Realizability and exceptionality of candidate surface branched covers: methods and results

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Abstract

Let $f:\widetilde{\Sigma}\xrightarrow[\Pi]{d:1}\Sigma$ denote a branched cover between closed, connected, and orientable surfaces, where d is the global degree, $\Pi=\{\Pi_1,\ldots,\Pi_n\}$, n is the number of branching points, and Π_i is the partition of d given by the local degrees over the i-th branching point. If $\ell(\Pi)$ is the total length of Π, then the Riemmann-Hurwitz formula asserts that $\chi(\widetilde{\Sigma})-\ell(\Pi)=d\cdot(\chi(\Sigma)-n)$. A *candidate branched cover* is a symbol $\widetilde{\Sigma}\overset{d:1}{\underset{\Pi}{\dashrightarrow}}\Sigma$ satisfying the same condition, and it is called *realizable* if there exists a corresponding $f:\widetilde{\Sigma}\xrightarrow[\Pi]{d:1}\Sigma$. The problem of determining which candidate covers are realizable is very old and still partially unsolved. In this paper we will review five different techniques employed in recent years to attack it, and we will state the main results obtained using them. Each technique will be exemplified through a proof that the candidate cover $S^2\overset{4:1}{\underset{(2,2),(2,2),(3,1)}{\dashrightarrow}}S^2$ is *exceptional*, namely non-realizable. We will also state some results for the non-orientable version of the problem, none of which is due to us.

1 Introduction

A *branched cover* is a map $f:\widetilde{\Sigma}\to\Sigma$, where $\widetilde{\Sigma}$ and Σ are closed connected surfaces and f is locally modelled on maps of the form $\mathbb{C}\ni z\mapsto z^k\in$

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$\mathbb{C}$ for some $k \geqslant 1$. The integer k is called the *local degree* at the point of $\widetilde{\Sigma}$ corresponding to 0 in the source $\mathbb{C}$. If $k > 1$ then the point of Σ corresponding to 0 in the target $\mathbb{C}$ is called a *branching point*. The branching points are isolated, hence there are finitely many, say n, of them. Removing the branching points in Σ and all their pre-images in $\widetilde{\Sigma}$, the restriction of f gives a genuine cover, whose degree we will denote by d. The collection of local degrees at the preimages of the i-th branching point of Σ is a partition Π_i of d, namely a set of positive integers summing up to d. Let us define $\ell(\Pi_i)$ as the length of this partition, $\Pi = \{\Pi_1, \ldots, \Pi_n\}$ and $\ell(\Pi) = \ell(\Pi_1) + \ldots + \ell(\Pi_n)$. The whole information on the branched cover f will be summarized by the symbol

$$f : \widetilde{\Sigma} \xrightarrow[\Pi]{d:1} \Sigma.$$

The following is known:

Proposition 1.1. *If* $f : \widetilde{\Sigma} \xrightarrow[\Pi_1,\ldots,\Pi_n]{d:1} \Sigma$ *is a surface branched cover then:*

(1) $\chi(\widetilde{\Sigma}) - \ell(\Pi) = d \cdot \big(\chi(\Sigma) - n\big)$;

(2) $n \cdot d - \ell(\Pi)$ *is even;*

(3) If Σ *is orientable then* $\widetilde{\Sigma}$ *is also orientable;*

(4) If Σ *is non-orientable and* d *is odd then* $\widetilde{\Sigma}$ *is also non-orientable;*

(5) If Σ *is non-orientable and* $\widetilde{\Sigma}$ *is orientable then each partition* Π_i *of* d *is a juxtaposition of two partitions of* $d/2$.

Condition (1) is the classical Riemann-Hurwitz formula. Condition (2) follows from (1) in the orientable case, and it is not too hard to establish in general, see [8]. Conditions (3) and (4) are obvious. Condition (5) is quite easy, see [8] again; note that d is even by condition (4).

Candidate surface branched covers Suppose we are given closed connected surfaces $\widetilde{\Sigma}$ and Σ, integers $n \geqslant 0$ and $d \geqslant 2$, and a collection $\Pi = (\Pi_1, \ldots, \Pi_n)$ of n partitions of d. We will say that these data define a *candidate surface branched cover*, and we will associate to them the symbol

$$\widetilde{\Sigma} \overset{d:1}{\underset{\Pi_1,\ldots,\Pi_n}{\dashrightarrow}} \Sigma$$

if the conditions of Proposition 1.1 hold.

