

On certain permutation groups and sums of two squares

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Abstract

We consider the question of existence of ramified covers over \mathbb{P}_1 matching certain prescribed ramification conditions. This problem has already been faced in a number of papers, but we discuss alternative approaches for an existence proof, involving elliptic curves and universal ramified covers with signature. We also relate the geometric problem with finite permutation groups and with the Fermat-Euler Theorem on the representation of a prime as a sum of two squares.
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Introduction

The present note does not contain new results. It may be considered as an *addendum* to [4], presenting another viewpoint for one of the results proved there. The paper [4] is concerned with the so-called *Hurwitz Existence Problem* for branched covers between real, closed, and connected surfaces. To any such cover one can associate certain permutations whose cycle decompositions lengths are the branching local degrees, and the problem asks to “describe” the cycle lengths coming from an actual cover. In [4] the authors discuss a number of situations, using an approach based on 2-orbifolds. More specifically, for various infinite families of cycles lengths they establish numerical criteria for the existence of the cover in terms of the (candidate) total degree.

Thanks to various results obtained over the time (see in particular [1] and the references quoted in [4]), the Hurwitz Existence Problem is only

open when the target of the (candidate) cover is the sphere, that one can view as the Riemann sphere $\mathbb{P}_1(\mathbb{C})$. Moreover, by the Riemann Existence Theorem, one knows that any topological branched cover over the sphere can be realized as a ramified cover of $\mathbb{P}_1(\mathbb{C})$ by a complex algebraic curve (see [2, 8]). One of our purposes here is precisely to present one of the conclusions of [4] in this last perspective.

We shall focus on an example related with representations of integers as sums of two squares. It follows directly from [4] that a positive integer d congruent to 1 modulo 4 is likewise representable if and only if there exist three permutations on d letters satisfying certain simple conditions (see Proposition 1.4 below). In turn, the complex-algebraic viewpoint will show that the existence of these permutations is related to the endomorphisms of the elliptic curve E with Weierstrass equation $y^2 = x^3 - x$, namely to the complex torus $E = \mathbb{C}/\mathbb{Z}[i]$. The link with representations of an integer as a sum of two squares becomes apparent, since the degree of an endomorphism of such an elliptic curve is always the sum of two squares. This is a well-known situation, and we also remark that the link already appears implicitly in [6] and especially in [2]. (The latter paper, among many other things, also contains explicit constructions of algebraic covers associated to the alluded permutations and several other similar ones.)

The second purpose of this note is to show the following somewhat surprising fact: putting together some of the different viewpoints on the Hurwitz Existence Problem one can get a proof of the Fermat-Euler theorem, which asserts that a prime p congruent to 1 modulo 4 is a sum of two squares. Of course, such a proof has a small interest in itself, since it relies on deep results, whereas many elegant and simple proofs are known. However the connection seems a striking one to us, and it raises the question whether a direct proof exists in purely combinatorial terms related to the said permutations.

The paper is organized as follows: in Section 1 we will recall one of the results from [4], connect it with the Riemann Existence Theorem, and interpret and reprove it in terms of $\text{End}(E)$. Then, in Section 2, we will deduce from this connection and interpretation a proof of the Fermat-Euler Theorem.

1 Certain branched covers of the Riemann Sphere

We start by recalling Theorem 0.4 from [4], that we reformulate using a slightly different language. To give the statement we will need the following definition. Let $\pi : \tilde{\Sigma} \rightarrow \Sigma$ be a branched topological cover between real, closed, and connected surfaces. We say that π has *branching type* (a_1, \dots, a_r) over a point $P \in \Sigma$, where a_1, \dots, a_r are positive integers, if $\pi^{-1}(P)$ consists of r distinct points Q_1, \dots, Q_r such that locally at Q_i the map π may be represented as $z \mapsto z^{a_i}$, on viewing Σ and $\tilde{\Sigma}$ as locally homeomorphic to a complex disk. Henceforth we will deal with the case $\Sigma = \mathbb{P}_1(\mathbb{C})$. We have:

Theorem 1.1. [4, Theorem 0.4] Suppose $d = 4k + 1$ for some $k \in \mathbb{N}$. The following conditions are equivalent:

(I) There exists a branched cover $\tilde{\Sigma} \rightarrow \mathbb{P}_1(\mathbb{C})$ of degree d , ramified over three points, with branching types

$$(1, 4, \dots, 4), \quad (1, 4, \dots, 4), \quad (1, 2, \dots, 2);$$

(II) $d = x^2 + y^2$ for some $x, y \in \mathbb{N}$.

