirichlet quotient $Q(t)$ defined by (3.16) is absolutely continuous in $(a,b)$, and*

$$Q'(t) \leq -\frac{|Au(t) - Q(t)u(t)|^2}{|u(t)|^2} + \frac{|g(t)|^2}{|u(t)|^2} \tag{3.18}$$

*for almost every $t \in (a,b)$.*

*(ii) Let us assume that (2.15) holds true for some constant $\nu > 0$. Then for every $d > 0$ the generalized Dirichlet quotient $Q_d(t)$ defined by (3.17) is absolutely continuous in $(a,b)$, and*

$$Q_d'(t) \leq -\nu Q_d(t) + 2(2+d)|u(t)|^d \cdot |Q_d(t)|^2 + (3+d)\frac{|g(t)|^2}{|u(t)|^{2+d}} \tag{3.19}$$

*for almost every $t \in (a,b)$.*

*Proof* The time-derivative of (3.16) is

$$Q'(t) = -2\frac{|Au(t) - Q(t)u(t)|^2}{|u(t)|^2} + 2\frac{\langle Au(t) - Q(t)u(t), g(t)\rangle}{|u(t)|^2}.$$

Since

$$2\langle Au(t) - Q(t)u(t), g(t)\rangle \leq |Au(t) - Q(t)u(t)|^2 + |g(t)|^2,$$

estimate (3.18) easily follows.

The time-derivative of (3.17) is

$$\begin{aligned} Q_d'(t) &= -2\frac{|Au(t)|^2}{|u(t)|^{2+d}} + (2+d)\frac{|A^{1/2}u(t))|^2}{|u(t)|^2}\cdot Q_d(t) \\ &\quad +2\frac{\langle Au(t), g(t)\rangle}{|u(t)|^{2+d}} - (2+d)\frac{\langle Q_d(t)u(t), g(t)\rangle}{|u(t)|^2} \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Now it is easy to see that

$$I_2 = (2+d)|u(t)|^d \cdot [Q_d(t)]^2, \qquad I_3 \leq \frac{|Au(t)|^2}{|u(t)|^{2+d}} + \frac{|g(t)|^2}{|u(t)|^{2+d}},$$

and

$$\begin{aligned} I_4 &\leq \frac{2+d}{|u(t)|^2}\left(|u(t)|^d \cdot [Q_d(t)]^2|u(t)|^2 + \frac{1}{|u(t)|^d}\cdot |g(t)|^2\right) \\ &= (2+d)|u(t)|^d \cdot [Q_d(t)]^2 + (2+d)\frac{|g(t)|^2}{|u(t)|^{2+d}}, \end{aligned}$$

so that

$$Q_d'(t) \leq -\frac{|Au(t)|^2}{|u(t)|^{2+d}} + 2(2+d)|u(t)|^d \cdot [Q_d(t)]^2 + (3+d)\frac{|g(t)|^2}{|u(t)|^{2+d}}.$$

The first term in the right-hand side is less than or equal to $-\nu Q_d(t)$ because of (2.15), and this proves (3.19). □

## 3.3 Proof of Theorem 2.1

Let us describe the scheme of the proof before entering into details. In the first section of the proof we get rid of the null solution. Indeed we prove that $u(T) = 0$ for some $T \geq 0$ if and only if $u(t) = 0$ for every $t \geq 0$. This is a result of forward and backward uniqueness of the null solution. After proving it, we can assume that

$$u(t) \neq 0 \qquad \forall t \geq 0, \tag{3.20}$$

which allows to consider the Dirichlet quotients for every $t \geq 0$. In the second section of the proof we assume that there exist a constant $c_1 > 0$ and a sequence $t_n \to +\infty$ such that

$$|A^{1/2}u(t_n)|^2 \leq c_1|u(t_n)|^{2+p} \qquad \forall n \in \mathbb{N}. \tag{3.21}$$

Under this assumption, we prove that a similar estimate holds true for all times, namely

$$|A^{1/2}u(t)|^2 \leq c_2|u(t)|^{2+p} \qquad \forall t \geq 0 \tag{3.22}$$

for a suitable constant $c_2 \geq c_1$. This is not yet (2.12), but in any case it shows that $|A^{1/2}u(t)|$ decays faster than $|u(t)|$. As already pointed out, this means that the solution $u(t)$ moves closer and closer to the kernel of $A$, and suggests that the terms with $Au(t)$ and $A^{1/2}u(t)$ in equation (2.1) and estimate (2.10) can be neglected. With this ansatz, we obtain that $|u'(t)| \leq K_0|u(t)|^{1+p}$, and it is easy to show that all nonzero solutions of this differential inequality are slow in the sense of (2.11). Finally, we improve (3.22) in order to obtain (2.12).

In the third and last section of the proof we are left with the case where (3.21) is false for every constant $c_1$ and every sequence $t_n \to +\infty$. This easily implies that there exists $T_0 \geq 0$ such that

$$|u(t)|^{2+p} \leq |A^{1/2}u(t)|^2 \qquad \forall t \geq T_0. \tag{3.23}$$

This means that now $u(t)$ is faraway from the kernel of $A$. Thus we are not allowed to ignore the operator $A$, but we can neglect the right-hand side of (2.1) because the exponents in (2.10) are larger than one. Therefore, a good approximation of (2.1) is now the linear homogeneous equation $u'(t) + Au(t) = 0$, whose solutions decay exponentially with possible rates corresponding to eigenvalues of $A$. The formal proof requires several steps. First of all, we provide exponential estimates from below and from above with non-optimal rates. Then we identify the exact rate, and finally we prove that (2.13) holds true.

We point out that the exponent $2+p$ is non-optimal both in (3.21) and in the opposite estimate (3.23). Indeed, a posteriori it turns out that (up to multiplicative constants) $|A^{1/2}u| \leq |u|^{1+p}$ in the case of slow solutions, and $|A^{1/2}u| \sim |u|$ in the case of fast solutions, so that the right exponents would be $2+2p$ and 2, respectively. Nevertheless, the intermediate exponent $2+p$ acts as a threshold separating the two different regimes, and leaving enough room on both sides to perform our estimates.

**Non-trivial solutions never vanish.**

*Forward uniqueness* We prove that $u(0) = 0$ implies that $u(t) = 0$ for every $t \geq 0$.

To this end, we set $z(t) := |u(t)|^2 + |A^{1/2}u(t)|^2$. A simple computation shows that

$$\begin{aligned} z'(t) &= -2|A^{1/2}u(t)|^2 - 2|Au(t)|^2 + 2\langle u(t), g(t)\rangle + 2\langle Au(t), g(t)\rangle \\ &\leq -2|A^{1/2}u(t)|^2 - 2|Au(t)|^2 + |u(t)|^2 + |g(t)|^2 + |Au(t)|^2 + |g(t)|^2 \\ &\leq |u(t)|^2 + 2|g(t)|^2. \end{aligned}$$

From (2.10) we obtain that

$$|g(t)|^2 \leq 2K_0^2 \left(|u(t)|^{2+2p} + |A^{1/2}u(t)|^{2+2q}\right), \tag{3.24}$$

hence

$$z'(t) \leq z(t) + 4K_0^2[z(t)]^{1+p} + 4K_0^2[z(t)]^{1+q} \qquad \forall t \geq 0.$$

All powers of $z(t)$ in the right-hand side of this scalar differential inequality have exponents greater than or equal to one. It follows that the right-hand side, as a function of $z(t)$, is Lipschitz continuous. This is enough to guarantee that necessarily $z(t) = 0$, hence $u(t) = 0$, for every $t \geq 0$.

*Backward uniqueness* We prove that $u(0) \neq 0$ implies that $u(t) \neq 0$ for every $t \geq 0$.

As in the classical references [1, 8], here we exploit the standard Dirichlet quotient $Q(t)$ defined in (3.16). To this end, we set

$$S := \sup \{t \geq 0 : u(\tau) \neq 0 \quad \forall \tau \in [0,t]\}.$$

Since $u(0) \neq 0$, a simple continuity argument shows that $S > 0$. Backward uniqueness is equivalent to saying that $S = +\infty$. So let us assume by contradiction that $S < +\infty$. By the maximality of $S$, this means that $u(S) = 0$. Now we show that this is not possible.