Remark 1.2. The symbol $\left(\widetilde{\Sigma}, \Sigma, n, d, (d_{ij})\right)$ and the name *compatible branch datum* are used in [8, 9] instead of the terminology of "candidate covers" we will use here.

A candidate surface branched cover $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ will be called *realizable* if there is an actual cover $f : \widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\longrightarrow}} \Sigma$ matching it, and *exceptional* otherwise. A classical problem dating back to Hurwitz [5] asks which candidate covers are realizable and which are exceptional. Many authors contributed to it, as we will now outline, but the problem in full generality is still unsolved.

Known results A thorough description of the partial solutions to the Hurwitz problem obtained over the time was given in [8]. Here we restrict ourselves to the main results. We start with the following theorem of Husemoller [6], proved also in [2]:

Theorem 1.3. *A candidate surface branched cover* $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ *is realizable if* Σ *is orientable and* $\chi(\Sigma) \leqslant 0$.

We next have the following result which combines theorems of Ezell [3] and Edmonds-Kulkarny-Stong [2]:

Theorem 1.4. *A candidate surface branched cover* $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ *is realizable if either* Σ *is non-orientable and* $\chi(\Sigma) \leqslant 0$*, or* Σ *is the projective plane and* $\widetilde{\Sigma}$ *is non-orientable.*

According to these results the problem of the realizability of $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ remains open only if $\widetilde{\Sigma}$ is orientable and Σ is either the sphere S^2 or the projective plane. However other results in [2] allow to reduce the latter case to the former one. For this reason we will restrict in the rest of this paper to the case $\Sigma = S^2$. Many exceptions are known in this case, the easiest of which is

$$S^2 \overset{4:1}{\underset{(2,2),(2,2),(3,1)}{\dashrightarrow}} S^2,$$

but the general pattern of realizability and exceptionality remains elusive. The following conjecture suggesting connections with number-theoretic facts was however proposed in [2]:

Conjecture 1.5. *If* $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} S^2$ *is a candidate surface branched cover and the degree* d *is a prime number then the candidate is realizable.*

Its validity was recently supported by the results and computer experiments of Zheng [12] and by the results of [9] and [7].

In the rest of this paper we will review five different techniques employed to attack the Hurwitz problem, using them we will give five independent proofs of the exceptionality of $S^2 \overset{4:1}{\underset{(2,2),(2,2),(3,1)}{\dashrightarrow}} S^2$, and we will state the main results they have led to. We address the reader to [8, 9, 7] for more details.

2 Permutations

Hurwitz himself already showed that the problem of realizability of a candidate surface branched cover can be reformulated, using the notion of monodromy, in terms of permutations. We will first provide an example of how this works and then we will sketch the general technique and the main achievements obtained using it.

Sample application We will now give our first proof of the exceptionality of the candidate surface branched cover $S^2 \overset{4:1}{\underset{(2,2),(2,2),(3,1)}{\dashrightarrow}} S^2$. Suppose, on the contrary, that there exists a branched cover $f : S^2 \to S^2$ realizing it. Let S^2_n denote the surface obtained from the sphere S^2 by removing n open discs with disjoint closures. Clearly, f induces a genuine cover $S^2_6 \to S^2_3$ such that each boundary component of S^2_3 is covered by two boundary components of S^2_6, and the degrees of the restrictions to these components are (2,2), (2,2), and (3,1).

Choose two simple proper disjoint arcs ϵ_1, ϵ_2 cutting S^2_3 into a 2-disc Δ, and denote by $\epsilon_{i,\pm}$ the arcs in the boundary of Δ corresponding to ϵ_i. Now note that a genuine cover over a disc is given by a disjoint union of discs, each of which is mapped homeomorphically to the target. Therefore each degree-4 cover over S^2_3 can be reconstructed in the following way. We first take the disjoint union of 4 copies $(\Delta^{(h)})_{h=1}^4$ of Δ, with the corresponding arcs $\epsilon^{(h)}_{i,\pm}$, and we choose two permutations $\theta_1, \theta_2 \in \mathfrak{S}_4$. Then we glue each $\epsilon^{(h)}_{i,-}$ to $\epsilon^{(\theta_i(h))}_{i,+}$ and we project to S^2_3 in the obvious way.

