Finding the exact decay rate of all solutions to some second order evolution equations with dissipation

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Abstract

We consider an abstract second order evolution equation with damping. The "elastic" term is represented by a self-adjoint nonnegative operator A with discrete spectrum, and the nonlinear term has order greater than one at the origin. We investigate the asymptotic behavior of solutions.

We prove the coexistence of slow solutions and fast solutions. Slow solutions live close to the kernel of A, and decay as negative powers of t as solutions of the first order equation obtained by neglecting the operator A and the second order time-derivatives in the original equation. Fast solutions live close to the range of A and decay exponentially as solutions of the linear homogeneous equation obtained by neglecting the nonlinear terms in the original equation.

The abstract results apply to semilinear dissipative hyperbolic equations.

Mathematics Subject Classification 2010 (MSC2010): 35L71, 35L90, 35B40.

Key words: semilinear hyperbolic equation, dissipative hyperbolic equation, decay rates, slow solutions, exponentially decaying solutions.

1 Introduction

In this paper we study the precise asymptotic behavior of decaying solutions to the second order evolution equation

$$u''(t) + 2\delta u'(t) + Au(t) = f(u(t)) \qquad \forall t \geq 0, \tag{1.1}$$

where $\delta > 0$ is a real parameter, A is a self-adjoint linear operator on a Hilbert space H, and f is a nonlinear term.

We assume that A is non-negative, but not necessarily strictly positive, and that its spectrum is a finite or countable set of eigenvalues without finite accumulation points. We also assume that the nonlinear term has order greater than one in the origin, in the sense that it satisfies inequalities such as

$$|f(u)| \leq K_0 \left(|u|^{1+p} + |A^{1/2}u|^{1+q}\right)$$

for some positive exponents p and q.

As model examples, have in mind such semilinear hyperbolic equations as

$$u_{tt} + u_t - \Delta u + |u|^p u = 0 \tag{1.2}$$

with Neumann boundary conditions in a bounded domain $\Omega \subseteq \mathbb{R}^n$, or

$$u_{tt} + u_t - \Delta u - \lambda_1(\Omega)u + |u|^p u = 0 \tag{1.3}$$

with Dirichlet boundary conditions in a bounded domain $\Omega \subseteq \mathbb{R}^n$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet Laplacian. We point out that in both cases the operator associated to the linear part has a nontrivial kernel.

This paper is the final step of a project started with [4] and [5]. In [5] we investigated the corresponding first order equation

$$u'(t) + Au(t) = f(u(t)) \tag{1.4}$$

under analogous assumptions. The main results obtained in the first order case are the following.

(1) *Slow-fast alternative.* All non-zero solutions to (1.4) which decay to 0 are either slow solutions, in the sense that they decay at most as $t^{-1/p}$, or fast solutions decaying to 0 exponentially.

(2) *Asymptotic profile of fast solutions.* Every fast solution $u(t)$ is asymptotic to a solution $v(t)$ to the corresponding homogeneous equation

$$v'(t) + Av(t) = 0 \tag{1.5}$$

in the sense that the difference $u(t) - v(t)$ decays faster than both u and v. More precisely, we can always take $v(t)$ to be a "pure" solution to (1.5) of the form $v(t) := v_0 e^{-\lambda t}$, where λ is an eigenvalue of A and v_0 is a corresponding eigenvector.

(3) *Existence of an open set of slow solutions.* If $\ker(A)$ is nontrivial, and f satisfies a natural sign condition, then there exists a nonempty open set of initial data giving rise to slow solutions. These solutions live close to the kernel of A, and decay as solutions to the ordinary differential equation

$$u'(t) = -|u(t)|^p u(t). \tag{1.6}$$

(4) *Existence of families of fast solutions.* For every small enough "pure" solution $v(t) = v_0 e^{-\lambda t}$ to the homogeneous equation (1.5), there exists a family of fast solutions to (1.4) asymptotic to $v(t)$. This family has the same structure as the family of solutions to (1.5) which are asymptotic to the given pure solution $v(t)$.

In the recent paper [4] we proved the existence of a nonempty open set of slow solutions for the second order equation (1.1) under the additional assumption that $f(u) = -\nabla F(u)$ for a suitable nonnegative functional $F(u)$. Once again, these solutions live close to the kernel of A and decay as the solutions to the first order ordinary differential equation (1.6). Roughly speaking, this means that in the slow regime both operator A and second order time-derivative can be neglected in (1.1). This result extends point (3) above from the first order equation (1.4) to a large class of second order equations (1.1).

In this paper we extend points (1), (2) and (4). In Theorem 3.1 we prove the slow-fast alternative and we describe the asymptotic profile of fast solutions, which now behave as solutions to the linear homogeneous equation obtained from (1.1) by neglecting the nonlinear term. In Theorem 3.6 we construct families of fast solutions with a given asymptotic profile. The main difference from [5] is that *in the first order case fast solutions can have infinitely many exponential decay rates*, corresponding to eigenvalues of A, while here *in the second order case only finitely many exponential decay rates can occur*, even if A has infinitely many distinct eigenvalues.

In the main results of this paper we never require a gradient structure on the nonlinear term. On the contrary, our assumptions do not even guarantee the existence of global solutions for all initial data in a neighborhood of the origin, and hence our abstract results apply also to those equations which exhibit coexistence of decaying solutions and solutions that do not globally exist. A typical example is provided by partial differential equations such as (1.2) or (1.3), but with the minus sign in front of the nonlinear term.

Concerning the technique, the literature seems to reveal a shortage of appropriate tools to tackle questions of this type, even in finite dimensions. For example, the classical linearization results à la Hartmann (see [10, 11, 12]) provide a good description of the dynamic in a neighborhood of a stationary point (the origin in this case). On the other hand, the linearization is realized through homeomorphisms which are just Hölder continuous, and therefore they do not preserve decay rates. More important, almost all these results seems to deal with the case where eigenvalues of the linearized equation have real part different from zero, while we know that a nontrivial kernel is exactly what produces slow solutions. Finally, these tools seem to require assumptions on the dynamics in a whole neighborhood of the origin.

For the same reasons, also geometric tools such as stable or center manifolds are unlikely to be helpful in answering questions (1) through (4) above. On the contrary, it is our classification of decay rates which seems to lead to a better description of the stable manifold $\mathcal{S}$, when it exists, since what we actually provide is a "stratification" of $\mathcal{S}$ into submanifolds $\mathcal{S}_1 \supset \mathcal{S}_2 \supset \mathcal{S}_3 \supset \ldots$ corresponding to different decay rates.

A first step in the classification of decay rates was done by the third author in [7], where the full answer is given in the case of the scalar ordinary differential equation

$$u''(t) + u'(t) + |u(t)|^p u(t) = 0. \tag{1.7}$$

The technique exploited in [7] seems to be specific to the scalar case . Nevertheless, this was enough to show the existence of both slow and fast solutions to the Neumann problem for (1.2), since it is enough to consider spatially homogeneous solutions depending only on t, for which (1.2) reduces to (1.7). It was later shown in [6] that the slow-fast alternative holds true for (1.2), and that the set of initial data producing fast solutions is closed with empty interior. Unfortunately, the proofs of these results seem to exploit in an essential way the fact that the kernel of the linear part (in this case the set of constant functions) is an invariant space also for the nonlinear equation. Without this assumption, which fails for example in the Dirichlet case, even the existence of a single slow solution was open until [4].

For all these reasons, in this paper we follow a different path, close to what we did in the first order case. We exploit two main tools, one linear and one nonlinear. The first one (see section 4.1) is a sharp analysis of the linear homogeneous equation obtained by replacing the right-hand side of (1.1) by a forcing term $g(t)$. The second one are what we call modified Dirichlet quotients (see section 4.2). They have been developed in completely different contexts, for example backward uniqueness results for parabolic differential equations, but they proved to be fundamental in [5] in order to establish the slow-fast alternative. Here we need a "hyperbolic version" of Dirichlet quotients, analogous to the quotients introduced in [4] and previously in [1, 3].

When we apply the results of [4] and of the present paper to the model examples we started with, we end up with a satisfactory description of the asymptotic behavior of solutions. More important, this description descends from an abstract framework which applies in the same way to both ordinary and partial differential equations, both Neumann and Dirichlet boundary conditions, both equations with the right and with the wrong sign.

Just as a further example of this flexibility, we present in the last part of this paper a simple application to a system of ordinary differential equations describing the so-called finite modes of a degenerate hyperbolic equation of Kirchhoff type, actually a quasi-linear equation.

This paper is organized as follows. In Section 2 we fix the notation and we introduce the terminology needed when dealing with decay rates of solutions to the linear homogeneous equation of order two. In section 3 we state our main abstract results and we comment on them. In Section 4 we develop our tools and we prove the main results. In Section 5 we present some simple applications of the abstract theory.

2 Notation and preliminaries

Throughout this paper H denotes a separable Hilbert space, $|x|$ denotes the norm of an element $x \in H$, and $\langle x, y\rangle$ denotes the scalar product of two elements x and y in H. We consider a self-adjoint linear operator A on H with dense domain $D(A)$. We assume that A is nonnegative, namely $\langle Au, u\rangle \geq 0$ for every $u \in D(A)$, so that for every $\alpha \geq 0$ the power $A^\alpha u$ is defined provided that u lies in a suitable domain $D(A^\alpha)$, which is itself a separable Hilbert space with norm

$$|u|_{D(A^\alpha)} := \left(|u|^2 + |A^\alpha u|^2\right)^{1/2}.$$

2.1 The notion of solution

Let us spend a few words on the notion of solutions to semilinear equations and inequalities. Let us start by considering the linear equation

$$u''(t) + 2\delta u'(t) + Au(t) = g(t), \tag{2.1}$$

with initial data

$$u(0) = u_0, \qquad u'(0) = u_1. \tag{2.2}$$

There are several ways to introduce a notion of weak solution to evolution problems, for example as uniform limits of strong solutions, or through integral formulations, but fortunately all of them coincide in the case of linear equations such as (2.1) with initial data in the so-called energy space $D(A^{1/2}) \times H$.

For our purposes we limit ourselves to forcing terms $g(t)$ which are defined for every $t \geq 0$ and continuous with values in H. We recall the following classical result just to state precisely the regularity of solutions and energy functions.

Theorem A (Linear equation – Existence)**.** *Let H be a separable Hilbert space, and let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$. Let us assume that $g \in C^0([0, +\infty), H)$ and $(u_0, u_1) \in D(A^{1/2}) \times H$.*

Then problem (2.1)–(2.2) has a unique (weak) solution

$$u \in C^0\left([0, +\infty), D(A^{1/2})\right) \cap C^1\left([0, +\infty), H\right). \tag{2.3}$$

Moreover, the function

$$E(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2 \tag{2.4}$$

is of class C^1 in $[0, +\infty)$, and

$$E'(t) = -4\delta|u'(t)|^2 + 2\langle u'(t), g(t)\rangle \qquad \forall t \geq 0. \tag{2.5}$$

Next step is considering semilinear equations of the form

$$u''(t) + 2\delta u'(t) + Au(t) = f(u(t)). \tag{2.6}$$

We assume that $f : B_{R_0} \to H$, where $R_0 > 0$ and

$$B_{R_0} := \left\{ u \in D(A^{1/2}) : |u|_{D(A^{1/2})} < R_0 \right\}. \tag{2.7}$$

We also assume that f is continuous in B_{R_0} with respect to the norm of $D(A^{1/2})$. We say that $u(t)$ is a solution to (2.6) for $t \geq 0$ if there exists $g \in C^0([0, +\infty), H)$ such that $u(t)$ is a solution to (2.1) in $[0, +\infty)$, and in addition

$$g(t) = f(u(t)) \qquad \forall t \geq 0.$$

Finally, we consider differential inequalities such as

$$\left| u''(t) + 2\delta u'(t) + Au(t) \right| \leq K_0 \left(|u(t)|^{1+p} + |A^{1/2}u(t)|^{1+q} \right). \tag{2.8}$$

The notion of solution to these inequalities can be introduced in analogy with the case of equation (2.6), as follows.