In the interval $[0, S)$ we have that $u(t) \neq 0$, hence $Q(t)$ is defined. Let us estimate its time-derivative as in (3.18). If we neglect the first term in the right-hand side, and we estimate the second one by means of (3.24), we obtain that

$$\begin{aligned} Q'(t) &\leq \frac{1}{|u(t)|^2} \cdot 2K_0^2 \left(|u(t)|^{2+2p} + |A^{1/2}u(t)|^{2+2q}\right) \\ &\leq 2K_0^2 |u(t)|^{2p} + 2K_0^2 |A^{1/2}u(t)|^{2q} \cdot Q(t). \end{aligned}$$

Due to the regularity of the solution, and in particular to (2.3), both $|u(t)|$ and $|A^{1/2}u(t)|$ are bounded on bounded time-intervals, and in particular in $[0, S)$. This is enough to conclude that also $Q(t)$ is bounded in $[0, S)$.

Now let us consider the function $y(t) := |u(t)|^2$. A simple computation shows that

$$y'(t) = -2|A^{1/2}u(t)|^2 + 2\langle u(t), g(t)\rangle \geq -2|A^{1/2}u(t)|^2 - 2|u(t)| \cdot |g(t)|. \tag{3.25}$$

By definition of Dirichlet quotient, assumption (2.10) can be rewritten as

$$|g(t)| \leq K_0 |u(t)| \left(|u(t)|^p + [Q(t)]^{1/2} |A^{1/2}u(t)|^q\right),$$

so that

$$\begin{aligned} y'(t) &\geq -2\frac{|A^{1/2}u(t)|^2}{|u(t)|^2} |u(t)|^2 - 2K_0 |u(t)|^2 \left(|u(t)|^p + [Q(t)]^{1/2} |A^{1/2}u(t)|^q\right) \\ &= -2\left(Q(t) + K_0 |u(t)|^p + K_0 [Q(t)]^{1/2} |A^{1/2}u(t)|^q\right) y(t). \end{aligned} \tag{3.26}$$

Since $|u(t)|$, $|A^{1/2}u(t)|$, and $Q(t)$ are bounded in $[0, S)$, we deduce that there exists $c_3$ such that $y'(t) \geq -c_3 y(t)$ for almost every $t \in [0, S)$, hence

$$y(t) \geq y(0) e^{-c_3 t} \qquad \forall t \in [0, S).$$

Since $y(0) > 0$, letting $t \to S^-$ we conclude that $y(S) > 0$, hence $u(S) \neq 0$. This contradicts the maximality of $S$, and completes the proof that $u(t)$ cannot vanish in a finite time.

**Slow solutions**

In this second part of the proof we consider the case where $u(t)$ in not the null solution and (3.21) holds true for some $c_1 > 0$ and some sequence $t_n \to +\infty$.

*Main estimate* We prove that there exists a constant $c_2$ such that (3.22) holds true.

This estimate is trivial if $A$ is the null operator. Otherwise, let $\nu > 0$ denote the smallest positive eigenvalue of $A$, which exists because we assumed that eigenvalues are an increasing sequence. With this choice, the operator $A$ satisfies assumption (2.15). Let us consider the modified Dirichlet quotient (3.17) with $d := p$, which is defined for every $t \geq 0$ by virtue of (3.20). From (3.19) it follows that

$$Q_p'(t) \leq -\nu Q_p(t) + 2(2+p)|u(t)|^p \cdot [Q_p(t)]^2 + (3+p)\frac{|g(t)|^2}{|u(t)|^{2+p}}.$$

Therefore, if we write (3.24) in the form

$$|g(t)|^2 \leq 2K_0^2|u(t)|^{2+p}\left(|u(t)|^p + [Q_p(t)]^{1+q}|u(t)|^{(2+p)q}\right),$$

we obtain that

$$\begin{aligned} Q_p'(t) \quad \leq \quad & -\nu Q_p(t) + 2(2+p)|u(t)|^p \cdot [Q_p(t)]^2 \\ & +2(3+p)K_0^2|u(t)|^p + 2(3+p)K_0^2 \cdot [Q_p(t)]^{1+q}|u(t)|^{(2+p)q}. \end{aligned} \tag{3.27}$$

Let us consider now the constant $c_1$ and the sequence $t_n \to +\infty$ of (3.21). From assumption (2.9) we have in particular that $|u(t)| \to 0$ as $t \to +\infty$. Therefore, there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} 2\nu c_1 \quad \geq \quad & 2(2+p)|u(t)|^p \cdot (2c_1)^2 + 2(3+p)K_0^2|u(t)|^p \\ & +2(3+p)K_0^2 \cdot (2c_1)^{1+q}|u(t)|^{(2+p)q} \end{aligned} \tag{3.28}$$

for every $t \geq t_{n_0}$. Now we claim that

$$Q_p(t) \leq 2c_1 \qquad \forall t \geq t_{n_0}. \tag{3.29}$$

Since $Q_p(t)$ is continuous, and consequently bounded, in the compact interval $[0, t_{n_0}]$, this is enough to establish (3.22).

In order to prove (3.29), we set

$$S := \sup\left\{t \geq t_{n_0} : Q_p(\tau) \leq 2c_1 \quad \forall \tau \in [t_{n_0}, t]\right\},$$

so that now (3.29) is equivalent to $S = +\infty$. To begin with, we observe that $Q_p(t_{n_0}) \leq c_1 < 2c_1$, hence a simple continuity argument yields that $S > t_{n_0}$. Let us assume by contradiction that $S < +\infty$. By the maximality of $S$, this means that $Q_p(S) = 2c_1$.

Now we show that this is impossible. We already know that $Q_p(t) \leq 2c_1$ for every $t \in [t_{n_0}, S)$. Plugging this estimate into (3.27), and exploiting (3.28), we obtain that

$$Q_p'(t) \leq -\nu Q_p(t) + 2\nu c_1 \qquad \forall t \in [t_{n_0}, S).$$

Integrating this differential inequality, and recalling once again that $Q_p(t_{n_0}) \leq c_1$, we obtain that

$$Q_p(t) \leq 2c_1 - c_1 \exp\left(-\nu(t - t_{n_0})\right) \qquad \forall t \in [t_{n_0}, S).$$

Letting $t \to S^-$, we conclude that $Q_p(S) < 2c_1$, which contradicts the maximality of $S$. This completes the proof of (3.22).

*Faster decay of the range component* Let us prove (2.12). Once again this is trivial if $A$ is the null operator. Otherwise, assumption (2.15) is satisfied with $\nu > 0$ equal to the smallest positive eigenvalue of $A$. Let us consider the modified Dirichlet quotient (3.17) with $d := 2p$, which is defined for every $t \geq 0$ because of (3.20). From (3.19) it follows that

$$Q_{2p}'(t) \leq -\nu Q_{2p}(t) + 4(1+p)|u(t)|^{2p} \cdot [Q_{2p}(t)]^2 + (3+2p)\frac{|g(t)|^2}{|u(t)|^{2+2p}}. \tag{3.30}$$

From (3.22) we have that

$$|u(t)|^{2p} \cdot [Q_{2p}(t)]^2 = |u(t)|^p \cdot Q_{2p}(t) \cdot Q_p(t) \leq c_2 |u(t)|^p \cdot Q_{2p}(t). \tag{3.31}$$

Moreover, now we can rewrite (3.24) in the form

$$|g(t)|^2 \leq 2K_0^2 |u(t)|^{2+2p} \left(1 + [Q_{2p}(t)] \cdot |A^{1/2}u(t)|^{2q}\right). \tag{3.32}$$

Plugging (3.31) and (3.32) into (3.30), we obtain that

$$Q_{2p}'(t) \leq -Q_{2p}(t) \cdot \left\{\nu - c_4|u(t)|^p - c_5|A^{1/2}u(t)|^{2q}\right\} + c_6.$$

Due to assumption (2.9), there exists $T_1 \geq 0$ such that

$$\nu - c_4|u(t)|^p - c_5|A^{1/2}u(t)|^{2q} \geq \frac{\nu}{2} \qquad \forall t \geq T_1,$$

hence

$$Q_{2p}'(t) \leq -\frac{\nu}{2} Q_{2p}(t) + c_6$$

for almost every $t \geq T_1$. Integrating this differential inequality we conclude that $Q_{2p}(t)$ is uniformly bounded for every $t \geq T_1$. Since $Q_{2p}(t)$ is continuous, and consequently bounded, in the compact interval $[0, T_1]$, this is enough to establish (2.12).