Remark 1.2. For a branched cover $\tilde{\Sigma} \rightarrow \mathbb{P}_1(\mathbb{C})$ of degree $d = 4k + 1$ and ramified over three points, the branching types $(1, 4, \dots, 4)$, $(1, 4, \dots, 4)$, and $(1, 2, \dots, 2)$ force $\tilde{\Sigma}$ to be $\mathbb{P}_1(\mathbb{C})$ too. In fact the types have lengths $k + 1$, $k + 1$, and $2k + 1$, and the Riemann-Hurwitz formula shows that if the genus of $\tilde{\Sigma}$ is g then $2(1 - g) - (k + 1) - (k + 1) - (2k + 1) = (4k + 1) \cdot (2 - 3)$, which implies that $g = 0$.

By this remark, from now on we only deal with the case $\tilde{\Sigma} = \mathbb{P}_1(\mathbb{C})$.

Remark 1.3. Up to an automorphism of the target $\mathbb{P}_1(\mathbb{C})$ one can suppose without loss of generality that, if a branched cover $\mathbb{P}_1(\mathbb{C}) \rightarrow \mathbb{P}_1(\mathbb{C})$ has three branching points, these points are 0, 1, and ∞ . For a cover as in Theorem 1.1 we will always assume that the branching types are $(1, 4, \dots, 4)$ over 0 and 1, and $(1, 2, \dots, 2)$ over ∞ .

The proofs in [4] employ the geometry of 2-obifolds. For Theorem 1.1 they exploit in particular the fact that $S^2(4, 4, 2)$, namely the sphere with three

cone points of orders 4, 4, and 2, bears a Euclidean geometric structure which is rigid up to rescaling. We will now sketch a proof of Theorem 1.1 in terms of branched covers of complex algebraic curves. We first note that considering the monodromy of a branched cover (a representation of the fundamental group of the complement of the branching points into the symmetric group \mathfrak{S}_d on the d letters $\{1, \dots, d\}$), one gets the following:

Proposition 1.4. *A cover as in the statement of Theorem 1.1 exists if and only if there exist permutations $\sigma_0, \sigma_1, \sigma_\infty \in \mathfrak{S}_d$ such that:*

- (i) $\sigma_0\sigma_1 = \sigma_\infty$;
- (ii) *The cycles in the decompositions of σ_0 and σ_1 have lengths $(1, 4, \dots, 4)$, while those in the decomposition of σ_∞ have lengths $(1, 2, \dots, 2)$;*
- (iii) *The subgroup of \mathfrak{S}_d generated by $\sigma_0, \sigma_1, \sigma_\infty$ is transitive on $\{1, \dots, d\}$.*

This permutation viewpoint was already employed in [3]; see also [6, 8]. We next spell out the following consequence of the Riemann Existence Theorem already anticipated in the Introduction:

Proposition 1.5. *If a topological cover of the sphere onto itself matching certain branching types exists, it can also be realized as a cover of algebraic curves $\pi : \mathbb{P}_1 \rightarrow \mathbb{P}_1$, defined over \mathbb{C} or even over the algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} in \mathbb{C} .*

Note that a map π as in this proposition will be a rational function with complex coefficients of a complex variable t , and the coefficients may actually be assumed, by specialization or operating with an automorphism of the domain, to lie in $\overline{\mathbb{Q}}$. This last fact implies that the absolute Galois group of $\overline{\mathbb{Q}}$ acts on the set of such covers, leading to Grothendieck's theory of *dessins d'enfants*, for which the interested reader is referred to [5].

Let us now concentrate on branching types as in Theorem 1.1. To discuss the existence of a corresponding map π as in Proposition 1.5 we further normalize the situation noting that, up to composition with an automorphism of the domain \mathbb{P}_1 , we can assume without loss of generality that 0 (respectively, 1, and ∞) is the unique unramified point above 0 (respectively, 1, and ∞). The branching conditions then imply that the map π , if any, can be expressed as

$$\pi(t) = \frac{tP^4(t)}{R^2(t)} = 1 + \frac{(t-1)Q^4(t)}{R^2(t)}$$

with $\deg P(t) = \deg Q(t) = k$, and $\deg R(t) = 2k$. More precisely, a cover as in Theorem 1.1 exists if and only if there exist polynomials $P(t), Q(t), R(t) \in \overline{\mathbb{Q}}[t]$ without multiple roots such that $tP(t)$, $(t-1)Q(t)$, and $R(t)$ are pairwise coprime, $\deg P(t) = \deg Q(t) = k$, $\deg R(t) = 2k$, and

$$tP^4(t) = (t-1)Q^4(t) + R^2(t). \quad (1)$$

Remark 1.6. When three such polynomials exist, they provide an *extremal example* of the *abc*-theorem for the field of rational functions in one variable (see, *e.g.*, [9] and [10]), a known analogue (due to Mason and Stothers) of the celebrated *abc*-conjecture of Masser and Oesterlé over number fields.

