CONSTRUCTION OF SOLUTIONS FOR A NONLINEAR ELLIPTIC PROBLEM ON RIEMANNIAN MANIFOLDS WITH BOUNDARY

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Abstract. Let (M,g) be a smooth compact, n dimensional Riemannian manifold, $n \geq 2$ with smooth $n-1$ dimensional boundary ∂M. We prove that the stable critical points of the mean curvature of the boundary generates $H^1(M)$ solutions for the singularly perturbed elliptic problem with Neumann boundary conditions

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = u^{p-1} & \text{in } M \\ u > 0 & \text{in } M \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M \end{cases}$$

when ε is small enough. Here p is subcritical.

1. Introduction

Let (M,g) be a smooth compact, n dimensional Riemannian manifold, $n \geq 2$ with boundary ∂M which is the union of a finite number of connected, smooth, boundaryless, $n-1$ submanifolds embedded in M. Here g denotes the Riemannian metric tensor. By Nash theorem [17] we can consider (M,g) as a regular submanifold embedded in $\mathbb{R}^N$.

We consider the following Neumann problem

$$\begin{cases} -\varepsilon^2 \Delta_g u + u = u^{p-1} & \text{in } M \\ u > 0 & \text{in } M \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial M \end{cases} \tag{1}$$

where $p > 2$ if $n = 2$ and $2 < p < 2^* = \frac{2n}{n-2}$ if $n \geq 3$, ν is the external normal to ∂M and ε is a positive parameter.

We are interested in finding solutions $u \in H^1(M)$ to problem (1), where

$$H^1(M) = \left\{ u : M \to \mathbb{R} : \int_M |\nabla u|_g^2 + u^2 d\mu_g < \infty \right\}$$

and μ_g denotes the volume form on M associated to g. More precisely, we want to show that, for ε sufficiently small, we can construct a solution which has a peak near a stable critical point of the scalar curvature of the boundary, as stated in the following.

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Definition 1. Let $f \in C^1(N,\mathbb{R})$, where (N,g) is a Riemannian manifold. We say that $K \subset N$ is a C^1-stable critical set of f if $K \subset \{x \in N : \nabla f(x) = 0\}$ and for any $\mu > 0$ there exists $\delta > 0$ such that, if $h \in C^1(N,\mathbb{R})$ with

$$\max_{d_g(x,K)\leq\mu} |f(x) - h(x)| + |\nabla f(x) - \nabla h(x)| \leq \delta,$$

then h has a critical point x_0 with $d_g(x_0, K) \leq \mu$. Here d_g denotes the geodesic distance associated to the Riemannian metric g.

Now we can state the main theorem.

Theorem 2. *Assume $K \subset \partial M$ is a C^1-stable critical set of the mean curvature of the boundary. Then there exists $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, Problem (1) has a solution $u_\varepsilon \in H^1(M)$ which concentrates at a point $\xi_0 \in K$ as ε goes to zero.*

Problem (1) in a flat domain has a long history. Starting from a problem of pattern formation in biology, Lin, Ni and Takagi [14, 18] showed the existence of a mountain pass solution for Problem (1) and proved that this solution has exactly one maximum point which lies on the boundary of the domain. Moreover in [19] the authors proved that the maximum point of the solution approaches the maximum point for the mean curvature of the boundary when the perturbation parameter ε goes to zero.

Thenceforth, many papers were devoted to the study of Problem (1) on flat domains. In particular, in [3, 20] it is proved that any stable critical point of the mean curvature of the boundary generated a single peaked solution whose peak approaches the critical point as ε vanishes. Moreover in [10, 13, 22, 11] the existence of multipeak solutions whose peaks lies on the boundary is studied. We also mention a series of works in which the authors proved the existence of solutions which have internal peaks [21, 12, 8, 9].

In the case of a manifold M, Problem (1) has been firstly studied in [2] where the authors prove the existence of a mountain pass solution when the manifold M is closed and when the manifold M has a boundary. They show that for ε small such a solution has a spike which approaches -as ε goes to zero- a maximum point of the scalar curvature when M is closed and a maximum point of the mean curvature of the boundary when M has a boundary.

In the case of M closed manifold, in [1] the authors show that Problem (1) has at least $\text{cat}M + 1$ non trivial positive solutions when ε goes to zero. Here $\text{cat}M$ denotes the Lusternik-Schnirelmann category of M. Moreover in [15] the effect of the geometry of the manifold (M, g) is examined. In fact it is shown that positive solution of the problem are generated by stable critical points of the scalar curvature of M.

More recently we proved in [6] in the case of a manifold M with boundary ∂M that Problem (1) has at least $\text{cat}\partial M$ non trivial positive solutions when ε goes to zero. We can compare the result of [6] with Theorem 2. In fact, in [7] the authors prove that generically with respect to the metric g, the mean curvature of the boundary has nondegenerate critical points. More precisely, the set of metrics for which the mean curvature has only nondegenerate critical points is an open dense set among all the C^k metrics on M, $k \geq 3$. Thus, generically with respect to the metric, the mean curvature has $P_1(\partial M)$ nondegenerate (hence stable) critical points, where $P_1(\partial M)$ is the Poincaré polynomial of ∂M, namely $P_t(\partial M)$, evaluated in $t = 1$. So, generically with respect to metric, Problem (1) has $P_1(\partial M)$ solution, and it holds $P_1(\partial M) \geq \text{cat}\partial M$, and in many cases the strict inequality holds.

The paper is organized as follows. In Section 2 some preliminary notions are introduced, which are necessary to the comprehension of the paper. In Section 3 we study the variational structure of the problem and we perform the finite dimensional reduction. In Section 4 the proof of Theorem 2 is sketched while the expansion of the reduced functional is postponed in Section 5. The Appendix collects some technical lemmas.

2. Preliminary results

In this section we give some general facts preliminary to our work. These results are widely present in literature, anyway, we refer mainly to [2, 4, 5, 15] and the reference therein.

First of all we need to define a suitable coordinate chart on the boundary.

We know that on the tangent bundle of any compact Riemannian manifold $\mathcal{M}$ it is defined the exponential map $\exp : T\mathcal{M} \to \mathcal{M}$ which is of class C^∞. Moreover there exists a constant $R_M > 0$, called radius of injectivity, and a finite number of $x_i \in \mathcal{M}$ such that $\mathcal{M} = \cup_{i=1}^{l} B_g(x_i, R_M)$ and $\exp_{x_i} : B(0, R_M) \to B_g(x_i, R_M)$ is a diffeormophism for all i. By choosing an orthogonal coordinate system $(y_1, \dots, y_n)$ of $\mathbb{R}^n$ and identifying $T_{x_0}\mathcal{M}$ with $\mathbb{R}^n$ for $x_0 \in \mathcal{M}$ we can define by the exponential map the so called normal coordinates. For $x_0 \in \mathcal{M}$, g_{x_0} denotes the metric read through the normal coordinates. In particular, we have $g_{x_0}(0) = \mathrm{Id}$. We set $|g_{x_0}(y)| = \det\left(g_{x_0}(y)\right)_{ij}$ and $g_{x_0}^{ij}(y) = \left(\left(g_{x_0}(y)\right)_{ij}\right)^{-1}$.

Definition 3. If q belongs to the boundary ∂M, let $\bar{y} = (y_1, \dots, y_{n-1})$ be Riemannian normal coordinates on the $n-1$ manifold ∂M at the point q. For a point $\xi \in M$ close to q, there exists a unique $\bar{\xi} \in \partial M$ such that $d_g(\xi, \partial M) = d_g(\xi, \bar{\xi})$. We set $\bar{y}(\xi) \in \mathbb{R}^{n-1}$ the normal coordinates for $\bar{\xi}$ and $y_n(\xi) = d_g(\xi, \partial M)$. Then we define a chart $\psi_q^\partial : \mathbb{R}^n_+ \to M$ such that $(\bar{y}(\xi), y_n(\xi)) = \left(\psi_q^\partial\right)^{-1}(\xi)$. These coordinates are called *Fermi coordinates* at $q \in \partial M$. The Riemannian metric $g_q\left(\bar{y}, y_n\right)$ read through the Fermi coordinates satisfies $g_q(0) = \mathrm{Id}$.

We note by d_g^∂ and $\exp^\partial$ respectively the geodesic distance and the exponential map on by ∂M. By compactness of ∂M, there is an R^∂ and a finite number of points $q_i \in \partial M$, $i = 1, \dots, k$ such that

$$I_{q_i}(R^\partial, R_M) := \left\{x \in M,\, d_g(x, \partial M) = d_g(x, \bar{\xi}) < R_M,\, d_g^\partial(q_i, \bar{\xi}) < R^\partial\right\}$$

form a covering of $(\partial M)_{R^M} := \{x \in M,\, d_g(x, \partial M) < R^M\}$ and on every I_{q_i} the Fermi coordinates are well defined. In the following we choose, $R = \min\left\{R^\partial, R_M\right\}$, such that we have a finite covering

$$M \subset \left\{\cup_{i=1}^{k} B(q_i, R)\right\} \bigcup \left\{\cup_{i=k+1}^{l} I_{\xi_i}(R, R)\right\}$$

where $k, l \in \mathbb{N}$, $q_i \in M \setminus \partial M$ and $\xi_i \in \partial M$.

For $p \in \partial M$, consider $\pi_p : T_pM \to T_p\partial M$ the projection on the tangent space $T_p\partial M$. For a pair of tangent vectors $X, Y \in T_p\partial M$ we define the second fundamental form $II_p(X, Y) := \nabla_X Y - \pi_p(\nabla_X Y)$. The mean curvature at the boundary H_p, where $p \in \partial M$ is the trace of the second fundamental form.

If we consider Fermi coordinates in a neighborhood of p, and we note by the matrix $(h_{ij})_{i,j=1,\dots,n-1}$ the second fundamental form, we have the well known formulas

$$g^{ij}(y) = \delta_{ij} + 2h_{ij}(0)y_n + O(|y|^2) \text{ for } i,j = 1,\dots n-1 \tag{2}$$

$$g^{in}(y) = \delta_{in} \tag{3}$$

$$\sqrt{g}(y) = 1 - (n-1)H(0)y_n + O(|y|^2) \tag{4}$$

where $(y_1,\dots,y_n)$ are the Fermi coordinates and, by definition of h_{ij},

$$H = \frac{1}{n-1}\sum_{i}^{n-1} h_{ii}. \tag{5}$$

Also, by Escobar [4, eq. (3.2)], we have that

$$\left.\frac{\partial^2}{\partial y_n \partial y_i}\sqrt{g}(y)\right|_{y=0} = -(n-1)\frac{\partial H}{\partial y_i}(0) \text{ for } i = 1,\dots,n-1 \tag{6}$$

It is well known that, in $\mathbb{R}^n$, there is a unique positive radially symmetric function $V(y) \in H^1(\mathbb{R}^n)$ satisfying

$$-\Delta V + V = V^{p-1} \text{ on } \mathbb{R}^n. \tag{7}$$

Moreover, the function V exponentially decays at infinity as well as its derivative, that is, for some $c > 0$

$$\lim_{|y|\to\infty} V(|y|)|y|^{\frac{n-1}{2}}e^{|y|} = c \qquad \lim_{|y|\to\infty} V'(|y|)|y|^{\frac{n-1}{2}}e^{|y|} = -c.$$

We can define on the half space $\mathbb{R}^n_+ = \{(y_1,\dots,y_n) \in \mathbb{R}^n \ , \ y_n \geq 0\}$ the function

$$U(y) = \left. V\right|_{y_n \geq 0}.$$

The function U satisfies the following Neumann problem in $\mathbb{R}^n_+$

$$\begin{cases} -\Delta U + U = U^{p-1} & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial U}{\partial y_n} = 0 & \text{on } \{y_n = 0\}. \end{cases} \tag{8}$$

We set $U_\varepsilon(y) = U\left(\frac{y}{\varepsilon}\right)$.