*Slow decay of the solution* Let us set as usual $y(t) := |u(t)|^2$. Since $Q_{2p}(t)$ and $|A^{1/2}u(t)|$ are uniformly bounded, from (3.32) we have now that

$$|g(t)| \leq c_7 |u(t)|^{1+p} \qquad \forall t \geq 0.$$

Plugging this estimate and (3.22) into (3.25) we obtain that

$$y'(t) \geq -2|A^{1/2}u(t)|^2 - 2|g(t)| \cdot |u(t)| \geq -c_8 |u(t)|^{2+p} = -c_8 [y(t)]^{1+p/2}.$$

Integrating this differential inequality we deduce (2.11).

### Spectral fast solutions

In this last section of the proof it remains to consider the case where (3.20) and (3.23) hold true. This implies in particular that $|A^{1/2}u(t)| \neq 0$ for every $t \geq T_0$, and therefore $A$ is not the null operator.

*Non-optimal exponential decay from above* Let $\nu$ be the smallest positive eigenvalue of $A$. We prove that there exists a constant $c_9$ such that

$$|A^{1/2}u(t)| \leq c_9 \exp\left(-\frac{\nu}{4}t\right) \qquad \forall t \geq 0. \tag{3.33}$$

As a consequence of (3.23), this implies also that

$$|u(t)| \leq c_{10} \exp\left(-\frac{\nu}{2(2+p)}t\right) \qquad \forall t \geq 0. \tag{3.34}$$

To this end, we consider the function $E(t) := |A^{1/2}u(t)|^2$, and we estimate its time-derivative as usual

$$E'(t) = -2|Au(t)|^2 + 2\langle Au(t), g(t)\rangle \leq -|Au(t)|^2 + |g(t)|^2. \tag{3.35}$$

For the first term we have that

$$-|Au(t)|^2 \leq -\nu |A^{1/2}u(t)|^2 = -\nu E(t).$$

For the second term, from (3.24) and (3.23) it follows that

$$|g(t)|^2 \leq 2K_0^2 |A^{1/2}u(t)|^2 \left(|u(t)|^p + |A^{1/2}u(t)|^{2q}\right) \qquad \forall t \geq T_0. \tag{3.36}$$

Plugging these estimates into (3.35) we obtain that

$$E'(t) \leq -\left(\nu - 2K_0^2|u(t)|^p - 2K_0^2|A^{1/2}u(t)|^{2q}\right) E(t).$$

Exploiting again assumption (2.9), we deduce that there exists $T_1 \geq T_0$ such that

$$\nu - 2K_0^2|u(t)|^p - 2K_0^2|A^{1/2}u(t)|^{2q} \geq \frac{\nu}{2} \qquad \forall t \geq T_1.$$

It follows that $E'(t) \leq -(\nu/2)E(t)$ for every $t \geq T_1$, hence

$$E(t) \leq E(T_1) \exp\left(-\frac{\nu}{2}(t - T_1)\right) \qquad \forall t \geq T_1,$$

which easily implies (3.33).

*Boundedness of the Dirichlet quotient* Let us consider once again the Dirichlet quotient $Q(t)$ defined by (3.16), which now is defined for every $t \geq 0$ because of (3.20). We prove that there exists a constant $c_{11}$ such that

$$Q(t) \leq c_{11} \qquad \forall t \geq 0. \tag{3.37}$$

To this end, we estimate $Q'(t)$ starting from (3.18). Exploiting (3.36) we obtain that

$$Q'(t) \leq \frac{|g(t)|^2}{|u(t)|^2} \leq 2K_0^2 Q(t) \left( |u(t)|^p + |A^{1/2}u(t)|^{2q} \right)$$

for almost every $t \geq T_0$. Integrating this differential inequality we find that

$$Q(t) \leq Q(T_0) \exp \left( 2K_0^2 \int_{T_0}^t \left( |u(s)|^p + |A^{1/2}u(s)|^{2q} \right) ds \right) \qquad \forall t \geq T_0.$$

Due to (3.34) and (3.33), the integral in the right-hand side is bounded independently of $t$. Since $Q(t)$ is continuous, and consequently bounded, in the compact interval $[0, T_0]$, this is enough to establish (3.37).

*Non-optimal exponential decay from below* We prove that there exist positive constants $c_{12}$ and $c_{13}$ such that

$$|u(t)| \geq c_{12} e^{-c_{13} t} \qquad \forall t \geq 0. \tag{3.38}$$

To this end, we consider once again the function $y(t) := |u(t)|^2$, and we estimate $y'(t)$ starting from (3.26). Now we have that $|u(t)|$ and $|A^{1/2}u(t)|$ are bounded independently of $t$ because of assumption (2.9), and $Q(t)$ is bounded independently of $t$ because of (3.37). Therefore, there exists a constant $c_{14}$ such that $y'(t) \geq -c_{14} y(t)$ for almost every $t \geq 0$. Since $y(0) > 0$, integrating this differential inequality we obtain (3.38).

*Exact exponential decay rate* Let us set

$$\lambda := \sup \left\{ \gamma > 0 : \lim_{t \to +\infty} |u(t)|_{D(A^{1/2})} e^{\gamma t} = 0 \right\}. \tag{3.39}$$

From (3.33) and (3.34) it follows that $\lambda$ is the supremum of a nonempty set. Moreover, from (3.38) it follows that $|u(t)|_{D(A^{1/2})} \geq c_{12} e^{-c_{13} t}$, which implies that $\lambda$ is finite. Therefore, $\lambda$ is a positive real number.

We claim that $\lambda$ is an eigenvalue of $A$. To this end, we write $H$ as an orthogonal direct sum

$$H := H_{\lambda,-} \oplus H_\lambda \oplus H_{\lambda,+}, \tag{3.40}$$

where

- $H_\lambda$ is the eigenspace relative to $\lambda$ if $\lambda$ is an eigenvalue of $A$, or $H_\lambda = \{0\}$ otherwise,

- $H_{\lambda,-}$ is the closure of the space generated by all eigenvectors relative to eigenvalues of $A$ less than $\lambda$ (if any),
- $H_{\lambda,+}$ is the closure of the space generated by all eigenvectors relative to eigenvalues of $A$ greater than $\lambda$ (if any).

These three subspaces of $H$ are $A$-invariant, and some of them might be the trivial subspace $\{0\}$ depending on the value of $\lambda$. Let $u_{\lambda,-}(t)$, $u_\lambda(t)$ and $u_{\lambda,+}(t)$ denote the components of $u(t)$ with respect to the decomposition (3.40), and let $g_{\lambda,-}(t)$, $g_\lambda(t)$ and $g_{\lambda,+}(t)$ be the corresponding components of $g(t)$. Let $\beta$ be the smallest eigenvalue of $A$ larger than $\lambda$ (if any), or $\beta = +\infty$ otherwise, and let $\delta := \min\{(1+p)\lambda, (1+q)\lambda\}$.

First of all, we observe that our definition of $\lambda$, combined with assumption (2.10), implies that

$$\lim_{t\to+\infty} |g(t)|e^{\gamma t} = 0 \qquad \forall \gamma < \delta. \tag{3.41}$$

We claim that

$$\lim_{t\to+\infty} |u_{\lambda,+}(t)|_{D(A^{1/2})}e^{\gamma t} = 0 \qquad \forall \gamma < \min\{\beta, \delta\}, \tag{3.42}$$

$$\lim_{t\to+\infty} |u_{\lambda,-}(t)|_{D(A^{1/2})}e^{\gamma t} = 0 \qquad \forall \gamma < \delta. \tag{3.43}$$

This is enough to conclude that $\lambda$ is an eigenvalue of $A$, because otherwise $u_\lambda(t) \equiv 0$, so that (3.42) and (3.43) would imply that

$$\lim_{t\to+\infty} |u(t)|_{D(A^{1/2})}e^{\gamma t} = 0 \qquad \forall \gamma < \min\{\beta, \delta\}, \tag{3.44}$$

and this would contradict the maximality of $\lambda$ because $\min\{\beta, \delta\} > \lambda$.

Let us prove (3.42). If all eigenvalues of $A$ are less than or equal to $\lambda$, then $H_{\lambda,+} = \{0\}$, so that (3.42) is trivial. Otherwise, we can apply Lemma 3.3 with

$$X := H_{\lambda,+}, \qquad w(t) := u_{\lambda,+}(t), \qquad \psi(t) := g_{\lambda,+}(t).$$

Indeed assumptions (3.8) and (3.9) follow from our definition of $\beta$ and from estimate (3.41). At this point, conclusion (3.10) of Lemma 3.3 is exactly (3.42).