Proof of (I) \Rightarrow (II) in Theorem 1.1. Suppose the relevant branched cover exists, so there are polynomials $P(t)$, $Q(t)$, and $R(t)$ satisfying equation (1) and the other conditions. Dividing by $Q^4(t)$ in equation (1) we obtain the $\overline{\mathbb{Q}}(t)$ -point

$$\left(\frac{P(t)}{Q(t)}, \frac{R(t)}{Q^2(t)} \right)$$

on the genus-1 curve over $\mathbb{Q}(t)$ with affine equation $y^2 = tx^4 - (t-1)$. Its points over $\mathbb{Q}(t)$ form a finitely generated group, and the involved degrees correspond to the values of a Néron-Tate height; this is a quadratic form, so a connection with sums of squares begins to emerge. In this particular case the (elliptic) curve turns out to have constant j -invariant (equal to 1728), so the curve can in fact be defined over the constant field \mathbb{Q} , which allows one to analyze the situation in a much simpler way than in more general circumstances.

Indeed, consider the curve which is the normalization of the closure in \mathbb{P}_2 of the affine curve $v^2 = u^4 - 1$. Since it has genus 1, it becomes an elliptic curve E if we choose, *e.g.*, the point $O := (0, i)$ as origin (where, here and in the sequel, $i = \sqrt{-1}$). Note that E is isomorphic, as a complex torus, to the quotient $\mathbb{C}/\mathbb{Z}[i]$, namely it admits an automorphism of order four fixing the origin, given by $(u, v) \mapsto (iu, v)$. We shall prove the following result:

Proposition 1.7. *Let $P(t), Q(t), R(t)$ be three polynomials satisfying (1), with $\deg P(t) = \deg Q(t) = k$, $\deg R(t) = 2k$, such that $t \cdot (t-1) \cdot P(t) \cdot Q(t) \cdot R(t)$ has no multiple roots. Then, up to replacing the polynomial $R(t)$ by $-R(t)$, the map $(u, v) \mapsto (x, y)$, where*

$$x = u \frac{P(t)}{Q(t)}, \quad y = v \frac{R(t)}{Q^2(t)}, \quad t = \frac{u^4}{v^2}$$

induces an endomorphism of E (as an elliptic curve) of degree $d = 4k + 1$.

Proof. From $v^2 = u^4 - 1$ and $t = u^4/v^2$ it immediately follows that $u^4 = \frac{t}{t-1}$ and $v^2 = \frac{1}{t-1}$. Substituting in the expression for x and y one obtains $x^4 = \frac{t}{t-1} \frac{P^4(t)}{Q^4(t)}$ and $y^2 = \frac{1}{t-1} \frac{R^2(t)}{Q^4(t)}$, which shows that the equality $x^4 - 1 = y^2$ is equivalent to (1). This proves that the map $(u, v) \mapsto (x, y)$ indeed sends E to itself. Since x vanishes at O , the morphism sends O either to itself or to the point $(0, -i)$, in which case we replace $R(t)$ by $-R(t)$, and then get that the morphism fixes the origin, so it is also an endomorphism of E in the sense of elliptic curves. Its degree is easily seen to be $d = 4k + 1$. \square

Now, condition (II) of the statement follows, since the ring $\text{End}(E)$ of the endomorphisms of E (as an elliptic curve) is well-known to be isomorphic to $\mathbb{Z}[i]$, with the degree given by the square of the absolute value. More precisely, the endomorphism in Proposition 1.7 corresponds to the multiplication by a Gaussian integer $a + ib$ and we have $d = a^2 + b^2$, concluding the proof. \square