Lemma 4. *The space solution of the linearized problem*

$$\begin{cases} -\Delta \varphi + \varphi = (p-1)U^{p-2}\varphi & \text{in } \mathbb{R}^n_+ \\ \dfrac{\partial \varphi}{\partial y_n} = 0 & \text{on } \{y_n = 0\}. \end{cases} \tag{9}$$

is generated by the linear combination of

$$\varphi^i = \frac{\partial U}{\partial y_i}(y) \text{ for } i = 1,\dots,n-1.$$

Proof. It is trivial that every linear combination of φ^i is a solution of (9). We notice that $\dfrac{\partial U}{\partial y_n}$ is not a solution of (9) because the derivative on $\{y_n = 0\}$ is not zero.

For the converse, suppose $\bar\varphi(y)$ be a solution of (9). Then, by even reflection around y_n, we can construct a solution $\tilde\varphi$ of

$$-\Delta\tilde\varphi + \tilde\varphi = (p-1)U^{p-2}\tilde\varphi \text{ in } \mathbb{R}^n \tag{10}$$

with $\dfrac{\partial\tilde{\varphi}}{\partial y_n} = 0$ on $y_n = 0$. But all solution of (10) with zero derivative on $y_n = 0$ are linear combination of $\dfrac{\partial V}{\partial y_j}$ with $j = 1, \cdots, n-1$. $\square$

We endow $H^1(M)$ with the scalar product $\langle u, v\rangle_\varepsilon := \dfrac{1}{\varepsilon^n}\displaystyle\int_M \varepsilon^2 g(\nabla u, \nabla v) + uv d\mu_g$ and the norm $\|u\|_\varepsilon = \langle u, u\rangle_\varepsilon^{1/2}$. We call H_ε the space H^1 equipped with the norm $\|\cdot\|_\varepsilon$. We also define L^p_ε as the space $L^p(M)$ endowed with the norm $|u|_{\varepsilon,p} = \left(\dfrac{1}{\varepsilon^n}\displaystyle\int_M u^p d\mu_g\right)^{1/p}$.

For any $p \in [2, 2^*)$ if $n \geq 3$ or for all $p \geq 2$ if $n = 2$, the embedding $i_\varepsilon : H_\varepsilon \hookrightarrow L_{\varepsilon,p}$ is a compact, continuous map, and it holds $|u|_{\varepsilon,p} \leq c\|u\|_\varepsilon$ for some constant c not depending on ε. We define the adjoint operator $i^*_\varepsilon : L_{\varepsilon,p'} :\hookrightarrow H_\varepsilon$ as

$$u = i^*_\varepsilon(v) \;\Leftrightarrow\; \langle u, \varphi\rangle_\varepsilon = \frac{1}{\varepsilon^n}\int_M v\varphi d\mu_g,$$

so we can rewrite problem (1) in an equivalent formulation

$$u = i^*_\varepsilon\left(\left(u^+\right)^{p-1}\right).$$

Remark 5. We have that $\|i^*_\varepsilon(v)\|_\varepsilon \leq c|v|_{p',\varepsilon}$

From now on we set, for sake of simplicity

$$f(u) = (u^+)^{p-1} \text{ and } f'(u) = (p-1)(u^+)^{p-2}$$

We want to split the space H_ε in a finite dimensional space generated by the solution of (9) and its orthogonal complement. Fixed $\xi \in \partial M$ and $R > 0$, we consider on the manifold the functions

$$Z^i_{\varepsilon,\xi} = \begin{cases} \varphi^i_\varepsilon\left(\left(\psi^\partial_\xi\right)^{-1}(x)\right)\chi_R\left(\left(\psi^\partial_\xi\right)^{-1}(x)\right) & x \in I_\xi(R) := I_\xi(R, R); \\ 0 & \text{elsewhere.} \end{cases} \tag{11}$$

where $\varphi^i_\varepsilon(y) = \varphi^i\left(\dfrac{y}{\varepsilon}\right)$ and $\chi_R : B^{n-1}(0, R) \times [0, R) \to \mathbb{R}^+$ is a smooth cut off function such that $\chi_R \equiv 1$ on $B^{n-1}(0, R/2) \times [0, R/2)$ and $|\nabla\chi| \leq 2$.

In the following, for sake of simplicity, we denote

$$D^+(R) = B^{n-1}(0, R) \times [0, R) \tag{12}$$

Let

$$K_{\varepsilon,\xi} := \text{Span}\left\{Z^1_{\varepsilon,\xi}, \cdots, Z^{n-1}_{\varepsilon,\xi}\right\}.$$

We can split H_ε in the sum of the $(n-1)$-dimensional space and its orthogonal complement with respect of $\langle\cdot,\cdot\rangle_\varepsilon$, i.e.

$$K^\perp_{\varepsilon,\xi} := \left\{u \in H_\varepsilon \;,\; \left\langle u, Z^i_{\varepsilon,\xi}\right\rangle_\varepsilon = 0.\right\}.$$

We solve problem (1) by a Lyapunov Schmidt reduction: defined

$$W_{\varepsilon,\xi}(x) = \begin{cases} U_\varepsilon\left(\left(\psi^\partial_\xi\right)^{-1}(x)\right)\chi_R\left(\left(\psi^\partial_\xi\right)^{-1}(x)\right) & x \in I_\xi(R) := I_\xi(R, R); \\ 0 & \text{elsewhere.} \end{cases}$$

we look for a function of the form $W_{\varepsilon,\xi}+\phi$ with $\phi\in K_{\varepsilon,\xi}^{\perp}$ such that

$$\Pi_{\varepsilon,\xi}^{\perp}\left\{W_{\varepsilon,\xi}+\phi-i_{\varepsilon}^{*}\left[f\left(W_{\varepsilon,\xi}+\phi\right)\right]\right\} = 0 \tag{13}$$
$$\Pi_{\varepsilon,\xi}\left\{W_{\varepsilon,\xi}+\phi-i_{\varepsilon}^{*}\left[f\left(W_{\varepsilon,\xi}+\phi\right)\right]\right\} = 0 \tag{14}$$

where $\Pi_{\varepsilon,\xi}:H_\varepsilon\to K_{\varepsilon,\xi}$ and $\Pi_{\varepsilon,\xi}^{\perp}:H_\varepsilon\to K_{\varepsilon,\xi}^{\perp}$ are, respectively, the projection on $K_{\varepsilon,\xi}$ and $K_{\varepsilon,\xi}^{\perp}$. We see that $W_{\varepsilon,\xi}+\phi$ is a solution of (1) if and only if $W_{\varepsilon,\xi}+\phi$ solves (13-14).

Hereafter we collect a series of results which will be useful in the paper.

Definition 6. Given $\xi_0\in\partial M$, using the normal coordinates on the sub manifold ∂M, we define

$$\mathcal{E}(y,x)=\left(\exp^{\partial}_{\xi(y)}\right)^{-1}(x)=\left(\exp^{\partial}_{\exp^{\partial}_{\xi_0}y}\right)^{-1}(\exp^{\partial}_{\xi_0}\bar\eta)=\tilde{\mathcal{E}}(y,\bar\eta)$$

where $x,\xi(y)\in\partial M$, $y,\bar\eta\in B(0,R)\subset\mathbb{R}^{n-1}$ and $\xi(y)=\exp^{\partial}_{\xi_0}y$, $x=\exp^{\partial}_{\xi_0}\bar\eta$. Using Fermi coordinates around ξ_0 in a similar way we define

$$\mathcal{H}(y,x)=\left(\psi^{\partial}_{\xi(y)}\right)^{-1}(x)=\left(\psi^{\partial}_{\exp^{\partial}_{\xi_0}y}\right)^{-1}\left(\psi^{\partial}_{\xi_0}(\bar\eta,\eta_n)\right)=\tilde{\mathcal{H}}(y,\bar\eta,\eta_n)=(\tilde{\mathcal{E}}(y,\bar\eta),\eta_n)$$

where $x\in M$, $\eta=(\bar\eta,\eta_n)$, with $\bar\eta\in B(0,R)\subset\mathbb{R}^{n-1}$ and $0\le\eta_n<R$, $\xi(y)=\exp^{\partial}_{\xi_0}y\in\partial M$ and $x=\psi^{\partial}_{\xi_0}(\eta)$.

Lemma 7. *Set $x=\psi^{\partial}_{\xi_0}(\varepsilon z)$ where $z=(\bar z,z_n)$ and $\xi(y)=\exp^{\partial}_{\xi_0}(y)$, for $j=1,\dots,n-1$ we have*

$$\left.\frac{\partial}{\partial y_j}W_{\varepsilon,\xi(y)}(x)\right|_{y=0}=\sum_{k=1}^{n-1}\left[\frac{1}{\varepsilon}\chi_R(\varepsilon z)\frac{\partial}{\partial z_k}U(z)+U(z)\frac{\partial}{\partial z_k}\chi_R(\varepsilon z)\right]\left.\frac{\partial}{\partial y_j}\tilde{\mathcal{E}}_k(y,\varepsilon\bar z)\right|_{y=0}.$$

We need some preliminaries in order to prove of Lemma 7.

Lemma 8. *It holds*

$$\begin{aligned}
\mathcal{E}(0,\bar\eta)=&\bar\eta \text{ for } \bar\eta\in\mathbb{R}^{n-1}\\
\frac{\partial\tilde{\mathcal{E}}_k}{\partial\eta_j}(0,\bar\eta)=&\delta_{jk} \text{ for } y\in\mathbb{R}^{n-1},\ j,k=1,\dots,n-1\\
\frac{\partial\tilde{\mathcal{E}}_k}{\partial y_j}(0,0)=&-\delta_{jk} \text{ for } j,k=1,\dots,n-1\\
\frac{\partial^2\tilde{\mathcal{E}}_k}{\partial y_j\partial\eta_h}(0,0)=&0 \text{ for } j,h,k=1,\dots,n-1
\end{aligned}$$

Proof. We recall that $\tilde{\mathcal{E}}(y,\bar\eta)=\left(\exp^{\partial}_{\xi(y)}\right)^{-1}(\exp^{\partial}_{\xi_0}\bar\eta)$, so the first claim is obvious. Let us introduce, for $y,\bar\eta\in B(0,R)\subset\mathbb{R}^{n-1}$

$$\begin{aligned}
F(y,\bar\eta)&=\left(\exp^{\partial}_{\xi_0}\right)^{-1}\left(\exp^{\partial}_{\xi(y)}(\bar\eta)\right)\\
\Gamma(y,\bar\eta)&=(y,F(y,\bar\eta)).
\end{aligned}$$

We notice that $\Gamma^{-1}(y,\beta),=(y,\tilde{\mathcal{E}}(y,\beta))$. We can easily compute the derivative of Γ. Given $\hat y,\hat\eta\in\mathbb{R}^{n-1}$ we have

$$\Gamma'(\hat y,\hat\eta)[y,\beta]=\begin{pmatrix}\mathrm{Id}_{\mathbb{R}^{n-1}} & 0\\ F'_y(\hat y,\hat\eta) & F'_\eta(\hat y,\hat\eta)\end{pmatrix}\begin{pmatrix}y\\ \beta\end{pmatrix},$$

thus

$$\left(\Gamma^{-1}\right)'(\hat{y},\hat{\eta})[y,\beta] = \begin{pmatrix} \mathrm{Id}_{\mathbb{R}^{n-1}} & 0 \\ -\left(F'_\eta(\hat{y},\hat{\eta})\right)^{-1} F'_y(\hat{y},\hat{\eta}) & \left(F'_\eta(\hat{y},\hat{\eta})\right)^{-1} \end{pmatrix} \begin{pmatrix} y \\ \beta \end{pmatrix}$$