Definition 2.1. We say that the function $u(t)$ is a global solution to the differential inequality (2.8) if

- $u(t)$ has the regularity stated in (2.3),
- there exists $g \in C^0([0, +\infty), H)$ such that $u(t)$ is a solution to (2.1) in $[0, +\infty)$,
- the forcing term $g(t)$ satisfies

$$|g(t)| \leq K_0 \left(|u(t)|^{1+p} + |A^{1/2}u(t)|^{1+q} \right) \qquad \forall t \geq 0. \tag{2.9}$$

In particular, the energy (2.4) of all solutions to (2.8) is of class C^1, and its time-derivative is given by (2.5).

2.2 Decay rates for the linear homogeneous equation

In the sequel we assume that the spectrum $\sigma(A)$ of A is a finite or countable set of eigenvalues without finite accumulation points. Under this assumption, the space H admits a finite or countable orthonormal system made by eigenvectors of A. We denote this system by $\{e_k\}$, where k ranges over some finite or countable set of indices $\mathcal{K}$. The corresponding eigenvalues of A are denoted by $\{\lambda_k\}$, so that

$$Ae_k = \lambda_k e_k \qquad \forall k \in \mathcal{K}.$$

We never assume that eigenvalues of A are simple or with finite multiplicity, so that λ_k's are not necessarily distinct, and it could even happen that λ_k is the same element of $\sigma(A)$ for infinitely many indices k.

For every $\lambda \in \sigma(A)$ we consider the polynomial

$$z^2 - 2\delta z + \lambda. \tag{2.10}$$

Then we define the set

$$\mathcal{D} := \{\Re(z) : z \in \mathbb{C} \text{ is a root of (2.10) for some } \lambda \in \sigma(A)\},$$

where $\Re(z)$ denotes the real part of z. With some standard algebra it is possible to list the elements of $\mathcal{D}$. Indeed, for all eigenvalues $\lambda < \delta^2$ the polynomial (2.10) has two distinct real roots, which thus provide two elements of $\mathcal{D}$, one less than δ and one greater than δ. All eigenvalues $\lambda \geq \delta^2$ produce the same element δ of $\mathcal{D}$, as a real root of multiplicity 2 if $\lambda = \delta^2$, and as the real part of the two distinct complex conjugate roots if $\lambda > \delta^2$. Since we assumed that $\sigma(A)$ has no finite accumulation point, from this list it follows that $\mathcal{D}$ is a *finite set*.

The set $\mathcal{D}$ is strongly related to decay rates of solutions to the homogeneous linear equation

$$u''(t) + 2\delta u'(t) + Au(t) = 0. \tag{2.11}$$

In order to make the relation more explicit, we start by considering *simple modes*, namely solutions to (2.11) of the form $u_k(t)e_k$, where e_k is one of the elements of the orthogonal system, and $u_k(t)$ is a solution to the ordinary differential equation

$$u_k''(t) + 2\delta u_k(t) + \lambda_k u_k(t) = 0. \tag{2.12}$$

The form of the solutions to (2.12) depends on the relative order of λ_k and δ^2, as follows. Let (u_{0k}, u_{1k}) be the initial data.

- If $\lambda_k > \delta^2$, the solution to (2.12) is

  $$u_k(t) = e^{-\delta t}\left(u_{0k}\cos(\phi_k t) + \frac{u_{1k} + \delta u_{0k}}{\phi_k}\sin(\phi_k t)\right), \tag{2.13}$$

  where ϕ_k denotes the imaginary part of the roots of the polynomial (2.10) with $\lambda = \lambda_k$. It always decays as $e^{-\delta t}$, and indeed in this case δ is the element of $\mathcal{D}$ corresponding to the eigenvalue λ_k of A.

- If $\lambda_k = \delta^2$, the solution to (2.12) is

  $$u_k(t) = e^{-\delta t}\left(u_{0k} + (u_{1k} + \delta u_{0k})t\right). \tag{2.14}$$

  Also in this case the exponential term in the decay rate is $e^{-\delta t}$, and δ is the element of $\mathcal{D}$ corresponding to λ_k.

- If $\lambda_k < \delta^2$, the solution to (2.12) is

$$u_k(t) = \frac{u_{1k} + r_{2,k}u_{0k}}{r_{2,k} - r_{1,k}} \cdot e^{-r_{1,k}t} - \frac{u_{1k} + r_{1,k}u_{0k}}{r_{2,k} - r_{1,k}} \cdot e^{-r_{2,k}t}, \tag{2.15}$$

  where

$$r_{1,k} := \delta - \sqrt{\delta^2 - \lambda_k}, \qquad r_{2,k} := \delta + \sqrt{\delta^2 - \lambda_k}. \tag{2.16}$$

  In this case the simple mode is actually the sum of two simple modes with different decay rates, described by $r_{1,k}$ and $r_{2,k}$, namely by the elements of $\mathcal{D}$ corresponding to λ_k in this range.

Therefore, if we limit ourselves to simple modes, all possible decay rates involve an exponential term of the form e^{-rt} for some $r \in \mathcal{D}$.

The general case is not so different. Indeed, any solution to (2.11) is the sum (or the series) of simple modes. If we group together all terms with the same exponential term, we obtain a decomposition of the form

$$u(t) = \sum_{r \in D} u_r(t). \tag{2.17}$$

We point out that this decomposition is unique and involves only a *finite number* of terms (some of which might be zero). Each term is itself a solution to (2.11), and it can be the series of countably many simple modes, all with the same exponential factor. We specify that terms of type $te^{-\delta t}$, which might come from (2.14), are grouped together with all other terms involving $e^{-\delta t}$.

The decomposition (2.17) motivates the following definitions.

Definition 2.2. Let $r_0 \in \mathcal{D}$, and let $u(t)$ be a solution to the homogeneous equation (2.11). Let $u_r(t)$ be the components of $u(t)$ in the decomposition (2.17). We say that

- $u(t)$ is r_0-*fast* if $u_r(t) \equiv 0$ for every $r \leq r_0$,
- $u(t)$ is r_0-*slow* if $u_r(t) \equiv 0$ for every $r > r_0$,
- $u(t)$ is r_0-*pure* if $u_r(t) \equiv 0$ for every $r \neq r_0$.

Definition 2.3. A pair $(u_0.u_1) \in D(A^{1/2}) \times H$ is called r_0-fast (respectively, r_0-slow or r_0-pure) if the solution to the homogeneous equation (2.11) with initial data (u_0, u_1) is r_0-fast (respectively, r_0-slow or r_0-pure).

Remark 2.4. One could equivalently say that a solution is r_0-fast if it is the sum (or the series) of simple modes with exponential factors e^{rt} with $r > r_0$. Analogously, a solution is r_0-slow if it is the sum (or the series) of simple modes with exponential factors e^{rt} with $r \leq r_0$. We point out that non-zero simple modes with $r = r_0$ are allowed in r_0-slow solutions, but not in r_0-fast solutions.

Remark 2.5. Let $u(t)$ be a solution to the homogeneous equation (2.11). Let us assume that $u(t)$ is r_0-pure or r_0-fast for some $r_0 \in \mathcal{D}$. Then it turns out that

$$\lim_{t\to+\infty} \left(|u'(t)| + |u(t)|_{D(A^{1/2})}\right) e^{\gamma t} = 0 \qquad \forall \gamma < r_0.$$

Conversely, if $u(t)$ is r_0-slow it turns out that

$$\liminf_{t\to+\infty} \left(|u'(t)| + |u(t)|_{D(A^{1/2})}\right) e^{r_0 t} > 0,$$

unless $u(t)$ is identically 0. These results follow in a standard way from the explicit expressions (2.13), (2.14), and (2.15) for the components of $u(t)$.

2.3 Second order equations as first order systems

In this section we present an alternative description of the set $\mathcal{D}$ and of the decomposition (2.17). Setting $U(t) := (u(t), u'(t))$, the second order equation (2.11) can be written as a first order system

$$U'(t) + \mathcal{A}U(t) = 0 \tag{2.18}$$

in the product space $\mathcal{H} := D(A^{1/2}) \times H$, where the operator $\mathcal{A}$ is defined by

$$\mathcal{A} := \begin{pmatrix} 0 & -I \\ A & 2\delta I \end{pmatrix} \tag{2.19}$$

(here I denotes the identity on H). It can be proved that the spectrum of $\mathcal{A}$ is the union of the roots of the polynomials (2.10) when λ ranges over $\sigma(A)$. Therefore, the set $\mathcal{D}$ is just the set of real parts of eigenvalues of $\mathcal{A}$.

Every element e_k of the orthonormal system in H gives rise to either an $\mathcal{A}$-invariant subspace $\mathcal{H}_k$ of $\mathcal{H}$ of dimension two, or two $\mathcal{A}$-invariant subspaces $\mathcal{H}_{1,k}$ and $\mathcal{H}_{2,k}$ of $\mathcal{H}$ of dimension one, depending on the corresponding eigenvalue λ_k.

- If $\lambda_k > \delta^2$, then $\mathcal{H}_k$ is the two-dimensional subspace generated by $(e_k, 0)$ and $(0, e_k)$, where the action of $\mathcal{A}$ has canonical form represented by the matrix

  $$\begin{pmatrix} \delta & \phi_k \\ -\phi_k & \delta \end{pmatrix},$$

  where ϕ_k is the same as in (2.13).

- If $\lambda_k = \delta^2$, then $\mathcal{H}_k$ is the two-dimensional subspace generated by $(e_k, 0)$ and $(0, e_k)$, where the action of $\mathcal{A}$ has canonical form represented by the matrix

  $$\begin{pmatrix} \delta & 1 \\ 0 & \delta \end{pmatrix}.$$

- If $\lambda_k < \delta^2$, then $\mathcal{H}_{1,k}$ is the subspace generated by $(e_k, -r_{1,k}e_k)$, where $\mathcal{A}$ acts as multiplication by $r_{1,k}$, and $\mathcal{H}_{2,k}$ is the subspace generated by $(e_k, -r_{2,k}e_k)$, where $\mathcal{A}$ acts as multiplication by $r_{2,k}$.

If we group together all subspaces corresponding to eigenvalues of $\mathcal{A}$ with the same real part, we end up with a decomposition of $\mathcal{H}$ of the form

$$\mathcal{H} = \bigoplus_{r\in\mathcal{D}} \mathcal{H}_r. \tag{2.20}$$

More precisely, $\mathcal{H}_r$ is the eigenspace of $\mathcal{A}$ relative to the real eigenvalue r if $r \neq \delta$, while $\mathcal{H}_\delta$ is the closure of the space generated by all pairs of the form $(e_k, 0)$ and $(0, e_k)$, with k ranging over all indices for which the real part of λ_k is greater than or equal to δ.

We point out that (2.20) is a *finite direct sum of closed subspaces.* In general it is not an orthogonal sum, the only reason being that the pair of spaces originating from each $\lambda_k < \delta^2$ are not orthogonal.

In this setting, a solution $u(t)$ to (2.11) is

- r_0-pure if $(u(t), u'(t)) \in \mathcal{H}_{r_0}$ for every $t \geq 0$,
- r_0-fast if $(u(t), u'(t)) \in \bigoplus_{r>r_0} \mathcal{H}_r$ for every $t \geq 0$,
- r_0-slow if $(u(t), u'(t)) \in \bigoplus_{r\leq r_0} \mathcal{H}_r$ for every $t \geq 0$.

3 Statement of main results

Our first result provides a classification of all possible decay rates for solutions to the differential inequality (2.8).

Theorem 3.1 (Classification of decay rates). *Let H be a separable Hilbert space, let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$, and let $u(t)$ be a global solution to the differential inequality (2.8) in the sense of Definition 2.1.*

Let us assume that

- *the spectrum of A is a finite or countable set of eigenvalues without finite accumulation points,*
- *$K_0 \geq 0$, $p > 0$, and $q > 0$,*
- *u is a decaying solution in the sense that*

$$\lim_{t\to+\infty} |u(t)|_{D(A^{1/2})} = 0. \tag{3.21}$$

Then one and only one of the following statements apply.