Let us prove (3.43). If all eigenvalues of $A$ are greater than or equal to $\lambda$, then $H_{\lambda,-} = \{0\}$, so that (3.43) is trivial. Otherwise, let $\alpha$ be the largest eigenvalue of $A$ less than $\lambda$. In this case we can apply Lemma 3.4 with

$$X := H_{\lambda,-}, \qquad w(t) := u_{\lambda,-}(t), \qquad \psi(t) := g_{\lambda,-}(t).$$

Indeed assumptions (3.11) and (3.12) follow from our definition of $\alpha$ and from estimate (3.41). From Lemma 3.4 we deduce that $e^{tA}u_{\lambda,-}(t)$ has a limit $x_0 \in H_{\lambda,-}$ as $t \to +\infty$. Recalling that $\alpha < \lambda$, from (3.39) we have that

$$|x_0| = \lim_{t\to+\infty} \left|e^{tA}u_{\lambda,-}(t)\right| \leq \lim_{t\to+\infty} e^{\alpha t}|u_{\lambda,-}(t)| \leq \lim_{t\to+\infty} |u(t)|e^{\alpha t} = 0,$$

so that $x_0 = 0$. At this point conclusion (3.14) of Lemma 3.4 holds true with $x_0 = 0$, and this is exactly to (3.43).

*Exact limit* Now we know that $\lambda$ is an eigenvalue of $A$, and that $u_{\lambda,-}(t)$ and $u_{\lambda,+}(t)$ decay faster than $e^{-\gamma t}$ for every $\gamma$ satisfying (2.14). It remains to consider the component $u_\lambda(t)$. To this end, we apply again Lemma 3.4, this time with

$$X := H_\lambda, \qquad w(t) := u_\lambda(t), \qquad \psi(t) := g_\lambda(t).$$

Now assumption (3.11) is trivially satisfied with $\alpha := \lambda$ because the operator $A$ is $\lambda$ times the identity in $H_\lambda$, while assumption (3.12) follows again from estimate (3.41).

Thus from Lemma 3.4 we deduce that $e^{\lambda t}u_\lambda(t)$ tends to some $v_0 \in H_\lambda$ as $t \to +\infty$, and

$$\lim_{t\to+\infty} \left|u_\lambda(t) - v_0 e^{-\lambda t}\right| e^{\gamma t} = 0 \qquad \forall \gamma < \delta. \tag{3.45}$$

We claim that $v_0 \neq 0$. Indeed otherwise (3.45), (3.42) and (3.43) would imply (3.44), and this would contradict the maximality of $\lambda$ because $\min\{\beta, \delta\} > \lambda$.

At this point (3.42), (3.43) and (3.45) imply (2.13) for every $\gamma < \min\{\beta, \delta\}$, hence for every $\gamma$ satisfying (2.14). $\square$

## 3.4 Proof of Theorem 2.6

Let us set

$$K_1 = \frac{4K_0^2(3+2p)}{\nu},$$

and let us choose $\sigma_0 > 0$ small enough so that the following two conditions are satisfied:

$$\sigma_0^2 + K_1\sigma_0^{2+2p} < R^2, \tag{3.46}$$

$$4(1+p)\sigma_0^{2p}K_1^2 + 2K_0^2(3+2p)K_1^{1+q}\sigma_0^{(2+2p)q} \leq 2K_0^2(3+2p). \tag{3.47}$$

Let $\mathcal{S}$ be the set of all $u_0 \in D(A^{1/2})$ such that

$$u_0 \neq 0, \qquad |u_0| < \sigma_0, \qquad |A^{1/2}u_0|^2 < K_1|u_0|^{2+2p}. \tag{3.48}$$

It is clear that these assumptions define an open set in $D(A^{1/2})$, which is nonempty because it contains at least all $u_0 \in \ker(A)$ with $u_0 \neq 0$ and $|u_0| < \sigma_0$.

Let $u_0 \in \mathcal{S}$, and let $u(t)$ be the unique local solution to problem (2.6)–(2.2) provided by Theorem B, defined in a maximal interval $[0, T)$. We claim that $T = +\infty$ and this solution is slow in the sense of (2.11).

*Basic estimate* We prove that

$$|u(t)| \leq |u_0| < \sigma_0 \qquad \forall t \in [0, T). \tag{3.49}$$

Indeed let us set $y(t) := |u(t)|^2$. A simple computation shows that

$$y'(t) = -2|A^{1/2}u(t)|^2 + 2\langle u(t), f(u(t))\rangle. \tag{3.50}$$

Thanks to (2.17) we have that $y'(t) \leq 0$, which proves (3.49).

*Boundedness of the generalized Dirichlet quotient* We prove that

$$u(t) \neq 0 \quad \text{and} \quad |A^{1/2}u(t)|^2 < K_1|u(t)|^{2+2p} \qquad \forall t \in [0,T). \tag{3.51}$$

To this end, we consider once again the modified Dirichlet quotient (3.17) with $d := 2p$, defined as long as $u(t) \neq 0$, and we set

$$S := \sup\left\{t \in [0,T) : |u(\tau)| \cdot (Q_{2p}(\tau) - K_1) < 0 \quad \forall \tau \in [0,t]\right\},$$

so that now (3.51) is equivalent to $S = T$. To begin with, we set $t = 0$ and we obtain that $|u(0)| \cdot (Q_{2p}(0) - K_1) < 0$ because of the first and third condition in (3.48). Therefore, a simple continuity argument gives that $S > 0$. Let us assume by contradiction that $S < T$. Then, by the maximality of $S$, this means that at least one of the following equalities is satisfied

$$u(S) = 0, \qquad Q_{2p}(S) = K_1. \tag{3.52}$$

Now we exclude both possibilities. First of all, our definition of $S$ implies that

$$u(t) \neq 0 \quad \text{and} \quad Q_{2p}(t) < K_1 \qquad \forall t \in [0,S). \tag{3.53}$$

In particular, keeping into account assumption (2.16) and (3.49), we obtain that

$$\begin{aligned} |f(u(t))| &\leq K_0|u(t)|^{1+p}\left(1 + [Q_{2p}(t)]^{(1+q)/2}|u(t)|^{(1+p)q}\right) \\ &\leq K_0|u(t)|^{1+p}\left(1 + K_1^{(1+q)/2}\sigma_0^{(1+p)q}\right) \end{aligned} \tag{3.54}$$

for every $t \in [0,S)$. Let us consider again the function $y(t) := |u(t)|^2$. We compute the time-derivative as in (3.50), and then we estimate the right-hand side exploiting (3.53) and (3.54). We obtain that

$$\begin{aligned} y'(t) &\geq -2|A^{1/2}u(t)|^2 - 2|u(t)| \cdot |f(u(t))| \\ &\geq -2Q_{2p}(t) \cdot |u(t)|^p \cdot |u(t)|^{2+p} - 2K_0|u(t)|^{2+p}\left(1 + K_1^{(1+q)/2}\sigma_0^{(1+p)q}\right) \\ &\geq -2\left(K_1\sigma_0^p + K_0 + K_0K_1^{(1+q)/2}\sigma_0^{(1+p)q}\right)|y(t)|^{1+p/2} \end{aligned} \tag{3.55}$$

for almost every $t \in [0,S)$. Integrating this differential inequality we conclude that $u(t)$ cannot vanish in a finite time, which rules out the first possibility in (3.52).

In order to exclude the second one, we compute the time-derivative of $Q_{2p}(t)$. From (3.19) with $d = 2p$ we obtain that

$$Q'_{2p}(t) \leq -\nu Q_{2p}(t) + 4(1+p)|u(t)|^{2p}[Q_{2p}(t)]^2 + (3+2p)\frac{|f(u(t))|^2}{|u(t)|^{2+2p}}.$$

From (3.49), (3.53) and (3.54) we have that

$$|u(t)|^{2p}[Q_{2p}(t)]^2 \leq \sigma_0^{2p}K_1^2, \qquad \frac{|f(u(t))|^2}{|u(t)|^{2+2p}} \leq 2K_0^2\left(1 + K_1^{1+q}\sigma_0^{(2+2p)q}\right),$$

so that

$$Q_{2p}'(t) \leq -\nu Q_{2p}(t) + 4(1+p)\sigma_0^{2p}K_1^2 + 2K_0^2(3+2p) + 2K_0^2(3+2p)K_1^{1+q}\sigma_0^{(2+2p)q}.$$

Keeping the smallness condition (3.47) into account, we finally deduce that

$$Q_{2p}'(t) \leq -\nu Q_{2p}(t) + 4K_0^2(3+2p) = -\nu(Q_{2p}(t) - K_1)$$

for almost every $t \in [0, S)$. Integrating this differential inequality we obtain that

$$Q_{2p}(t) \leq K_1 + (Q_{2p}(0) - K_1)e^{-\nu t} \qquad \forall t \in [0, S).$$

Letting $t \to S^-$, and recalling that $Q_{2p}(0) < K_1$, we conclude that $Q_{2p}(S) < K_1$, which rules out the second possibility in (3.52).