Here $y, \beta \in \mathbb{R}^{n-1}$. Now, by direct computation we have that

$$F'_\eta(0,\hat{\eta}) = \mathrm{Id}_{\mathbb{R}^{n-1}} \text{ and } F'_y(\hat{y},0) = \mathrm{Id}_{\mathbb{R}^{n-1}},$$

so $\frac{\partial \tilde{\mathcal{E}}_k}{\partial \eta_j}(0,\hat{\eta}) = \left(\left(F'_\eta(0,\hat{\eta})\right)^{-1}\right)_{jk} = \delta_{jk}$ and $\frac{\partial \tilde{\mathcal{E}}_k}{\partial y_j}(0,0) = \left(-\left(F'_\eta(0,0)\right)^{-1} F'_y(0,0)\right)_{jk} = -\delta_{jk}$. For the last claim we refer to [15, Lemma 6.4] □

Lemma 9. *We have that*

$$\begin{aligned}
\tilde{\mathcal{H}}(0,\bar{\eta},\eta_n) =& (\bar{\eta},\eta_n) \textit{ for } \bar{\eta} \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+ \\
\frac{\partial \tilde{\mathcal{H}}_k}{\partial y_j}(0,0,\eta_n) =& -\delta_{jk} \textit{ for } j,k = 1,\dots,n-1, \eta_n \in \mathbb{R}_+ \\
\frac{\partial \tilde{\mathcal{H}}_n}{\partial y_j}(y,\bar{\eta},\eta_n) =& 0 \textit{ for } j = 1,\dots,n-1, y,\bar{\eta} \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+ \\
\frac{\partial \tilde{\mathcal{H}}_k}{\partial \eta_n}(y,\bar{\eta},\eta_n) =& 0 \textit{ for } j,k = 1,\dots,n-1, \bar{\eta} \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+ \\
\frac{\partial^2 \tilde{\mathcal{H}}_k}{\partial \eta_n \partial y_j}(y,\bar{\eta},\eta_n) =& 0 \textit{ for } j,k = 1,\dots,n-1, \bar{\eta} \in \mathbb{R}^{n-1}, \eta_n \in \mathbb{R}_+
\end{aligned}$$

Proof. The first three claim follows immediately by Definition 6 and Lemma 8. For the last two claims, observe that $\tilde{\mathcal{H}}_k(y,\bar{\eta},\eta_n) = \tilde{\mathcal{E}}_k(y,\bar{\eta})$ which does not depends on η_n as well as its derivatives. □

We now prove the claimed result.

Proof of Lemma 7. By definition 6, set $x = \psi^\partial_{\xi_0}(\eta) = \psi^\partial_{\xi_0}(\bar{\eta},\eta_n)$ with $\eta = (\bar{\eta},\eta_n) \in \mathbb{R}^n$, and $\xi(y) = \exp^\partial_{\xi_0}(y)$ where $y \in \mathbb{R}^{n-1}$, we have that

$$W_{\varepsilon,\xi(y)}(x) = U\left(\frac{\tilde{\mathcal{H}}(y,\eta)}{\varepsilon}\right)\chi_R(\tilde{\mathcal{H}}(y,\eta)).$$

Fixed j, by Lemma 9,

$$\begin{aligned}
\left.\frac{\partial}{\partial y_j} W_{\varepsilon,\xi(y)}(x)\right|_{y=0} &= \sum_{k=1}^{n} \frac{\partial}{\partial v_k}\left[\chi_R(\tilde{\mathcal{H}}(y,\eta)) U_\varepsilon(\tilde{\mathcal{H}}(y,\eta))\right]\Big|_{\tilde{\mathcal{H}}(0,\eta)} \left.\frac{\partial}{\partial y_i}\tilde{\mathcal{H}}_k(y,\eta)\right|_{y=0} \\
&= \sum_{k=1}^{n-1} \frac{\partial}{\partial \eta_k}\left[\chi_R(\eta) U_\varepsilon(\eta)\right] \left.\frac{\partial}{\partial y_j}\tilde{\mathcal{E}}_k(y,\bar{\eta})\right|_{y=0} \\
&= \sum_{k=1}^{n-1} \frac{\partial}{\partial z_k}\left[\chi_R(\varepsilon z) U(z)\right] \left.\frac{\partial}{\partial y_j}\tilde{\mathcal{E}}_k(y,\varepsilon\bar{z})\right|_{y=0}
\end{aligned}$$

Because $\tilde{\mathcal{H}}_k(y,\bar{\eta},\eta_n) = \mathcal{E}_k(y,\bar{\eta})$ for $k = 1,\dots,n-1$. Using the change of variables $\eta = \varepsilon z = (\varepsilon\bar{z},\varepsilon z_n)$, we get the claim. □

3. Reduction to finite dimensional space

In this section we find a solution for equation (13). In particular, we prove that for all $\varepsilon > 0$ and for all $\xi \in \partial M$ there exists $\phi_{\varepsilon,\xi} \in K^{\perp}_{\varepsilon,\xi}$ solving (13). Here and in the hereafter, all the proof are similar to [15]. So, for the sake of simplicity, we will underline the parts where differences appear, and sketch the remains of the proofs (we will provide precise references for each proof).

We introduce the linear operator $L_{\varepsilon,\xi} : K^{\perp}_{\varepsilon,\xi} \to K^{\perp}_{\varepsilon,\xi}$

$$L_{\varepsilon,\xi}(\phi) := \Pi^{\perp}_{\varepsilon,\xi}\left\{\phi - i^*_\varepsilon\left[f'(W_{\varepsilon,\xi})\phi\right]\right\}$$

thus we can rewrite equation (13) as

$$L_{\varepsilon,\xi}(\phi) = N_{\varepsilon,\xi}(\phi) + R_{\varepsilon,\xi}$$

where $N_{\varepsilon,\xi}(\phi)$ is the nonlinear term

$$N_{\varepsilon,\xi} := \Pi^{\perp}_{\varepsilon,\xi}\left\{i^*_\varepsilon\left[f(W_{\varepsilon,\xi}+\phi) - f(W_{\varepsilon,\xi}) - f'(W_{\varepsilon,\xi})\phi\right]\right\}$$

and $R_{\varepsilon,\xi}$ is a remainder term

$$R_{\varepsilon,\xi} := \Pi^{\perp}_{\varepsilon,\xi}\left\{i^*_\varepsilon\left[f(W_{\varepsilon,\xi})\right] - W_{\varepsilon,\xi}\right\}.$$

The first step is to prove that the linear term is invertible.

Lemma 10. *There exist ε_0 and $c > 0$ such that, for any $\xi \in \partial M$ and $\varepsilon \in (0, \varepsilon_0)$*

$$\|L_{\varepsilon,\xi}\|_\varepsilon \geq c\|\phi\|_\varepsilon \text{ for any } \phi \in K^{\perp}_{\varepsilon,\xi}.$$

The proof of this Lemma is postponed to the Appendix. We estimate now the remainder term $R_{\varepsilon,\xi}$.

Lemma 11. *There exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\xi \in \partial M$ and for all $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\|R_{\varepsilon,\xi}\|_\varepsilon \leq c\varepsilon^{1+\frac{n}{p'}}.$$

Proof. We proceed as in [15, Lemma 3.3]. We define on M the function $V_{\varepsilon,\xi}$ such that $W_{\varepsilon,\xi} = i^*_\varepsilon(V_{\varepsilon,\xi})$, thus $-\varepsilon^2\Delta_g W_{\varepsilon,\xi} + W_{\varepsilon,\xi} = V_{\varepsilon,\xi}$.

It is well known[1], by definition of Laplace-Beltrami operator, that in a local chart it holds

$$-\Delta v = -\Delta_g v + (g^{ij}_\xi - \delta_{ij})\frac{\partial^2}{\partial x_i \partial x_j}v - g^{ij}_\xi \Gamma^k_{ij}\frac{\partial}{\partial x_k}v$$

where Δ is the euclidean Laplace operator. Thus, defined

$$\tilde{V}_{\varepsilon,\xi}(y) = V_{\varepsilon,\xi}\left(\psi^\partial_\xi(y)\right),\ y \in D^+(R)$$

we have

$$\begin{aligned}
\tilde{V}_{\varepsilon,\xi}(y) &= -\varepsilon^2\Delta_g(U_\varepsilon\chi_R) + U_\varepsilon\chi_R = \\
&= U_\varepsilon^{p-1}\chi_R - \varepsilon^2 U_\varepsilon\Delta\chi_R - 2\varepsilon^2\nabla U_\varepsilon\nabla\chi_R \\
&\quad -\varepsilon^2(g^{ij}_\xi - \delta_{ij})\frac{\partial^2}{\partial y_i\partial y_j}(U_\varepsilon\chi_R) + \varepsilon^2 g^{ij}_\xi\Gamma^k_{ij}\frac{\partial}{\partial y_k}(U_\varepsilon\chi_R)
\end{aligned} \tag{15}$$

Also, we remind that, by Remark 5 and by definition of $R_{\varepsilon,\xi}$, it holds

$$\|R_{\varepsilon,\xi}\|_\varepsilon \leq \|i^*_\varepsilon f(W_{\varepsilon,\xi}) - W_{\varepsilon,\xi}\|_\varepsilon \leq c\left|W^{p-1}_{\varepsilon,\xi} - V_{\varepsilon,\xi}\right|_{p',\varepsilon}.$$

[1] [16, page134]

Finally, by definition of $W_{\varepsilon,\xi}$ and by (15) we get

$$\begin{aligned}\left|W_{\varepsilon,\xi}^{p-1}-V_{\varepsilon,\xi}\right|_{p',\varepsilon}^{p'}=&\int_{D^+(R)}\left|U_\varepsilon^{p-1}(y)\chi_R^{p-1}(y)-\tilde{V}_{\varepsilon,\xi}(y)\right|^{p'}|g_\xi(y)|^{1/2}dy\\ \leq& c\int_{D^+(R)}\left|U_\varepsilon^{p-1}(y)\left(\chi_R^{p-1}(y)-\chi_R(y)\right)\right|^{p'}dy\\ &+c\varepsilon^{2p'}\int_{D^+(R)}U_\varepsilon^{p'}|\Delta\chi_R|^{p'}dy+c\varepsilon^{2p'}\int_{D^+(R)}|\nabla U_\varepsilon\cdot\nabla\chi_R|^{p'}dy\\ &+c\varepsilon^{2p'}\int_{D^+(R)}\left|g_\xi^{ij}(y)-\delta_{ij}\right|^{p'}\left|\frac{\partial^2}{\partial y_i\partial y_j}(U_\varepsilon\chi_R)(y)\right|^{p'}dy\\ &+c\varepsilon^{2p'}\int_{D^+(R)}\left|g_\xi^{ij}(y)\Gamma_{ij}^k(y)\right|^{p'}\left|\frac{\partial}{\partial y_k}(U_\varepsilon\chi_R)(y)\right|^{p'}dy\end{aligned}$$

By exponential decay and by definition of χ_r, using (2) and (3) we have

$$\begin{aligned}&\varepsilon^{2p'}\int_{D^+(R)}\left|g_\xi^{ij}(y)-\delta_{ij}\right|^{p'}\left|\frac{\partial^2}{\partial y_i\partial y_j}(U_\varepsilon\chi_R)(y)\right|^{p'}dy\\ &\quad=\varepsilon^{2p'}\int_{D^+(R)}\left|g_\xi^{ij}(y)-\delta_{ij}\right|^{p'}\left|\frac{\partial^2}{\partial y_i\partial y_j}U_\varepsilon(y)\right|^{p'}dy+O(\varepsilon^{n+p'})\\ &\quad\leq\varepsilon^{n}\int_{\mathbb{R}^n_+}\left|g_\xi^{ij}(\varepsilon z)-\delta_{ij}\right|^{p'}\left|\frac{\partial^2}{\partial z_i\partial z_j}U(z)\right|^{p'}dz+O(\varepsilon^{n+p'})\\ &\quad\leq\varepsilon^{n+p'}\int_{\mathbb{R}^n_+}\left|\frac{\partial^2}{\partial z_i\partial z_j}U(z)\right|^{p'}dz+O(\varepsilon^{n+p'})=O(\varepsilon^{n+p'}).\end{aligned}$$

The other terms can be estimate in a similar way. □

By fixed point theorem and by implicit function theorem we can solve equation (13).