(1) (Null solution) *The solution is the zero-solution* $u(t) \equiv 0$ *for every* $t \geq 0$.

(2) (Slow solutions) *There exist* $T_0 \geq 0$, *and positive constants* M_1 *and* M_2, *such that*

$$|u(t)| \geq \frac{M_1}{(1+t)^{1/p}} \qquad \forall t \geq T_0, \tag{3.22}$$

$$|u'(t)| + |A^{1/2}u(t)| \leq M_2|u(t)|^{1+p} \qquad \forall t \geq T_0. \tag{3.23}$$

(3) (Fast solutions) *There exist* $r_0 \in \mathcal{D}$, *with* $r_0 > 0$, *and a nontrivial* r_0*-pure solution* $v_0(t)$ *to the linear homogeneous equation (2.11), such that*

$$\lim_{t\to+\infty} \left(|u'(t) - v_0'(t)| + |u(t) - v_0(t)|_{D(A^{1/2})}\right) e^{\gamma t} = 0 \tag{3.24}$$

for every

$$\gamma < \min\left\{\beta_0, (1+p)r_0, (1+q)r_0\right\}, \tag{3.25}$$

where

$$\beta_0 := \begin{cases} \min\{r \in \mathcal{D} : r > r_0\} & \textit{if } r_0 < \max\mathcal{D}, \\ +\infty & \textit{otherwise.} \end{cases} \tag{3.26}$$

Remark 3.2. When the kernel of A is non-trivial, a differential inequality such as (2.8) does not guarantee that all its solutions in a neighborhood of the origin tend to 0. As an example, we can think to the ordinary differential equation $u'' + u' = u^3 + 3u^5$, which has $u(t) = (1-2t)^{-1/2}$, and all its time-translations, among its solutions. This is the reason why we need assumption (3.21).

In other words, there might be coexistence of solutions that decay to 0 and solutions that do not decay, or even do not globally exist. When this is the case, our result classifies all possible decay rates of decaying solutions, regardless of non-decaying ones.

Remark 3.3. Concerning the null solution, Theorem 3.1 implies that every solution which decays faster than $e^{-2\delta t}$ is actually the null solution. This follows from the fact that all elements of $\mathcal{D}$ are less than or equal to 2δ. In sharp contrast with the first order case (see [5]), now there is a maximal possible decay rate for non-zero solutions.

Remark 3.4. Concerning slow solutions, first of all we remark that in general (3.22) and (3.23) involve a time $T_0 \geq 0$. This is because $u(t)$ could even vanish many times, of course not together with $u'(t)$, before becoming slow "eventually". This point contrasts with the parabolic case, where the corresponding estimates hold true for every $t \geq 0$.

For the rest, the slow regime is completely analogous to the first order case. For example, estimates (3.22) and (3.23) involve uniquely the exponent p of the differential inequality (2.8), while q is irrelevant provided it is positive. Roughly speaking, this happens because slow solutions move closer and closer to the kernel of A, as suggested by the otherwise unnatural estimate (3.23) in which $|A^{1/2}u(t)|$ is controlled by a higher power of $|u(t)|$. Close to the kernel of A, the term $|A^{1/2}u(t)|$ can be neglected, and this justifies the disappearance of q in the final decay rate.

For the same reason, the slowness of $u(t)$ is due uniquely to its component with respect to $\ker(A)$. Indeed, let us write $u(t)$ as the sum of its projection $P_K u(t)$ into $\ker(A)$, and its "range component" $u(t) - P_K u(t)$ orthogonal to $\ker(A)$. Since the operator A is coercive when restricted to the range of A, estimate (3.23) implies that there exists a constant c such that

$$|u(t) - P_K u(t)| \leq c|A^{1/2}u(t)| \leq cM_2|u(t)|^{1+p}.$$

Therefore, when $u(t)$ decays to 0, its range component always decays faster. This is consistent with previous results (see [8]), and shows also that slow solutions cannot exist when the operator A is coercive.

Finally, we show that the exponent $(1+p)$ in (3.23) is optimal, both for $|u'(t)|$ and for $|A^{1/2}u(t)|$. Indeed, let us consider the case where $H = \mathbb{R}^2$, $p = 2$, and the evolution problem reduces to the following system of ordinary differential equations

$$\begin{cases} x''(t) + x'(t) = -x^3(t) + 3x^5(t), \\ y''(t) + y'(t) + y(t) = x^3(t) - 3x^5(t) + 15x^7(t). \end{cases}$$

A solution of this system is $x(t) = (1+2t)^{-1/2}$ and $y(t) = (1+2t)^{-3/2}$. Therefore, in this case it turns out that $u(t) = (x(t), y(t))$ decays as $x(t)$, hence as $t^{-1/2}$, while both $u'(t) = (x'(t), y'(t))$ and $A^{1/2}u(t) = (0, y(t))$, decay as $t^{-3/2}$, hence as $|u(t)|^{1+p}$.

Remark 3.5. Concerning fast solutions, we show that (3.25) is optimal. This can be seen by considering the case where $H = \mathbb{R}^2$ and the evolution problem reduces to the following system of ordinary differential equations

$$\begin{cases} x''(t) + x'(t) + (r_0 - r_0^2)x(t) = 0, \\ y''(t) + y'(t) + (\beta_0 - \beta_0^2)y(t) = |x(t)|^{1+p} + |x(t)|^{1+q}, \end{cases}$$

with parameters $0 < r_0 < \beta_0 < 1/2$. It is not difficult to see that in this case $\mathcal{D} = \{r_0, \beta_0, 1 - \beta_0, 1 - r_0\}$, and a solution of the first equation is $x(t) = e^{-r_0 t}$. At this point, solutions of the second equation can decay as $e^{-\eta t}$, where η is the right-hand side of (3.25), or even as $te^{-\eta t}$ in case of resonance.

The slow-fast alternative alone does not guarantee the existence of both slow and fast solutions, and actually it does not guarantee the existence of solutions at all.

As mentioned in the introduction, the existence of slow solutions was addressed in [4], where an affirmative answer was given assuming that the kernel of A is nontrivial and that $f(u) = -\nabla F(u)$ for a suitable nonnegative potential $F(u)$ such that $\langle \nabla F(u), u \rangle \geq 0$ in a neighborhood of the origin. We refer for the details to Theorem 2.3 in [4]. Existence of slow solutions without this gradient structure remains an open problem, as well as the slow-fast alternative when assumption (3.21) is weakened by considering just the norm in H instead of $D(A^{1/2})$.

In the next result we show the existence of families of fast solution with prescribed asymptotic profile.

Theorem 3.6 (Existence of fast solutions)**.** *Let H be a separable Hilbert space, and let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$. Let $f : B_{R_0} \to H$ be a function, with $R_0 > 0$ and B_{R_0} defined by (2.7).*

Let us assume that

(i) the spectrum of A is a finite or countable set of eigenvalues without finite accumulation points,

(ii) there exist $p > 0$ and $L \geq 0$ such that

$$|f(u) - f(v)| \leq L \left(|u|^p_{D(A^{1/2})} + |v|^p_{D(A^{1/2})} \right) |u - v|_{D(A^{1/2})} \tag{3.27}$$

for every u and v in B_{R_0}, and in addition

$$f(0) = 0. \tag{3.28}$$

Then for every $r_0 \in \mathcal{D}$, with $r_0 > 0$, there exists $\varepsilon_0 > 0$ with the following property. For every r_0-pure pair $(v_0, v_1) \in D(A^{1/2}) \times H$, and for every r_0-fast pair $(z_0, z_1) \in D(A^{1/2}) \times H$ such that

$$|v_1| + |v_0|_{D(A^{1/2})} + |z_1| + |z_0|_{D(A^{1/2})} \leq \varepsilon_0, \tag{3.29}$$

there exists an r_0-slow pair $(w_0, w_1) \in D(A^{1/2}) \times H$ such that equation (2.6) with initial data

$$u_0 := v_0 + z_0 + w_0, \qquad u_1 := v_1 + z_1 + w_1, \tag{3.30}$$

has a global solution $u(t)$ satisfying

$$\lim_{t \to +\infty} \left(|u'(t) - v'(t)| + |u(t) - v(t)|_{D(A^{1/2})} \right) e^{r_0 t} = 0, \tag{3.31}$$

where $v(t)$ denotes the r_0-pure solution to the linear homogeneous equation (2.11) with initial data (v_0, v_1).

Remark 3.7. Once we know that $u(t)$ satisfies (3.31), then we can always apply Theorem 3.1 to it, and improve the exponent r_0 to a better exponent.

Remark 3.8. Theorem 3.6 can be seen as an existence result with a mix of conditions at $t = 0$ and at $t = +\infty$. Indeed we fixed (z_0, z_1), which at the end are the r_0-fast components of the initial condition (u_0, u_1) (these components are not modified when adding (v_0, v_1) and (w_0, w_1)), and through (v_0, v_1) we also fixed $v(t)$, which at the end is the asymptotic profile of the solution as $t \to +\infty$.

In the case of the linear homogeneous equation (2.11) it is not difficult to see that, for every r_0-pure asymptotic profile and every r_0-fast component of initial condition, there exists a (unique) solution satisfying these conditions. Theorem 3.6 shows that the existence part is still true in the nonlinear case, at least if we limit ourselves to a neighborhood of the origin.

4 Proofs

4.1 Estimates for non-homogeneous linear equations

The following result is the linear core of this paper. In a few words, we prove existence and uniqueness of a solution to the linear non-homogeneous equation (2.1) satisfying two conflicting constraints. The first one is that this solution decays almost as the forcing term, the second one is that its initial data are as slow as possible.

Proposition 4.1 (Special solution to the non-homogeneous equation)**.** *Let H be a separable Hilbert space, and let A be a self-adjoint linear operator on H with dense domain $D(A)$, whose spectrum $\sigma(A)$ is a finite or countable set of eigenvalues without finite accumulation points. Let $g \in C^0([0,+\infty), H)$, and let us assume that there exist real numbers $K_g \geq 0$ and $\gamma_0 > 0$, with $\gamma_0 \notin \mathcal{D}$, such that*

$$|g(t)| \leq K_g e^{-\gamma_0 t} \qquad \forall t \geq 0. \tag{4.1}$$

Then there exists a constant Γ_0, depending only on γ_0, δ and $\sigma(A)$, for which the following statements hold true.

(1) If $\gamma_0 > \min \mathcal{D}$, and we set

$$\alpha_0 := \max\{r \in \mathcal{D} : r < \gamma_0\}, \tag{4.2}$$

then the non-homogeneous equation (2.1) admits a unique solution $w(t)$ such that its initial condition is an α_0-slow pair and

$$\lim_{t \to +\infty} \left(|w'(t)| + |w(t)|_{D(A^{1/2})}\right) e^{\alpha_0 t} = 0. \tag{4.3}$$

Moreover, this solution satisfies the stronger decay estimate

$$|w'(t)| + |w(t)|_{D(A^{1/2})} \leq \Gamma_0 K_g e^{-\gamma_0 t} \qquad \forall t \geq 0. \tag{4.4}$$

(2) If $\gamma_0 < \min \mathcal{D}$, then the solution to the non-homogeneous equation (2.1) with initial data $w(0) = w'(0) = 0$ satisfies (4.4).

Proof

Uniqueness In the case $\gamma_0 < \min \mathcal{D}$ uniqueness is trivial because both initial data are given. In the case $\gamma_0 > \min \mathcal{D}$, let $w_1(t)$ and $w_2(t)$ be two solutions, and let $v(t) := w_1(t) - w_2(t)$ denote their difference. Clearly $v(t)$ is a solution to the corresponding homogeneous equation. Moreover, it is α_0-slow because its initial data are the difference of two α_0-slow pairs, and it satisfies

$$\lim_{t \to +\infty} \left(|v'(t)| + |v(t)|_{D(A^{1/2})}\right) e^{\alpha_0 t} = 0 \tag{4.5}$$

because the same is true for $w_1(t)$ and $w_2(t)$. On the other hand, the unique α_0-slow solution for which (4.5) holds true is the null solution, as pointed out in Remark 2.5. This proves that $w_1(t) = w_2(t)$.