*Global existence and slow decay* We show that $T = +\infty$. Let us assume indeed that $T < +\infty$. Letting $t \to T^-$ in (3.49) and (3.51), and taking into account the smallness assumption (3.46), we obtain that

$$\lim_{t \to T^-} \left(|u(t)|^2 + |A^{1/2}u(t)|^2\right) \leq \sigma_0^2 + K_1\sigma_0^{2+2p} < R^2,$$

which contradicts (2.8).

Since we have proved that $S = T = +\infty$, the differential inequality in (3.55) now holds true for every $t \geq 0$. A simple integration of this differential inequality proves that $u(t)$ satisfies (2.11) for a suitable positive constant $M_1$, depending only on $K_0$, $K_1$, $\sigma_0$, $p$, $q$. □

## 3.5 Proof of Theorem 2.9

Let us sketch the strategy of the proof, based on a fixed point argument, before entering into details. We begin by observing that our definition of $H_-$ implies that $H_- \subseteq D(A)$ and

$$|Au| \leq \lambda|u| \qquad \forall u \in H_-, \tag{3.56}$$

while our definition of $H_+$ implies that there exists $\beta > \lambda$ such that

$$|Au| \geq \beta|u| \qquad \forall u \in H_+ \cap D(A). \tag{3.57}$$

More precisely, the last inequality holds true with $\beta$ equal to the smallest eigenvalue of $A$ greater than $\lambda$ (if any), or with any $\beta > \lambda$ if the spectrum of $A$ is finite and $\lambda$ is its maximum, in which case $H_+ = \{0\}$.

Now let us choose a constant $\delta$ such that $\lambda < \delta < \min\{\beta, (1+p)\lambda\}$, let us set

$$r_1 := 2\left(1 + \frac{1}{\beta}\right)^{1/2} r_0, \tag{3.58}$$

and let us assume that $r_0$ is small enough so that

$$2r_1 < R, \tag{3.59}$$

$$2L(2r_1)^p\left(\frac{\sqrt{\lambda+1}}{\delta-\lambda}+\frac{\sqrt{\beta+1}}{\beta-\delta}\right)\leq\frac{1}{2}. \tag{3.60}$$

Let us consider the space

$$\mathbb{X} := \left\{g \in C^0\left([0,+\infty); D(A^{1/2})\right) : |g(t)|_{D(A^{1/2})} \leq r_1 \quad \forall t \geq 0\right\}.$$

It is well-known that $\mathbb{X}$ is a complete metric space with respect to the distance

$$\operatorname{dist}(g_1, g_2) := \sup\left\{|g_1(t) - g_2(t)|_{D(A^{1/2})} : t \geq 0\right\}.$$

For every $g \in \mathbb{X}$ we set

$$\varphi_g(t) := f\left(v_0 e^{-\lambda t} + g(t)e^{-\delta t}\right) \qquad \forall t \geq 0, \tag{3.61}$$

and we define $\varphi_{g,-}(t)$ and $\varphi_{g,+}(t)$ as the projections of $\varphi_g(t)$ into $H_-$ and $H_+$, respectively. Then we define $u_{g,-} : [0,+\infty) \to H_-$ and $u_{g,+} : [0,+\infty) \to H_+$ as

$$u_{g,-}(t) := v_0 e^{-\lambda t} - \int_t^{+\infty} e^{(s-t)A} \cdot \varphi_{g,-}(s)\, ds, \tag{3.62}$$

$$u_{g,+}(t) := e^{-tA} w_0 + \int_0^t e^{(s-t)A} \cdot \varphi_{g,+}(s)\, ds, \tag{3.63}$$

and $u_g(t) := u_{g,-}(t) + u_{g,+}(t)$. Finally, we set

$$\overline{g}(t) := \left(u_g(t) - v_0 e^{-\lambda t}\right) e^{\delta t} \qquad \forall t \geq 0. \tag{3.64}$$

We claim that the following three statements hold true, provided that the smallness assumptions (3.59) and (3.60) are satisfied.

- *Well-posedness of the construction.* The functions $\varphi_g$ and $u_g$ are well-defined for every $g \in \mathbb{X}$. Moreover, $u_g$ is a solution to the *linear* equation

$$u_g'(t) + Au_g(t) = \varphi_g(t) = f\left(v_0 e^{-\lambda t} + g(t)e^{-\delta t}\right) \qquad \forall t \geq 0 \tag{3.65}$$

  in the sense of Theorem A.

- *Closedness.* We have that $\overline{g} \in \mathbb{X}$ for every $g \in \mathbb{X}$.

- *Contractivity.* The map $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ defined by $\mathcal{F}(g) = \overline{g}$ is a contraction.

If we prove the three claims, then the conclusion easily follows. Indeed the contractivity implies that $\mathcal{F}$ has a fixed point, namely there exists $g \in \mathbb{X}$ such that $\overline{g} = g$. Thus from (3.64) we have that

$$u_g(t) = v_0 e^{-\lambda t} + \overline{g}(t) e^{-\delta t} = v_0 e^{-\lambda t} + g(t) e^{-\delta t}, \tag{3.66}$$

so that (2.21) follows from the boundedness of $g$ and the fact that $\delta > \lambda$.

Moreover, the projection of $u_g(0)$ into $H_+$, namely $u_{g,+}(0)$, is $w_0$ because of (3.63), hence the initial condition $u_g(0)$ is of the form $w_1 + w_0$ for some $w_1 \in H^-$.

Finally, exploiting (3.66) once again, we have that

$$f(u_g(t)) = f\left(v_0 e^{-\lambda t} + g(t) e^{-\delta t}\right),$$

so that saying that $u_g$ is a solution to (3.65) is equivalent to saying that $u_g$ is a solution to (2.6).

*Well-posedness of the construction* From (2.20) and our definition of $\mathbb{X}$ we have that

$$\left|v_0 e^{-\lambda t} + g(t) e^{-\delta t}\right|_{D(A^{1/2})} \leq |v_0|_{D(A^{1/2})} e^{-\lambda t} + |g(t)|_{D(A^{1/2})} e^{-\delta t} \leq r_0 e^{-\lambda t} + r_1 e^{-\delta t}.$$

Since $\delta > \lambda$ and $r_0 \leq r_1$, it follows that

$$\left|v_0 e^{-\lambda t} + g(t) e^{-\delta t}\right|_{D(A^{1/2})} \leq 2 r_1 e^{-\lambda t} \qquad \forall t \geq 0. \tag{3.67}$$

Due to the smallness assumption (3.59), we have in particular that

$$\left|v_0 e^{-\lambda t} + g(t) e^{-\delta t}\right|_{D(A^{1/2})} < R \qquad \forall t \geq 0,$$

which proves that $\varphi_g(t)$ is well-defined.

Setting $v = 0$ into (2.18), from (2.19) we obtain that

$$|f(u)| \leq L |u|_{D(A^{1/2})}^{1+p} \qquad \forall u \in B_R.$$

Therefore, recalling that $\delta < (1+p)\lambda$, from (3.61) and (3.67) we deduce that

$$|\varphi_g(t)| \leq L (2 r_1)^{1+p} e^{-\delta t} \qquad \forall t \geq 0. \tag{3.68}$$

Let us examine the integrand in (3.62). From (3.56) it turns out that the operator $e^{(s-t)A}$ is bounded in $H_-$ with norm less than or equal to $e^{(s-t)\lambda}$. Since the right-hand side of (3.68) is of course an estimate also for $|\varphi_{g,-}(t)|$, we deduce that

$$\left|e^{(s-t)A} \varphi_{g,-}(s)\right| \leq e^{(s-t)\lambda} |\varphi_{g,-}(s)| \leq e^{(s-t)\lambda} L (2 r_1)^{1+p} e^{-\delta s} \qquad \forall s \geq t \geq 0. \tag{3.69}$$

Since $\delta > \lambda$, this proves that the integral in the right-hand side of (3.62) converges for every $t \geq 0$, hence $u_{g,-}(t)$ is well-defined.

