ON THE CONVERGENCE OF A FULLY DISCRETE SCHEME OF LES TYPE TO PHYSICALLY RELEVANT SOLUTIONS OF THE INCOMPRESSIBLE NAVIER-STOKES

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Abstract. Obtaining reliable numerical simulations of turbulent fluids is a challenging problem in computational fluid mechanics. The Large Eddy Simulations (LES) models are efficient tools to approximate turbulent fluids and an important step in the validation of these models is the ability to reproduce relevant properties of the flow. In this paper we consider a fully discrete approximation of the Navier-Stokes-Voigt model by an implicit Euler algorithm (with respect to the time variable) and a Fourier-Galerkin method (in the space variables). We prove the convergence to weak solutions of the incompressible Navier-Stokes equations satisfying the natural local entropy condition, hence selecting the so-called *physically relevant* solutions.

1. Introduction

We consider the incompressible Navier-Stokes Equations (NSE) with periodic boundary conditions

$$\begin{aligned} \partial_t u - \Delta u + (u\cdot\nabla)\,u + \nabla p &= 0 \qquad \text{in } (0,T)\times\mathbb{T}^3,\\ \nabla\cdot u &= 0 \qquad \text{in } (0,T)\times\mathbb{T}^3,\\ u|_{t=0} &= u_0 \qquad \text{on } \mathbb{T}^3, \end{aligned} \tag{1.1}$$

where $T>0$ is arbitrary and $\mathbb{T}^3$ the three dimensional flat torus. Here the velocity field $u\in\mathbb{R}^3$ as well as the pressure $p\in\mathbb{R}$ are both space periodic and with zero mean value. Even if turbulent phenomena arise for large values of the Reynolds number, we set here the viscosity equal to one and the external force equal to zero, since these assumptions do not affect the main result.

Obtaining an accurate prediction (of averaged quantities) of turbulent flows is a central difficulty in computational fluid mechanics and we recall that direct numerical simulations have –at present– unaffordable computational costs to perform this task. The most promising tools to perform accurate simulations of turbulent fluids are given by the Large Eddy Simulations (LES) models. LES models are based on the idea that in many practical situations it is enough to simulate the mean characteristics of the flow by averaging/filtering the equations. A very popular LES model is given by the Navier-Stokes-Voigt equations, whose Cauchy problem reads as follows:

$$\begin{aligned} \partial_t(u^\alpha_t - \alpha^2\Delta u^\alpha) - \Delta u^\alpha + (u^\alpha\cdot\nabla)\,u^\alpha + \nabla p^\alpha &= 0 \qquad \text{in } (0,T)\times\mathbb{T}^3,\\ \nabla\cdot u^\alpha &= 0 \qquad \text{in } (0,T)\times\mathbb{T}^3,\\ u^\alpha|_{t=0} &= u_0 \qquad \text{on } \mathbb{T}^3. \end{aligned} \tag{1.2}$$

Here, the parameter $\alpha>0$ has the dimension of a length and –roughly speaking– the scales smaller than α are truncated. It is also well-known that for system (1.2) one can prove global existence and uniqueness of solutions. We refer to [9, 19] for the analysis of the Cauchy problem (1.2) and for the interpretation of the results. In particular, the regularization

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introduced by the operator $-\partial_t\Delta$ is of hyperbolic type (not an extra dissipation as in eddy viscosity models) and the system is of pseudo-parabolic type. To assess the model from the mathematical point of view one important question is to show that the solutions, in the limit as $\alpha \to 0$, produce weak solutions of the Navier-Stokes equations which satisfy the local energy inequality

$$\partial_t\left(\frac{|u|^2}{2}\right) + \nabla\cdot\left(\left(\frac{|u|^2}{2}+p\right)u\right) - \Delta\left(\frac{|u|^2}{2}\right) + |\nabla u|^2 \leq 0, \tag{1.3}$$

in the sense of distributions over $(0,T)\times\mathbb{T}^3$.

We recall that starting with the results on global existence of weak solutions for the NSE by Leray [20] and Hopf [18] a still unsolved problem is that of uniqueness and regularity of these solutions. Moreover, among weak solutions those satisfying the local energy inequality (1.3) are of particular importance because for them holds true the celebrated partial regularity theorem of Caffarelli-Kohn-Nirenberg [8]. Finally, we notice that the inequality (1.3) is a natural request that solutions constructed by numerical methods should satisfy, see Guermond [16, 17]. A weak solution of (1.1) satisfying (1.3) is known in literature a suitable weak solution. The first existence result of suitable weak solutions is due to Caffarelli-Kohn-Nirenberg [8]. Then, the convergence to suitable weak solutions has been proved for different methods, see [1, 2, 6, 12], but the approximation methods are of all of "*infinite dimensional type*", that is obtained by approximating the NSE (1.1) by another system of partial differential equations, and few results are available when the approximation methods are finite dimensional as in numerical methods. In [14, 15] Guermond proved the convergence to a suitable weak solution for numerical solutions obtained by using some finite element Galerkin methods (only with respect to the space variables), while some conditional results on Fourier based Galerkin methods on the torus are proved in [7]. In particular, the convergence to a suitable weak solution of the standard Fourier-Galerkin method is still an interesting open problem and the space-periodic setting and the use of Fourier series expansion is not an assumption to simplify the technicalities. From the numerical point of view another important issue is the time discretization. In [5] it is proved that solutions of periodic Navier-Stokes equations constructed by the standard implicit Euler algorithm are suitable. The result has been later extended to a general domain in assuming at the boundary slip vorticity based conditions, which are important in the vanishing viscosity problem [3, 4]. The case of Dirichlet boundary conditions is treated in Gigli and Mosconi [13] with a semigroup approach.

The aim of this paper is to perform a space-time full discretization of (1.2) and to prove the convergence (varying the parameters of the numerical discretization and as $\alpha \to 0$) to approach weak solutions of Navier-Stokes equations satisfying the local energy inequality

In order to discretize in time (1.2) we use the implicit Euler algorithm, while in space we use the spectral Galerkin methods, based on Fourier series expansion

$$d_t(u_n^{\alpha,m} - \alpha^2\Delta u_n^{\alpha,m}) - \Delta u_n^{\alpha,m} + P_n((u_n^{\alpha,m}\cdot\nabla)\,u_n^{\alpha,m}) = 0, \tag{1.4}$$

where d_t denotes the backword finite difference operator and where P_n is the projection over the space of Fourier modes smaller of equal than n, see Section 3 for the precise formulations of the discretization. Here, we only point out that the output of this Euler-Fourier-Galerkin type of approximation is a triple $(u_n^{\alpha,M}, v_n^{\alpha,M}, p_n^{\alpha,M})$, where $M\in\mathbb{N}$ is the parameter defining the time-step $\kappa = T/M$. The main result of this paper is the following theorem. See Section 2 for the notations concerning the spaces.