Proposition 12. *There exists $\varepsilon_0>0$ and $c>0$ such that for any $\xi\in\partial M$ and for all $\varepsilon\in(0,\varepsilon_0)$ there exists a unique $\phi_{\varepsilon,\xi}=\phi(\varepsilon,\xi)\in K_{\varepsilon,\xi}^{\perp}$ which solves (13). Moreover*

$$\|\phi_{\varepsilon,\xi}\|_\varepsilon<c\varepsilon^{1+\frac{n}{p'}}.$$

Finally, $\xi\mapsto\phi_{\varepsilon,\xi}$ is a C^1 map.

Proof. The proof is similar to Proposition 3.5 of [15], which we refer to for all details. We want to solve (13) by a fixed point argument. We define the operator

$$\begin{aligned}T_{\varepsilon,\xi}&:&K_{\varepsilon,\xi}^{\perp}\to K_{\varepsilon,\xi}^{\perp}\\ T_{\varepsilon,\xi}(\phi)&=&L_{\varepsilon,\xi}^{-1}\left(N_{\varepsilon,\xi}(\phi)+R_{\varepsilon,\xi}\right)\end{aligned}$$

By Lemma 10, $T_{\varepsilon,\xi}$ is well defined and it holds

$$\begin{aligned}\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon&\leq&c\left(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon+\|R_{\varepsilon,\xi}\|_\varepsilon\right)\\ \|T_{\varepsilon,\xi}(\phi_1)-T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon&\leq&c\left(\|N_{\varepsilon,\xi}(\phi_1)-N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon\right)\end{aligned}$$

for some suitable constant $c>0$. By the mean value theorem (and by the properties of i^*) we get

$$\|N_{\varepsilon,\xi}(\phi_1)-N_{\varepsilon,\xi}(\phi_2)\|_\varepsilon \le c\,|f'(W_{\varepsilon,\xi}+\phi_2+t(\phi_1-\phi_2))-f'(W_{\varepsilon,\xi})|_{\frac{p}{p-2},\varepsilon}\|\phi_1-\phi_2\|_\varepsilon.$$

By [15, Remark 3.4], we have that $|f'(W_{\varepsilon,\xi}+\phi_2+t(\phi_1-\phi_2))-f'(W_{\varepsilon,\xi})|_{\frac{p}{p-2},\varepsilon}<<1$ provided $\|\phi_1\|_\varepsilon$ and $\|\phi_2\|_\varepsilon$ small enough. Thus there exists $0<C<1$ such that $\|T_{\varepsilon,\xi}(\phi_1)-T_{\varepsilon,\xi}(\phi_2)\|_\varepsilon\le C\|\phi_1-\phi_2\|_\varepsilon$. Also, with the same estimates we get

$$\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon \le c\left(\|\phi\|_\varepsilon^2+\|\phi\|_\varepsilon^{p-1}\right).$$

This, combined with Lemma 11 gives us

$$\|T_{\varepsilon,\xi}(\phi)\|_\varepsilon \le c\left(\|N_{\varepsilon,\xi}(\phi)\|_\varepsilon+\|R_{\varepsilon,\xi}\|_\varepsilon\right)\le c\left(\|\phi\|_\varepsilon^2+\|\phi\|_\varepsilon^{p-1}+\varepsilon^{1+\frac{n}{p'}}\right).$$

So, there exists $c>0$ such that $T_{\varepsilon,\xi}$ maps a ball of center 0 and radius $c\varepsilon^{1+\frac{n}{p'}}$ in $K^\perp_{\varepsilon,\xi}$ into itself and it is a contraction. So there exists a fixed point $\phi_{\varepsilon,\xi}$ with norm $\|\phi_{\varepsilon,\xi}\|_\varepsilon\le\varepsilon^{1+\frac{n}{p'}}$.

The regularity of $\phi_{\varepsilon,\xi}$ with respect to ξ is proved via implicit function theorem. Let us define the functional

$$\begin{aligned} G &: \partial M\times H_\varepsilon\to\mathbb{R}\\ G(\xi,u) &:= \Pi^\perp_{\varepsilon,\xi}\left\{W_{\varepsilon,\xi}+\Pi^\perp_{\varepsilon,\xi}u+i^*_\varepsilon\left[f\left(W_{\varepsilon,\xi}+\Pi^\perp_{\varepsilon,\xi}u\right)\right]\right\}+\Pi_{\varepsilon,\xi}u.\end{aligned}$$

We have that $G(\xi,\phi_{\varepsilon,\xi})=0$ and that the operator $\dfrac{\partial}{\partial u}G(\xi,\phi_{\varepsilon,\xi}):H_\varepsilon\to H_\varepsilon$ is invertible. This concludes the proof. □

4. Sketch of the proof of Theorem 2

In section 3, Proposition 12 we found a function $\phi_{\varepsilon,\xi}$ solving (13). In order to solve (14) we define the functional $J_\varepsilon:H^1(M)\to\mathbb{R}$

$$J_\varepsilon(u)=\frac{1}{\varepsilon^n}\int_M\frac12\varepsilon^2|\nabla u|_g^2+\frac12u^2-\frac1p(u^+)^pd\mu_g.$$

In which follows we will often use the notation $F(u)=\frac1p(u^+)^p$.

By J_ε we define the reduced functional $\tilde J_\varepsilon$ on ∂M as

$$\tilde J_\varepsilon(\xi)=J_\varepsilon(W_{\varepsilon,\xi}+\phi_{\varepsilon,\xi})$$

where $\phi_{\varepsilon,\xi}$ is uniquely determined by Proposition 12.

Remark 13. Our goal is to find critical points for $\tilde J_\varepsilon$, since any critical point ξ for $\tilde J_\varepsilon$ corresponds to a function $\phi_{\varepsilon,\xi}+W_{\varepsilon,\xi}$ which solves equation (14).

At this point we give the expansion for the functional $\tilde J_\varepsilon$ with respect to ε. By Lemma 14 and Lemma 15 it holds

$$\tilde J_\varepsilon(\xi)=C-\varepsilon H(\xi)+o(\varepsilon) \tag{16}$$

C^1 uniformly with respect to $\xi\in\partial M$ as ε goes to zero. Here $H(\xi)$ is the mean curvature of the boundary ∂M at ξ. If ξ_0 is a C^1-stable critical point for H, in light of (16) and by definition of C^1-stability, we have that, for ε small enough there exists ξ_ε close to ξ_0 critical point for $\tilde J_\varepsilon$, and we can prove Theorem 2.

5. Asymptotic expansion of the reduced functional

In this we study the asymptotic expansion of $\tilde{J}_\varepsilon(\xi)$ with respect to ε.

Lemma 14. *It holds*

$$\tilde{J}_\varepsilon(\xi) = J_\varepsilon(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}) = J_\varepsilon(W_{\varepsilon,\xi}) + o(\varepsilon) \tag{17}$$

uniformly with respect to $\xi \in \partial M$ as ε goes to zero.

Moreover, setting $\xi(y) = \exp_\xi^\partial(y)$, $y \in B^{n-1}(0,r)$ it holds

$$\begin{aligned}\left(\frac{\partial}{\partial y_h}\tilde{J}_\varepsilon(\xi(y))\right)_{|y=0} &= \left(\frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\right)_{|y=0} = \\ &= \left(\frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)})\right)_{|y=0} + o(\varepsilon) \end{aligned}\tag{18}$$

Proof. We split the proof in several steps.

Step 1: we prove (17). Using (13) we get

$$\begin{aligned}\tilde{J}_\varepsilon(\xi) - J_\varepsilon(W_{\varepsilon,\xi}) =& \frac{1}{2}\|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \frac{1}{\varepsilon^n}\int_M \varepsilon^2 g(\nabla W_{\varepsilon,\xi}, \nabla\phi_{\varepsilon,\xi}) + W_{\varepsilon,\xi}\phi_{\varepsilon,\xi} - f\left(W_{\varepsilon,\xi}\right)\phi_{\varepsilon,\xi} d\mu_g \\ &- \frac{1}{\varepsilon^n}\int_M F\left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}\right) - F\left(W_{\varepsilon,\xi}\right) - f\left(W_{\varepsilon,\xi}\right)\phi_{\varepsilon,\xi} \\ =& -\frac{1}{2}\|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \frac{1}{\varepsilon^n}\int_M \left[f\left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}\right) - f\left(W_{\varepsilon,\xi}\right)\right]\phi_{\varepsilon,\xi} d\mu_g \\ &- \frac{1}{\varepsilon^n}\int_M F\left(W_{\varepsilon,\xi} + \phi_{\varepsilon,\xi}\right) - F\left(W_{\varepsilon,\xi}\right) - f\left(W_{\varepsilon,\xi}\right)\phi_{\varepsilon,\xi}\end{aligned}$$

By the mean value theorem we obtain that

$$\left|\tilde{J}_\varepsilon(\xi) - J_\varepsilon(W_{\varepsilon,\xi})\right| \le \frac{1}{2}\|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \left|\frac{1}{\varepsilon^n}\int_M f'\left(W_{\varepsilon,\xi} + t_1\phi_{\varepsilon,\xi}\right)\phi_{\varepsilon,\xi}^2\right| + \left|\frac{1}{\varepsilon^n}\int_M f'\left(W_{\varepsilon,\xi} + t_2\phi_{\varepsilon,\xi}\right)\phi_{\varepsilon,\xi}^2\right|$$

for some $t_1, t_2 \in (0,1)$. Now, by the properties of f' we can conclude that

$$\left|\tilde{J}_\varepsilon(\xi) - J_\varepsilon(W_{\varepsilon,\xi})\right| \le c\left(\|\phi_{\varepsilon,\xi}\|_\varepsilon^2 + \|\phi_{\varepsilon,\xi}\|_\varepsilon^p\right)$$

and in light of Proposition 12 we obtain (17).