It is easy to show that the degrees of the covers over the boundary components of S^2_3 are then given by the lengths of the cycles of θ_1, θ_2, and $\theta_1 \cdot \theta_2$. (Moreover the covering surface is connected if and only if θ_1 and θ_2 generate a subgroup of $\mathfrak{S}_4$ acting transitively on $\{1, \ldots, 4\}$, but we will not need this fact here).

We conclude that if the candidate $S^2 \xrightarrow[(2,2),(2,2),(3,1)]{4:1} S^2$ is realizable then there exist permutations $\theta_1, \theta_2 \in \mathfrak{S}_4$ such that the lengths of the cycles of θ_1, θ_2 and $\theta_1 \cdot \theta_2$ are respectively $(2,2)$, $(2,2)$, and $(3,1)$. However the set of all elements of $\mathfrak{S}_4$ with cycles of lengths $(2,2)$, together with the trivial permutation, is a subgroup of $\mathfrak{S}_4$, whence a contradiction and the conclusion that $S^2 \xrightarrow[(2,2),(2,2),(3,1)]{4:1} S^2$ is exceptional.

General technique The above argument can be generalized to obtain the following result of Hurwitz [5]. Again Σ_n denotes Σ with n discs removed. We confine ourselves to the statement for the orientable case, a small additional but rather technical condition being required in the general case.

Theorem 2.1. *A candidate surface branched cover* $\widetilde{\Sigma} \xrightarrow[\Pi_1,\ldots,\Pi_n]{d:1} \Sigma$ *with orientable* Σ *is realizable if and only if there exists a representation* $\theta : \pi_1(\Sigma_n) \to \mathfrak{S}_d$ *such that:*

1. *The image of* θ *acts transitively on* $\{1, \ldots, d\}$*;*
2. *The image under* θ *of the element of* $\pi_1(\Sigma_n)$ *corresponding to the* i*-th boundary component of* Σ_n *has cycles of lengths* Π_i.

It is precisely this technique that was employed in [6, 2, 3] and led to the results obtained therein and stated in the Introduction. The proofs are not elementary at all, but the underlying philosophy is easy to explain. The main restrictions imposed by Theorem 2.1 are on the images under θ of the peripheral elements of $\pi_1(\Sigma_n)$. But if $\chi(\Sigma) \leqslant 0$ then $\pi_1(\Sigma_n)$ is *not* generated by these peripheral elements, so one has more flexibility for the choice of θ.

3 Dessins d'enfants

The notion of dessin d'enfant was introduced by Grothendieck [4] for matters related to the Hurwitz problem but not strictly equivalent to it. Again before giving the general definition we show in a concrete case how the notion works.

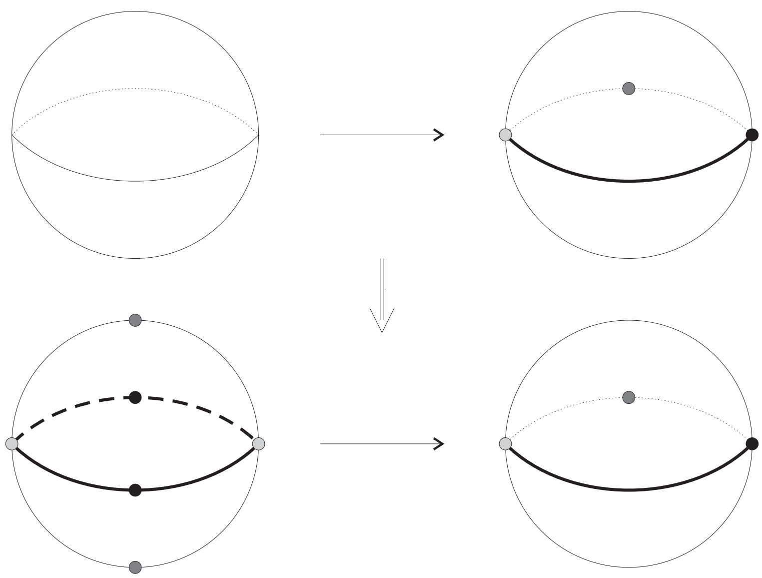

Figure 1: Dessin d'enfant for a cover of degree 4 with two partitions (2,2)