Theorem 1.1. *Let $u_0 \in H^2_{0,\sigma}$ and let $\{(u_n^{\alpha,M}, v_n^{\alpha,M}, p_n^{\alpha,M})\}_{(n,\alpha,M)}$ be a sequence of solutions of the approximating Euler-Fourier-Galerkin scheme of* (1.4). *Let $\{M_n\}_n \subset \mathbb{N}$ be any monotone sequence converging to infinity and let $\alpha_n \subset (0,1)$ be any monotone sequence converging to zero and such that*

$$\lim_{n\to\infty} n\,\alpha_n^3 = 0. \tag{1.5}$$

Then, there exists

$$(u,p) \in L^\infty(0,T;L^2_{0,\sigma}) \cap L^2(0,T;H^1_{0,\sigma}) \times L^{5/3}((0,T)\times\mathbb{T}^3),$$

such that, up to a subsequence not relabelled, the following convergence holds true as $n \to \infty$*:*

$$\begin{aligned}
&v_n^{\alpha_n,M_n} \to u \text{ strongly in } L^2((0,T)\times\mathbb{T}^3),\\
&u_n^{\alpha_n,M_n} \to u \text{ strongly in } L^2((0,T)\times\mathbb{T}^3),\\
&\nabla u_n^{\alpha_n,M_n} \rightharpoonup \nabla u \text{ weakly in } L^2((0,T)\times\mathbb{T}^3),\\
&p_n^{\alpha_n,M_n} \rightharpoonup p \text{ weakly in } L^{5/3}((0,T)\times\mathbb{T}^3).
\end{aligned}$$

Moreover, (u,p) *is a suitable weak solution of* (1.1) *in the sense of Definition 2.2.*

Remark 1.2. The assumption on the initial datum can be relaxed, by an appropriate regularization. We do not state and prove Theorem 1.1 under this more general hypothesis in order to avoid further technicalities.

Remark 1.3. We note that, while the sequence $\{\alpha_n\}_n$ is related to n by (1.5), the sequence $\{M_n\}_n$ is arbitrary. This means that there is no need to link the time and the space approximation in order to have convergence of the scheme. Theorem 1.1 may be equivalently stated in terms of a double sequence $\{(u_n^{\alpha_n,M}, v_n^{\alpha_n,M}, p_n^{\alpha_n,M}\}_{(M,n)}$ and the convergences hold as $(M,n)\to\infty$.

The convergence of Fourier-Galerkin method of (1.2) to a suitable weak solutions of (1.1), without the time discretization, but with α_n satisfying (1.5) has been proved as one of the results in [6]. Here new difficulties arise from the non trivial combinations of the time discretization and the proof of certain discrete *a priori* estimates, which are counterpart of those obtained in [6].

The problem of the convergence of numerical schemes to solutions satisfying local energy-type balance is present also in several other equations in fluid mechanics. Among them we want to cite the case of the two dimensional Euler equations with vorticity in L^p. In this case, satisfying the local energy balance is almost equivalent to solve the vorticity equations in the renormalized sense and the additional information obtained is that the solution obtained is Lagrangian, we refer to [10, 11] for further details.

Plan of the paper. In Section 2 we fix the notation that we use in the paper, we recall the main definitions regarding the NSE (1.1), and the tools used. In Section 3 we introduce and describe in detail the space-time discretization we consider. In Section 4 we prove the main *a priori* estimates needed to study the convergence and finally in Section 5 we prove Theorem 1.1.

2. Preliminaries

In this section we give details on the functional setting and then we recall the main definitions concerning weak solutions of incompressible Navier-Stokes equations.

2.1. **Notations.** We introduce the notations typical of space-periodic problems. The three dimensional torus is defined by $\mathbb{T}^3 := \mathbb{R}^3/2\pi\mathbb{Z}^3$. We denote with $C^\infty_c(I;C^\infty(\mathbb{T}^3))$ the space of smooth functions or vectors which are compactly supported on the interval $I\subset\mathbb{R}$ and 2π-periodic with respect to the space variables. In the sequel we shall use the customary Lebesgue spaces $L^p(\mathbb{T}^3)$ and Sobolev spaces $W^{k,p}(\mathbb{T}^3)$ and we shall denote their norms by $\|\cdot\|_p$ and $\|\cdot\|_{W^{k,p}}$. Moreover, in the case $p=2$ we use the notation $H^s(\mathbb{T}^3) := W^{s,2}(\mathbb{T}^3)$ and, for simplicity, we shall not distinguish between scalar and vector valued functions. Finally, we use $(\cdot,\cdot)$ to denote the $L^2(\mathbb{T}^3)$ paring. Since we are working in the periodic setting we denote by the subscript "$_0$" the subspaces of zero average vectors of $L^2(\mathbb{T}^3)$ and $H^s(\mathbb{T}^3)$, for any exponent $s\geq 0$.

The divergence-free constraint is also directly included in the function spaces in the analysis of the NSE and, as usual, we define

$$L^2_{0,\sigma} := \left\{ w : \mathbb{T}^3 \to \mathbb{R}^3,\ w \in L^2(\mathbb{T}^3), \quad \nabla \cdot w = 0 \quad \int_{\mathbb{T}^3} w\, dx = 0 \right\},$$
$$H^s_{0,\sigma} := \left\{ w : \mathbb{T}^3 \to \mathbb{R}^3,\ w \in H^s(\mathbb{T}^3), \quad \nabla \cdot w = 0 \quad \int_{\mathbb{T}^3} w\, dx = 0 \right\},$$

and we recall that the divergence condition can be easily defined in terms of the Fourier coefficients. For any $s > 0$ we denote by $H^{-s} := (H^s_{0,\sigma})'$.

Finally, the space $L^p(0,T;X)$, where X is a Banach space, is the classical Bochner spaces endowed with its natural norm denoted by $\|\cdot\|_{L^p(X)}$.

2.2. **Leray-Hopf and Suitable Weak Solutions.** We start by recalling the definition of weak solution of the initial value problem (1.1), as introduced by Leray and Hopf.

Definition 2.1 (Leray-Hopf Weak Solutions). *The vector field $u \in L^\infty(0,T;L^2_{0,\sigma}) \cap L^2(0,T;H^1_{0,\sigma})$ is a Leray-Hopf weak solution of* (1.1) *if:*

(1) u satisfies the following identity

$$\int_0^T (u, \partial_t \varphi) - (\nabla u, \nabla \varphi) - ((u \cdot \nabla)\, u, \varphi)\, dt + (u_0, \varphi(0)) = 0,$$

for all smooth, periodic and divergence-free functions $\varphi \in C^\infty_c([0,T);C^\infty(\mathbb{T}^3))$ with zero mean value over $\mathbb{T}^3$.

(2) The following energy inequality holds true:

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2\, ds \leq \frac{1}{2}\|u_0\|_2^2 \qquad \text{for all } t \in [0,T].$$

We remark that u attains the initial datum in the strong sense, namely

$$\lim_{t \to 0^+} \|u(t) - u_0\|_2 = 0.$$

Suitable weak solutions are a particular subclass of Leray-Hopf weak solutions. They were introduced by Scheffer in [22] and Caffarelli-Kohn-Nirenberg in [8]. The definition in the periodic setting is the following.