Proof of (II) \Rightarrow (I) in Theorem 1.1. Since for every Gaussian integer $a + ib$ there exists an endomorphism of E of degree $d = a^2 + b^2$, we must show that every endomorphism φ of E of odd degree d can be obtained as above for some polynomials $P(t)$, $Q(t)$, and $R(t)$. (Similar considerations are valid for even degrees, leading to slightly different analogue conclusions.) Let $\varphi \in \text{End}(E)$ be an endomorphism of degree $d = 4k + 1$. We have an expression of the form $\varphi(u, v) = (x(u, v), y(u, v))$ for suitable rational functions $x, y \in \mathbb{C}(E)$. Now, the degree-8 map $t : E \rightarrow \mathbb{P}_1$ given by $t = u^4/v^2$ is clearly invariant under the action of the subgroup G of the automorphisms of E (as algebraic curves) generated by

$$\alpha : (u, v) \mapsto (iu, v), \quad \beta : (u, v) \mapsto (u, -v).$$

Note that α generates the isotropy group of O , whereas β may be also described as the map $p \mapsto \delta - p$ where $\delta := (0, -i)$; of course, α has order four and β has order two; note that β is central in G , and $\alpha^2 \circ \beta : p \mapsto p + \delta$ is the (only) central translation in the automorphism group of E . Since G has order 8, t generates the field of invariants for G . On the other hand, it is easy to check that $t \circ \varphi = x^4/y^2$ is invariant under the action of G , therefore it is a rational function $Z(t)$ of t .

The function t has divisor on E of the shape $4((O) + (\delta)) - 2((Q_1) + (Q_i) + (Q_{-1}) + (Q_{-i}))$, where $Q_l = (l, 0)$. Also, the divisor of $u - i^s$ (for $s = 1, \dots, 4$)

is $2(Q_{i^*}) - (\infty_+) - (\infty_-)$, that of $v + u^2$ is $2((\infty_+) - (\infty_-))$ for some labeling of the poles of u, v , and finally that of $v - i - u^2$ is $2(\delta) - 2(\infty_+)$. It easily follows that δ, ∞_\pm have order 2 on E whereas the Q_i 's have order 4.

With this information, considering zeros and multiplicities, we see that $Z(t) = x^4/y^2 = tP^4(t)/R^2(t)$ for suitable polynomials $P(t)$ and $R(t)$, where $\deg(tP^4(t)) > \deg R^2(t)$ —here we use the fact that d is odd, so φ fixes δ and sends the set of poles of t to itself. Similarly, we have $x^4/y^2 - 1 = 1/y^2$, that we can rewrite as $(t - 1)Q^4(t)/R^2(t)$. Finally, the equation for E shows that $P(t)$, $Q(t)$, and $R(t)$ satisfy (1) and thus lead to a cover as in part (I) of the statement. \square

Remark 1.8. As a byproduct of our argument we have obtained a correspondence between permutations as in Proposition 1.4 and polynomials satisfying relation (1). Our proof actually also produces a relevant field of definition for the coefficients, as in [2].

We recall in passing that a Weierstrass model of the curve E employed above is obtained by the inverse transformations $\eta := \frac{u}{v-i}$, $\xi := \frac{u^2}{v-i} = u\eta$ and $u = \frac{\xi}{\eta}$, $v = i + \frac{\xi}{\eta^2}$, that lead to the equation $\eta^2 = \frac{1}{2i}(\xi^3 - \xi)$.

Galois structure We conclude this paragraph with some extra considerations on the constructions we encountered so far. First of all we prove the following:

Proposition 1.9. *With the above notation (in particular $G = \langle \alpha, \beta \rangle$ is the group defined in the previous proof), the map $t \circ \varphi = Z(t) =: z$ defines a Galois cover $E \rightarrow \mathbb{P}_1$, whose Galois group Γ (of order $8d$) is*

$$\Gamma = \{p \mapsto gp + \kappa : g \in G, \kappa \in \text{Ker } \varphi\}. \quad (2)$$

We begin with a preliminary result:

Lemma 1.10. *Let $G = \langle \alpha, \beta \rangle$ be the group defined above. Let $\varphi : E \rightarrow E$ be an isogeny of odd degree. Then for every $g \in G$ one has $g \circ \varphi = \varphi \circ g$.*