Estimates in the product space Let us interpret the non-homogeneous equation (2.1) as a first order system, as we did in section 2.3 for the homogeneous equation. Setting $W(t) := (w(t), w'(t))$ it turns out that $w(t)$ solves (2.1) in H if and only if $W(t)$ solves the first order system

$$W'(t) + \mathcal{A}W(t) = (0, g(t)) \tag{4.6}$$

in the product space $\mathcal{H} := D(A^{1/2}) \times H$, with the operator $\mathcal{A}$ defined by (2.19).

Let us assume for simplicity that

$$\min \mathcal{D} < \gamma_0 < \max \mathcal{D}, \tag{4.7}$$

and let us define α_0 as in (4.2), and β_0 as the smallest element of $\mathcal{D}$ greater than γ_0. Let us write

$$\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+, \tag{4.8}$$

where

$$\mathcal{H}_- := \bigoplus_{\substack{r\in\mathcal{D} \\ r<\gamma_0}} \mathcal{H}_r = \bigoplus_{\substack{r\in\mathcal{D} \\ r\leq\alpha_0}} \mathcal{H}_r, \qquad\qquad \mathcal{H}_+ := \bigoplus_{\substack{r\in\mathcal{D} \\ r>\gamma_0}} \mathcal{H}_r = \bigoplus_{\substack{r\in\mathcal{D} \\ r\geq\beta_0}} \mathcal{H}_r.$$

Since (2.20) is a direct sum of closed subspaces, the projection onto each $\mathcal{H}_r$ is continuous with a norm depending only on $\mathcal{D}$. As a consequence, the projections P_- and P_+ onto $\mathcal{H}_-$ and $\mathcal{H}_+$, respectively, are continuous, hence there exist two constants Γ_1 and Γ_2, depending only on $\mathcal{D}$, such that

$$|P_+(x,y)|_{\mathcal{H}} \leq \Gamma_1\,|(x,y)|_{\mathcal{H}} \qquad \forall (x,y) \in \mathcal{H}, \tag{4.9}$$

$$|P_-(x,y)|_{\mathcal{H}} \leq \Gamma_2\,|(x,y)|_{\mathcal{H}} \qquad \forall (x,y) \in \mathcal{H}, \tag{4.10}$$

where of course $|(x,y)|_{\mathcal{H}}^2 := |x|_{D(A^{1/2})}^2 + |y|^2$.

Now let us consider the semigroup $S(t)$ generated on $\mathcal{H}$ by the first order system (2.18). From the explicit solutions (2.13) through (2.15) it follows that

$$|S(t)(x,y)|_{\mathcal{H}} \leq \Gamma_3(1+t)e^{-\beta_0 t}\,|(x,y)|_{\mathcal{H}} \qquad \forall t \geq 0, \quad \forall (x,y) \in \mathcal{H}_+, \tag{4.11}$$

where Γ_3 depends only on $\mathcal{D}$ and $\sigma(A)$.

The operators $S(t)$ of the semigroup are invertible, and $S(-t)$ corresponds to solving the second order equation

$$u''(t) - 2\delta u'(t) + Au(t) = 0,$$

in which we just reversed the sign of the damping term. Explicit solutions in this case are analogous to (2.13) through (2.15), just with the opposite sign in the argument of all exponentials. As a consequence, $S(-t)$ can be estimates in $\mathcal{H}_-$ as follows:

$$|S(-t)(x,y)|_{\mathcal{H}} \leq \Gamma_4(1+t)e^{\alpha_0 t}\,|(x,y)|_{\mathcal{H}} \qquad \forall t \geq 0, \quad \forall (x,y) \in \mathcal{H}_-, \tag{4.12}$$

where Γ_4 depends only on $\mathcal{D}$ and $\sigma(A)$.

When assumption (4.7) is not satisfied, the situation is even simpler. If $\gamma_0 > \max \mathcal{D}$, then $\mathcal{H}_- = \mathcal{H}$ and $\mathcal{H}_+ = \{0\}$, and (4.12) still holds true with α_0 given by (4.2). If $\gamma_0 < \min \mathcal{D}$, then $\mathcal{H}_- = \{0\}$ and $\mathcal{H}_+ = \mathcal{H}$, and (4.11) still holds true with the same β_0.

Existence We are now ready to prove existence of the solution with the required properties. Let us set

$$W_+(t) := \int_0^t S(t-s)P_+(0,g(s))\,ds, \tag{4.13}$$

$$W_-(t) := -\int_t^{+\infty} S(t-s)P_-(0,g(s))\,ds. \tag{4.14}$$

Let us assume for a while that $W_\pm(t)$ are well-defined, namely that the integrals are convergent. Then they are solutions to

$$W_\pm'(t) + \mathcal{A}W_\pm(t) = P_\pm(0,g(t)),$$

hence their sum $W(t) := W_+(t) + W_-(t)$ is a solution to (4.6), which is equivalent to saying that $W(t)$ is of the form $(w(t), w'(t))$ for some solution $w(t)$ to (2.1). Moreover, the initial condition $W(0)$ is α_0-slow, because its α_0-fast component $W_+(0)$ vanishes due to (4.13) with $t = 0$.

Therefore, we are left to proving that $W_\pm(t)$ are well-defined and satisfy suitable decay estimates. Let us start with $W_+(t)$. Since $t - s \geq 0$ in (4.13), from (4.11), (4.9), and (4.1) it follows that

$$\begin{aligned}
|S(t-s)P_+(0,g(s))| &\leq \Gamma_3(1+t-s)e^{-\beta_0(t-s)} \cdot |P_+(0,g(s))|_{\mathcal{H}} \\
&\leq \Gamma_3(1+t-s)e^{-\beta_0(t-s)} \cdot \Gamma_1|g(s)| \\
&\leq \Gamma_3(1+t-s)e^{-\beta_0(t-s)} \cdot \Gamma_1 K_g e^{-\gamma_0 s} \\
&= \Gamma_1\Gamma_3 K_g \cdot e^{-\gamma_0 t} \cdot (1+t-s)e^{-(\beta_0-\gamma_0)(t-s)},
\end{aligned}$$

and hence

$$|W_+(t)| \leq \Gamma_1\Gamma_3 K_g \cdot e^{-\gamma_0 t} \cdot \int_0^t (1+t-s)e^{-(\beta_0-\gamma_0)(t-s)}\,ds \leq \Gamma_5 K_g e^{-\gamma_0 t}. \tag{4.15}$$

Now let us consider $W_-(t)$. Since $t - s \leq 0$ in (4.14), from (4.12), (4.10), and (4.1) it follows that

$$\begin{aligned}
|S(t-s)P_-(0,g(s))| &\leq \Gamma_4(1+s-t)e^{\alpha_0(s-t)} \cdot |P_-(0,g(s))|_{\mathcal{H}} \\
&\leq \Gamma_4(1+s-t)e^{\alpha_0(s-t)} \cdot \Gamma_2|g(s)| \\
&\leq \Gamma_4(1+s-t)e^{\alpha_0(s-t)} \cdot \Gamma_2 K_g e^{-\gamma_0 s} \\
&= \Gamma_2\Gamma_4 K_g \cdot e^{-\gamma_0 t} \cdot (1+s-t)e^{-(\gamma_0-\alpha_0)(s-t)}.
\end{aligned}$$

Since $\gamma_0 > \alpha_0$, this proves that the integral in (4.14) converges. In addition, it turns out that

$$|W_-(t)| \leq \Gamma_2\Gamma_4 K_g \cdot e^{-\gamma_0 t} \cdot \int_t^{+\infty} (1+s-t)e^{-(\gamma_0-\alpha_0)(s-t)}\,ds \leq \Gamma_6 K_g e^{-\gamma_0 t}. \tag{4.16}$$

Summing (4.15) and (4.16) it follows that

$$|W(t)|_{\mathcal{H}} \leq (\Gamma_5 + \Gamma_6)K_g e^{-\gamma_0 t} \qquad \forall t \geq 0,$$

which is equivalent to (4.4). $\square$

Remark 4.2. Proposition 4.1 can also be proved without relying on the product space $\mathcal{H}$. It is enough to define the components $w_k(t)$ of the solution as suitable integrals involving the components $g_k(t)$ of the forcing term and the fundamental solutions of the homogeneous equation. This requires to distinguish several cases. For example, when $\lambda_k > \delta^2$ the component is given by

$$w_k(t) := \frac{1}{(\lambda_k - \delta^2)^{1/2}} \int_I e^{-\delta(t-s)} \sin\left((\lambda_k - \delta^2)^{1/2}(t-s)\right) g_k(s)\, ds,$$

where the integration region is $I := [0, t]$ if $\delta > \gamma_0$, and $I := [t, +\infty)$ if $\delta < \gamma_0$.

4.2 Energies and Dirichlet quotients

In this section we introduce the energies which we are going to exploit in the proof of Theorem 3.1. To begin with, we consider the usual "hyperbolic energy" $E(t)$ defined in (2.4), and the following *generalized Dirichlet quotient*

$$G_d(t) := \frac{|u'(t)|^2 + |A^{1/2}u(t)|^2}{|u(t)|^{2+d}} = \frac{E(t)}{|u(t)|^{2+d}}, \tag{4.17}$$

defined for every $d \geq 0$ provided that $u(t) \neq 0$. This is the hyperbolic version of the quotient

$$Q_d(t) := \frac{|A^{1/2}u(t)|^2}{|u(t)|^{2+d}},$$

introduced in [5] when dealing with the semilinear parabolic problem.

We need also a modified version of the hyperbolic energy and of the generalized Dirichlet quotient, which we define in the following way. Let $Q : H \to H$ denote the orthogonal projection onto $\ker(A)^{\perp}$. Thanks to our assumptions on the spectrum of A, there exists a constant $\nu > 0$ such that

$$|Qu|^2 \leq \frac{1}{\nu}|A^{1/2}u|^2 \qquad \forall u \in D(A^{1/2}). \tag{4.18}$$

More precisely, we can take ν equal to any positive number if A is the null operator, and ν equal to the smallest positive eigenvalue of A otherwise. Now let us set

$$\mu := \min\left\{\frac{1}{2}, \frac{\nu}{2}, \frac{\delta}{2}, \frac{\nu}{5\delta}\right\}, \tag{4.19}$$

and let us define the *modified hyperbolic energy*

$$\widehat{E}(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2 + 2\mu\langle u'(t), Qu(t)\rangle, \tag{4.20}$$

and finally the *modified generalized Dirichlet quotient*

$$\widehat{G}_d(t) := \frac{|u'(t)|^2 + |A^{1/2}u(t)|^2 + 2\mu\langle u'(t), Qu(t)\rangle}{|u(t)|^{2+d}} = \frac{\widehat{E}(t)}{|u(t)|^{2+d}}. \tag{4.21}$$

Next result provides estimates on these quantities and their time-derivatives in the case where $u(t)$ is a solution of a non-homogeneous linear equation such as (2.1).

Lemma 4.3 (Energies and generalized Dirichlet quotients)**.** *Let H be a separable Hilbert space, and let A be a self-adjoint nonnegative operator on H with dense domain $D(A)$. Let $g \in C^0([0,+\infty),H)$, and let $u(t)$ be a solution to (2.1) in $[0,+\infty)$ in the sense of Theorem A.*

Then the energies $E(t)$, $\widehat{E}(t)$, $G_d(t)$, $\widehat{G}_d(t)$ satisfy the following estimates.