Definition 2.2 (Suitable Weak Solutions). *A pair (u,p) is a Suitable Weak Solution to the Navier-Stokes equation* (1.1) *if u is a Leray-Hopf weak solution, $p \in L^{\frac{5}{3}}((0,T) \times \mathbb{T}^3)$, and the local energy balance holds true*

$$\int_0^T \int_{\mathbb{T}^3} |\nabla u|^2 \phi\, dxdt \leq \int_0^T \int_{\mathbb{T}^3} \left[\frac{|u|^2}{2} (\partial_t \phi + \Delta \phi) + \left(\frac{|u|^2}{2} + p \right) u \cdot \nabla \phi \right] dxdt, \tag{2.1}$$

for all $\phi \in C^\infty_0(0,T;C^\infty(\mathbb{T}^3))$ such that $\phi \geq 0$.

3. Time-Discrete Fourier-Galerkin Methods

In this section we introduce the space-time full discretization of the Navier-Stokes-Voigt equations (1.2) we are going to analyze. Let P denote the Leray projector of $L^2_0(\mathbb{T}^3)$ onto $L^2_{0,\sigma}$, which explicitly reads in the orthogonal Hilbert basis of complex exponentials as follows:

$$P: \ g(x) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \hat{g}_k\, \mathrm{e}^{ik\cdot x} \ \mapsto \ Pg(x) = \sum_{k \in \mathbb{Z}^3 \setminus \{0\}} \left[\hat{g}_k - \frac{(\hat{g}_k \cdot k)k}{|k|^2} \right] \mathrm{e}^{ik \cdot x}.$$

Then, for any $n \in \mathbb{N}$, we denote by P_n the projector of $L^2_0(\mathbb{T}^3)$ on the finite-dimensional sub-space $V_n := P_n(L^2_{0,\sigma})$, given by the following expression

$$P_n : g(x) = \sum_{k\in\mathbb{Z}^3\setminus\{0\}} \hat{g}_k \,\mathrm{e}^{ik\cdot x} \;\mapsto\; P_n g(x) = \sum_{0<|k|\leq n} \left[\hat{g}_k - \frac{(\hat{g}_k\cdot k)k}{|k|^2}\right] \mathrm{e}^{ik\cdot x}.$$

The (space) approximate Fourier-Galerkin method to (1.2) is given by the following system

$$\begin{aligned} \partial_t(u^\alpha_n - \alpha^2\Delta u^\alpha_n) - \Delta u^\alpha_n + P_n((u^\alpha_n\cdot\nabla)\,u^\alpha_n) = 0 \qquad &\text{in } (0,T)\times\mathbb{T}^3,\\ u^\alpha_n|_{t=0} = P_n u_0 \qquad\qquad &\text{in } \mathbb{T}^3, \end{aligned} \tag{3.1}$$

where

$$u^\alpha_n(t,x) = \sum_{0<|k|\leq n} \hat{u}^\alpha_{n,k}(t)\,\mathrm{e}^{ik\cdot x}, \qquad \text{with} \quad k\cdot\hat{u}^\alpha_{n,k} = 0. \tag{3.2}$$

We note that the divergence-free condition is encoded in (3.2) and (3.1) is a (finite dimensional) system of ODEs in the unknowns $\hat{u}^\alpha_{n,k}(t)$.

Next, we proceed by performing the time discretization of (3.1) by finite differences in time. Let $M \in \mathbb{N}$ and $\kappa = T/M$. We consider the net $I^M = \{t_m\}_{m=0}^M$ with $t_0 = 0$ and $t_m = m\kappa$ and discretize (3.1) by using the implicit Euler algorithm: Set $u^{\alpha,0}_n = P_n u_0$. For any $m = 1, ..., M$, given $u^{\alpha,m-1}_n \in V_n$ find $u^{\alpha,m}_n \in V_n$ by solving

$$d_t(u^{\alpha,m}_n - \alpha^2\Delta u^{\alpha,m}_n) - \Delta u^{\alpha,m}_n + P_n((u^{\alpha,m}_n\cdot\nabla)\,u^{\alpha,m}_n) = 0, \tag{3.3}$$

where

$$u^{\alpha,m}_n(x) = \sum_{0<|k|\leq n} \hat{u}^{\alpha,m}_{n,k}\,\mathrm{e}^{ik\cdot x}, \qquad \text{with} \quad k\cdot\hat{u}^{\alpha,m}_{n,k} = 0, \tag{3.4}$$

and

$$d_t u^{\alpha,m}_n := \frac{u^{\alpha,m}_n - u^{\alpha,m-1}_n}{\kappa}.$$

We point out that again the divergence-free condition is enforced by (3.4) and now, for each $m = 1, \ldots, M$, the system (3.3) is a finite dimensional nonlinear (algebraic) system in the unknowns $\hat{u}^{\alpha,m}_{n,k} \in \mathbb{R}$.

Finally, since we are considering the periodic setting we can define the associated approximation for the pressure by solving the Poisson problem

$$-\Delta p^{\alpha,m}_n = \nabla\cdot\big(\nabla\cdot(u^{\alpha,m}_n \otimes u^{\alpha,m}_n)\big) \qquad m = 1, \ldots, M, \tag{3.5}$$

with periodic boundary conditions and zero mean value on $p^{\alpha,m}_n$. Moreover, in order to prove the convergence to a suitable weak solution, it will turn out to be convenient to (re)formulate the equations (3.3) as follows

$$d_t(u^{\alpha,m}_n - \alpha^2\Delta u^{\alpha,m}_n) - \Delta u^{\alpha,m}_n + (u^{\alpha,m}_n\cdot\nabla)\,u^{\alpha,m}_n - Q_n((u^{\alpha,m}_n\cdot\nabla)\,u^{\alpha,m}_n) + \nabla p^{\alpha,m}_n = 0. \tag{3.6}$$

where the operator Q_n is defined by $Q_n := P - P_n$.

As usual in the study of finite difference numerical schemes, we can now rephrase the problem (3.3) on $(0,T)\times\mathbb{T}^3$, by introducing the following time dependent functions

$$\begin{aligned} u^{\alpha,M}_n(t) &= \begin{cases} u^{\alpha,m}_n & \text{for } t\in[t_{m-1},t_m),\\ u^{\alpha,M}_n & \text{for } t = t_M, \end{cases}\\ v^{\alpha,M}_n(t) &= \begin{cases} u^{\alpha,m-1}_n + \dfrac{t-t_{m-1}}{\kappa}(u^{\alpha,m}_n - u^{\alpha,m-1}_n) & \text{for } t\in[t_{m-1},t_m),\\ u^{\alpha,M}_n & \text{for } t = t_M, \end{cases}\\ p^{\alpha,M}_n(t) &= \begin{cases} p^{\alpha,m}_n & \text{for } t\in[t_{m-1},t_m),\\ p^{\alpha,M}_n & \text{for } t = t_M. \end{cases} \end{aligned} \tag{3.7}$$

Then, equations (3.3) read as follows

$$\partial_t(v_n^{\alpha,M} - \alpha^2 \Delta v_n^{\alpha,M}) - \Delta u_n^{\alpha,M} + P_n((u_n^{\alpha,M}\cdot\nabla)\, u_n^{\alpha,M}) = 0, \tag{3.8}$$

and Eq. (3.6) on $(0,T)\times\mathbb{T}^3$ becomes

$$\partial_t(v_n^{\alpha,M} - \alpha^2 \Delta v_n^{\alpha,M}) - \Delta u_n^{\alpha,M} + (u_n^{\alpha,M}\cdot\nabla)\, u_n^{\alpha,M} - Q_n((u_n^{\alpha,M}\cdot\nabla)\, u_n^{\alpha,M}) + \nabla p_n^{\alpha,M} = 0. \tag{3.9}$$

We stress that in order to prove the convergence to a suitable weak solution, it is crucial to prove that the term involving Q_n goes to zero as $n\to\infty$. To this end we recall the following lemma, which is proved as one of the main steps in [7, Lemma 4.4].