Proof. Clearly φ commutes with α , since both are isogenies. To prove that φ commutes with β , we shall prove the equivalent fact that φ commutes with $\alpha^2 \circ \beta$, which is the translation by δ . To this end, it is useful to think in terms of the actions of φ and $\alpha^2 \circ \beta$ on the complex plane \mathbb{C} . The latter corresponds

to the multiplication by $\frac{1+i}{2}$, while φ corresponds to the multiplication by a Gaussian integer $a+ib$ with $a^2+b^2 \equiv 1 \pmod{2}$. Now, we have to prove that the functions $\mathbb{C} \ni z \mapsto (a+ib)z + \frac{1+i}{2}$ and $\mathbb{C} \ni z \mapsto (a+ib)(z + \frac{1+i}{2})$ coincide modulo $\mathbb{Z}[i]$. This is equivalent to the fact that $(a+ib)\frac{1+i}{2} - \frac{1+i}{2} \in \mathbb{Z}[i]$, which easily follows from the hypothesis that $a^2 + b^2$ is odd. \square

Proof of Proposition 1.9. The map $z = t \circ \varphi : E \rightarrow \mathbb{P}_1$ induces a field extension $\overline{\mathbb{Q}}(E)/\overline{\mathbb{Q}}(z)$ of degree $8d$. Let us prove that it is invariant under the action of the group Γ defined above. Let $p \in E$, $g \in G$ and $\kappa \in \text{Ker } \varphi$. Clearly, $\varphi(gp + \kappa) = \varphi(gp)$; now, by Lemma 1.10, we have $\varphi(gp) = g(\varphi(p))$ and since t is invariant by G we have $z(p) = (t \circ \varphi)(p) = z(gp + \kappa)$ as wanted. It remains to prove that Γ has order $8d$; this is due to the fact that φ has odd degree, so $\text{Ker } \varphi$ has odd order. Then the subgroup of translations in Γ has order $2 \deg \varphi$; more precisely Γ is also given as the extension

$$\{0\} \rightarrow \langle \delta \rangle \oplus \text{Ker } \varphi \rightarrow \Gamma \rightarrow \{1, i, -1, -i\} \rightarrow \{1\},$$

where the map $\Gamma \rightarrow \{1, i, -1, -i\}$ denotes the action on the invariant differentials. From this representation, it is clear that its order is $8d$. Hence Γ is the Galois group of the cover $z = t \circ \varphi : E \rightarrow \mathbb{P}_1$. \square

Proposition 1.9 implies in particular that the Galois closure of the equation in t over $\overline{\mathbb{Q}}(z)$ given by $Z(t) = z$ is contained in the above extension $\overline{\mathbb{Q}}(E)/\overline{\mathbb{Q}}(z)$. However the Galois closure, whose Galois group is generated by the permutations $\sigma_0, \sigma_1, \sigma_\infty$ corresponding to our cover as in Proposition 1.4, is actually smaller: in fact, as already noticed, the element $\gamma := \alpha^2 \circ \beta$ acts on (u, v) as $\gamma(u, v) = (-u, -v)$, namely as a translation by δ , and hence fixes the field $\overline{\mathbb{Q}}(t)$. We have also already remarked that γ is in the center of Γ . Therefore the said Galois closure is contained in the fixed field of γ , and is in fact equal to it, because no subgroup of G larger than $\langle \gamma \rangle$ is normal in Γ . This fixed field of γ is easily seen to be $\overline{\mathbb{Q}}(u^2, v^2, uv)$. If we set $\sigma := v/u^3$ and $\tau := -1/u^2$ we find that σ and τ generate this field and $\sigma^2 = \tau^3 - \tau$. This is a Weierstrass equation for an elliptic curve E^* (again isomorphic to E) which is the quotient of E by the order-2 group of automorphisms generated by γ .

Remark 1.11. Equation (2) yields an explicit representation of the group generated by our three permutations $\sigma_0, \sigma_1, \sigma_\infty$, which is isomorphic to $\Gamma/\langle \gamma \rangle$, and has order $4d$.

Remark 1.12. Alternative proofs based on techniques similar to those employed here are possible also for Theorems 0.5 and 0.6 in [4].

2 Sums of two squares

To proceed we spell out the following consequence of the results established in Section 1 (and essentially contained in Theorem 1.1):

Proposition 2.1. *Given a positive integer d congruent to 1 modulo 4, there exist permutations $\sigma_0, \sigma_1, \sigma_\infty \in \mathfrak{S}_d$ satisfying the conditions (i), (ii), and (iii) of Proposition 1.4 if and only if d is a sum of two squares. If they exist, $\sigma_0, \sigma_1, \sigma_\infty$ generate a group of order $4d$.*

We will now specialize to the case where d is a prime number and construct the permutations explicitly.