Sample application Our second proof of the exceptionality of the sample candidate $S^2 \overset{4:1}{\underset{(2,2),(2,2),(3,1)}{\dashrightarrow}} S^2$ is based on the following observation. Suppose that some $S^2 \overset{4:1}{\underset{(2,2),(2,2),\Pi_3}{\dashrightarrow}} S^2$ is realized by a map f. Denote the branching points by x_1, x_2, x_3, with the partitions $(2,2)$ associated to x_1 and x_2. Let α be a simple arc in the base S^2 that joins x_1 to x_2 and avoids x_3. Then $f^{-1}(\alpha)$ is a graph D in the covering S^2 whose vertices are the pre-images of x_1 and x_2. However the valence of any such vertex v is the local degree of f at v, which is 2, so D is a union of circles.

Now note that the complement of α in the base S^2 is an open disc containing only one branching point. This implies that the complement of D in the covering S^2 is a union of two discs, each containing one element of $f^{-1}(x_3)$, so D is a single circle (with 4 vertices, even if one cannot see them). Moreover the local degree at the element of $f^{-1}(x_3)$ contained in a disc is half the number of vertices of D that the disc is incident to, therefore it is 2. This shows that Π_3 is forced to be $(2,2)$, so $S^2 \overset{4:1}{\underset{(2,2),(2,2),(3,1)}{\dashrightarrow}} S^2$ is exceptional. Our argument is pictorially illustrated in Figure 1.

General technique Grothendieck's original dessins d'enfants [4, 11] arise precisely as in the argument we have just given, when one considers a branched cover $f:\widetilde{\Sigma}\to S^2$ with three branching points and one defines the dessin D as $f^{-1}(\alpha)$, where α is an arc joining two of the branching points and avoiding the third one. This technique was generalized in [8] to the case of an arbitrary number of branching points, as we will now explain.

Definition 3.1. A *dessin d'enfant* on $\widetilde{\Sigma}$ is a graph $D\subset\widetilde{\Sigma}$ where:

1. For some $n\geqslant 3$ the set of vertices of D is split as $V_1\sqcup\ldots\sqcup V_{n-1}$ and the set of edges of D is split as $E_1\sqcup\ldots\sqcup E_{n-2}$;
2. For $i=1,\ldots,n-2$ each edge in E_i joins a vertex of V_i to one of V_{i+1};
3. For $i=2,\ldots,n-2$ any vertex of V_i has even valence and going around the vertex one alternatively encounters edges from E_{i-1} and from E_i;
4. $\widetilde{\Sigma}\setminus D$ consists of open discs.

The following was established in [8].

Proposition 3.2. *The realizations of a candidate surface branched cover* $\widetilde{\Sigma}\xrightarrow[\Pi_1,\ldots,\Pi_n]{d:1}\Sigma$ *correspond to the dessins d'enfants* $D\subset\widetilde{\Sigma}$ *with the set of vertices split as* $V_1\sqcup V_2\sqcup\ldots\sqcup V_{n-1}$ *such that:*

- *For* $i=1$ *and* $i=n-1$ *the vertices in* V_i *have valences* Π_i*;*
- *For* $i=2,\ldots,n-2$ *the vertices in* V_i *have valences* $2\cdot\Pi_i$*;*
- *The numbers of vertices (with multiplicity) that the discs in* $\widetilde{\Sigma}\setminus D$ *are incident to are* $2(n-2)\cdot\Pi_n$*.*

Some results Using Proposition 3.2 one can analyze the realizability of several infinite series of candidate surface branched covers. For instance the following results were establishes in [8]:

Proposition 3.3. *Let* $d\geqslant 8$ *be even and consider a candidate surface branched cover of the form*

$$\widetilde{\Sigma}\xrightarrow[(2,\ldots,2),(5,3,2,\ldots,2),\Pi_3]{d:1}S^2.$$

- *If $\widetilde{\Sigma}$ is the torus, whence $\ell(\Pi_3) = 2$, the candidate is realizable if and only if* $\Pi_3 \neq (d/2, d/2)$*;*
- *If $\widetilde{\Sigma}$ is the sphere S^2, whence $\ell(\Pi_3) = 4$, the candidate is realizable if and only if* Π_3 *does not have the form* (a, a, b, b) *or* $(3a, a, a, a)$ *for* $a, b \in \mathbb{N}$.