Lemma 3.1. *Let be given $\phi \in C^\infty((0,T)\times\mathbb{T}^3)$ and let u^n be defined as*

$$u^n(t,x) := \sum_{0<|k|\le n} \widehat{U}_k^n(t)\, e^{ik\cdot x}.$$

Then, there exists a constant c, depending only on ϕ (but independent of $n\in\mathbb{N}$), such that

$$\|Q_n(u^n(t)\phi(t))\|_\infty^2 \le c\left(n^2 \sum_{|k|\ge \frac{n}{2}} |\widehat{U}_k^n(t)|^2 + \frac{1}{n}\sum_{k\in\mathbb{Z}^3} |\widehat{U}_k^n(t)|^2\right).$$

4. A Priori Estimates

In this section we prove the *a priori* estimates needed to prove the convergence to (1.1). We start with the following basic discrete energy inequality.

Lemma 4.1. *Let be given $u_0 \in H^2_{0,\sigma}$. Let $u_n^{\alpha,m}$ be a solution of (3.3). Then, the following discrete energy equality holds true for all $M\in\mathbb{N}$ and $m = 1,..,M$*

$$\begin{aligned} \|u_n^{\alpha,m}\|_2^2 + &\sum_{i=1}^m \|u_n^{\alpha,i} - u_n^{\alpha,i-1}\|_2^2 + 2\kappa \sum_{i=1}^m \|\nabla u_n^{\alpha,i}\|_2^2 \\ &+ \alpha^2 \|\nabla u_n^{\alpha,m}\|_2^2 + \alpha^2 \sum_{i=1}^m \|\nabla u_n^{\alpha,i} - \nabla u_n^{\alpha,i-1}\|_2^2 = \|u_0\|_2^2 + \alpha^2\|\nabla u_0\|_2^2. \end{aligned} \tag{4.1}$$

Proof. Fix $M\in\mathbb{N}$ and $m = 1,...,M$. Consider the equations (3.1) for $i = 1,...,m$ and multiply (3.1) by $u_n^{\alpha,i}$. Then, after integration by parts over $\mathbb{T}^3$, we get

$$\left(\frac{u_n^{\alpha,i} - u_n^{\alpha,i-1}}{\kappa}, u_n^{\alpha,i}\right) + \alpha^2\left(\frac{\nabla u_n^{\alpha,i} - \nabla u_n^{\alpha,i-1}}{\kappa}, \nabla u_n^{\alpha,i}\right) + \|\nabla u_n^{\alpha,i}\|_2^2 = 0,$$

where we used that fact that since $u_n^{\alpha,i}\in V_n$ then

$$(P_n((u_n^{\alpha,i}\cdot\nabla)\, u_n^{\alpha,i}), u_n^{\alpha,i}) = 0.$$

By using the elementary equality

$$(a, b-a) = \frac{|a|^2}{2} - \frac{|b|^2}{2} + \frac{|a-b|^2}{2}, \tag{4.2}$$

the terms involving the discrete derivative become the following:

$$\begin{aligned} (u_n^{\alpha,i} - u_n^{\alpha,i-1}, u_n^{\alpha,i}) &= \frac{1}{2}(\|u_n^{\alpha,i}\|_2^2 - \|u_n^{\alpha,i-1}\|_2^2) + \frac{1}{2}\|u_n^{\alpha,i} - u_n^{\alpha,i-1}\|_2^2, \\ (\nabla u_n^{\alpha,i} - \nabla u_n^{\alpha,i-1}, \nabla u_n^{\alpha,i}) &= \frac{1}{2}(\|\nabla u_n^{\alpha,i}\|_2^2 - \|\nabla u_n^{\alpha,i-1}\|_2^2) + \frac{1}{2}\|\nabla u_n^{\alpha,i} - \nabla u_n^{\alpha,i-1}\|_2^2. \end{aligned}$$

Finally, by summing up over $i = 1,...,m$ we get (4.1). □

The next lemma regards two weighted estimates on higher derivatives of solutions of (3.3) and they will be useful when proving the convergence to a suitable weak solution. The results in the following lemma are a discrete counterpart of those proved in [6].

Lemma 4.2. *Let $u_0 \in H^2_{0,\sigma}$ and $\alpha \le 1$. Let $M \in \mathbb{N}$ and $m = 1, ..., M$. Let $u_n^{\alpha,m}$ be a solution of (3.3). Then, there exists $c > 0$, independent of $\alpha > 0$, $M \in \mathbb{N}$, and $n \in \mathbb{N}$, such that*

$$\alpha^3 \kappa \sum_{m=1}^{M} \|d_t u_n^{\alpha,m}\|_2^2 \le c, \tag{4.3}$$

$$\alpha^6 \kappa \sum_{m=1}^{M} \|\Delta u_n^{\alpha,m}\|_2^2 \le c. \tag{4.4}$$

Proof. Let $M \in \mathbb{N}$ and $m = 1, ..., M$. We multiply (3.3) by $\alpha^3 d_t u_n^{\alpha,m}$. After integrating by parts over $\mathbb{T}^3$ we get

$$\alpha^3(d_t\nabla u_n^{\alpha,m}, \nabla u_n^{\alpha,m}) + \alpha^3\|d_t u_n^{\alpha,m}\|_2^2 + \alpha^5\|d_t\nabla u_n^{\alpha,m}\|_2^2 + \alpha^3(P_n((u_n^{\alpha,m}\cdot\nabla)\, u_n^{\alpha,m}, d_t u_n^{\alpha,m}) = 0.$$