Step 2: in order to prove (18), consider that

$$\begin{aligned}&\frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - \frac{\partial}{\partial y_h}J_\varepsilon(W_{\varepsilon,\xi(y)}) \\ &= J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\left[\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)} + \frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\right] - J_\varepsilon'(W_{\varepsilon,\xi(y)})\left[\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right] \\ &\qquad= \left[J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)}) - J_\varepsilon'(W_{\varepsilon,\xi(y)})\right]\left[\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right] \\ &\qquad\qquad + J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\left[\frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\right] = L_1 + L_2.\end{aligned}$$

Step 3: we estimate L_2. We have, by (13), that

$$J_\varepsilon'(W_{\varepsilon,\xi(y)} + \phi_{\varepsilon,\xi(y)})\left[\frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\right] = \sum_{l=1}^{n-1} c_\varepsilon^l \left\langle Z_{\varepsilon,\xi(y)}^l, \frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\right\rangle_\varepsilon.$$

We prove that

$$\sum_{l=1}^{n-1}|c_\varepsilon^l| = O(\varepsilon). \tag{19}$$

Indeed we have, by (13) and (32), for some positive constant C,
(20)

$$J'_\varepsilon(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})\left[Z^s_{\varepsilon,\xi(y)}\right] = \sum_{l=1}^{n-1} c_\varepsilon^l \left\langle Z^l_{\varepsilon,\xi(y)}, Z^s_{\varepsilon,\xi(y)}\right\rangle_\varepsilon = C\sum_{l=1}^{n-1} c_\varepsilon^l(\delta_{ls}+o(1)).$$

Also, since $\phi_{\varepsilon,\xi(y)} \in K^\perp_{\varepsilon,\xi(y)}$, we have

$$\begin{aligned} J'_\varepsilon(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})&\left[Z^s_{\varepsilon,\xi(y)}\right] \\ = \frac{1}{\varepsilon^n}\int_M &\varepsilon^2 g(\nabla W_{\varepsilon,\xi(y)}, \nabla Z^s_{\varepsilon,\xi(y)}) + W_{\varepsilon,\xi(y)}Z^s_{\varepsilon,\xi(y)} - f(W_{\varepsilon,\xi(y)})Z^s_{\varepsilon,\xi(y)}d\mu_g \\ &- \frac{1}{\varepsilon^n}\int_M \left[f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)})\right] Z^s_{\varepsilon,\xi(y)}d\mu_g. \end{aligned}$$

By (2), (3) and (4), after a change of variables we have

$$\begin{aligned} \frac{1}{\varepsilon^n}\int_M \varepsilon^2 g(\nabla W_{\varepsilon,\xi(y)}, \nabla Z^s_{\varepsilon,\xi(y)}) + W_{\varepsilon,\xi(y)}Z^s_{\varepsilon,\xi(y)} - f(W_{\varepsilon,\xi(y)})Z^s_{\varepsilon,\xi(y)}d\mu_g \\ = \int_{\mathbb{R}^n_+} \nabla U\nabla\varphi^l + U\varphi^l - f(U)\varphi^l dz + O(\varepsilon) = O(\varepsilon). \end{aligned}$$

Besides, by the mean value theorem, for some $t\in(0,1)$,

$$\begin{aligned} &\left|\frac{1}{\varepsilon^n}\int_M \left[f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)})\right] Z^s_{\varepsilon,\xi(y)}d\mu_g\right| \\ &\qquad = \left|\frac{1}{\varepsilon^n}\int_M \left[f'(W_{\varepsilon,\xi(y)}+t\phi_{\varepsilon,\xi(y)})\right] Z^s_{\varepsilon,\xi(y)}\phi_{\varepsilon,\xi(y)}d\mu_g\right| \\ &\qquad \le c\frac{1}{\varepsilon^n}\int_M \left(\left|W_{\varepsilon,\xi(y)}\right|^{p-2} + \left|\phi_{\varepsilon,\xi(y)}\right|^{p-2}\right)\left|Z^s_{\varepsilon,\xi(y)}\right|\left|\phi_{\varepsilon,\xi(y)}\right|d\mu_g \\ &\qquad \le c\left(\|W_{\varepsilon,\xi(y)}\|_\varepsilon^{p-2} + \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon^{p-2}\right)\|Z^s_{\varepsilon,\xi(y)}\|_\varepsilon\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon = O(\varepsilon^{1+\frac{n}{p'}}) = o(\varepsilon). \end{aligned}$$

Hence $J'_\varepsilon(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)})\left[Z^s_{\varepsilon,\xi(y)}\right] = O(\varepsilon)$ and, comparing with (20), we get (19). At this point we have, by (27), (19) and by Proposition (12), that

$$\begin{aligned} |L_2| &\le \left|\sum_{l=1}^{n-1} c_\varepsilon^l \left\langle Z^l_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h}\phi_{\varepsilon,\xi(y)}\right\rangle_\varepsilon\right| = \left|\sum_{l=1}^{n-1} c_\varepsilon^l \left\langle \frac{\partial}{\partial y_h}Z^l_{\varepsilon,\xi(y)}, \phi_{\varepsilon,\xi(y)}\right\rangle_\varepsilon\right| \\ &\le \left(\sum_{l=1}^{n-1}|c_\varepsilon^l|\right)\left\|\frac{\partial}{\partial y_h}Z^l_{\varepsilon,\xi(y)}\right\|_\varepsilon \left\|\phi_{\varepsilon,\xi(y)}\right\|_\varepsilon \le O(\varepsilon^{1+\frac{n}{p'}}) = o(\varepsilon). \end{aligned}$$

Step 4: we estimate L_1. We have

$$
\begin{aligned}
L_1 =& \left\langle \phi_{\varepsilon,\xi(y)}, \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\rangle_\varepsilon - \frac{1}{\varepsilon^n}\int_M \left[f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)}) \right] \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} d\mu_g \\
=& \left\langle \phi_{\varepsilon,\xi(y)} - i^*_\varepsilon \left[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)} \right], \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\rangle_\varepsilon \\
& - \frac{1}{\varepsilon^n}\int_M \left[f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)}) - f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)} \right] \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} d\mu_g \\
=& \left\langle \phi_{\varepsilon,\xi(y)} - i^*_\varepsilon \left[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)} \right], \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} + \frac{1}{\varepsilon} Z^h_{\varepsilon,\xi(y)} \right\rangle_\varepsilon \\
& - \frac{1}{\varepsilon} \left\langle \phi_{\varepsilon,\xi(y)}, Z^h_{\varepsilon,\xi(y)} - i^*_\varepsilon \left[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)} \right] \right\rangle_\varepsilon \\
& - \frac{1}{\varepsilon^n}\int_M \left[f(W_{\varepsilon,\xi(y)}+\phi_{\varepsilon,\xi(y)}) - f(W_{\varepsilon,\xi(y)}) - f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)} \right] \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} d\mu_g \\
=& A_1 + A_2 + A_3
\end{aligned}
$$

For the first term we have, by (29)

$$
\begin{aligned}
A_1 \leq& \|\phi_{\varepsilon,\xi(y)} - i^*_\varepsilon \left[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)} \right] \|_\varepsilon \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} + \frac{1}{\varepsilon} Z^h_{\varepsilon,\xi(y)} \right\|_\varepsilon \\
\leq& c\varepsilon \|\phi_{\varepsilon,\xi(y)} - i^*_\varepsilon \left[f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)} \right] \|_\varepsilon \\
\leq& c\varepsilon \left(\|\phi_{\varepsilon,\xi(y)}\|_\varepsilon + -|f'(W_{\varepsilon,\xi(y)})\phi_{\varepsilon,\xi(y)}|_{p',\varepsilon} \right) \leq \varepsilon \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon = o(\varepsilon).
\end{aligned}
$$

For the second term, in light of Proposition 12 and Equation (26), we have

$$
A_2 \leq \frac{1}{\varepsilon} \|\phi_{\varepsilon,\xi(y)}\|_\varepsilon \|Z^h_{\varepsilon,\xi(y)} - i^*_\varepsilon \left[f'(W_{\varepsilon,\xi(y)}) Z^h_{\varepsilon,\xi(y)} \right] \|_\varepsilon = O(\varepsilon^{1+2\frac{n}{p'}}) = o(\varepsilon).
$$

In order to estimate the last term, we have to consider separately case $2 \leq p < 3$ and $p \geq 3$.

We recall ([15, Remark 3.4]) that

$$
|f'(W_{\varepsilon,\xi}+v) - f'(W_{\varepsilon,\xi})| \leq \begin{cases} c(p)|v|^{p-2} & 2 < p < 3 \\ c(p)[W^{p-3}_{\varepsilon,\xi}|v| + |v|^{p-2}] & p \geq 3 \end{cases}
$$

For $p \geq 3$, we have, by the growth properties of f , and using (27), we get

$$
\begin{aligned}
A_3 \leq& \frac{c}{\varepsilon^n}\int_M \left[|W_{\varepsilon,\xi(y)}|^{p-3}\phi^2_{\varepsilon,\xi(y)} + |\phi_{\varepsilon,\xi(y)}|^{p-1} \right] \left| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right| d\mu_g \\
\leq& \|\phi_{\varepsilon,\xi(y)}\|^2_\varepsilon \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\|_\varepsilon + \|\phi_{\varepsilon,\xi(y)}\|^{p-1}_\varepsilon \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\|_\varepsilon \\
\leq& O(\varepsilon^{1+2\frac{n}{p'}}) + O(\varepsilon^{p-2+(p-1)\frac{n}{p'}}) = o(\varepsilon)
\end{aligned}
$$

since $p \geq 3$.

For $2 < p < 3$, in a similar way, we get

$$
A_3 \leq \frac{c}{\varepsilon^n}\int_M \phi^{p-1}_{\varepsilon,\xi(y)} \left| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right| d\mu_g \leq \|\phi_{\varepsilon,\xi(y)}\|^{p-1}_\varepsilon \left\| \frac{\partial}{\partial y_h} W_{\varepsilon,\xi(y)} \right\|_\varepsilon = o(\varepsilon)
$$

which concludes the proof. □

Lemma 15. *It holds*

$$J_\varepsilon(W_{\varepsilon,\xi}) = C - \varepsilon\alpha H(\xi) + o(\varepsilon)$$

C^0-uniformly with respect to ξ as ε goes to zero, where

$$\begin{aligned}
C &:= \int_{\mathbb{R}^n_+} \frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)dz \\
\alpha &:= \frac{(n-1)}{2}\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_n^3 dz
\end{aligned}$$

Proof. By definition of J_ε we have

$$\begin{aligned}
J_\varepsilon(W_{\varepsilon,\xi}) =& \frac{1}{2}\int_{D^+(R/\varepsilon)}\sum_{i,j=1}^n g^{ij}(\varepsilon z)\frac{\partial\left(U(z)\chi_{R/\varepsilon}(z)\right)}{\partial z_i}\frac{\partial\left(U(z)\chi_{R/\varepsilon}(z)\right)}{\partial z_j}|g(\varepsilon z)|^{\frac{1}{2}}dz \\
&+\int_{D^+(R/\varepsilon)}\left[\frac{1}{2}\left(U(z)\chi_{R/\varepsilon}(z)\right)^2 - \frac{1}{p}\left(U(z)\chi_{R/\varepsilon}(z)\right)^p\right]|g(\varepsilon z)|^{\frac{1}{2}}dz.
\end{aligned}$$