Proposition 3.4. *Let d be even. Then a candidate surface branched cover of one of the forms*

$$S^2 \xrightarrow[(2,\ldots,2),(3,3,2,\ldots,2),\Pi_3]{d:1} S^2, \qquad S^2 \xrightarrow[(2,\ldots,2),(3,2,\ldots,2,1),\Pi_3]{d:1} S^2.$$

is realizable if and only if the largest element of Π_3 *is not $d/2$.*

The proofs of these propositions are not too difficult, the idea being that a vertex of valence 2 is actually not a true vertex of a graph. So in all cases one has to deal with a small number of topological types of dessins d'enfant, and a slightly larger number of inequivalent embeddings of these graphs in the relevant surface $\widetilde{\Sigma}$. Then one has to analyze how the (invisible) valence-2 vertices are placed along the graph and to analyze what partitions Π_3 arise. A somewhat more complicated argument yields the following result.

Theorem 3.5. *Suppose that a candidate surface branched cover* $S^2 \xrightarrow[\Pi_1,\ldots,\Pi_n]{d:1} S^2$ *is realizable and each entry of* Π_1 *and* Π_2 *is a multiple of some k with $1 < k < d$, so also d is. Then each entry of each* Π_i *with $i \geqslant 3$ is at most d/k.*

4 Checkerboard graphs

This technique was introduced by Baránski [1] for the case of candidate covers $S^2 \dashrightarrow S^2$ and extended to the case $\widetilde{\Sigma} \dashrightarrow S^2$ in [9]. As above we start with an example and then we outline the general method.

Sample application Here we present the third proof of the exceptionality of $S^2 \xrightarrow[(2,2),(2,2),(3,1)]{4:1} S^2$. Suppose that the candidate is realized by some map f. Let us identify S^2 with $\mathbb{C} \cup \{\infty\}$ and assume that the branching points are the third roots of 1 in $\mathbb{C}$, labelled 1, 2 and 3. Let Δ be the unit disc in $\mathbb{C}$ and

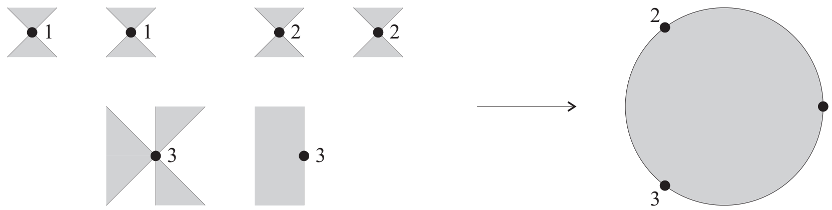

Figure 2: Vertices of a checkerboard graph for a degree-4 cover of S^2 with partitions (2,2), (2,2), and (3,1)

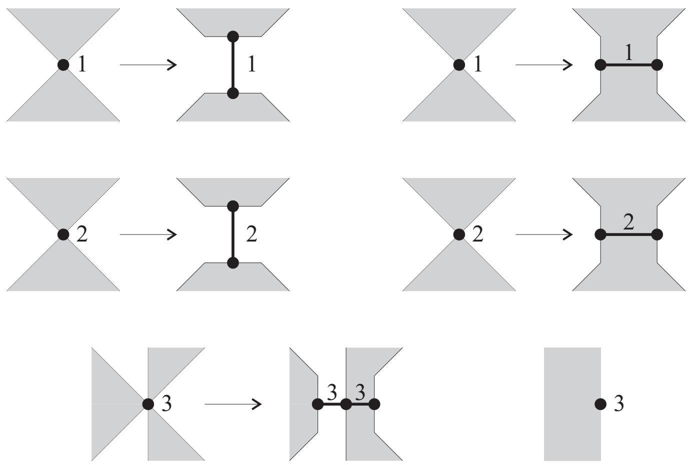

Figure 3: Moves employed to merge the triangles

let us colour it black and its complement white. Then $f^{-1}(\partial\Delta)$ is a graph with 6 vertices, of valences 4, 4, 4, 4, 6, and 2, each with one of the labels 1, 2 or 3 equal to that of its image. Moreover the complement of the graph consists of 4 black and 4 white discs (because the degree of f is 4). Each disc is actually a (curvilinear) triangle, and whenever two triangles share an edge they have different colours (as in a checkerboard, whence the name of the technique). See Figure 2.