By using (4.2) and multiplying by κ we then get

$$\begin{aligned}\frac{\alpha^3}{2}(\|\nabla u_n^{\alpha,m}\|_2^2 - \|\nabla u_n^{\alpha,m-1}\|_2^2) + \frac{\alpha^3}{2}\|\nabla u_n^{\alpha,m} - \nabla u_n^{\alpha,m-1}\|_2^2 \\ + \alpha^3\kappa\|d_t u_n^{\alpha,m}\|_2^2 + \alpha^5\kappa\|d_t\nabla u_n^{\alpha,m}\|_2^2 \le \alpha^3\kappa|((u_n^{\alpha,m}\cdot\nabla)\, u_n^{\alpha,m}, d_t u_n^{\alpha,m})|,\end{aligned} \tag{4.5}$$

where we used the fact that $d_t u_n^{\alpha,m} \in V_n$. By using Hölder and Gagliardo-Nirenberg inequalities we estimate the right hand side as follows

$$\begin{aligned}\alpha^3\kappa|((u_n^{\alpha,m}\cdot\nabla)\, u_n^{\alpha,m}, d_t u_n^{\alpha,m})| &\le \alpha^3\kappa\|u_n^{\alpha,m}\|_4\|\nabla u_n^{\alpha,m}\|_2\|d_t u_n^{\alpha,m}\|_4 \\ &\le c\alpha^3\kappa\|u_n^{\alpha,m}\|_2^{\frac14}\|\nabla u_n^{\alpha,m}\|_2^{\frac74}\|\nabla d_t u_n^{\alpha,m}\|_2^{\frac34}\|d_t u_n^{\alpha,m}\|_2^{\frac14} \\ &\le c\alpha^3\kappa(\|u_0\|_2^2 + \alpha^2\|\nabla u_0\|_2^2)^{\frac18}\|\nabla u_n^{\alpha,m}\|_2^{\frac74}\|\nabla d_t u_n^{\alpha,m}\|_2^{\frac34}\|d_t u_n^{\alpha,m}\|_2^{\frac14},\end{aligned}$$

where in the second line we used (4.1). By using Young inequality with $p_1 = 2$, $p_2 = \frac{8}{3}$, and $p_3 = 8$ and we get

$$\begin{aligned}\alpha^3\kappa|((u_n^{\alpha,m}\cdot\nabla)\, u_n^{\alpha,m}, d_t u_n^{\alpha,m})| \le c\kappa(\|u_0\|_2^2 + \alpha^2\|\nabla u_0\|_2^2)^{\frac14}\alpha^{\frac32}\|\nabla u_n^{\alpha,m}\|_2^{\frac32}\|\nabla u_n^{\alpha,m}\|_2^2 \\ + \frac{\alpha^3}{2}\kappa\|d_t u_n^{\alpha,m}\|_2^2 + \frac{\alpha^5}{2}\kappa\|\nabla d_t u_n^{\alpha,m}\|_2^2.\end{aligned} \tag{4.6}$$

Then, by using again (4.1) we have that $\alpha^{\frac32}\|\nabla u_n^{\alpha,m}\|_2^{\frac32} \le \left(\|u_0\|_2^2 + \alpha^2\|\nabla u_0\|_2^2\right)^{\frac34}$, and then inequality (4.5) becomes

$$\begin{aligned}\alpha^3\|\nabla u_n^{\alpha,m}\|_2^2 - \alpha^3\|\nabla u_n^{\alpha,m-1}\|_2^2 + \alpha^3\|\nabla u_n^{\alpha,m} - \nabla u_n^{\alpha,m-1}\|_2^2 \\ + \alpha^3\kappa\|d_t u_n^{\alpha,m}\|_2^2 + \alpha^5\kappa\|_2^2 d_t\nabla u_n^{\alpha,m}\|_2^2 \le c\kappa\|\nabla u_n^{\alpha,m}\|_2^2,\end{aligned}$$

where c is a positive constant depending only on the initial datum u_0. By summing up over $m = 1, ..., M$ we get (4.3).

To prove (4.4) we multiply by $-\Delta u_n^{\alpha,m}$ the equations (3.1) and after integration by parts in space we get

$$(d_t\nabla u_n^{\alpha,m}, \nabla u_n^{\alpha,m}) + \alpha^2(d_t\Delta u_n^{\alpha,m}, \Delta u_n^{\alpha,m}) + \|\Delta u_n^{\alpha,m}\|_2^2 - (P_n((u_n^{\alpha,m}\cdot\nabla)\, u_n^{\alpha,m})\cdot\Delta u_n^{\alpha,m}) = 0.$$

By using (4.2), the fact that $\Delta u_n^{\alpha,m} \in V_n$, and Hölder inequality we get

$$\begin{aligned}\|\nabla u_n^{\alpha,m}\|_2^2 + \alpha^2\|\Delta u_n^{\alpha,m}\|_2^2 - \|\nabla u_n^{\alpha,m-1}\|_2^2 - \alpha^2\|\Delta u_n^{\alpha,m-1}\|_2^2 \\ + \|\nabla u_n^{\alpha,m} - \nabla u_n^{\alpha,m-1}\|_2^2 + \alpha^2\|\Delta u_n^{\alpha,m} - \Delta u_n^{\alpha,m-1}\|_2^2 \\ + 2\kappa\|\Delta u^n\|_2^2 \le 2\kappa\|u_n^{\alpha,m}\|_4\|\nabla u_n^{\alpha,m}\|_4\|\Delta u_n^{\alpha,m}\|_2.\end{aligned} \tag{4.7}$$

Then, by Gagliardo Nirenberg inequality and Young inequality we have that

$$\begin{aligned}\kappa\|u_n^{\alpha,m}\|_4\|\nabla u_n^{\alpha,m}\|_4\|\Delta u_n^{\alpha,m}\|_2 &\leq \kappa\|u_n^{\alpha,m}\|_2^{\frac{1}{4}}\|\nabla u_n^{\alpha,m}\|_2\|\Delta u_n^{\alpha,m}\|_2^{\frac{7}{4}} \\ &\leq c\kappa\|u_n^{\alpha,m}\|_2^2\|\nabla u_n^{\alpha,m}\|_2^8+\frac{\kappa\|\Delta u_n^{\alpha,m}\|_2^2}{2}.\end{aligned} \tag{4.8}$$

Then, by inserting (4.8) in (4.7) and using (4.1) we get

$$\begin{aligned}\|\nabla u_n^{\alpha,m}\|_2^2+\alpha^2\|\Delta u_n^{\alpha,m}\|_2^2-\|\nabla u_n^{\alpha,m-1}\|_2^2-\alpha^2\|\Delta u_n^{\alpha,m-1}\|_2^2 \\ +\|\nabla u_n^{\alpha,m}-\nabla u_n^{\alpha,m-1}\|_2^2+\alpha^2\|\Delta u_n^{\alpha,m}-\Delta u_n^{\alpha,m-1}\|_2^2 \\ +\kappa\|\Delta u^n\|_2^2\leq c\kappa\|u_n^{\alpha,m}\|_2^2\|\nabla u_n^{\alpha,m}\|_2^8.\end{aligned} \tag{4.9}$$

By multiplying the previous inequality on both side by α^6, using again (4.1), and summing up over $m=1,...,M$ we get (4.4) with a constant c independent of α, n and of M, thus ending the proof. ☐

Finally, we prove an *a priori* estimate on the approximate pressure, which is, as usual, a crucial step when considering the local energy inequality.