We easily get, by (2), (3) and (4)

$$\begin{aligned}
J_\varepsilon(W_{\varepsilon,\xi}) &= \int_{D^+(R/\varepsilon)}\frac{1}{2}\sum_{i=1}^n\frac{\partial\left(U(z)\chi_{R/\varepsilon}(z)\right)}{\partial z_i}\frac{\partial\left(U(z)\chi_{R/\varepsilon}(z)\right)}{\partial z_i}dz \\
&\quad+\int_{D^+(R/\varepsilon)}\frac{1}{2}\left(U(z)\chi_{R/\varepsilon}(z)\right)^2 - \frac{1}{p}\left(U(z)\chi_{R/\varepsilon}(z)\right)^p dz \\
&\quad+\varepsilon\int_{D^+(R/\varepsilon)}\sum_{i,j=1}^{n-1}h^{ij}(0)z_n\frac{\partial\left(U(z)\chi_{R/\varepsilon}(z)\right)}{\partial z_i}\frac{\partial\left(U(z)\chi_{R/\varepsilon}(z)\right)}{\partial z_j}dz \\
&\quad-\frac{n-1}{2}\varepsilon\int_{D^+(R/\varepsilon)}H(\xi)z_n\sum_{i=1}^n\frac{\partial\left(U(z)\chi_{R/\varepsilon}(z)\right)}{\partial z_i}\frac{\partial\left(U(z)\chi_{R/\varepsilon}(z)\right)}{\partial z_i}dz \\
&\quad-\frac{n-1}{2}\varepsilon\int_{D^+(R/\varepsilon)}H(\xi)z_n\left(U(z)\chi_{R/\varepsilon}(z)\right)^2 dz \\
&\quad+\frac{n-1}{p}\varepsilon\int_{D^+(R/\varepsilon)}H(\xi)z_n\left(U(z)\chi_{R/\varepsilon}(z)\right)^p dz + o(\varepsilon) = \\
&= \int_{\mathbb{R}^n_+}\frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)dz \\
&\quad-\varepsilon(n-1)H(\xi)\int_{\mathbb{R}^n_+}z_n\left(\frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)\right)dz \\
&\quad+\varepsilon\sum_{i,j=1}^{n-1}h^{ij}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_i z_j z_n dz + o(\varepsilon).
\end{aligned}$$

Using Lemma 17 finally we have

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) &= \int_{\mathbb{R}^n_+} \frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)dz \\ &\quad -\varepsilon(n-1)H(\xi)\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_n^3 dz \\ &\quad +\varepsilon\sum_{i,j=1}^{n-1} h^{ij}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_i z_j z_n dz + o(\varepsilon). \end{aligned}$$

Now, by symmetry arguments and by (5) we have that

$$\begin{aligned} \sum_{i,j=1}^{n-1} h^{ij}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_i z_j z_n dz &= \sum_{i,j=1}^{n-1} h^{ij}(0)\delta_{ij}\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_i z_j z_n dz = \\ &= \sum_{i=1}^{n-1} h^{ii}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_i^2 z_n dz = \\ &= (n-1)H(\xi)\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_1^2 z_n dz, \end{aligned}$$

and, by simple computation in polar coordinates,

$$\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_1^2 z_n dz = \frac{1}{2}\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_n^3 dz. \tag{21}$$

Concluding, we get

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,\xi}) = &\int_{\mathbb{R}^n_+} \frac{1}{2}|\nabla U(z)|^2 + \frac{1}{2}U^2(z) - \frac{1}{p}U^p(z)dz \\ &- \varepsilon H(\xi)\left[\frac{(n-1)}{2}\int_{\mathbb{R}^n_+}\left(\frac{U'(|z|)}{|z|}\right)^2 z_n^3 dz\right] + o(\varepsilon), \end{aligned}$$

and we have the proof. □

Lemma 16. *Let $\xi(y) = \exp_\xi^\partial(y)$, $y \in B^{n-1}(0,r)$ it holds*

$$\left.\frac{\partial}{\partial y_h} J_\varepsilon(W_{\varepsilon,\xi(y)})\right|_{y=0} = -\varepsilon\alpha\left.\left(\frac{\partial}{\partial y_h}H(\xi(y))\right)\right|_{y=0} + o(\varepsilon)$$

uniformly with respect to ξ as ε goes to zero.

Proof. For simplicity, we prove the claim for $h = 1$. Cases $h = 2, \dots, n-1$ are straightforward. Let us consider first

$$\left.\frac{\partial}{\partial y_1}\int_{I_\xi(R)} \frac{1}{2\varepsilon^n} W^2_{\varepsilon,\xi(y)} d\mu_g\right|_{y=0} = \int_{I_\xi(R)} \frac{1}{\varepsilon^n} W_{\varepsilon,\xi} \frac{\partial}{\partial y_1} \left. W_{\varepsilon,\xi(y)}\right|_{y=0} d\mu_g$$

by Lemma 7 and by exponential decay of U we have

$$
\begin{aligned}
&\frac{\partial}{\partial y_1}\int \frac{1}{2\varepsilon^n} W^2_{\varepsilon,\xi(y)} d\mu_g\Big|_{y=0}\\
&= \int_{\mathbb{R}^n_+} U(z)\chi_R(\varepsilon z)\left[U(z)\frac{\partial \chi_R}{\partial z_k}(\varepsilon z)+\frac{1}{\varepsilon}\frac{\partial U}{\partial z_k}(z)\chi_R(\varepsilon z)\right]\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}(y,\varepsilon z)\Big|_{y=0}|g(\varepsilon z)|^{1/2}dz\\
&\qquad= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+} U(z)\frac{\partial U}{\partial z_k}(z)\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}(y,\varepsilon z)\Big|_{y=0}|g(\varepsilon z)|^{1/2}dz+o(\varepsilon)\\
&\qquad\qquad= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+} U(z)\frac{U'(z)}{|z|}z_k\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}_k(y,\varepsilon z)\Big|_{y=0}|g(\varepsilon z)|^{1/2}dz+o(\varepsilon)
\end{aligned}
$$

where $\tilde{\mathcal{E}}$ is defined in Definition 6. Expanding in ε, by Lemma 8 and by (4) we obtain

$$
\begin{aligned}
&\frac{\partial}{\partial y_1}\int \frac{1}{2\varepsilon^n} W^2_{\varepsilon,\xi(y)} d\mu_g\Big|_{y=0}\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+} U(z)\frac{U'(z)}{|z|}z_k(-\delta_{1k}+\frac{1}{2}\varepsilon^2E^k_{ij}z_iz_j)(1-\varepsilon(n-1)Hz_n+\frac{1}{2}\varepsilon^2G_{ls}z_lz_s)dz+o(\varepsilon)\\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+} U(z)\frac{U'(z)}{|z|}\left[-z_1(1-\varepsilon(n-1)Hz_n+\frac{1}{2}\varepsilon^2G_{ls}z_lz_s)+\frac{1}{2}z_k\varepsilon^2E^k_{ij}z_iz_j\right]dz+o(\varepsilon)
\end{aligned}
$$

where $E^k_{ij}=\frac{\partial^2}{\partial z_i\partial z_j}\frac{\partial}{\partial y_1}\tilde{\mathcal{E}}_k(y,z)\Big|_{y=0,z=0}$ and $G_{ls}=\frac{\partial^2}{\partial z_i\partial z_j}|g(z)|^{1/2}\Big|_{z=0}$. By symmetry reason the only terms remaining are the ones containing $z_r^2z_n$, thus

$$
\frac{\partial}{\partial y_1}\int \frac{1}{2\varepsilon^n} W^2_{\varepsilon,\xi(y)} d\mu_g\Big|_{y=0} = \varepsilon\int_{\mathbb{R}^n_+} U(z)\frac{U'(z)}{|z|}\left[-z_1G_{n1}z_nz_1+z_kE^k_{kn}z_kz_n\right]dz+o(\varepsilon).
$$

By (6) we have that $G_{n1}(0)=-(n-1)\frac{\partial H}{\partial z_1}(0)$ and in light of Lemma 9 we get $E^k_{kn}=0$. We conclude that

$$
\begin{aligned}
\frac{\partial}{\partial y_1}\int \frac{1}{2\varepsilon^n} W^2_{\varepsilon,\xi(y)} d\mu_g\Big|_{y=0} &= \varepsilon\int_{\mathbb{R}^n_+} U(z)\frac{U'(z)}{|z|}(n-1)\left(\frac{\partial H}{\partial z_1}(0)\right)z_nz_1^2dz+o(\varepsilon)\\
&= \varepsilon\int_{\mathbb{R}^n_+}\frac{\partial}{\partial z_1}\left(\frac{1}{2}U^2(z)\right)(n-1)\left(\frac{\partial H}{\partial z_1}(0)\right)z_nz_1dz+o(\varepsilon)\\
&= -\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\frac{1}{2}U^2(z)z_ndz+o(\varepsilon)
\end{aligned}
$$

In the same way we get that

$$
\begin{aligned}
\frac{\partial}{\partial y_1}\int \frac{1}{p\varepsilon^n} W^p_{\varepsilon,\xi(y)} d\mu_g\Big|_{y=0} &= \varepsilon\int_{\mathbb{R}^n_+} U^{p-1}(z)\frac{U'(z)}{|z|}(n-1)\left(\frac{\partial}{\partial z_1}H(0)\right)z_nz_1^2dz+o(\varepsilon)\\
&= -\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\frac{1}{p}U^p(z)z_ndz+o(\varepsilon)
\end{aligned}
$$

Now we look at the last term

$$
I:=\frac{\partial}{\partial y_1}\int_{I_\xi(R)}\frac{\varepsilon^2}{2\varepsilon^n}\left|\nabla W_{\varepsilon,\xi(y)}\right|^2_g d\mu_g\Big|_{y=0}=\int_{I_\xi(R)}\frac{\varepsilon^2}{\varepsilon^n}g\left(\nabla W_{\varepsilon,\xi}\nabla\frac{\partial}{\partial y_1}W_{\varepsilon,\xi(y)}\right)\Big|_{y=0}d\mu_g
$$

and again, using Lemma 7 and the decay of U we have

$$I=\frac{1}{\varepsilon}\int_{\mathbb{R}^n_+} g^{ij}(\varepsilon z)\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\left(\frac{\partial U}{\partial z_k}\frac{\partial}{\partial y_1}\ \tilde{\mathcal{E}}_k(y,\varepsilon z))\Big|_{y=0}\right)|g(\varepsilon z)|^{1/2}dz+o(\varepsilon)$$

Recalling (2) (3) and (4), and set, with abuse of language, $h_{in}=h_{nj}=0$ for all $i,j=1,\dots,n$ we have

$$\begin{aligned} I &= \frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}(\delta_{ij}+2\varepsilon h_{ij}z_n+\frac{1}{2}\varepsilon^2\gamma^{ij}_{rt}z_rz_t)(1-\varepsilon(n-1)Hz_n+\frac{1}{2}\varepsilon^2G_{ls}z_lz_s)\times \\ &\quad\times\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\left(\frac{\partial U}{\partial z_k}(-\delta_{1k}+\frac{1}{2}\varepsilon^2E^k_{vw}z_vz_w)\right)dz+o(\varepsilon)\end{aligned}$$

where $E^k_{vw}=\frac{\partial^2}{\partial z_v\partial z_w}\frac{\partial}{\partial y_1}\ \tilde{\mathcal{E}}_k(y,z)\Big|_{y=0,z=0}$, $G_{ls}=\frac{\partial^2}{\partial z_l\partial z_s}|g(z)|^{1/2}\Big|_{z=0}$ and $\gamma^{ij}_{rt}=\frac{\partial^2}{\partial z_r\partial z_t}g^{ij}(z)\Big|_{z=0}$.