We now modify our graph by merging together all the black triangles into one black disc, and all the white triangles into one white disc. This is done by moves such as those of Figure 3. The moves are applied successively and in such a way that each time the pairs of local germs of discs merged together belong to different global discs. In addition to performing this merging, as also shown in Figure 3, we insert some vertices and arcs labelled 1, 2 or 3, with obvious conventions. Note that the set of moves to apply is not unique

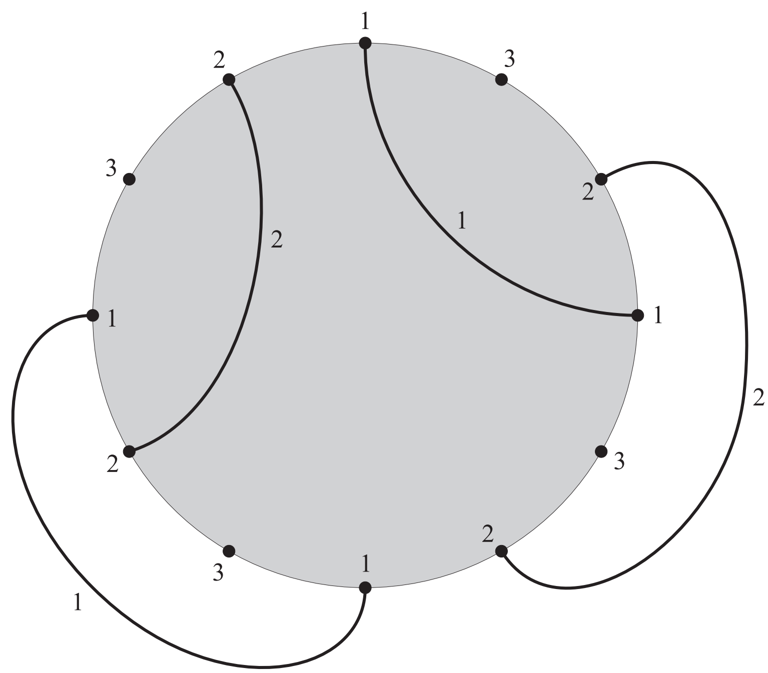

Figure 4: Merged black and white discs with labelled arcs

(for instance, a different set could perhaps give rise to two arcs labelled 1 both contained in the black merged disc, or similar other variations), but the number of actually distinct possibilities is finite and small.

It is not difficult to prove that, after all the merging, on the boundary of the single black disc there are 12 vertices, with labels $1, 2, 3$ repeated cyclically 4 times. In addition there are two arcs labelled 1 with vertices 1 at the ends, two similar arcs labelled 2, and an arc labelled 3 with two vertices labelled 3 at the ends and one in the middle. There are a few different possibilities for the background colours of the arcs, and by examining all of them one sees that this triple actually cannot exist. One of the situations to consider is shown as an example in Figure 4, where after insertion of the arcs labelled 1 and 2 it is impossible to draw the arc labelled 3, because the arcs must be mutually disjoint. This shows that the initial assumption about the existence of f was absurd, which implies once again that $S^2 \underset{(2,2),(2,2),(3,1)}{\overset{4:1}{\dashrightarrow}} S^2$ is exceptional.

General technique To describe the general form of this approach, we need a preliminary notion. We call *checkerboard graph* in a surface $\widetilde{\Sigma}$ a finite 1-subcomplex G of $\widetilde{\Sigma}$ whose complement consists of open discs each bearing a color black or white, so that each edge separates black from white.