Let us examine the right-hand side of (3.63). Now the integration is over a bounded interval, and the operator $e^{(s-t)A}$ is a contraction because $s \leq t$. Therefore, also $u_{g,+}(t)$ is well-defined.

Finally, both the regularity (2.3) of $u_g$, and the fact that it is a solution to (3.65), follow from definitions (3.62) and (3.63).

*Closedness* To begin with, we observe that $\overline{g} : [0, +\infty) \to D(A^{1/2})$ is a continuous map because of the regularity of $u_g$. Now we claim that

$$\left|u_{g,-}(t) - v_0 e^{-\lambda t}\right|_{D(A^{1/2})} \leq L(2r_1)^{1+p} \frac{\sqrt{\lambda+1}}{\delta - \lambda} e^{-\delta t} \qquad \forall t \geq 0, \tag{3.70}$$

$$|u_{g,+}(t)|_{D(A^{1/2})} \leq L(2r_1)^{1+p} \frac{\sqrt{\beta+1}}{\beta - \delta} e^{-\delta t} + \frac{r_1}{2} e^{-\delta t} \qquad \forall t \geq 0. \tag{3.71}$$

Plugging these estimates into (3.64) we obtain that

$$\begin{aligned}
|\overline{g}(t)|_{D(A^{1/2})} &= \left|u_{g,-}(t) + u_{g,+}(t) - v_0 e^{-\lambda t}\right|_{D(A^{1/2})} e^{\delta t} \\
&\leq \left|u_{g,-}(t) - v_0 e^{-\lambda t}\right|_{D(A^{1/2})} e^{\delta t} + |u_{g,+}(t)|_{D(A^{1/2})} e^{\delta t} \\
&\leq L(2r_1)^{1+p} \left( \frac{\sqrt{\lambda+1}}{\delta - \lambda} + \frac{\sqrt{\beta+1}}{\beta - \delta} \right) + \frac{r_1}{2}.
\end{aligned}$$

The smallness assumption (3.60) is equivalent to saying that the right-hand side is less than or equal to $r_1$. This proves that $\overline{g} \in \mathbb{X}$.

Let us prove (3.70). From (3.69) we have that

$$\begin{aligned}
\left|u_{g,-}(t) - v_0 e^{-\lambda t}\right| &\leq \int_t^{+\infty} \left|e^{(s-t)A} \varphi_{g,-}(s)\right| ds \\
&\leq L(2r_1)^{1+p} \int_t^{+\infty} e^{(s-t)\lambda} \cdot e^{-\delta s} \, ds \\
&= L(2r_1)^{1+p} \frac{1}{\delta - \lambda} e^{-\delta t},
\end{aligned}$$

so that (3.70) follows by simply remarking that

$$|w|_{D(A^{1/2})} \leq \sqrt{\lambda + 1} \cdot |w| \qquad \forall w \in H_-. \tag{3.72}$$

Let us prove (3.71). To this end, we set $E(t) := |A^{1/2} u_{g,+}(t)|^2$. Its time-derivative is

$$\begin{aligned}
E'(t) &= -2|Au_{g,+}(t)|^2 + 2\langle Au_{g,+}(t), \varphi_{g,+}(t)\rangle \\
&\leq -2|Au_{g,+}(t)|^2 + \frac{\beta - \delta}{\beta} |Au_{g,+}(t)|^2 + \frac{\beta}{\beta - \delta} |\varphi_{g,+}(t)|^2 \\
&\leq -\frac{\delta + \beta}{\beta} |Au_{g,+}(t)|^2 + \frac{\beta}{\beta - \delta} |\varphi_g(t)|^2.
\end{aligned}$$

Let us estimate the first term using (3.57), and the second term using (3.68). We deduce that

$$E'(t) \leq -(\delta + \beta) E(t) + \frac{\beta}{\beta - \delta} L^2 (2r_1)^{2+2p} e^{-2\delta t}.$$

Integrating this differential inequality we obtain that

$$E(t) \leq |A^{1/2}w_0|^2 e^{-(\delta+\beta)t} + \frac{\beta}{(\beta-\delta)^2} L^2(2r_1)^{2+2p} e^{-2\delta t} \qquad \forall t \geq 0.$$

Since $\delta + \beta > 2\delta$, and since $|A^{1/2}w_0| \leq r_0$ because of assumption (2.20), this easily implies that

$$\left|A^{1/2}u_{g,+}(t)\right| \leq r_0 e^{-\delta t} + \frac{\sqrt{\beta}}{\beta-\delta} L(2r_1)^{1+p} e^{-\delta t} \qquad \forall t \geq 0,$$

so that (3.71) follows by simply recalling our definition (3.58) of $r_1$, and the fact that

$$|w|_{D(A^{1/2})} \leq \left(1 + \frac{1}{\beta}\right)^{1/2} |A^{1/2}w| \qquad \forall w \in D(A^{1/2}) \cap H_+. \tag{3.73}$$

*Contractivity* Let $g_1$ and $g_2$ be two elements of $\mathbb{X}$. From (2.18) we have that

$$\begin{aligned} |\varphi_{g_1}(t) - \varphi_{g_2}(t)| &= \left|f\left(v_0 e^{-\lambda t} + g_1(t)e^{-\delta t}\right) - f\left(v_0 e^{-\lambda t} + g_2(t)e^{-\delta t}\right)\right| \\ &\leq L\left(\left|v_0 e^{-\lambda t} + g_1(t)e^{-\delta t}\right|^p_{D(A^{1/2})} + \left|v_0 e^{-\lambda t} + g_2(t)e^{-\delta t}\right|^p_{D(A^{1/2})}\right) \times \\ &\quad \times |g_1(t) - g_2(t)|_{D(A^{1/2})} e^{-\delta t}. \end{aligned}$$

Therefore, applying inequality (3.67) to $g_1$ and $g_2$, we deduce that

$$|\varphi_{g_1}(t) - \varphi_{g_2}(t)| \leq 2L(2r_1)^p \cdot \mathrm{dist}(g_1, g_2) \cdot e^{-\delta t} \qquad \forall t \geq 0. \tag{3.74}$$

The right-hand side of (3.74) is of course an estimate also for the projections of $\varphi_{g_1} - \varphi_{g_2}$ into $H_-$ and $H_+$. In particular, for the component with respect to $H_-$ we obtain that

$$\begin{aligned} |u_{g_1,-}(t) - u_{g_2,-}(t)| &\leq \int_t^{+\infty} \left|e^{(s-t)A}(\varphi_{g_1,-}(s) - \varphi_{g_2,-}(s))\right| ds \\ &\leq \int_t^{+\infty} e^{(s-t)\lambda} |\varphi_{g_1}(s) - \varphi_{g_2}(s)|\, ds \\ &\leq 2L(2r_1)^p \cdot \mathrm{dist}(g_1, g_2) \int_t^{+\infty} e^{(s-t)\lambda} \cdot e^{-\delta s}\, ds \\ &\leq 2L(2r_1)^p \cdot \mathrm{dist}(g_1, g_2) \cdot \frac{1}{\delta-\lambda} e^{-\delta t}. \end{aligned}$$

Keeping (3.72) into account, this proves that

$$|u_{g_1,-}(t) - u_{g_2,-}(t)|_{D(A^{1/2})} \leq 2L(2r_1)^p \cdot \mathrm{dist}(g_1, g_2) \cdot \frac{\sqrt{\lambda+1}}{\delta-\lambda} e^{-\delta t}. \tag{3.75}$$

Let us consider now the component with respect to $H_+$. To this end, we set once again $E(t) := |A^{1/2}(u_{g_1,+}(t) - u_{g_2,+}(t))|^2$. Taking the time-derivative, and arguing as we did in the previous paragraph, we obtain that

$$E'(t) \leq -\frac{\delta+\beta}{\beta}\left|A(u_{g_1,+}(t) - u_{g_2,+}(t))\right|^2 + \frac{\beta}{\beta-\delta}|\varphi_{g_1}(t) - \varphi_{g_2}(t)|^2.$$