Lemma 4.3. *Let $u_0\in H^2_{0,\sigma}$. Let $M\in\mathbb{N}$ and $m=1,...,M$. Let $u_n^{\alpha,m}$ be a solution of* (3.5). *Then, there exists $c>0$, independent of $\alpha>0$, of $n\in\mathbb{N}$, and of $M\in\mathbb{N}$ such that*

$$\kappa\sum_{m=1}^{M}\|p_n^{\alpha,m}\|_{\frac{5}{3}}^{\frac{5}{3}}\leq c. \tag{4.10}$$

Proof. The proof is rather standard. We recall that by Gagliardo-Nirenberg inequality we have

$$\|u_n^{\alpha,m}\|_{\frac{10}{3}}\leq\|u_n^{\alpha,m}\|_2^{\frac{2}{5}}\|\nabla u_n^{\alpha,m}\|_2^{\frac{3}{5}}. \tag{4.11}$$

By using the L^q-elliptic estimates applied to (3.5) we have that

$$\|p_n^{\alpha,m}\|_{\frac{5}{3}}\leq c\|u_n^{\alpha,m}\|_{\frac{10}{3}}^2.$$

Then, by using (4.11) we obtain

$$\|p_n^{\alpha,m}\|_{\frac{5}{3}}^{\frac{5}{3}}\leq\|u_n^{\alpha,m}\|_2^{\frac{4}{3}}\|\nabla u_n^{\alpha,m}\|_2^2\leq c\|\nabla u_n^{\alpha,m}\|_2^2, \tag{4.12}$$

where we used (4.1). By multiplying both sides of (4.12) by κ, by summing up over $m=1,...,M$, and by using again the equality (4.1) we get (4.10). ☐

At this point we re-state the *a priori* estimates proved in Lemmas 4.1-4.3 in terms of the (time-dependent) functions defined in (3.7).

Proposition 4.4. *Let $u_0\in H^2_{0,\sigma}$. There exists $c>0$, independent of $\alpha>0$, of $M\in\mathbb{N}$ and of $n\in\mathbb{N}$, such that*

$$\|v_n^{\alpha,M}\|_{L^\infty(L^2)\cap L^2(H^1)}\leq c, \tag{4.13}$$

$$\|\partial_t v_n^{\alpha,M}\|_{L^{4/3}(H^{-2})}\leq c, \tag{4.14}$$

$$\|u_n^{\alpha,M}\|_{L^\infty(L^2)\cap L^2(H^1)}\leq c, \tag{4.15}$$

$$\|p_n^{\alpha,M}\|_{L^{5/3}(L^{5/3})}\leq c, \tag{4.16}$$

$$\alpha\|\nabla v_n^{\alpha,M}\|_{L^2(L^2)}\leq c, \tag{4.17}$$

$$\alpha^{\frac{3}{2}}\|\partial_t v_n^{\alpha,M}\|_{L^2(L^2)}\leq c, \tag{4.18}$$

$$\alpha^3\|\Delta u_n^{\alpha,M}\|_{L^2(L^2)}\leq c. \tag{4.19}$$

Moreover, we also have the following identities

$$\|v_n^{\alpha,M} - u_n^{\alpha,M}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))} = \frac{\kappa}{3}\sum_{m=1}^{M} \|u_n^{\alpha,m} - u_n^{\alpha,m-1}\|_2^2, \tag{4.20}$$

$$\|\nabla u_n^{\alpha,M} - \nabla v_n^{\alpha,M}\|^2_{L^2(0,T;L^2(\mathbb{T}^3))} = \frac{\kappa}{3}\sum_{m=1}^{M} \|\nabla u_n^{\alpha,m} - \nabla u_n^{\alpha,m-1}\|_2^2. \tag{4.21}$$

Proof. The bound (4.13) follows from Lemma 4.1 and the definition (3.7). We remark that in order to get the bound in $L^2(0,T;H^1_{0,\sigma})$ we need $u_0 \in H^1_{0,\sigma}$. The bounds (4.15), (4.16), and (4.17) follow from the definitions in (3.7) and Lemma 4.1. Finally, the bound (4.14) follows by a simple comparison argument on (3.8). The bounds (4.18) and (4.19) follows by Lemma 4.2 and (3.7); the identities (4.20) and (4.21) follow by a direct calculation. □

5. Proof of the main Theorem

In this section we give the proof of Theorem 1.1. We divide the proof in two main steps: a) the convergence to a Leray-Hopf weak solution and b) the convergence to a suitable weak solution. Let $\{M_n\}_n \subset \mathbb{N}$ and $\{\alpha_n\}_n \subset (0,1)$ be two sequences as in the statement of Theorem 1.1. We recall that $\{\alpha_n\}_n$ is chosen such that

$$\lim_{n\to+\infty} n\alpha_n^3 = 0. \tag{5.1}$$

Step 1: Convergence to a Leray-Hopf weak solution

Let $\varphi \in C_c^\infty([0,T);C^\infty(\mathbb{T}^3))$ with $\nabla\cdot\varphi = 0$ and zero mean value. It is easy to show that there exists a sequence $\{\varphi_n\}_n \subset C^1([0,T);V_n)$ such that

$$\sup_{t\in(0,T)} \|\varphi_n - \varphi\|_{H^1} + \|\partial_t(\varphi_n - \varphi)\|_{H^1} \to 0, \text{ as } n\to\infty. \tag{5.2}$$

In order to simplify the exposition we use the following abbreviations:

$$v^n := v_n^{\alpha_n,M_n}, \qquad u^n := u_n^{\alpha_n,M_n}, \quad \text{and} \quad p^n := p_n^{\alpha_n,M_n}.$$

Then, (3.8) reads as follows

$$\partial_t(v^n - \alpha^2\Delta v^n) - \Delta u^n + (u^n\cdot\nabla)\,u^n - Q_n((u^n\cdot\nabla)\,u^n) + \nabla p^n = 0. \tag{5.3}$$

We recall from (4.13) and (4.14) that (with bounds independent of n)

$$v^n \in L^\infty(0,T;L^2_{0,\sigma}) \cap L^2(0,T;H^1_{0,\sigma}),$$
$$\partial_t v^n \subset L^{\frac{4}{3}}(0,T;H^{-2}).$$

Then, there exists $v \in L^\infty(0,T;L^2_{0,\sigma}) \cap L^2(0,T;H^1_{0,\sigma})$ such that, up to a subsequence not relabelled,

$$v^n \to v \text{ strongly in } L^2(0,T;L^2_{0,\sigma}), \text{ as } n\to\infty.$$

Next, from (4.15) there exists $u \in L^\infty(0,T;L^2_{0,\sigma}) \cap L^2(0,T;H^1_{0,\sigma})$ such that, up to a subsequence not relabelled,

$$u^n \rightharpoonup u \text{ weakly in } L^2(0,T;H^1_{0,\sigma}), \text{ as } n\to\infty. \tag{5.4}$$

Finally, by using (4.20) we have

$$\int_0^T \|u^n - v^n\|_2^2\,dt = \frac{T}{3M_n}\sum_{m=1}^{M_n} \|u_n^{\alpha,m} - u_n^{\alpha,m-1}\|_2^2 \le \frac{T}{3M_n}(\|u_0\|_2^2 + \alpha_n^2\|\nabla u_0\|_2^2),$$

where we used Lemma 4.1. We have then that

$$u^n - v^n \to 0 \text{ strongly in } L^2(0,T;L^2_{0,\sigma}), \text{ as } n\to\infty. \tag{5.5}$$

Hence, it follows that $u = v$ and also that

$$\begin{aligned} &u^n \to u \text{ strongly in } L^2(0,T;L^2_{0,\sigma}), \text{ as } n \to \infty,\\ &v^n \to u \text{ strongly in } L^2(0,T;L^2_{0,\sigma}), \text{ as } n \to \infty. \end{aligned} \tag{5.6}$$