Proposition 2.2. *Let p be a prime congruent to 1 modulo 4. Then the group \mathbb{F}_p^* has an element ℓ of order 4. Consider the affine automorphisms L and T of the line \mathbb{A}^1 over \mathbb{F}_p defined by $L(x) = \ell x$ and $T(x) = x + 1$, and the permutations*

$$\sigma_0 := L, \quad \sigma_1 = T^{-1}LT, \quad \sigma_\infty := LT^{-1}LT$$

of the set $\mathbb{F}_p = \mathbb{A}^1(\mathbb{F}_p)$. Then $\sigma_0, \sigma_1, \sigma_\infty$ satisfy the conditions (i), (ii), and (iii) of Proposition 1.4 with $d = p$.

Proof. Existence of $\ell \in \mathbb{F}_p^*$ of order 4 readily follows from the assumption $p \equiv 1 \pmod{4}$. Let us proceed and prove that the permutations $\sigma_0, \sigma_1, \sigma_\infty$ defined in the statement satisfy the conditions; (i) asserts that $\sigma_\infty = \sigma_0\sigma_1$, which is indeed true by definition.

The cycle type of σ_0 is clearly $(1, 4, \dots, 4)$, because $\ell^2 = -1$, whence L and L^2 have 0 as a fixed point and act injectively on \mathbb{F}_p^* . Since σ_1 is conjugate to σ_0 , it also has such a cycle type. Turning to σ_∞ , and using again the fact that $\ell^2 = -1$, we see that σ_∞ takes the form $\sigma_\infty(x) = -x + c$, for a suitable $c \in \mathbb{F}_p$ (actually $c = -\ell - 1$). Therefore it is an (affine) involution of the line \mathbb{A}^1 , and since $p \neq 2$ its cycle type is of the form $(1, 2, \dots, 2)$, which completes the proof of condition (ii).

To establish (iii) we note that the commutator $[\sigma_0, \sigma_1]$ is a nontrivial translation, so it acts transitively on \mathbb{F}_p . \square

Combining Propositions 2.1 and 2.2 one readily deduces the well-known:

Theorem 2.3 (Fermat-Euler). *If p is a prime congruent to 1 modulo 4 then p is a sum of two squares.*

As mentioned in the Introduction, the resulting proof of this classical result is extraordinarily demanding: a closer look shows that, in addition to the construction of the permutations in Proposition 2.2, it depends also on:

- (A) The topological construction of a finite cover of $\mathbb{P}_1(\mathbb{C}) \setminus \{0, 1, \infty\}$ such that lifting three simple disjoint loops based at a point P_0 and encircling $0, 1, \infty$ one obtains the given permutations $\sigma_0, \sigma_1, \sigma_\infty$ on the fiber over P_0 . This construction appears, *e.g.*, in [8, Theorems 4.27 and 4.31]; it may be proved by patching local covers or taking a suitable quotient of the universal cover of $\mathbb{P}_1(\mathbb{C}) \setminus \{0, 1, \infty\}$.
- (B) The Riemann Existence Theorem, used to realize the said topological cover as the unramified part of a ramified cover of complex algebraic curves. This step is delicate and requires fairly hard analysis, based either on the Dirichlet principle or on the vanishing of suitable cohomology of holomorphic sheaves on Riemann surfaces. (See again [8, Theorem 4.27].)
- (C) The structure of the endomorphism ring of the elliptic curve E of Section 1, namely its identification with $\mathbb{Z}[i]$. This may be established as follows. First, one notes that $\text{End}(E)$ is a commutative ring, of rank at most 2 over \mathbb{Z} : this may be seen by viewing E as \mathbb{C}/Λ , where Λ is a lattice, and by realizing $\text{End}(E)$ as a ring of multiplications by complex numbers μ such that $\mu\Lambda \subset \Lambda$. In our case $\text{End}(E)$ contains a ring isomorphic to $\mathbb{Z}[i]$, because it contains \mathbb{Z} and the order-4 automorphism denoted by α in Section 1. Hence it must be equal to $\mathbb{Z}[i]$ because it is commutative, of rank 2 and integral over \mathbb{Z} . An alternative way is to note that the elliptic curve E' corresponding to $\mathbb{C}/\mathbb{Z}[i]$ has vanishing Weierstrass invariant g_3 (an easy direct computation which amounts to showing that $\sum_{\omega \in \mathbb{Z}[i] \setminus \{0\}} \omega^{-6}$ is 0 —see [7] for careful proofs of all of these facts). Hence a Weierstrass equation for E' has the shape $y^2 = 4x^3 + g_2x$, therefore E' is isomorphic to E , under a substitution $x \mapsto cx$ for suitable c .