(1) It turns out that

$$\frac{1}{2}E(t) \leq \widehat{E}(t) \leq 2E(t) \qquad \forall t \geq 0. \tag{4.22}$$

Moreover, the function $\widehat{E}(t)$ is of class C^1 in $[0,+\infty)$ and

$$\widehat{E}'(t) \leq -\frac{\mu}{2}\widehat{E}(t) + \frac{2}{\delta}|g(t)|^2 \qquad \forall t \geq 0. \tag{4.23}$$

(2) Let us assume that $u(t) \neq 0$ for every t in some time-interval (a,b). Then the generalized Dirichlet quotients are well defined in (a,b), and for every $d \geq 0$ it turns out that

$$\frac{1}{2}G_d(t) \leq \widehat{G}_d(t) \leq 2G_d(t) \qquad \forall t \in (a,b). \tag{4.24}$$

Moreover, the function $\widehat{G}_d(t)$ is of class C^1 in (a,b) and

$$\widehat{G}_d'(t) \leq -\frac{\mu}{2}\widehat{G}_d(t) + \frac{2}{\delta}\frac{|g(t)|^2}{|u(t)|^{2+d}} + (2+d)|u(t)|^{d/2}\cdot[G_d(t)]^{1/2}\cdot\widehat{G}_d(t) \tag{4.25}$$

for every $t \in (a,b)$.

Proof From (4.18) we obtain that

$$|2\langle u'(t), Qu(t)\rangle| \leq 2|u'(t)|\cdot|Qu(t)| \leq |u'(t)|^2 + |Qu(t)|^2 \leq |u'(t)|^2 + \frac{1}{\nu}|A^{1/2}u(t)|^2.$$

Since $\mu \leq 1/2$ and $\mu \leq \nu/2$, it follows that

$$|2\mu\langle u'(t), Qu(t)\rangle| \leq \frac{1}{2}|u'(t)|^2 + \frac{1}{2}|A^{1/2}u(t)|^2,$$

which proves both (4.22) and (4.24).

The time-derivative of $\widehat{E}(t)$ is

$$\begin{aligned} \widehat{E}'(t) \;=\;& -4\delta|u'(t)|^2 - 2\mu\langle A^{1/2}u(t), A^{1/2}Qu(t)\rangle + 2\mu\langle u'(t), Qu'(t)\rangle \\ & - 4\mu\delta\langle u'(t), Qu(t)\rangle + 2\langle u'(t), g(t)\rangle + 2\mu\langle Qu(t), g(t)\rangle. \end{aligned} \tag{4.26}$$

Let $I_1, \ldots, I_6$ denote the six terms in the right-hand side, which we now estimate separately. From the definition of Q it follows that

$$I_2 = -2\mu|A^{1/2}u(t)|^2 \tag{4.27}$$

and $\langle u'(t), Qu'(t)\rangle \leq |u'(t)|^2$. Since $2\mu \leq \delta$, we deduce that

$$I_3 \leq \delta|u'(t)|^2. \tag{4.28}$$

Since

$$I_4 \leq 4\mu\delta|u'(t)|\cdot|Qu(t)| \leq 2\mu\delta\left(\frac{|u'(t)|^2}{2\mu} + 2\mu|Qu(t)|^2\right),$$

from (4.18) it follows that

$$I_4 \leq \delta|u'(t)|^2 + \frac{4\mu^2\delta}{\nu}|A^{1/2}u(t)|^2. \tag{4.29}$$

As for the last two terms, it turns out that

$$I_5 \leq \delta|u'(t)|^2 + \frac{1}{\delta}|g(t)|^2, \tag{4.30}$$

and

$$I_6 \leq 2\mu|Qu(t)|\cdot|g(t)| \leq \mu\left(\mu\delta|Qu(t)|^2 + \frac{1}{\mu\delta}|g(t)|^2\right),$$

so that from (4.18) it follows that

$$I_6 \leq \frac{\mu^2\delta}{\nu}|A^{1/2}u(t)|^2 + \frac{1}{\delta}|g(t)|^2. \tag{4.31}$$

Plugging (4.27) through (4.31) into (4.26), we obtain that

$$\widehat{E}'(t) \leq -\delta|u'(t)|^2 - \mu\left(2 - \frac{5\mu\delta}{\nu}\right)|A^{1/2}u(t)|^2 + \frac{2}{\delta}|g(t)|^2.$$

Keeping into account that $\mu \leq \delta$ and $5\mu\delta \leq \nu$, we conclude that

$$\widehat{E}'(t) \leq -\mu\left(|u'(t)|^2 + |A^{1/2}u(t)|^2\right) + \frac{2}{\delta}|g(t)|^2.$$

At this point (4.23) follows from (4.22).

It remains to compute the time-derivative of $\widehat{G}_d(t)$, which turns out to be

$$\widehat{G}_d'(t) = \frac{\widehat{E}'(t)}{|u(t)|^{2+d}} - (2+d)\frac{\langle u'(t), u(t)\rangle}{|u(t)|^2}\cdot\widehat{G}_d(t). \tag{4.32}$$

From (4.23) it follows that

$$\frac{\widehat{E}'(t)}{|u(t)|^{2+d}} \leq -\frac{\mu}{2}\frac{\widehat{E}(t)}{|u(t)|^{2+d}} + \frac{2}{\delta}\frac{|g(t)|^2}{|u(t)|^{2+d}}, \tag{4.33}$$

and from the definition of $G_d(t)$ it follows that

$$\frac{\langle u'(t), u(t)\rangle}{|u(t)|^2} \leq \frac{|u'(t)|}{|u(t)|^{1+d/2}}\cdot|u(t)|^{d/2} \leq [G_d(t)]^{1/2}\,|u(t)|^{d/2}. \tag{4.34}$$

Plugging (4.33) and (4.34) into (4.32), we obtain (4.25). □

4.3 Proof of Theorem 3.1

Let us describe the scheme and the heuristic ideas behind the proof before entering into details. In the first section of the proof we get rid of the null solution. Indeed we prove that $(u(T), u'(T)) = (0, 0)$ for some $T \geq 0$ if and only if $(u(t), u'(t)) = (0, 0)$ for every $t \geq 0$. This is a result of forward and backward uniqueness of the null solution. After proving it, we can assume that

$$(u(t), u'(t)) \neq (0, 0) \qquad \forall t \geq 0. \tag{4.35}$$

In the second section of the proof we assume that there exist a constant $c_1 > 0$ and a sequence $t_n \to +\infty$ such that

$$|u'(t_n)|^2 + |A^{1/2}u(t_n)|^2 \leq c_1|u(t_n)|^{2+p} \qquad \forall n \in \mathbb{N}. \tag{4.36}$$

Under this assumption, we prove that a similar estimate holds true eventually, namely there exists $n_0 \in \mathbb{N}$ such that

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 \leq 4c_1|u(t)|^{2+p} \qquad \forall t \geq t_{n_0}. \tag{4.37}$$

This is not yet (3.23), but in any case it shows that $|A^{1/2}u(t)|$ decays faster than $|u(t)|$. As already pointed out, this means that $u(t)$ moves closer and closer to the kernel of A, and suggests that the terms with $Au(t)$ and $A^{1/2}u(t)$ in the differential inequality (2.8) can be neglected. Moreover, since we expect solutions decaying as negative powers of t, it seems reasonable to neglect also $u''(t)$, which for negative powers of t decays faster than $u'(t)$. With this ansatz, the second order differential inequality (2.8) has become the first order differential inequality $|u'(t)| \leq K_0|u(t)|^{1+p}$, whose nonzero solutions are slow in the sense of (3.22). The formal proof requires a sharp analysis of the Dirichlet quotients of section 4.2, first with $d := p$ and then with $d := 2p$.

In the third and last section of the proof we are left with the case where (4.36) is false for every constant c_1 and every sequence $t_n \to +\infty$. This easily implies the existence of $T_1 \geq 0$ such that

$$|u(t)|^{2+p} \leq |u'(t)|^2 + |A^{1/2}u(t)|^2 \qquad \forall t \geq T_1. \tag{4.38}$$

In this case we are not allowed to ignore the operator A, but we can neglect the right-hand side of (2.8) because the exponents are larger than one. Therefore, a good approximation of (2.8) is now the linear homogeneous equation (2.11), whose solutions decay exponentially with possible rates corresponding to elements of $\mathcal{D}$. The formal proof requires several steps. First of all, we provide exponential estimates from below and from above with non-optimal rates. Then we identify the exact rate, and finally we prove that (3.24) holds true. The basic tool in this part of the proof is Proposition 4.1.

We point out that the exponent $2 + p$ is non-optimal both in (4.36) and in the opposite estimate (4.38). Indeed, a posteriori it turns out that (up to multiplicative constants) $|A^{1/2}u| \leq |u|^{1+p}$ in the case of slow solutions, and $|A^{1/2}u| \sim |u|$ in the case of fast solutions, so that the right exponents would be $2 + 2p$ in the slow regime and 2 in the fast regime. Nevertheless, the intermediate exponent $2 + p$ acts as a threshold separating the two different regimes, and leaving enough room on both sides to perform our estimates.

Non-trivial solutions never vanish in the phase space

We prove that either $(u(t), u'(t)) = (0, 0)$ for every $t \geq 0$ or $(u(t), u'(t)) \neq (0, 0)$ for every $t \geq 0$.

To this end, we consider the energy

$$F(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2 + |u(t)|^2.$$

Its time-derivative is

$$F'(t) = -4\delta|u'(t)|^2 + 2\langle u'(t), g(t)\rangle + 2\langle u'(t), u(t)\rangle,$$

with $g(t)$ as in Definition 2.1. From assumption (2.9) it follows that

$$|g(t)|^2 \leq 2K_0^2 \left(|u(t)|^{2(1+p)} + |A^{1/2}u(t)|^{2(1+q)}\right) \qquad \forall t \geq 0, \tag{4.39}$$

and hence in this case

$$|g(t)|^2 \leq 2K_0^2 \left([F(t)]^{1+p} + [F(t)]^{1+q}\right) \qquad \forall t \geq 0.$$

Thus it follows that

$$\begin{aligned} |F'(t)| &\leq 4\delta|u'(t)|^2 + |u'(t)|^2 + |g(t)|^2 + |u'(t)|^2 + |u(t)|^2 \\ &\leq (4\delta + 2)F(t) + 2K_0^2 \left([F(t)]^{1+p} + [F(t)]^{1+q}\right) \end{aligned}$$

for every $t \geq 0$. The exponents of $F(t)$ in the right-hand side are all greater than or equal to 1. Therefore, this differential inequality guarantees that either $F(t) = 0$ for every $t \geq 0$ or $F(t) > 0$ for every $t \geq 0$, which is equivalent to what we had to prove.

Slow solutions

In this second part of the proof we consider the case where (4.35) holds true and $u(t)$ satisfies (4.36) for some $c_1 > 0$ and some sequence $t_n \to +\infty$.

Main estimate Let ν be the constant which appears in (4.18), and let μ be defined as in (4.19). Due to assumption (3.21), there exists $n_0 \in \mathbb{N}$ such that

$$2(2+p)\sqrt{c_1}|u(t)|^{p/2} \leq \frac{\mu}{4} \qquad \forall t \geq t_{n_0}, \tag{4.40}$$

$$\frac{4K_0^2}{\delta} \left(|u(t)|^p + (4c_1)^{1+q}|u(t)|^{(2+p)q}\right) \leq \frac{\mu}{4}c_1 \qquad \forall t \geq t_{n_0}. \tag{4.41}$$

We claim that (4.37) holds true with this choice of n_0, and that in addition

$$|u(t)| > 0 \qquad \forall t \geq t_{n_0}. \tag{4.42}$$

To this end, let us consider the generalized Dirichlet quotient (4.17), and its modified version (4.21), with $d := p$. To begin with, we observe that $u(t_{n_0}) \neq 0$, because if not we could deduce from (4.36) that $(u(t_{n_0}), u'(t_{n_0})) = (0,0)$, and this would contradict assumption (4.35). As a consequence, $G_p(t)$ and $\widehat{G}_p(t)$ are defined at least in a neighborhood of t_{n_0}. Moreover, since $G_p(t_{n_0}) \leq c_1 < 4c_1$, by a continuity argument it follows that $G_p(t) < 4c_1$ in a suitable neighborhood of t_{n_0}. Let us set

$$S := \sup \left\{ t > t_{n_0} : |u(\tau)| > 0 \text{ and } G_p(\tau) \leq 4c_1 \quad \forall \tau \in [t_{n_0}, t] \right\},$$

so that (4.37) and (4.42) are now equivalent to showing that $S = +\infty$.