More explicitly

$$\begin{aligned} I=&-\frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\frac{\partial U}{\partial z_1}dz+\int_{\mathbb{R}^n_+}(n-1)Hz_n\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\frac{\partial U}{\partial z_1}dz \\ &-\frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}G_{ls}z_lz_s\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\frac{\partial U}{\partial z_1}dz-2\int_{\mathbb{R}^n_+}h_{ij}z_n\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\frac{\partial U}{\partial z_1}dz \\ &+2\varepsilon\int_{\mathbb{R}^n_+}(n-1)Hh_{ij}z_n^2\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\frac{\partial U}{\partial z_1}dz-\frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\gamma^{ij}_{rt}z_rz_t\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_j}\frac{\partial U}{\partial z_1}dz \\ &+\frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\left(\frac{\partial U}{\partial z_k}E^k_{vw}z_vz_w\right)dz+o(\varepsilon) \\ &:=I_1+I_2+I_3+I_4+I_5+I_6+I_7.\end{aligned}$$

Easily we have

$$I_1=-\frac{1}{\varepsilon}\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial}{\partial z_i}\frac{\partial U}{\partial z_1}dz=-\frac{1}{2\varepsilon}\int_{\mathbb{R}^n_+}\frac{\partial}{\partial z_1}|\nabla U|^2dz=0,$$

and, in a similar way, by integration by parts

$$I_2=(n-1)H(0)\frac{1}{2}\int_{\mathbb{R}^n_+}z_n\frac{\partial}{\partial z_1}|\nabla U|^2dz=0$$

$$I_3=-\frac{1}{4}\varepsilon\int_{\mathbb{R}^n_+}G_{ls}(0)z_lz_s\frac{\partial}{\partial z_1}|\nabla U|^2dz=\varepsilon\frac{1}{2}\int_{\mathbb{R}^n_+}G_{1s}(0)z_s|\nabla U|^2dz.$$

Moreover, by symmetry reasons, the only non zero contribution comes from the term containing z_n, so, by (6),

$$I_3=\varepsilon\frac{1}{2}\int_{\mathbb{R}^n_+}G_{1n}(0)z_n|\nabla U|^2dz=-\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\frac{1}{2}|\nabla U|^2z_ndz.$$

Since h_{ij} is symmetric, we have

$$I_4=-\int_{\mathbb{R}^n_+}h_{ij}(0)z_n\frac{\partial}{\partial z_1}\left(\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_j}\right)dz=0$$

by integration by parts and, in a similar way, we obtain also that $I_5=0$.

For I_7 it holds

$$I_7=\frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_i\partial z_k}E^k_{vw}z_vz_wdz+\frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_k}\left(E^k_{vi}z_v+E^k_{iw}z_w\right)dz$$

and, since $\frac{\partial U}{\partial z_i}=\frac{U'(z)}{|z|}z_i$

$$\int_{\mathbb{R}^n_+}\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_k}E^k_{vi}z_v=\int_{\mathbb{R}^n_+}\left(\frac{U'(z)}{|z|}\right)^2E^k_{vi}z_iz_kz_vdz. \tag{22}$$

The only non zero integral are the ones of the form $z_r^2z_n$ and, since $E^n_{rr}=0$ (Lemma 9), all the terms in (22) are 0. With a similar argument, since $\frac{\partial U}{\partial z_i\partial z_k}=\frac{U'(z)}{|z|}\delta_{ik}+\frac{U''(z)|z|-U'}{|z|^3}z_kz_i$ we can conclude that $I_7=0$.

Finally let us consider I_6. Since g^{ij} is symmetric, integrating by parts we have

$$I_6=-\frac{1}{4}\varepsilon\int_{\mathbb{R}^n_+}\gamma^{ij}_{rt}z_rz_t\frac{\partial}{\partial z_1}\left(\frac{\partial U}{\partial z_i}\frac{\partial U}{\partial z_j}\right)dz=\frac{1}{2}\varepsilon\int_{\mathbb{R}^n_+}\gamma^{ij}_{1t}\left(\frac{U'(z)}{|z|}\right)^2z_iz_jz_tdz$$

By (3) we have that $\gamma^{nj}_{1t}=\gamma^{in}_{1t}=0$ for all $i,j,t=1,\dots,n$. In addition, for symmetry reasons, only the term which contains $z_r^2z_n$ gives a non zero contribution, and by (2) and (21)

$$\begin{aligned}I_6&=\frac{1}{2}\varepsilon\sum_{i=1}^{n-1}\int_{\mathbb{R}^n_+}\gamma^{ii}_{1n}\left(\frac{U'(z)}{|z|}\right)^2z_i^2z_ndz=\varepsilon\sum_{i=1}^{n-1}\frac{\partial h^{ii}}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(z)}{|z|}\right)^2z_i^2z_ndz.\\&=\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(z)}{|z|}\right)^2z_1^2z_ndz=\frac{1}{2}\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(z)}{|z|}\right)^2z_n^3dz.\end{aligned}$$

Concluding we have

$$\begin{aligned}\left(\frac{\partial}{\partial y_1}J_\varepsilon(W_{\varepsilon,\xi(y)})\right)_{|y=0}&=-\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\left[\frac{1}{2}|\nabla U|^2+\frac{1}{2}|U|^2-\frac{1}{p}|U|^p\right]z_ndz\\&\quad+\frac{1}{2}\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(z)}{|z|}\right)^2z_n^3dz.\end{aligned}$$

and, by Lemma 17,

$$\left.\frac{\partial}{\partial y_1}J_\varepsilon(W_{\varepsilon,\xi(y)})\right|_{y=0}=-\frac{1}{2}\varepsilon(n-1)\frac{\partial H}{\partial z_1}(0)\int_{\mathbb{R}^n_+}\left(\frac{U'(z)}{|z|}\right)^2z_n^3dz$$

which completes the proof. □

Appendix A. Technical lemmas

Here we collect a series of estimates that we used in the paper as well as the proof of some Lemma which was previously claimed.

Lemma 17. *It holds*

$$\begin{aligned}\int_{\mathbb{R}^n_+}\left(\partial_{z_n}U\right)^2z_ndz&=\int_{\mathbb{R}^n_+}\left(\frac{U'(z)}{|z|}\right)^2z_n^3dz\\&=\frac{1}{2}\int_{\mathbb{R}^n_+}|\nabla U|^2z_ndz+\frac{1}{2}\int_{\mathbb{R}^n_+}U^2z_ndz-\frac{1}{p}\int_{\mathbb{R}^n_+}U^pz_ndz\end{aligned}$$

Proof. We multiply $-\Delta U$ by $z_n^2\partial_{z_n}U$, we integrate over $\mathbb{R}^n_+$ and we integrate by parts, obtaining

$$\begin{aligned}
-\int_{\mathbb{R}^n_+}\Delta U z_n^2\partial_{z_n}U dz = &\int_{\mathbb{R}^n_+}(\Delta\partial_{z_n}U)\, z_n^2 U dz + 2\int_{\mathbb{R}^n_+}\Delta U z_n U dz = \\
&= -\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}\partial_{z_n}U)\, z_n^2\partial_{z_i}U dz - 2\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}\partial_{z_n}U)\, z_n\delta_{in}U dz \\
&\quad - 2\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}U)\, z_n\partial_{z_i}U dz - 2\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}U)\,\delta_{in}U dz = \\
&= -\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}\partial_{z_n}U)\, z_n^2\partial_{z_i}U dz - 2\int_{\mathbb{R}^n_+}\left(\partial^2_{z_n}U\right) z_n U dz \\
&\quad - 2\int_{\mathbb{R}^n_+}|\nabla U|^2 z_n dz - 2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)\, U dz.
\end{aligned}$$

Now, again by integration by parts

$$\begin{aligned}
-\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_i}\partial_{z_n}U)\, z_n^2\partial_{z_i}U dz &= \int_{\mathbb{R}^n_+}(\partial_{z_n}U)\, z_n^2\Delta U dz + 2\int_{\mathbb{R}^n_+}\sum_{i=1}^n(\partial_{z_n}U)\,\delta_{in}z_n\partial_{z_i}U dz \\
&= \int_{\mathbb{R}^n_+}(\partial_{z_n}U)\, z_n^2\Delta U dz + 2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)^2 z_n dz,
\end{aligned}$$

thus

$$\begin{aligned}
-\int_{\mathbb{R}^n_+}\Delta U z_n^2\partial_{z_n}U dz = &\int_{\mathbb{R}^n_+}(\partial_{z_n}U)\, z_n^2\Delta U dz + 2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)^2 z_n dz \\
&- 2\int_{\mathbb{R}^n_+}\left(\partial^2_{z_n}U\right) z_n U dz - 2\int_{\mathbb{R}^n_+}|\nabla U|^2 z_n dz - 2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)\, U dz
\end{aligned}$$

that is

$$\begin{aligned}
-\int_{\mathbb{R}^n_+}\Delta U z_n^2\partial_{z_n}U dz = &\int_{\mathbb{R}^n_+}(\partial_{z_n}U)^2 z_n dz - \int_{\mathbb{R}^n_+}\left(\partial^2_{z_n}U\right) z_n U dz \\
&- \int_{\mathbb{R}^n_+}|\nabla U|^2 z_n dz - \int_{\mathbb{R}^n_+}(\partial_{z_n}U)\, U dz.
\end{aligned}$$

Now

$$0 = \int_{\mathbb{R}^n_+}\partial_{z_n}\left(z_n U\partial_{z_n}U\right) dz = \int_{\mathbb{R}^n_+}U\partial_{z_n}U + z_n\left(\partial_{z_n}U\right)^2 + z_n U\partial^2_{z_n}U dz,$$

and we get

$$-\int_{\mathbb{R}^n_+}\Delta U z_n^2\partial_{z_n}U dz = 2\int_{\mathbb{R}^n_+}(\partial_{z_n}U)^2 z_n dz - \int_{\mathbb{R}^n_+}|\nabla U|^2 z_n dz. \tag{23}$$

In a similar way we prove that

$$\int_{\mathbb{R}^n_+}U z_n^2\partial_{z_n}U dz = -\int_{\mathbb{R}^n_+}U^2 z_n dz, \tag{24}$$

$$\int_{\mathbb{R}^n_+}U^{p-1} z_n^2\partial_{z_n}U dz = -\frac{2}{p}\int_{\mathbb{R}^n_+}U^p z_n dz. \tag{25}$$

Now, multiplicating by $z_n^2\partial_{z_n}U$ both terms of (8), integrating over $\mathbb{R}^n_+$ and using (23), (24), (25) we finally obtain the claim. □

The following lemma collects several estimates on $Z^j_{\varepsilon,\xi}$.

Lemma 18. *There exists $\varepsilon_0>0$ and $c>0$ such that, for any $\xi_0\in\partial M$ and for any $\varepsilon\in(0,\varepsilon_0)$ it holds*

$$\|Z^h_{\varepsilon,\xi}-i^*\left[f'(W_{\varepsilon,\xi})Z^h_{\varepsilon,\xi}\right]\|_\varepsilon\le c\varepsilon^{1+\frac{N}{p'}} \tag{26}$$

$$\left\|\frac{\partial}{\partial y_h}Z^l_{\varepsilon,\xi(y)}\right\|_\varepsilon=O\left(\frac{1}{\varepsilon}\right),\quad \left\|\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right\|_\varepsilon=O\left(\frac{1}{\varepsilon}\right), \tag{27}$$

$$\left\langle Z^l_{\varepsilon,\xi_0},\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|_{y=0}}\right\rangle=-\frac{1}{\varepsilon}c\delta_{lh}+o\left(\frac{1}{\varepsilon}\right), \tag{28}$$

$$\left\|\frac{1}{\varepsilon}Z^h_{\varepsilon,\xi_0}+\left(\frac{\partial}{\partial y_h}W_{\varepsilon,\xi(y)}\right)_{|_{y=0}}\right\|_\varepsilon\le c\varepsilon \tag{29}$$

for $h=1,\dots,n-1$, $l=1,\dots,n$

Proof. The proof of (26) is similar to Lemma 11 and will be omitted. The other three estimates are similar to Lemma 6.1, Lemma 6.2 and Lemma 6.3 of [15], which we refer to for the proof of the claim. □

Proof of Lemma 10. By contradiction we assume that there exist sequences $\varepsilon_k\to0$, $\xi_k\in\partial M$ with $\xi_k\to\xi\in\partial M$ and $\phi_k\in K^\perp_{\varepsilon_k,\xi_k}$ with $\|\phi\|_{\varepsilon_k}=1$ such that

$$L_{\varepsilon_k,\xi_k}(\phi_k)=\psi_k \text{ with } \|\psi_k\|_{\varepsilon_k}\to0 \text{ for } k\to+\infty.$$