In conclusion, all of these steps involve some nontrivial mathematics, and (B) is particularly delicate. Combining the (self-contained) proof of Proposition 2.2 with the arguments used in [4] to establish Theorem 1.1, one gets instead a proof of Theorem 2.3 based on item (A) above and on the existence and rigidity (up to scaling) of a Euclidean structure on the orbifold $S^2(4, 4, 2)$.

On the other hand there exist many few-lines self-contained proofs of the Fermat-Euler result. Nevertheless, one cannot say that the proof given above *contains*, from the logical viewpoint, any classical proof, as for instance the argument based on the unique factorization of $\mathbb{Z}[i]$. In fact, although this ring plays an implicit role in item (C) above, its factorization properties are not employed, neither explicitly nor implicitly.

Ramified covers with signature An alternative approach to Theorem 1.1, to which the argument in [4] is closer and which does not require items (B) and (C) above, is described in a sketchy but complete fashion in [6, pp. 60-63]. This avoids the viewpoint of complex algebraic curve altogether, being based on the notion of *ramified cover with signature* which, roughly speaking, consists of a topological cover of a space deprived of finitely many points, together with a ramified structure above the remaining points, of the same type as a map of the shape $z \mapsto z^n$.

In our case we have a *universal covering with signature* $(4, 4, 2)$, meaning a space Y which is obtained by suitably completing the quotient of the universal cover of $\mathbb{P}_1(\mathbb{C}) \setminus \{0, 1, \infty\}$ by the normal subgroup N of $\pi_1(\mathbb{P}_1(\mathbb{C}) \setminus \{0, 1, \infty\})$ generated by c_0^4, c_1^4, c_∞^2 , where c_0, c_1, c_∞ are the simple disjoint loops already mentioned above. As stated in [6, p. 63], one realizes Y as the Euclidean plane \mathbb{C} , with covering group Γ given by the rigid motions of the plane preserving the orientation and the lattice $\mathbb{Z}[i]$; we have $\Gamma \cong \pi_1(\mathbb{P}_1(\mathbb{C}) \setminus \{0, 1, \infty\})/N$.

An explicit description of the elements of Γ as affine transformations of the complex line is as follows:

$$\Gamma = \left\{ \mathbb{C} \ni z \mapsto uz + \lambda : u \in \{1, i, -1, -i\}, \lambda \in \mathbb{Z}[i] \right\}. \quad (3)$$

There are three orbits of points in \mathbb{C} having non trivial stabilizers: the first one is $\mathbb{Z}[i]$, where each point has a stabilizer of order four; the second one is $\frac{1+i}{2} + \mathbb{Z}[i]$, also having a stabilizer of order four; the third one is $\left(\frac{1}{2} + \mathbb{Z}[i]\right) \cup$

$(\frac{i}{2} + \mathbb{Z}[i])$, having a stabilizer of order two. They correspond to the pre-images of $0, 1, \infty$.

Theorem 2.4. *The group Γ defined by (3) is the universal group of type $(4, 4, 2)$.*

Proof. We confine ourselves to a sketch. Let G be a group of type $(4, 4, 2)$, so G is generated by two elements α, β with $\alpha^4 = \beta^4 = (\alpha\beta)^2 = 1$. It is immediate from this presentation that the congruence class modulo four of the length of a word representing an element of G only depends on that element. Also, it is easily checked that the words of length divisible by four commute and can be generated by $u := \alpha^3\beta$ and $v := \alpha\beta^3$. Hence we always have a group homomorphism $G \rightarrow \mathbb{Z}/4\mathbb{Z}$ whose kernel is an Abelian (normal) subgroup generated by two elements. So we have an exact sequence $\{0\} \rightarrow \langle u, v \rangle \rightarrow G \rightarrow \mathbb{Z}/4\mathbb{Z}$, where $\langle u, v \rangle \cong (\mathbb{Z}/a\mathbb{Z}) \times (\mathbb{Z}/b\mathbb{Z})$ for integers a, b (possibly zero or one) and the last morphism (which need not be surjective) is the reduction modulo four of the word length. We note that the action (by conjugation) of G on $\langle u, v \rangle$ is uniquely determined by the initial relations $\alpha^4 = \beta^4 = (\alpha\beta)^2 = 1$. Coming back to our group Γ , let us consider the three elements c_0, c_1, c_∞ of Γ defined by $c_0(z) = iz$, $c_1(z) = 1 + iz$, $c_\infty(z) = -z + i$. Then $c_0^4 = c_1^4 = c_\infty^2 = 1$ and $c_0c_1 = c_\infty$. Clearly, $\tilde{u} := c_0^3c_1$ and $\tilde{v} := c_0c_1^3$ act as $\tilde{u}(z) = z + i$ and $\tilde{v}(z) = z - 1$, so they generate the subgroup of translations, isomorphic to $\mathbb{Z}[i] \cong \mathbb{Z}^2$. Hence we have the exact sequence