Let us assume by contradiction that $S < +\infty$. By the maximality of S, this means that either $u(S) = 0$ or $G_p(S) = 4c_1$. Now we show that both options are impossible. In order to exclude the first one, we observe that

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 \leq 4c_1 |u(t)|^{2+p} \qquad \forall t \in [t_{n_0}, S).$$

If $|u(S)| = 0$, then letting $t \to S^-$ we deduce that also $|u'(S)| = 0$, which contradicts again (4.35).

It remains to exclude that $G_p(S) = 4c_1$. Setting $d := p$ in (4.25) we obtain that

$$\widehat{G}_p'(t) \leq -\frac{\mu}{2}\widehat{G}_p(t) + (2+p)|u(t)|^{p/2} \cdot [G_p(t)]^{1/2} \cdot \widehat{G}_p(t) + \frac{2}{\delta}\frac{|g(t)|^2}{|u(t)|^{2+p}}.$$

Therefore, since (4.39) implies that

$$|g(t)|^2 \leq 2K_0^2 |u(t)|^{2+p} \left(|u(t)|^p + [G_p(t)]^{1+q} |u(t)|^{(2+p)q} \right),$$

we find that $\widehat{G}_p'(t)$ is less than or equal to

$$-\left(\frac{\mu}{2} - (2+p)|u(t)|^{p/2} [G_p(t)]^{1/2} \right) \widehat{G}_p(t) + \frac{4K_0^2}{\delta} \left(|u(t)|^p + |u(t)|^{(2+p)q} \, [G_p(t)]^{1+q} \right).$$

If we keep into account that $G_p(t) \leq 4c_1$ for every $t \in [t_{n_0}, S)$, and the smallness assumptions (4.40) and (4.41), we conclude that

$$\widehat{G}_p'(t) \leq -\frac{\mu}{4}\widehat{G}_p(t) + \frac{\mu}{4}c_1 \qquad \forall t \in [t_{n_0}, S),$$

and hence

$$\widehat{G}_p(t) \leq \left(\widehat{G}_p(t_{n_0}) - c_1 \right) \exp\left(-\frac{\mu}{4}(t - t_{n_0}) \right) + c_1 \qquad \forall t \in [t_{n_0}, S).$$

Now from (4.24) and (4.36) we know that $\widehat{G}_p(t_{n_0}) \leq 2G_p(t_{n_0}) \leq 2c_1$, so that

$$\widehat{G}_p(t) \leq c_1 \left(1 + \exp\left(-\frac{\mu}{4}(t - t_{n_0}) \right) \right) \qquad \forall t \in [t_{n_0}, S).$$

Letting $t \to S^-$, we obtain that $\widehat{G}_p(S) < 2c_1$, hence $G_p(S) \leq 2\widehat{G}_p(S) < 4c_1$ because of (4.24). This contradicts the maximality of S and completes the proof of (4.37).

Faster decay of the range component Let us prove (3.23). To this end, we consider the generalized Dirichlet quotients (4.17) and (4.21) with $d := 2p$. They are defined at least for every $t \geq t_{n_0}$ because of (4.42). Setting $d := 2p$ in (4.25) we obtain that

$$\widehat{G}'_{2p}(t) \leq -\frac{\mu}{2}\widehat{G}_{2p}(t) + 2(1+p)|u(t)|^p \cdot [G_{2p}(t)]^{1/2} \cdot \widehat{G}_{2p}(t) + \frac{2}{\delta}\frac{|g(t)|^2}{|u(t)|^{2+2p}}. \tag{4.43}$$

From (4.37) we deduce that

$$|u(t)|^p \cdot [G_{2p}(t)]^{1/2} = |u(t)|^{p/2} \cdot [G_p(t)]^{1/2} \leq 2\sqrt{c_1}|u(t)|^{p/2}, \tag{4.44}$$

while from (4.24) and (4.39) we deduce that

$$\begin{aligned} |g(t)|^2 &\leq 2K_0^2|u(t)|^{2+2p}\left(1 + G_{2p}(t) \cdot |A^{1/2}u(t)|^{2q}\right) \\ &\leq 2K_0^2|u(t)|^{2+2p}\left(1 + 2\widehat{G}_{2p}(t) \cdot |A^{1/2}u(t)|^{2q}\right). \end{aligned} \tag{4.45}$$

Plugging (4.44) and (4.45) into (4.43), we obtain that

$$\widehat{G}'_{2p}(t) \leq -\widehat{G}_{2p}(t) \cdot \left\{\frac{\mu}{2} - c_2|u(t)|^{p/2} - c_3|A^{1/2}u(t)|^{2q}\right\} + c_4$$

for suitable constants c_2, c_3 and c_4. Due to assumption (3.21), there exists $T_0 \geq t_{n_0}$ such that

$$\frac{\mu}{2} - c_2|u(t)|^{p/2} - c_3|A^{1/2}u(t)|^{2q} \geq \frac{\mu}{4} \qquad \forall t \geq T_0,$$

and hence

$$\widehat{G}'_{2p}(t) \leq -\frac{\mu}{4}\widehat{G}_{2p}(t) + c_4 \qquad \forall t \geq T_0.$$

Integrating this differential inequality we conclude that $\widehat{G}_{2p}(t)$ is uniformly bounded for every $t \geq T_0$. Due to (4.24), also $G_{2p}(t)$ is uniformly bounded for every $t \geq T_0$, and this is enough to establish (3.23).

Slow decay of the solution Let us consider the function $y(t) := |u(t)|^2$. Since

$$|y'(t)| = 2|\langle u'(t), u(t)\rangle| \leq 2|u'(t)| \cdot |u(t)| \leq 2G_{2p}(t)^{1/2}|u(t)|^{2+p},$$

from the uniform bound on $G_{2p}(t)$ we obtain that there exists a constant c_5 such that

$$|y'(t)| \leq c_5 y(t)^{1+p/2} \qquad \forall t \geq T_0,$$

and in particular

$$y'(t) \geq -c_5 y(t)^{1+p/2} \qquad \forall t \geq T_0.$$

Since $y(T_0) \neq 0$, integrating this differential inequality we deduce (3.22).

Fast solutions

In this last section of the proof it remains to consider the case where (4.35) holds true and $u(t)$ satisfies (4.38) for some $T_1 \geq 0$. The constants $c_6, \ldots, c_{16}$ which we introduce in the sequel are positive and independent of time.

Non-optimal exponential decay from below We prove that

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 \geq c_6 e^{-c_7 t} \qquad \forall t \geq T_1. \tag{4.46}$$

To this end, let us consider the usual hyperbolic energy $E(t)$ defined in (2.4). Its time-derivative satisfies

$$E'(t) = -4\delta|u'(t)|^2 + 2\langle g(t), u'(t)\rangle \geq -(4\delta+1)|u'(t)|^2 - |g(t)|^2. \tag{4.47}$$

Let us estimate $|g(t)|$. Due to (4.38), inequality (4.39) implies that

$$|g(t)|^2 \leq 2K_0^2 \left(|u(t)|^p + |A^{1/2}u(t)|^{2q}\right) E(t) \qquad \forall t \geq T_1. \tag{4.48}$$

Since $|u(t)|$ and $|A^{1/2}u(t)|$ are uniformly bounded because of assumption (3.21), it follows that $|g(t)|^2 \leq c_8 E(t)$. Plugging this estimate into (4.47), we deduce that

$$E'(t) \geq -c_9 E(t) \qquad \forall t \geq T_1.$$

Integrating this differential inequality we obtain (4.46).

Non-optimal exponential decay from above We prove that there exists $T_2 \geq T_1$ such that

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 \leq c_{10} e^{-c_{11} t} \qquad \forall t \geq T_2. \tag{4.49}$$

To this end, let us consider the modified hyperbolic energy $\widehat{E}(t)$ defined in (4.20). From (4.23), (4.48), and (4.22) it follows that

$$\begin{aligned} \widehat{E}'(t) &\leq -\frac{\mu}{2}\widehat{E}(t) + c_{12}\left(|u(t)|^p + |A^{1/2}u(t)|^{2q}\right) E(t) \\ &\leq -\left[\frac{\mu}{2} - 2c_{12}\left(|u(t)|^p + |A^{1/2}u(t)|^{2q}\right)\right]\widehat{E}(t) \end{aligned}$$

for every $t \geq T_1$. Due to assumption (3.21), there exists $T_2 \geq T_1$ such that

$$\frac{\mu}{2} - 2c_{12}\left(|u(t)|^p + |A^{1/2}u(t)|^{2q}\right) \geq \frac{\mu}{4} \qquad \forall t \geq T_2,$$

and hence

$$\widehat{E}'(t) \leq -\frac{\mu}{4}\widehat{E}(t) \qquad \forall t \geq T_2.$$

Integrating this differential inequality, and keeping (4.22) into account, we deduce that

$$E(t) \leq 2\widehat{E}(t) \leq 2\widehat{E}(T_2)\exp\left(-\frac{\mu}{4}(t-T_2)\right) \qquad \forall t \geq T_2,$$

which proves (4.49).

Exact exponential decay rate To begin with, we observe that

$$c_{13}e^{-c_{14}t} \leq |u'(t)| + |A^{1/2}u(t)| + |u(t)| \leq c_{15}e^{-c_{16}t} \qquad \forall t \geq T_2. \tag{4.50}$$

Indeed, the estimate from below is an immediate consequence of (4.46), while the estimate from above follows from (4.49) and (4.38). Now let us set

$$r_0 := \sup\left\{r \geq 0 : \lim_{t\to+\infty}\left(|u'(t)| + |u(t)|_{D(A^{1/2})}\right)e^{rt} = 0\right\}. \tag{4.51}$$

From (4.50) it follows that r_0 is finite and strictly positive. We claim that $r_0 \in \mathcal{D}$, and that there exists a nontrivial r_0-pure solution to the homogeneous equation (2.11) for which (3.24) holds true.

To begin with, we set

$$\beta_1 := \min\{(1+p)r_0, (1+q)r_0\},$$

so that from assumption (2.9) we know now that

$$\lim_{t\to+\infty}|g(t)|e^{\gamma t} = 0 \qquad \forall \gamma < \beta_1. \tag{4.52}$$

Let β_0 be defined by (3.26), and let γ be any real number such that

$$r_0 < \gamma < \min\{\beta_1, \beta_0\}. \tag{4.53}$$

Since $\gamma < \beta_1$, from (4.52) it follows that

$$|g(t)| \leq K_{g,\gamma}e^{-\gamma t} \qquad \forall t \geq 0$$

for a suitable constant $K_{g,\gamma}$. Moreover, (4.53) implies that $\gamma \notin \mathcal{D}$. Therefore, we can apply Proposition 4.1 with $\gamma_0 := \gamma$. We deduce that there exists a solution $w_\gamma(t)$ to the non-homogeneous equation (2.1) such that

$$|w_\gamma'(t)| + |w_\gamma(t)|_{D(A^{1/2})} \leq \Gamma_0 K_{g,\gamma}e^{-\gamma t} \qquad \forall t \geq 0, \tag{4.54}$$

and whose initial conditions satisfy suitable constraints.

We claim that $w_\gamma(t)$ does not depend on γ as long as (4.53) holds true. This is almost trivial when $r_0 < \min\mathcal{D}$, because in this case $\gamma < \beta_0 = \min\mathcal{D}$ and hence w_γ has initial conditions $w_\gamma(0) = w_\gamma'(0) = 0$. If $r_0 \geq \min\mathcal{D}$, and α_0 denotes the largest element of $\mathcal{D}$ less than or equal to r_0, then $w_\gamma(t)$ is uniquely characterized by the limit (4.3) and by having α_0-slow initial data, and both conditions do not depend on γ in the range (4.53).