By definition of L_{ε_k,ξ_k}, there exists $\zeta_k\in K_{\varepsilon_k,\xi_k}$ such that

$$\phi_k-i^*_{\varepsilon_k}\left[f'(W_{\varepsilon_k,\xi_k})\phi_k\right]=\psi_k+\zeta_k. \tag{30}$$

We prove that $\|\zeta_k\|_{\varepsilon_k}\to0$ for $k\to+\infty$. Let $\zeta_k=\sum_{j=1}^{n-1}a^k_jZ^j_{\varepsilon_k,\xi_k}$, $Z^i_{\varepsilon,\xi}$ being defined in (11). By (30), using that $\phi_k,\psi_k\in K^\perp_{\varepsilon_k,\xi_k}$ we have

$$\begin{aligned}\sum_{j=1}^{n-1}a^k_j\left\langle Z^j_{\varepsilon_k,\xi_k},Z^h_{\varepsilon_k,\xi_k}\right\rangle_{\varepsilon_k}&=-\left\langle i^*_{\varepsilon_k}\left[f'(W_{\varepsilon_k,\xi_k})\phi_k\right],Z^h_{\varepsilon_k,\xi_k}\right\rangle_{\varepsilon_k}=\\&=-\frac{1}{\varepsilon_k^n}\int_Mf'(W_{\varepsilon_k,\xi_k})\phi_kZ^h_{\varepsilon_k,\xi_k}d\mu_g.\end{aligned}\tag{31}$$

By elementary properties of φ^j we have that

$$\left\langle Z^j_{\varepsilon_k,\xi_k},Z^j_{\varepsilon_k,\xi_k}\right\rangle_{\varepsilon_k}=C\delta_{jh}+o(1)\qquad\text{for all } j,h=1,\dots,n-1 \tag{32}$$

where C is a positive constant.

We set

$$\tilde\phi_k:=\begin{cases}\phi_k\left(\psi^\partial_{\xi_k}(\varepsilon_kz)\right)\chi_R(\varepsilon_kz) & \text{if } z\in D^+(R)\\ 0 & \text{otherwise}\end{cases}$$

Easily we get that $\|\tilde{\phi}_k\|_{H^1(\mathbb{R}^n)} \le c\|\phi_k\|_{\varepsilon_k} \le c$ for some positive constant c. Thus, there exists $\tilde{\phi} \in H^1(\mathbb{R}^n)$ such that $\tilde{\phi}_k \to \tilde{\phi}$ weakly in $H^1(\mathbb{R}^n)$ and strongly in $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for all $2 \le p < 2^*$ if $n \ge 3$ or $p \ge 2$ if $n = 2$.

We recall that $\phi_k \in K^{\perp}_{\varepsilon_k,\xi_k}$, so

$$
\begin{aligned}
-\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})\phi_k, Z^h_{\varepsilon_k,\xi_k} d\mu_g &= \left\langle \phi_k, Z^h_{\varepsilon_k,\xi_k}\right\rangle_\varepsilon - \frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})\phi_k, Z^h_{\varepsilon_k,\xi_k} d\mu_g \\
&= \frac{1}{\varepsilon_k^n}\int_M \left[\varepsilon_k^2 \nabla\phi_k \nabla Z^h_{\varepsilon_k,\xi_k} + Z^h_{\varepsilon_k,\xi_k}\phi_k - f'(W_{\varepsilon_k,\xi_k})\phi_k, Z^h_{\varepsilon_k,\xi_k}\right] d\mu_g \\
&= \int_{D^+(R/\varepsilon_k)} \left[\sum_{l,m=1}^n g^{lm}(\varepsilon_k z)\frac{\partial \tilde{\phi}_k}{\partial z_l}\frac{\partial\left(\varphi^h(z)\right)}{\partial z_m} + \tilde{\phi}_k \varphi^h(z)\right] |g(\varepsilon_k z)|^{\frac{1}{2}} dy \\
&\quad - \int_{D^+(R/\varepsilon_k)} f'(U(z)\chi_R(\varepsilon_k z))\tilde{\phi}_k \varphi^h(z) |g(\varepsilon_k z)|^{\frac{1}{2}} dz + o(1) \\
&= \int_{\mathbb{R}^n} \nabla\tilde{\phi}\nabla\varphi^h + \tilde{\phi}\varphi^h - f'(U)\tilde{\phi}\varphi^h dz + o(1) = o(1)
\end{aligned}
$$

because φ^h is a weak solution of the linearized problem (9). So we can rewrite (31), obtaining

$$
ca_h^k + o(1) = \sum_{j=1}^{n-1} a_j^k \left\langle Z^j_{\varepsilon_k,\xi_k}, Z^h_{\varepsilon_k,\xi_k}\right\rangle_{\varepsilon_k} = -\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})\phi_k, Z^h_{\varepsilon_k,\xi_k} d\mu_g = o(1),
$$

so $a_h^k \to 0$ for all h while $k \to +\infty$, thus $\|\zeta_k\|_{\varepsilon_k} \to 0$ for $k \to +\infty$.

Setting $u_k := \phi_k - \psi_k - \zeta_k$, (30) can be read as

$$
\left\{
\begin{array}{ll}
-\varepsilon_k^2 \Delta_g u_k + u_k = f'(W_{\varepsilon_k,\xi_k})u_k + f'(W_{\varepsilon_k,\xi_k})(\psi_k + \zeta_k) & \text{in } M \\
\dfrac{\partial u_k}{\partial \nu} = 0 & \text{on } \partial M.
\end{array}
\right. \tag{33}
$$

Multiplying (33) by u_k and integrating by parts we get

$$
\|u_k\|_{\varepsilon_k} = \frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})u_k^2 + f'(W_{\varepsilon_k,\xi_k})(\psi_k + \zeta_k)u_k. \tag{34}
$$

By Holder inequality, and recalling that $|u|_{\varepsilon,p} \le c\|u\|_\varepsilon$, we have

$$
\begin{aligned}
\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})(\psi_k + \zeta_k)u_k &\le \left(\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})^{\frac{n}{2}}\right)^{\frac{2}{n}} |u_k|^{\frac{n-2}{2n}}_{\varepsilon_k,\frac{2n}{n-2}} |\psi_k + \zeta_k|^{\frac{n-2}{2n}}_{\varepsilon_k,\frac{2n}{n-2}} \\
&\le c\left(\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})^{\frac{n}{2}}\right)^{\frac{2}{n}} \|u_k\|^{\frac{n-2}{2n}}_{\varepsilon_k} \|\psi_k + \zeta_k\|^{\frac{n-2}{2n}}_{\varepsilon_k}.
\end{aligned} \tag{35}
$$

Now,

$$
\begin{aligned}
\frac{1}{\varepsilon_k^n}\int_M f'(W_{\varepsilon_k,\xi_k})^{\frac{n}{2}} d\mu_g &\le \frac{1}{\varepsilon_k^n}\int_{I_{\xi_k}(R)} \left(U_\varepsilon\left(\left(\psi^\partial_{\xi_k}\right)^{-1}(x)\right)\right)^{\frac{n(p-2)}{2}} d\mu_g \\
&\le c\int_{D^+(R/\varepsilon)} (U(z))^{\frac{n(p-2)}{2}} dz \le c
\end{aligned} \tag{36}
$$

for some positive constant c.

Combining (34), (35), (36), and recalling that $\|u_k\|_{\varepsilon_k} \to 1$, $\|\psi_k + \zeta_k\|_{\varepsilon_k} \to 0$

while $k \to +\infty$,we get

$$\frac{1}{\varepsilon_k^n} \int_M f'(W_{\varepsilon_k,\xi_k}) u_k^2 \to 1 \text{ while } k \to +\infty. \tag{37}$$

We will see how this leads us to a contradiction.

We set

$$\tilde{u}_k(z) := u_k \left(\psi_{\xi_k}^{\partial}(\varepsilon_k z)\right) \chi_R(\varepsilon_k z) \text{ for } z \in \mathbb{R}_+^n$$

We have that

$$\|\tilde{u}_k\|_{H^1(\mathbb{R}^n)} \le c \|u_k\|_{\varepsilon_k} \le c,$$

so, up to subsequence, there exists $\tilde{u} \in H^1(\mathbb{R}_+^n)$ such that $\tilde{u}_k \to \tilde{u}$ weakly in $H^1(\mathbb{R}_+^n)$and strongly in $L^p_{\mathrm{loc}}(\mathbb{R}_+^n)$, $p \in (2, 2^*)$ if $n \ge 3$ or $p > 2$ if $n = 2$. By (33) we deduce that

$$\begin{cases} -\Delta \tilde{u} + \tilde{u} = f'(U)\tilde{u} & \text{in } \mathbb{R}_+^n \\ \\ \dfrac{\partial \tilde{u}}{\partial x_n} = 0 & \text{on } \{x_n = 0\}. \end{cases} \tag{38}$$

We prove also that

$$\left\langle \varphi^h, \tilde{u} \right\rangle_{H^1} = 0 \text{ for all } h \in 1, \dots, n-1. \tag{39}$$

In fact, since $\phi_k, \psi_k \in K_{\varepsilon,\xi}^{\perp}$ and $\|\zeta_k\|_{\varepsilon_k} \to 0$, we have

$$\left| \left\langle Z_{\varepsilon_k,\xi_k}^h, u_k \right\rangle_{\varepsilon_k} \right| = \left| \left\langle Z_{\varepsilon_k,\xi_k}^h, \zeta_k \right\rangle_{\varepsilon_k} \right| \le \|Z_{\varepsilon_k,\xi_k}^h\|_{\varepsilon_k} \|\zeta_k\|_{\varepsilon_k} = o(1). \tag{40}$$

On the other hand, by direct computation, we get

$$\begin{aligned} \left\langle Z_{\varepsilon_k,\xi_k}^h, u_k \right\rangle_{\varepsilon_k} &= \frac{1}{\varepsilon_k^n} \int_M \varepsilon_k^2 g(\nabla Z_{\varepsilon_k,\xi_k}^h \nabla u_k) + Z_{\varepsilon_k,\xi_k}^h u_k \\ &= \int_{D^+(R/\varepsilon_k)} \sum_{l,m=1}^n g^{lm}(\varepsilon_k z) \frac{\partial \left(\varphi^h(z)\chi_R(\varepsilon_k z)\right)}{\partial z_l} \frac{\partial \tilde{u}_k}{\partial z_m} |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\ &\quad + \int_{D^+(R/\varepsilon_k)} \varphi^h(z)\chi_R(\varepsilon_k z)\tilde{u}_k |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\ &= \int_{\mathbb{R}_+^n} \left(\nabla\varphi^h \nabla\tilde{u} + \varphi^h \tilde{u}\right) dz + o(1). \end{aligned} \tag{41}$$

So, by (40) and (41) we obtain (39).

Now (39) and (38) imply that $\tilde{u} = 0$. Thus

$$\begin{aligned} \frac{1}{\varepsilon_k^n} \int_M f'(W_{\varepsilon_k,\xi_k}(x)) u_k^2(x) d\mu_g &\le \frac{1}{\varepsilon_k^n} \int_{I_g(R)} f'\left(U_\varepsilon\left(\left(\psi_{\xi_k}^{\partial}\right)^{-1}(x)\right)\right) u_k^2(x) d\mu_g \\ &= c \int_{D^+(R/\varepsilon_k)} f'(U(z)) \tilde{u}_k^2(z) = o(1) \end{aligned}$$

which contradicts (37). This concludes the proof. □

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