Thus from (3.57) and (3.74) it follows that

$$E'(t) \leq -(\delta+\beta)E(t) + \frac{\beta}{\beta-\delta}\cdot 4L^2(2r_1)^{2p}\cdot \mathrm{dist}^2(g_1,g_2)\cdot e^{-2\delta t}.$$

Since now $E(0) = 0$, integrating this differential inequality we deduce that

$$E(t) \leq 4L^2(2r_1)^{2p}\cdot \mathrm{dist}^2(g_1,g_2)\cdot \frac{\beta}{(\beta-\delta)^2}e^{-2\delta t} \qquad \forall t \geq 0.$$

Keeping (3.73) into account, this proves that

$$|u_{g_1,+}(t) - u_{g_2,+}(t)|_{D(A^{1/2})} \leq 2L(2r_1)^p\cdot \mathrm{dist}(g_1,g_2)\cdot \frac{\sqrt{\beta+1}}{\beta-\delta}e^{-\delta t}. \tag{3.76}$$

From (3.64), (3.75) and (3.76) we conclude that

$$\begin{aligned}
|\overline{g_1}(t) - \overline{g_2}(t)|_{D(A^{1/2})} &= |u_{g_1}(t) - u_{g_2}(t)|_{D(A^{1/2})}\, e^{\delta t} \\
&\leq |u_{g_1,-}(t) - u_{g_2,-}(t)|_{D(A^{1/2})}\, e^{\delta t} + |u_{g_1,+}(t) - u_{g_2,+}(t)|_{D(A^{1/2})}\, e^{\delta t} \\
&\leq 2L(2r_1)^p\left(\frac{\sqrt{\lambda+1}}{\delta-\lambda} + \frac{\sqrt{\beta+1}}{\beta-\delta}\right)\cdot \mathrm{dist}(g_1,g_2)
\end{aligned}$$

for every $t \geq 0$. Taking the supremum over all $t \geq 0$, and keeping into account the smallness assumption (3.60), we conclude that

$$\mathrm{dist}(\overline{g_1},\overline{g_2}) \leq \frac{1}{2}\,\mathrm{dist}(g_1,g_2),$$

which proves that the map $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ is a contraction. $\Box$

# 4 Applications

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary (or any other condition which guarantees Sobolev embeddings). We consider semilinear parabolic equations of the form

$$u_t - \Delta u + \psi(u) = 0 \tag{4.1}$$

in $\Omega \times [0, +\infty)$, with homogeneous Neumann boundary conditions, and initial datum $u(0) = u_0$. We also consider semilinear parabolic equations of the form

$$u_t - \Delta u - \lambda u + \psi(u) = 0 \tag{4.2}$$

in $\Omega \times [0, +\infty)$, with homogeneous Dirichlet boundary conditions, and initial datum $u(0) = u_0$. In this case we assume that $\lambda \leq \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions in $\Omega$.

The classical approach to these problems consists in setting $H := L^2(\Omega)$ and considering the operator $Au := -\Delta u$ with domain $D(A) := H^2(\Omega)$ in the Neumann case, or the operator $Au := -\Delta u - \lambda u$ with domain $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$ in the Dirichlet case. In both cases $A$ is a self-adjoint operator on $H$. It is a coercive operator in the subcritical Dirichlet case with $\lambda < \lambda_1(\Omega)$, but it is just a nonnegative operator both in the Neumann case (where the kernel of $A$ is the space of constant functions), and in the critical Dirichlet case with $\lambda = \lambda_1(\Omega)$ (where the kernel of $A$ is the eigenspace of $-\Delta$ relative to the eigenvalue $\lambda_1(\Omega)$).

As for the nonlinear term, we assume that $\psi : \mathbb{R} \to \mathbb{R}$ is a function of class $C^1$ such that

$$\psi(0) = 0, \tag{4.3}$$

and

$$|\psi'(\sigma)| \leq C_1 |\sigma|^p \qquad \forall \sigma \in \mathbb{R}, \tag{4.4}$$

$$\psi(\sigma)\sigma \geq C_2 |\sigma|^{2+p} \qquad \forall \sigma \in \mathbb{R}, \tag{4.5}$$

for suitable positive constants $C_1$, $C_2$, $p$.

Now equations (4.1) and (4.2) can be written in the abstract form (1.1) provided that we set

$$[f(u)](x) := -\psi(u(x)) \qquad \forall x \in \Omega \tag{4.6}$$

Therefore, it is crucial to know when $f$ satisfies the assumptions of our main abstract results. Since clearly $f(0) = 0$, the key point is the verification of (2.18), which we already know to imply both (2.7) and (2.16) with $p = q$. The verification of (2.18) is quite standard, and requires that $D(A^{1/2})$ is contained into $L^{2+2p}(\Omega)$, which in turn is equivalent to the Sobolev embedding $H^1(\Omega) \subseteq L^{2+2p}(\Omega)$.

Here we skip the details, for which the interested reader is referred to Section 4.1 of [11]. The final result is the following.

**Proposition 4.1.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary. Let $p$ be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$. Let $H := L^2(\Omega)$, and let $A$ be the operator associated to the Neumann or Dirichlet problem. For every $R > 0$, let $B_R$ be the open ball in $D(A^{1/2})$, defined as in Theorem B. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$ satisfying (4.3) and (4.4).*

*Then there exist $R > 0$ and $L$ such that (4.6) defines a function $f : B_R \to H$ satisfying (2.18).*

We point out that in Proposition 4.1 above we need no assumption on the sign or monotonicity of $\psi$. Sign and monotonicity conditions are important when looking for global solutions for all initial data $u_0 \in L^2(\Omega)$. For the convenience of the reader, we quote the following well-known result for the case with the "right sign".

**Theorem C** (Right sign – Global existence)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary. Let $p > 0$, and let $\psi : \mathbb{R} \to \mathbb{R}$ be a continuous nondecreasing function satisfying (4.5).*

*Then the following statement applies to both the Neumann problem for equation (4.1), and to the Dirichlet problem for equation (4.2) with $\lambda \leq \lambda_1(\Omega)$. For every $u_0 \in L^2(\Omega)$, the problem admits a unique global solution*

$$u \in C^0([0, +\infty), H) \cap C^0((0, +\infty), D(A)) \cap C^1((0, +\infty), H), \tag{4.7}$$

*and there exists a constant $M$ such that*

$$|u(t)| \leq \frac{M}{t^{1/p}} \qquad \forall t > 0, \tag{4.8}$$

$$|A^{1/2}u(t)| \leq \frac{|u_0|}{t^{1/2}} \qquad \forall t > 0. \tag{4.9}$$

The proof of Theorem C is quite classical. Indeed, both the Neumann and the Dirichlet problem are the gradient-flow of a strictly convex functional, hence global existence, regularity and estimate (4.9) follow, for example, from the well-established theory of maximal monotone operators (see [5]). As for the decay estimate (4.8), it holds true for all solutions of the abstract equation (1.1), provided that the nonlinear term $f(u)$ satisfies a sign condition such as

$$\langle f(u), u \rangle \leq -K|u|^{2+p} \qquad \forall u \in D(A^{1/2}) \tag{4.10}$$

for a suitable constant $K$. Indeed, setting as usual $y(t) := |u(t)|^2$, and computing its time-derivative as in (3.50), we obtain that

$$y'(t) = -2|A^{1/2}u(t)|^2 + 2\langle u(t), f(u(t))\rangle \leq -2K|u(t)|^{2+p} = -2K[y(t)]^{1+p/2},$$

so that (4.8) easily follows by integrating this differential inequality. In the concrete example, (4.10) holds true whenever $\psi(\sigma)$ satisfies (4.5).

We point out that Theorem C holds true for every $n \geq 1$ and $p > 0$. Under the more restrictive assumptions of Proposition 4.1, we can apply our theory as follows.

**Theorem 4.2** (Right sign – Decay rates)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $p$ be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function of class $C^1$ satisfying (4.3) through (4.5).*

*Then the following three statements apply to both the Neumann problem for equation (4.1), and to the Dirichlet problem for equation (4.2) with $\lambda = \lambda_1(\Omega)$.*

*(1)* (Classification of decay rates) *All non-zero solutions are either slow or fast in the sense of Theorem 2.1.*

*(2)* (Existence of slow solutions) *There exists a nonempty open set* $\mathcal{S} \subseteq L^2(\Omega)$ *such that all solutions with* $u(0) \in S$ *are slow.*

*(3)* (Existence of fast solutions) *There exists families of fast solutions in the sense of Theorem 2.9.*

Of course the theory applies also to the Dirichlet problem with $\lambda < \lambda_1(\Omega)$, but in that case the operator is coercive and we have only fast solutions.