Therefore, in the sequel we denote $w_\gamma(t)$ just by $w(t)$ and, since (4.54) holds true for every γ in the range (4.53), we deduce that

$$\lim_{t\to+\infty}\left(|w'(t)| + |w(t)|_{D(A^{1/2})}\right)e^{\gamma t} = 0 \qquad \forall \gamma < \min\{\beta_1, \beta_0\}. \tag{4.55}$$

Now let us set $v(t) := u(t) - w(t)$. Since $v(t)$ is a solution to the homogeneous equation (2.11), it can be written as a finite sum of r-pure solutions $v_r(t)$ to the same homogeneous equation, with r ranging over $\mathcal{D}$. All terms $v_r(t)$ with $r < r_0$ are necessarily equal to 0, because otherwise the supremum in (4.51) would be less than r_0.

We are now ready to prove our conclusions. Let us assume by contradiction that $r_0 \notin \mathcal{D}$. In this case $v(t)$ is the sum of r-pure solutions $v_r(t)$ with $r > r_0$, hence also $r \geq \beta_0$, and therefore

$$\lim_{t\to+\infty} \left(|v'(t)| + |v(t)|_{D(A^{1/2})}\right) e^{\gamma t} = 0 \qquad \forall \gamma < \beta_0. \tag{4.56}$$

But (4.55) and (4.56) imply that

$$\lim_{t\to+\infty} \left(|u'(t)| + |u(t)|_{D(A^{1/2})}\right) e^{\gamma t} = 0 \qquad \forall \gamma < \min\{\beta_1, \beta_0\},$$

so that the supremum in (4.51) would be greater that r_0. This proves that $r_0 \in \mathcal{D}$.

Now $v(t)$ is the sum of an r_0-pure solution, which we denote by $v_0(t)$, and possibly some other r-pure solutions $v_r(t)$ with $r > r_0$, and hence $r \geq \beta_0$. As a consequence, it turns out that

$$\lim_{t\to+\infty} \left(|v'(t) - v_0'(t)| + |v(t) - v_0(t)|_{D(A^{1/2})}\right) e^{\gamma t} = 0 \qquad \forall \gamma < \beta_0. \tag{4.57}$$

At this point (3.24) follows from (4.55) and (4.57). Finally, $v_0(t)$ is not identically 0, because if not (4.56) would be true once again, and together with (4.55) this would contradict the maximality of r_0, exactly as before. $\square$

4.4 Proof of Theorem 3.6

Let us first describe the plan of the proof, based on a fixed point argument. Let us define β_0 as in (3.26), and let us choose once for all a constant s_0 such that

$$s_0 < r_0 < (1+p)s_0 < \min\{(1+p)r_0, \beta_0\}. \tag{4.58}$$

Let $v(t)$ be the r_0-pure solution to the homogeneous equation (2.11) with initial data (v_0, v_1). Let $z(t)$ be the r_0-fast solution to the homogeneous equation (2.11) with initial data (z_0, z_1). Let K_1 and K_2 be two constants such that

$$|v(t)|_{D(A^{1/2})} \leq K_1 \left(|v_1| + |v_0|_{D(A^{1/2})}\right) e^{-s_0 t} \qquad \forall t \geq 0, \tag{4.59}$$

$$|z(t)|_{D(A^{1/2})} \leq K_2 \left(|z_1| + |z_0|_{D(A^{1/2})}\right) e^{-s_0 t} \qquad \forall t \geq 0. \tag{4.60}$$

The constants K_1 and K_2 exist because the left-hand side of (4.59) decays at least as $(1+t)e^{-r_0 t}$, and the left-hand side of (4.60) decays at least as $(1+t)e^{-\beta_0 t}$ (or it is identically 0 if $\beta_0 = +\infty$).

Let Γ_0 be the constant which appears in Proposition 4.1 when $\gamma_0 := (1+p)s_0$, and let us assume that $\varepsilon_0 > 0$ is small enough so that

$$(K_1 + K_2 + 1)\varepsilon_0 < R_0, \tag{4.61}$$

$$2L\Gamma_0(K_1 + K_2 + 1)^{1+p}\varepsilon_0^p < 1. \tag{4.62}$$

Let us consider the space

$$\mathbb{X} := \left\{\psi \in C^0\left([0, +\infty); D(A^{1/2})\right) : |\psi(t)|_{D(A^{1/2})} \leq \varepsilon_0 \quad \forall t \geq 0\right\}.$$

It is well-known that $\mathbb{X}$ is a complete metric space with respect to the distance

$$\operatorname{dist}(\psi_1, \psi_2) := \sup\left\{|\psi_1(t) - \psi_2(t)|_{D(A^{1/2})} : t \geq 0\right\}.$$

For every $\psi \in \mathbb{X}$ we set

$$g_\psi(t) := f\left(v(t) + z(t) + \psi(t)e^{-\gamma_0 t}\right) \qquad \forall t \geq 0,$$

and we consider the non-homogeneous linear equation

$$w''(t) + 2\delta w'(t) + Aw(t) = g_\psi(t). \tag{4.63}$$

We claim that this equation admits a unique solution $w_\psi(t)$ such that

$$\lim_{t \to +\infty}\left(|w_\psi'(t)| + |w_\psi(t)|_{D(A^{1/2})}\right)e^{r_0 t} = 0, \tag{4.64}$$

and whose initial data are r_0-slow. Finally, we set

$$\overline{\psi}(t) := w_\psi(t)e^{\gamma_0 t} \qquad \forall t \geq 0, \tag{4.65}$$

and we claim that the following three statements hold true, provided that the smallness assumptions (4.61) and (4.62) are satisfied.

- *Well-posedness of the construction.* The functions g_ψ and w_ψ are well-defined for every $\psi \in \mathbb{X}$.
- *Closedness.* It turns out that $\overline{\psi} \in \mathbb{X}$ for every $\psi \in \mathbb{X}$.
- *Contractivity.* The map $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ defined by $\mathcal{F}(\psi) = \overline{\psi}$ is a contraction.

If we prove the three claims, then the conclusion follows. Indeed the contractivity implies that $\mathcal{F}$ has a fixed point, namely there exists $\psi \in \mathbb{X}$ such that $\overline{\psi} = \psi$. If ψ is the fixed point, then the function defined by

$$u_\psi(t) := v(t) + z(t) + w_\psi(t)$$

is the solution we were looking for.

Indeed, (3.31) holds true because both $z(t)$ and $w_\psi(t)$ decay faster than $e^{-r_0 t}$. The initial conditions of $u_\psi(t)$ are of the form (3.30), where (w_0, w_1) are the initial data of $w_\psi(t)$, which are r_0-slow. Finally, since $\overline{\psi}(t) = \psi(t)$, from (4.65) it follows that

$$v(t) + z(t) + \psi(t)e^{-\gamma_0 t} = v(t) + z(t) + \overline{\psi}(t)e^{-\gamma_0 t} = v(t) + z(t) + w_\psi(t) = u_\psi(t),$$

and hence

$$g_\psi(t) = f\left(v(t) + z(t) + \psi(t)e^{-\gamma_0 t}\right) = f(u_\psi(t)).$$

Since $v(t)$ and $z(t)$ are solutions to the homogeneous equation, we conclude that

$$u''_\psi(t) + 2\delta u'_\psi(t) + Au_\psi(t) = w''_\psi(t) + 2\delta w'_\psi(t) + Aw_\psi(t) = g_\psi(t) = f(u_\psi(t)),$$

which proves that $u_\psi(t)$ is a solution to (2.6).

Well-posedness of the construction From (4.59), (4.60), and our definition of $\mathbb{X}$, it turns out that

$$\begin{aligned} \left|v(t) + z(t) + \psi(t)e^{-\gamma_0 t}\right|_{D(A^{1/2})} \leq\ & K_1\left(|v_1| + |v_0|_{D(A^{1/2})}\right)e^{-s_0 t} \\ & + K_2\left(|z_1| + |z_0|_{D(A^{1/2})}\right)e^{-s_0 t} + \varepsilon_0 e^{-\gamma_0 t}. \end{aligned}$$

Since $s_0 < \gamma_0$, from (3.29) we obtain that

$$\left|v(t) + z(t) + \psi(t)e^{-\gamma_0 t}\right|_{D(A^{1/2})} \leq (K_1 + K_2 + 1)\varepsilon_0 e^{-s_0 t} \qquad \forall t \geq 0. \tag{4.66}$$

Due to the smallness assumption (4.61), this means in particular that

$$\left|v(t) + z(t) + \psi(t)e^{-\gamma_0 t}\right|_{D(A^{1/2})} \leq (K_1 + K_2 + 1)\varepsilon_0 < R_0,$$

which proves that $g_\psi(t)$ is well-defined.

Setting $v = 0$ into (3.27), from (3.28) we obtain that

$$|f(u)| \leq L|u|^{1+p}_{D(A^{1/2})} \qquad \forall u \in B_{R_0}.$$

Therefore, from (4.66) we deduce that

$$|g_\psi(t)| \leq L(K_1 + K_2 + 1)^{1+p}\varepsilon_0^{1+p}e^{-(1+p)s_0 t}. \tag{4.67}$$

Now we apply Proposition 4.1 with $\gamma_0 := (1 + p)s_0$. Due to (4.58), we are in the case where $\gamma_0 > \min \mathcal{D}$, and r_0 is the largest element of $\mathcal{D}$ smaller than γ_0 (namely what was called α_0 in Proposition 4.1). We obtain that (4.63) has a unique solution $w_\psi(t)$ with r_0-slow initial data and such that (4.64) holds true. This proves that $w_\psi(t)$ is well-defined.

Closedness To begin with, we observe that $\overline{\psi} : [0, +\infty) \to D(A^{1/2})$ is a continuous map because of the regularity of w_ψ and f. Due to estimate (4.67) and the smallness assumption (4.62), from Proposition 4.1 we obtain that

$$|w_\psi(t)|_{D(A^{1/2})} \leq \Gamma_0 L(K_1 + K_2 + 1)^{1+p} \varepsilon_0^{1+p} e^{-(1+p)s_0 t} \leq \varepsilon_0 e^{-\gamma_0 t},$$

from which we conclude that $|\overline{\psi}(t)|_{D(A^{1/2})} \leq \varepsilon_0$ for every $t \geq 0$. This proves that $\overline{\psi} \in \mathbb{X}$.

Contractivity Let ψ_1 and ψ_2 be two elements of $\mathbb{X}$. Estimate (4.66) holds true also with ψ_1 and ψ_2 instead of ψ. Therefore, from (3.27) we deduce that

$$\begin{aligned} |g_{\psi_1}(t) - g_{\psi_2}(t)|_{D(A^{1/2})} &\leq 2L(K_1 + K_2 + 1)^p \varepsilon_0^p e^{-ps_0 t} \, |\psi_1(t) - \psi_2(t)|_{D(A^{1/2})} \, e^{-\gamma_0 t} \\ &\leq 2L(K_1 + K_2 + 1)^p \varepsilon_0^p e^{-\gamma_0 t} \cdot \operatorname{dist}(\psi_1, \psi_2). \end{aligned}$$

Let $w_{\psi_1}(t)$ and $w_{\psi_2}(t)$ denote the corresponding solutions to (4.63) in the sense of Proposition 4.1. Since $w_{\psi_1}(t) - w_{\psi_2}(t)$ solves the same equation with forcing term $g_{\psi_1}(t) - g_{\psi_2}(t)$, now (4.4) reads as

$$|w_{\psi_1}(t) - w_{\psi_2}(t)|_{D(A^{1/2})} \leq \Gamma_0 \cdot 2L(K_1 + K_2 + 1)^p \varepsilon_0^p \cdot e^{-\gamma_0 t} \cdot \operatorname{dist}(\psi_1, \psi_2).$$

Multiplying by $e^{\gamma_0 t}$, and taking the supremum for $t \geq 0$, we finally obtain that

$$\operatorname{dist}(\overline{\psi}_1, \overline{\psi}_2) \leq 2\Gamma_0 L(K_1 + K_2 + 1)^p \varepsilon_0^p \cdot \operatorname{dist}(\psi_1, \psi_2),$$

so that the smallness assumption (4.62) implies that the map $\mathcal{F} : \mathbb{X} \to \mathbb{X}$ is a contraction. □

5 Applications

5.1 Semilinear dissipative hyperbolic equations

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded connected open set with Lipschitz boundary (or any other condition which guarantees Sobolev embeddings). As a model case, we consider dissipative hyperbolic equations of the form

$$u_{tt} + 2\delta u_t - \Delta u \pm |u|^p u = 0 \qquad \text{in } \Omega \times [0, +\infty), \tag{5.1}$$

with homogeneous Neumann boundary conditions, or of the form

$$u_{tt} + 2\delta u_t - \Delta u - \lambda u \pm |u|^p u = 0 \qquad \text{in } \Omega \times [0, +\infty), \tag{5.2}$$

with homogeneous Dirichlet boundary conditions. In both cases, $|u|$ denotes the absolute value of u. In the case of equation (5.2), we assume that $\lambda \leq \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ denotes the first eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions in Ω.