$$\{0\} \rightarrow \mathbb{Z}^2 \rightarrow \Gamma \rightarrow \mathbb{Z}/4\mathbb{Z} \rightarrow \{0\}.$$

Note that the action of Γ on $\mathbb{Z}[i]$ is compatible with the action of G on $\langle u, v \rangle$, in the natural sense: for instance we have in G the relation $\alpha u \alpha^{-1} = v^{-1}$, which corresponds in Γ to the relation $c_0 \tilde{u} c_0^{-1} = \tilde{v}^{-1}$. From this fact it easily follows that G is a quotient of Γ . \square

Let us get back to the setting of Theorem 1.1, and let us use the universal covering with signature $(4, 4, 2)$, denoted by Y as above. A cover X of the Riemann sphere with the relevant branching types exists if and only if it can be realized as the quotient of Y by a subgroup Δ of Γ , of *odd* index d in Γ . The permutations $\sigma_0, \sigma_1, \sigma_\infty$ then correspond to the images of c_0, c_1, c_∞ in the permutation representation of Γ on the right cosets Γ/Δ . One easily sees that if Δ exists then it must contain an element σ of order 4, which must be a rotation of $\pi/2$ around some point. The orbit of the origin by Δ is a

lattice stable under σ , which corresponds to an ideal in $\mathbb{Z}[i]$. This ideal is principal, and we find again the conclusion that d is a sum of two squares. As a matter of fact, to conclude one can also avoid invoking the principal-ideal ring structure, by observing that the said lattice must have a basis of type $v, \sigma v$, whence its index is necessarily a sum of two squares.

Remark 2.5. When the degree is an odd prime p , this approach also allows one to elucidate the structure of the group G generated by the permutations $\sigma_0, \sigma_1, \sigma_\infty$. In fact, as stated in the proof of Theorem 2.4, the group generated by the words of length 4 in σ_0, σ_1 is commutative. Hence G has an Abelian subgroup G_0 of index at most 4. Since G is a transitive subgroup of \mathfrak{S}_p it contains a p -cycle g , which must lie in G_0 . Then G_0 must be the group generated by g .

References

- [1] A. L. EDMONDS – R. S. KULKARNI – R. E. STONG, *Realizability of branched coverings of surfaces*, Trans. Amer. Math. Soc. **282** (1984), 773-790.
- [2] R. M. GURALNICK – P. MÜLLER – J. SAXL, *The rational function analogue of a question of Schur and exceptionality of permutation representations*, Mem. Amer. Math. Soc. **162** (2003), viii+79.
- [3] A. HURWITZ, *Riemann'sche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. **39** (1891), 1-61.
- [4] M. A. PASCALI – C. PETRONIO, *Surface branched covers and geometric 2-orbifolds*, preprint arXiv:0709:2026, to appear in Trans. Amer. Math. Soc.
- [5] L. SCHNEPS (Ed.), “The Grothendieck Theory of Dessins d’Enfants,” London Math. Soc. Lecture Note Series, Vol. 200, Cambridge Univ. Press, Cambridge, 1994.
- [6] J.-P. SERRE, “Topics in Galois Theory,” Research Notes in Mathematics, Vol. 1, Jones and Bartlett, Boston MA, 1992.
- [7] J. SILVERMAN, “The Arithmetic of Elliptic Curves,” Graduate Texts in Mathematics, Vol. 106, Springer-Verlag, New York, 1985.

- [8] H. VÖLKLEIN, “Groups as Galois Groups, an Introduction”, Cambridge Studies in Advanced Mathematics, Vol. 53, Cambridge Univ. Press, Cambridge, 1996.
- [9] U. ZANNIER, *A note on the S-unit equation over function fields*, Acta Arith. **64** (1993), 87-98.
- [10] U. ZANNIER, *On Davenport’s lower bound and Riemann Existence Theorem*, Acta Arith. **71** (1995), 107-137.

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