The proof of Theorem 4.2 is a straightforward application of our abstract results. We just point out that here the set $\mathcal{S}$ of initial data originating slow solutions is claimed to be open in $H$ and not just in $D(A^{1/2})$ as in Theorem 2.6. This improvement relies on the regularizing effect. Indeed, it is enough to take the set $\mathcal{S}' \subseteq D(A^{1/2})$ provided by Theorem 2.6, and then considering the set $\mathcal{S} \subseteq H$ of all initial data of solutions which end up in $\mathcal{S}'$ at time $t = 1$. This set in non-empty and open due to the regularizing effect.

**Remark 4.3.** For the heat equation, it is possible to use Theorem 2.1 in conjunction with additional properties of the heat flow such as smoothing effect in arbitrary Lebesgue-based Sobolev spaces to obtain a major improvement with respect to what was already known. In next result we obtain a slow-exponential alternative for all values of $p$ and $n$. We are confident that a relevant refinement of our techniques would yield in this general case also the remaining results, for example the existence of an *open set* of slow solutions or the existence of fast solutions with arbitrary spectral decay rate.

**Theorem 4.4** (Arbitrary power functions – The alternative)**.** *Let* $\Omega \subseteq \mathbb{R}^n$ *be a bounded open set with Lipschitz boundary. Let* $p$ *be any positive exponent, and let* $\psi(\sigma) := c|\sigma|^p\sigma$ *for some* $c > 0$.

*Then both Neumann problem for equation (4.1), and Dirichlet problem for equation (4.2) with* $\lambda = \lambda_1(\Omega)$ *satisfy the following decay alternative: all non-zero solutions with bounded initial data satisfy either (2.11) or (2.13) for some* $\gamma > \lambda$.

*Proof* Here global existence of solutions is well-known in the sense of $L^\infty(\Omega)$ for initial data in the same space, and solutions are in fact strong solutions with values in $D(A^{1/2})$ as a consequence of smoothing effect. In order to apply Theorem 2.1 we observe that

$$|f(u)|_H = \|f(u)\|_{L^2(\Omega)} \leq \|u\|^{p-r}_{L^\infty(\Omega)} \cdot \|u\|^{1+r}_{L^{2r+2}(\Omega)}.$$

Now along the trajectory, say for $t \geq 1$, we have a uniform bound on $u$ in $L^\infty(\Omega)$. It is therefore enough to choose $r \in (0, p)$ small enough so that $D(A^{1/2})$ is contained into $L^{2r+2}(\Omega)$ and obtain that

$$|f(u)| \leq c_2 |u|^{1+r}_{D(A^{1/2})}.$$

Then Theorem 2.1 is applicable and we obtain the alternative with $p$ replaced by $r$ in the lower estimate of slow solutions, but from the results of either [2] for Neumann's case or [3] for Dirichlet's case we know that any solution which does not satisfy the "slow condition" corresponding to $p$ tends to 0 faster than any negative power of $t$. The result follows immediately by exclusion. □

When $\psi$ has the wrong sign, for example when $\psi(\sigma) = -|\sigma|^p\sigma$, global existence can fail (see [7]). Nevertheless, exploiting the so-called potential well, one can obtain global existence for all initial data which are small enough with respect to the norm of $D(A^{1/2})$. This technique requires that the operator is coercive and controls the nonlinear term (which means Sobolev embeddings). Since the coerciveness of the operator is essential, this theory applies neither to the Neumann case, nor to the critical Dirichlet case. In other words, the potential well applies only to the subcritical Dirichlet case, in which case we obtain the following result.

**Theorem 4.5** (Wrong sign, with potential well). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $p$ be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$ satisfying (4.3) and (4.4). Let us consider the Dirichlet problem for equation (4.2) with $\lambda < \lambda_1(\Omega)$.*

*Then there exists $R > 0$ such that, for every $u_0 \in B_R$ (defined as in Theorem B), the problem has a unique global solution, which satisfies (4.7), (4.8), and (4.9).*

*Moreover, every non-zero solution is fast in the sense of Theorem 2.1, and there exist families of fast solutions parametrized in the sense of Theorem 2.9.*

When there is no potential well, Theorem 2.1 keeps on classifying all possible decay rates of solutions which exist globally and decay. On the other hand, nothing in this case guarantees decay, or even global existence, of solutions.

Nevertheless, there is one notable exception. Theorem 2.9 provides families of global solutions with exponential decay without assuming neither the coercivity of the operator, nor sign conditions on the nonlinear term. Therefore, even in the Neumann case and in the critical Dirichlet case, we obtain the following existence result.

**Theorem 4.6** (Wrong sign, without potential well). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $p$ be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a function of class $C^1$ satisfying (4.3) and (4.4).*

*Let us consider the Neumann problem for equation (4.1) or the Dirichlet problem for equation (4.2) with $\lambda = \lambda_1(\Omega)$.*

*Then there exist families of fast solutions parametrized in the sense of Theorem 2.9.*

We conclude by pointing out that our abstract results apply also to equations with second order operators with non-constant coefficients, or with higher order operators such as $\Delta^2$. We also allow more general nonlinear terms depending on $x$ and $t$, or even non-local nonlinear terms.

# References


[1] C. Bardos, L. Tartar; Sur l'unicité rétrograde des équations paraboliques et quelques questions voisines. (French) *Arch. Rational Mech. Anal.* **50** (1973), 10–25.

[2] I. Ben Arbi; Rate of decay to 0 of the solutions to a nonlinear parabolic equation. *Port. Math.* **69** (2012), no. 1, 23–39.

[3] I. Ben Arbi. A. Haraux; Slow and fast decaying solutions to a critical semilinear parabolic equation. *J. Abstr. Differ. Equ. Appl.* **4** (2013), no. 1, 1–10.

[4] I. Ben Arbi. A. Haraux; A sufficient condition for slow decay of a solution to a semilinear parabolic equation. *Anal. Appl. (Singap.)* **10** (2012), no. 4, 363–371.

[5] H. Brezis; *Opérateures maximaux monotones et semigroups de contraction dans les espaces de Hilbert*, North-Holland Mathematics Studies (1973).

[6] C. Foias, J.-C. Saut; Asymptotic behavior, as $t \to +\infty$, of solutions of Navier-Stokes equations and nonlinear spectral manifolds. *Indiana Univ. Math. J.* **33** (1984), no. 3, 459–477.

[7] H. Fujita; On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. *J. Fac. Sci. Univ. Tokyo Sect. I* **13** 1966, 109–124 (1966).

[8] J.-M. Ghidaglia; Some backward uniqueness results. *Nonlinear Anal.* **10** (1986), no. 8, 777–790.

[9] M. Ghisi; Global solutions for dissipative Kirchhoff strings with non-Lipschitz nonlinear term. *J. Differential Equations* **230** (2006), no. 1, 128–139.

[10] M. Ghisi, M. Gobbino; Hyperbolic-parabolic singular perturbation for mildly degenerate Kirchhoff equations: time-decay estimates. *J. Differential Equations* **245** (2008), no. 10, 2979–3007.

[11] M. Ghisi, M. Gobbino, A. Haraux; Optimal decay estimates for the general solution to a class of semi-linear dissipative hyperbolic equations. Preprint. `arXiv:1306.3644`.

[12] A. Haraux, M. A. Jendoubi, O. Kavian; Rate of decay to equilibrium in some semilinear parabolic equations. *J. Evol. Equ.* **3** (2003), no. 3, 463–484.

[13] A. Haraux; Decay rate of the range component of solutions to some semilinear evolution equations. *NoDEA Nonlinear Differential Equations Appl.* **13** (2006), no. 4, 435–445.

[14] P. Hartman; On local homeomorphisms of Euclidean spaces. *Bol. Soc. Mat. Mexicana (2)* **5** (1960), 220–241.

[15] P. Hartman; A lemma in the theory of structural stability of differential equations. *Proc. Amer. Math. Soc.* **11** (1960), 610–620.

[16] H. M. Rodrigues, J. Solà-Morales; Linearization of class $C^1$ for contractions on Banach spaces. *J. Differential Equations* **201** (2004), no. 2, 351–382.