The functional setting is the classical one, namely $H := L^2(\Omega)$ and $Au := -\Delta u$ with a suitable domain depending on boundary conditions. We refer to [4] or [5] for further details. We just point out that A is a coercive operator in the subcritical Dirichlet case where $\lambda < \lambda_1(\Omega)$, but it is just a nonnegative operator both in the Neumann case (where the kernel of A is the space of constant functions), and in the critical Dirichlet case where $\lambda = \lambda_1(\Omega)$ (where the kernel of A is the eigenspace of $-\Delta$ relative to the eigenvalue $\lambda_1(\Omega)$).

As for the nonlinear term, we set

$$[f(u)](x) := \mp|u(x)|^p u(x) \qquad \forall x \in \Omega.$$

The function f satisfies the assumptions of our abstract results provided that the Sobolev embedding $H^1(\Omega) \subseteq L^{2+2p}(\Omega)$ holds true. In turn, this condition is satisfied for every $p > 0$ if $n \in \{1, 2\}$, and when $0 < p \leq 2/(n-2)$ if $n \geq 3$. We refer to Section 4.1 of [4] for a proof of these basic facts, which are independent of the sign in (5.1) or (5.2).

The sign becomes relevant when looking for global solutions for all initial data in the energy space. With the "right sign", from the results of [4] and of the present paper, we can prove the following statement.

Theorem 5.1 (Right sign). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and let p be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$.*

Then the following five statements apply to both the Neumann problem for equation (5.1) with the plus sign, and to the Dirichlet problem for equation (5.2) with the plus sign and $\lambda = \lambda_1(\Omega)$. It is intended that $D(A^{1/2}) = H^1(\Omega)$ in the Neumann case, and $D(A^{1/2}) = H^1_0(\Omega)$ in the Dirichlet case.

(1) (Global existence and uniqueness) *For every $(u_0, u_1) \in D(A^{1/2}) \times H$ there exists a unique global solution with the regularity (2.3).*

(2) (Decay estimate from above) *All solutions satisfy*

$$\|u(t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t)^{1/p}} \qquad \forall t \geq 0 \tag{5.3}$$

and

$$\|u'(t)\|_{L^2(\Omega)} + \|\nabla u(t)\|_{L^2(\Omega)} \leq \frac{C}{(1+t)^{1+1/p}} \qquad \forall t \geq 0 \tag{5.4}$$

for a suitable constant C (depending on the solution).

(3) (Classification of decay rates) *All non-zero solutions are either slow or fast in the sense of Theorem 3.1.*

(4) (Existence of slow solutions) *There exists a nonempty open set $\mathcal{S} \subseteq D(A^{1/2}) \times H$ such that all solutions with initial data in S are slow.*

(5) (Existence of fast solutions) *There exists families of fast solutions parametrized in the sense of Theorem 3.6.*

Of course the theory applies also to the Dirichlet problem with $\lambda < \lambda_1(\Omega)$, but in that case the operator is coercive and we have only fast solutions.

Let us spend a few words on the proof of Theorem 5.1. Statement (1) is a well-known result. Estimate (5.3) was proved in Theorem 2.2 of [4] together with a weaker version of (5.4), in which the exponent $(1+1/p)$ is replaced by $(1/2+1/p)$. This weaker estimate is enough to conclude that $u(t)$ decays to 0 in $D(A^{1/2})$, which allows to apply Theorem 3.1 of the present paper to all solutions. Thus we obtain statement (3), and also estimate (5.4) with the correct exponent, which follows from (3.23) in the case of slow solutions, and is trivially true both for the null solution and for fast solutions, which decay exponentially. Statement (4) is a consequence of Theorem 2.3 of [4]. Statement (5) follows from Theorem 3.6 of the present paper.

When the nonlinear term has the wrong sign, global existence is known only in special cases, for example when the origin falls in the so-called potential well. This technique requires that the operator is coercive and controls the nonlinear term (which means Sobolev embeddings). Since the coerciveness of the operator is essential, this theory applies neither to the Neumann case, nor to the critical Dirichlet case. In other words, the potential well applies only to the subcritical Dirichlet case, in which case we obtain the following result.

Theorem 5.2 (Wrong sign, with potential well)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let p be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$. Let us consider the Dirichlet problem for equation (5.2) with the minus sign and $\lambda < \lambda_1(\Omega)$.*

Then there exists $R_0 > 0$ such that, for every $u_0 \in B_{R_0}$ (defined as in (2.7)), the problem has a unique global solution with the regularity (2.3).

Moreover, every non-zero solution in B_{R_0} is fast in the sense of Theorem 3.1, and there exist families of fast solutions parametrized in the sense of Theorem 3.6.

When there is no potential well, Theorem 3.1 keeps on classifying all possible decay rates of those solutions which exist globally and decay. On the other hand, nothing in this case guarantees decay, or even global existence, of solutions.

Nevertheless, there is one notable exception. Theorem 3.6 provides families of global solutions with exponential decay without assuming neither the coercivity of the operator, nor sign conditions on the nonlinear term. Thus we obtain the following existence result.

Theorem 5.3 (Wrong sign, without potential well)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let p be a positive exponent, with no further restriction if $n \in \{1, 2\}$, and $p \leq 2/(n-2)$ if $n \geq 3$.*

Let us consider the Neumann problem for equation (5.1) or the Dirichlet problem for equation (5.2) with $\lambda = \lambda_1(\Omega)$.

Then there exist families of fast solutions parametrized in the sense of Theorem 3.6, independently of the sign in the nonlinear term.

We conclude by pointing out that our abstract results apply also to equations with second order operators with non-constant coefficients, or with higher order operators such as Δ^2. We also allow more general nonlinear terms depending on x and t.

5.2 Degenerate Kirchhoff equations in finite dimension

In this final section we present a different application of our theory. We consider a degenerate Kirchhoff equation

$$u''(t) + u'(t) + \left|B^{1/2}u(t)\right|^{2\alpha} Bu(t) = 0, \tag{5.5}$$

where α is a positive real number and B is a self-adjoint operator on a Hilbert space H. Equations of this type have long been considered in the literature, but only partial results are known. As for global existence, the main result is that a global solution exists provided that initial data $(u_0, u_1) \in D(B) \times D(B^{1/2})$ satisfy the nondegeneracy condition $B^{1/2}u_0 \neq 0$, and a suitable smallness assumption. This was proved in [9] in the case $\alpha \geq 1$, and then in [1] in the case $0 < \alpha < 1$.

As for decay estimates, let us assume that the operator B is coercive, because if not solutions do not necessarily decay to 0 (just think to the limit case where B is the null operator). Under this coerciveness assumption, it is know that solutions provided in literature satisfy

$$\frac{C_1}{(1+t)^{1/(2\alpha)}} \leq |u(t)| \leq \frac{C_2}{(1+t)^{1/(2\alpha)}} \tag{5.6}$$

for suitable positive constants C_1 and C_2, which means that these solutions are slow. Analogous estimates hold true for $|B^{1/2}u(t)|$ and $|Bu(t)|$. This was proved in [9, 3, 2].

On the other hand, there exist solutions to (5.5) which are not slow. For example, if we limit ourselves to simple modes, namely solutions of the form $u(t) := u_k(t)e_k$, where e_k is an eigenvector of B corresponding to a positive eigenvalue λ_k, then (5.5) reduces to the ordinary differential equation

$$u_k''(t) + u_k'(t) + \lambda_k^{\alpha+1}|u_k(t)|^{2\alpha}u_k(t) = 0,$$

and it is well-known after [7] that this equation admits both slow solutions decaying as $t^{-1/(2\alpha)}$ and fast solutions decaying as e^{-t}.

Now we can say that this alternative holds true more generally for solutions with a finite number of modes, or more generally for solutions living in a subspace of H where (the restriction of) B is a bounded operator.

Theorem 5.4. *Let H be a separable Hilbert space, and let B be a linear operator on H. Let us assume that B is bounded, symmetric and coercive.*

Then for every $\alpha > 0$ the following conclusions hold true.

(1) (Global existence and uniqueness) *For every $(u_0, u_1) \in H \times H$, problem (5.5)–(2.2) admits a unique global solution $u \in C^2([0, +\infty), H)$.*

(2) (Classification of decay rates) *Every non-zero solution is either a slow solution satisfying (5.6) for suitable positive constants* C_1 *and* C_2*, or a fast solution for which there exists* $v_0 \in H$*, with* $v_0 \neq 0$*, such that*

$$\lim_{t \to +\infty} \left(|u'(t) + v_0 e^{-t}| + |u(t) - v_0 e^{-t}| \right) e^{\gamma t} = 0 \qquad \forall \gamma < 1 + 2\alpha. \tag{5.7}$$

(3) (Existence of slow solutions) *There exists a nonempty open set* $\mathcal{S} \subseteq H \times H$ *of initial data originating slow solutions.*

(4) (Existence of fast solutions) *For every* $v_0 \in H$*, small enough but different from 0, there exists at least one solution satisfying (5.7).*

Let us sketch the proof, which is just an application of our theory. Statement (1) follows from the boundedness of B and the fact that the usual Hamiltonian

$$|u'(t)|^2 + \frac{1}{\alpha + 1} |B^{1/2} u(t)|^{2(\alpha+1)}$$

is constant along trajectories.

Now let us rewrite (5.5) in the form

$$u''(t) + u'(t) = -\left| B^{1/2} u(t) \right|^{2\alpha} Bu(t) =: f(u(t)).$$

It can be seen that $f(u) = -\nabla F(u)$ with $F(u) := (\alpha+1)^{-1} |B^{1/2} u|^{2(\alpha+1)}$, and that $|f(u)| \leq K_0 |u|^{2\alpha+1}$ for a suitable constant K_0 because the norms in H, $D(B^{1/2})$ or $D(B)$ are equivalent due to the boundedness and coerciveness of B. Therefore, equation (5.5) fits in the abstract framework of [4] and of the present paper with A equal to the null operator and $p := 2\alpha$. At this point, from Theorem 2.2 of [4] it follows that all solutions satisfy

$$|u(t)| \leq \frac{C_3}{(1+t)^{1/(2\alpha)}} \qquad \forall t \geq 0$$

for a suitable constant C_3. In particular, all solutions decay to 0, and hence we can apply Theorem 3.1 of the present paper, which gives the slow-fast alternative. As for the asymptotic profile of fast solutions, it is enough to remark that now the associated homogeneous equation is $u'' + u' = 0$, so that the only positive element of $\mathcal{D}$ is $r_0 = 1$, and r_0-pure solutions are of the form $v_0(t) = v_0 e^{-t}$ for some $v_0 \in H$. This proves statement (2).

Statement (3) follows from Theorem 2.3 of [4].

Statement (4) follows from Theorem 3.6 of the present paper applied with $r_0 = 1$, after observing the structure of r_0-pure solutions and the fact that the only r_0-fast initial datum is $(0, 0)$.

Acknowledgments

This work has been done while the first two authors were visiting the Laboratoire Jacques Louis Lions of the UPMC (Paris VI). The stay was partially supported by the FSMP (Fondation Sciences Mathématiques de Paris). The first two authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

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