Let φ_n satisfy (5.2). By multiplying (5.3) by φ_n and by integrating by parts with respect to space and time we get

$$\int_0^T (v^n, \partial_t \varphi_n) - \alpha_n^2 (\nabla v^n, \partial_t \nabla \varphi_n) + (u^n \otimes u^n, \nabla \varphi_n) - (\nabla u^n, \nabla \varphi_n)\, dt = (P_n u_0, \varphi_0(0)).$$

By using (4.17), we have then

$$\alpha_n^2 \int_0^T \|\nabla v^n\|_2^2 \, dt \leq c.$$

This implies, in particular, that

$$\alpha_n^2 \int_0^T (\Delta \partial_t v^n, \varphi_n)\, dt \to 0, \text{ as } n \to \infty.$$

Then, by using (5.2), (5.4), and (5.6) it is now straightforward to prove the convergence to a Leray-Hopf weak solution. We omit further details.

Step 2: Convergence to a Suitable Weak Solution

We prove now the most original part of this work, namely that the limit of the approximate solutions satisfy the local energy inequality. By using (4.16) we can infer there exists $p \in L^{\frac{5}{3}}((0,T) \times \mathbb{T}^3)$ such that (again up to a subsequence)

$$p^n \rightharpoonup p \text{ weakly in } L^{\frac{5}{3}}((0,T) \times \mathbb{T}^3), \text{ as } n \to \infty. \tag{5.7}$$

In order to prove that (u,p) is a suitable weak solution we only need to prove that (u,p) satisfies the local energy inequality (2.1). To this end we consider the equations (3.9) that we rewrite for the reader's convenience

$$\partial_t v^n - \alpha^2 \partial_t \Delta v^n - \Delta u^n + (u^n \cdot \nabla)\, u^n - Q_n((u^n \cdot \nabla)\, u^n) + \nabla p^n = 0. \tag{5.8}$$

By testing (5.8) by $u^n \phi$ with $\phi \in C_c^\infty((0,T); C^\infty(\mathbb{T}^3))$, $\phi \geq 0$, and after several integration by parts we get

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^3} |\nabla u^n|^2 \phi \, dxdt = &- \int_0^T (\partial_t v^n, u^n \phi)\, dt + \alpha_n^2 \int_0^T (\partial_t \Delta v^n, u^n \phi)\, dt \\ &+ \int_0^T \left(\frac{|u^n|^2}{2}, \Delta \phi \right) dt + \int_0^T \int_{\mathbb{T}^3} \left(\frac{|u^n|^2}{2} + p \right) u^n \cdot \nabla \phi \, dxdt \\ &+ \int_0^T (Q_n (u^n \cdot \nabla)\, u^n), u^n \phi)\, dt =: \sum_{i=1}^5 I_i^n. \end{aligned} \tag{5.9}$$

We treat all the terms on the right-hand side of (5.9) separately. We start by I_1^n.

$$\begin{aligned} I_1^n &= - \int_0^T (\partial_t v^n, u^n \phi)\, dt = - \int_0^T (\partial_t v^n, v^n \phi) + \int_0^T (\partial_t v^n, (v^n - u^n)\phi)\, dt \\ &= \int_0^T \left(\frac{|v^n|^2}{2}, \partial_t \phi \right) + \sum_{m=1}^{M_n} \int_{t_{m-1}}^{t_m} (\partial_t v^n, (v^n - u^n)\phi)\, dt. \end{aligned}$$

By using that u^n is constant over $[t_{m-1}, t_m)$, we infer that

$$\sum_{m=1}^{M_n}\int_{t_{m-1}}^{t_m}(\partial_t v^n,(v^n-u^n)\phi)\,dt=\sum_{m=1}^{M_n}\int_{t_{m-1}}^{t_m}(\partial_t(v^n-u^n),(v^n-u^n)\phi)\,dt$$
$$=-\sum_{m=1}^{M_n}\int_{t_{m-1}}^{t_m}\left(\frac{|v^n-u^n|^2}{2},\partial_t\phi\right)\,dt,$$

and we point out that there are no boundary terms arising in integration by parts due to the fact that $v^n(t_m) = u^n(t_m)$ for any $m = 1, ..., M_n$ and ϕ is compactly supported in time. Then,

$$I_1^n=\int_0^T\left(\frac{|v^n|^2}{2}-\frac{|v^n-u^n|^2}{2},\partial_t\phi\right)\,dt,$$

and by using (5.6) and (5.5) it follows

$$I_1^n\to\int_0^T\left(\frac{|u|^2}{2},\partial_t\phi\right)\,dt,\ \text{ as } n\to\infty. \tag{5.10}$$

Let us consider now the term I_2^n. We have

$$I_2^n=\alpha_2^2\int_0^T(\partial_t\Delta v^n,u^n\phi)\,dt=\alpha_n^2\int_0^T(\partial_t\Delta v^n,(u^n-v^n)\phi)\,dt+\alpha_n^2\int_0^T(\partial_t\Delta v^n,v^n\phi)\,dt$$
$$=:I_{2,1}^n+I_{2,2}^n.$$

We estimate the term $I_{2,1}^n$ in a way similar to the term $I_{1,2}^n$. Indeed, by using that u^n is constant over the interval $[t_{m-1}, t_m)$ we get

$$\begin{aligned}I_{2,1}^n&=-\alpha_n^2\sum_{m=1}^{M_n}\int_{t_{m-1}}^{t_m}(\partial_t\nabla v^n,\nabla(v^n-u^n)\phi)\,dt\\&=\alpha_n^2\sum_{m=1}^{M_n}\int_{t_{m-1}}^{t_m}(\partial_t\nabla(v^n-u^n),\nabla(v^n-u^n)\phi)\,dt\\&=-\alpha_n^2\sum_{m=1}^{M_n}\int_{t_{m-1}}^{t_m}\left(\frac{|\nabla(v^n-u^n)|^2}{2},\partial_t\phi\right)\,dt\\&=-\alpha_n^2\int_0^T\left(\frac{|\nabla(v^n-u^n)|^2}{2},\partial_t\phi\right)\,dt,\end{aligned}$$

where we used that $\nabla v^n(t_m) = \nabla u^n(t_m)$ for any $m = 1, ..., M_n$ and again that ϕ is compactly supported in time. By using (4.21) we have (for a constant c depending only on ϕ)

$$\begin{aligned}|I_{2,1}^n|&\le c\,\alpha_n^2\int_0^T\|\nabla v^n-\nabla u^n\|_2^2\\&=\frac{cT}{3M_n}\alpha_n^2\sum_{m=1}^{M_n}\|\nabla u_n^{\alpha,m}-\nabla u_n^{\alpha,m-1}\|_2^2\\&\le\frac{cT}{3M_n}(\|u_0\|_2^2+\alpha_n^2\|\nabla u_0\|_2^2)\to 0,\ \text{ as } n\to\infty.\end{aligned}$$

Now we consider the term $I_{2,2}^n$. By standard manipulations involving integrations by parts we get that

$$
\begin{aligned}
I_{2,2}^n &= \alpha_n^2 \int_0^T \int_{\mathbb{T}^3} \Delta \partial_t v^n v^n \phi \, dxdt \\
&= \alpha_n^2 \int_0^T \int_{\mathbb{T}^3} \left[\frac{|\nabla v^n|^2}{2} \partial_t \phi + \nabla v^n \nabla \phi \, \partial_t v^n - \frac{|v^n|^2}{2} \Delta \, \partial_t \phi \right] dxdt \\
&\leq \frac{\alpha_n^2}{2} \int_0^T \int_{\mathbb{T}^3} |\nabla v^n|^2 |\partial_t \, \phi| \, dxdt + \frac{\alpha_n^2}{2} \int_0^T \int_{\mathbb{T}^3} |v^n|^2 |\Delta \, \partial_t \phi| \, dxdt \\
&\quad + \alpha_n^2 \int_0^T \int_{\mathbb{T}^3} |\partial_t v^n| \, |\nabla v^n| \, |\nabla \phi| \, dxdt \\
&\leq c\alpha_n^2 + c\alpha_n^2 \int_0^T \|\partial_t v^n\|_2 \|\nabla v^n\|_2 \, dt,
\end{aligned}
$$

where we used (4.13), Hölder inequality, and the fact that $\phi \in C_c^\infty((0,T) \times \mathbb{T}^3)$. Then,

$$
\begin{aligned}
|I_{2,2}^n| &\leq c\alpha_n^2 + c\alpha_n^2 \int_0^T \|\partial_t v^n\|_2 \|\nabla v^n\|_2 \, dt \\
&\leq c\alpha_n^2 + c\alpha_n^{\frac{1}{2}} \left(\int_0^T \alpha_n^3 \|\partial_t v^n\|_2^2 \right)^{\frac{1}{2}} \left(\int_0^T \|\nabla v^n\|_2^2 \right)^{\frac{1}{2}} \\
&\leq c\,(\alpha_n^2 + \alpha_n^{\frac{1}{2}}) \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

where we used Hölder inequality in time and (4.18). In particular, we have just proved that

$$
|I_2^n| \leq |I_{2,1}^n| + |I_{2,2}^n| \to 0, \text{ as } n \to \infty. \tag{5.11}
$$

Concerning the term I_3^n and I_4^n we recall that from (5.4) and (5.6)

$$
u^n \to u \text{ strongly in } L^3(0,T;L^3(\mathbb{T}^3)), \text{ as } n \to \infty. \tag{5.12}
$$

Then, (5.12) and (5.7) are enough to prove that

$$
I_3^n = \int_0^T \left(\frac{|u^n|^2}{2}, \Delta \phi \right) dt \;\to\; \int_0^T \left(\frac{|u|^2}{2}, \Delta \phi \right) dt, \text{ as } n \to \infty, \tag{5.13}
$$

$$
I_4^n = \int_0^T \left(\left(\frac{|u^n|^2}{2} + p^n \right) u^n, \nabla \phi \right) dt \;\to\; \int_0^T \left(\left(\frac{|u|^2}{2} + p \right) u, \nabla \phi \right) dt, \text{ as } n \to \infty. \tag{5.14}
$$

We are left with the term I_5^n. We have

$$
\begin{aligned}
I_5^n &= \int_0^T (Q_n((u^n \cdot \nabla)\, u^n, u^n \phi) \, dt = \int_0^T ((u^n \cdot \nabla)\, u^n, Q_n(u^n \phi)) \, dt \\
&\leq \int_0^T \|u^n(t)\|_2 \|\nabla u^n(t)\|_2 \|Q_n(u^n(t)\phi(t))\|_\infty \, dt \\
&\leq c \left(\int_0^T \|Q_n(u^n(t)\phi(t))\|_\infty^2 \, dt \right)^{\frac{1}{2}},
\end{aligned}
$$

where in the last line we used Hölder inequality and (4.15). Then, from (3.4) and (3.7) we have that u^n has the following representation in Fourier series expansion

$$
u^n(t,x) = \sum_{0<|k|\leq n} \sum_{m=1}^{M_n} \chi_{[t_{m-1},t_m)}(t) \hat{u}_{n,k}^{\alpha_n,m} \, \mathrm{e}^{ik\cdot x}.
$$

By defining

$$\widehat{U}_k^n(t) := \sum_{m=1}^{M_n} \chi_{[t_{m-1},t_m)}(t)\hat{u}_{n,k}^{\alpha_n,m},$$

we can write

$$u^n(t,x) = \sum_{0<|k|\leq n} \widehat{U}_k^n(t)\, \mathrm{e}^{ik\cdot x}.$$

Then, by using Lemma 3.1 we have that

$$\int_0^T \|Q_n(u^n(t)\phi(t))\|_\infty^2\, dt \leq \frac{c}{n}\sum_{k\in\mathbb{Z}^3}|\widehat{U}_n^k(t)|^2\, dt + c\int_0^T n^2 \sum_{|k|\geq\frac{n}{2}}|\widehat{U}_n^k(t)|^2 =: I_{5,1}^n + I_{5,2}^n.$$

Regarding the term $I_{5,1}^n$ it follows by (4.15) that

$$|I_{5,1}^n| \leq \frac{c}{n} \to 0, \text{ as } n\to\infty.$$

For the term $I_{5,2}^n$ we have

$$\begin{aligned}
\int_0^T n^2 \sum_{|k|\geq\frac{n}{2}}|\widehat{U}_k^n(t)|^2 &= \frac{n^2\alpha_n^6}{n^2\alpha_n^6}\int_0^T \sum_{|k|\geq\frac{n}{2}} n^2|\widehat{U}_k^n(t)|^2\, dt \\
&\leq 4\frac{\alpha_n^6}{n^2\alpha_n^6}\int_0^T \sum_{|k|\geq\frac{n}{2}} |k|^4|\widehat{U}_k^n(t)|^2\, dt \\
&\leq \frac{4}{n^2\alpha_n^6}\ \alpha_n^6\int_0^T \sum_{k\in\mathbb{Z}^3\setminus\{0\}} |k|^4|\widehat{U}_k^n(t)|^2\, dt \\
&\leq \frac{c}{n^2\alpha_n^6}\ \alpha_n^6\int_0^T \|\Delta u^n\|_2^2\, dt \leq \frac{c}{n^2\alpha_n^6},
\end{aligned}$$

where in the last inequality we have used (4.19). Then, by (5.1) we get that $|I_{5,2}^n| \to 0$ as $n\to\infty$ and then

$$|I_5^n| \to 0, \text{ as } n\to\infty. \tag{5.15}$$

Finally, by using (5.4) we have that

$$\int_0^T\int_{\mathbb{T}^3}|\nabla u|^2\phi\, dxdt \leq \liminf_{n\to\infty}\int_0^T\int_{\mathbb{T}^3}|\nabla u^n|^2\phi\, dxdt. \tag{5.16}$$

By inserting (5.16), (5.10), (5.11), (5.13), (5.14), and (5.15) in (5.9) we have finally proved the local energy inequality (2.1).

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