The strategy used in the example extends as follows to the general situation. Suppose we have a map realizing a candidate surface branched cover $\widetilde{\Sigma} \xrightarrow[\Pi_1,\ldots,\Pi_n]{d:1} S^2$. Then we arrange the branching points to be the n-th roots of 1 and we consider the graph $f^{-1}(\partial\Delta)$. Giving black colour to the complementary discs mapped to Δ, and white to the other discs, we see that $f^{-1}(\partial\Delta)$ is a checkerboard graph in $\widetilde{\Sigma}$. Putting labels on the vertices and performing moves as those in Figure 3 (each consisting in the merging of two discs and the insertion of a labelled arc), we end up with a checkerboard graph whose complement consists of a single black and a single white disc, together with a collection of labelled vertices and disjoint trees satisfying a long list of conditions in terms of the partitions $\Pi_1, \ldots, \Pi_n$. The precise list would be too long to give here, so we address the reader to [9]. The main point is however that the whole process is reversible, namely from a checkerboard graph and a family of labelled vertices and trees satisfying the conditions corresponding to some candidate branched surface cover $\widetilde{\Sigma} \xrightarrow[\Pi_1,\ldots,\Pi_n]{d:1} S^2$ one can construct a map realizing the candidate. Thus checkerboard graphs give a necessary and sufficient criterion for realizability.

Results The following main theorem was established in [9] by means of the checkerboard graphs realizability criterion. The proof was carried out using some inductive constructions.

Theorem 4.1. *Consider a candidate surface branched cover of the form*

$$\widetilde{\Sigma} \xrightarrow[(d-2,2),\Pi_2,\Pi_3]{d:1} S^2.$$

- *If $\widetilde{\Sigma}$ is the sphere S^2 then the candidate is exceptional if it has one of the forms*

  $$S^2 \xrightarrow[(2k-2,2),(2,\ldots,2),(k+1,1,\ldots,1)]{2k:1} S^2, \qquad S^2 \xrightarrow[(2k,2),(2,\ldots,2),(2,\ldots,2)]{2k+2:1} S^2$$

  for $k \geqslant 2$, and realizable otherwise. In particular, it is always realizable if the degree d is odd.

- *If $\widetilde{\Sigma}$ is the torus $S^1 \times S^1$ then the candidate is always realizable, with the single exception of*

$$S^1 \times S^1 \xrightarrow[(4,2),(3,3),(3,3)]{6:1} S^2.$$

- *If $\widetilde{\Sigma}$ has genus at least 2 then the candidate is always realizable.*

5 Factorization

A composition of branched covers is a branched cover, so if a candidate can be expressed as a "candidate composition" of two realizable covers then it is realizable. On the other hand if one has a candidate cover and one can show that a map realizing it, if any, should be the composition of two maps one of which realizes an exceptional cover, one deduces that the original cover is exceptional too.

Sample application We will now give our fourth proof of the exceptionality of $S^2 \xrightarrow[(2,2),(2,2),(3,1)]{4:1} S^2$. The proof is actually slightly incomplete, but see below. Suppose we want to construct a map f realizing some candidate cover $S^2 \xrightarrow[(2,2),(2,2),\Pi_3]{4:1} S^2$ and we decide to do this stepwise. Let us denote by x_1, x_2, x_3 the would-be branching points of f. We start by realizing the local degrees 2 at x_1 and x_2 via a degree-2 cover $g : S^2 \to S^2$, but there is essentially just one of them, whose only non-trivial automorphism is a rotation of angle π through two antipodal points, see the right portion of Figure 5. Note that $g^{-1}(x_3)$ consists of two points. Now we want to find another degree-2 cover $h : S^2 \to S^2$ such that $f = g \circ h$. But we know that Π_3 must have length 2, so the two points in $g^{-1}(x_3)$ must be the branching points of h (which coincides with g up to automorphisms of S^2). So the situation is that Figure 5, which implies that $\Pi_3 = (2, 2)$.

This shows that no map f realizing $S^2 \xrightarrow[(2,2),(2,2),(3,1)]{4:1} S^2$ can be constructed using the stepwise method we have outlined. As we will see below, one can show that the candidate satisfies properties ensuring that if it is realizable then there exists a stepwise realization, so again we conclude that it is exceptional.

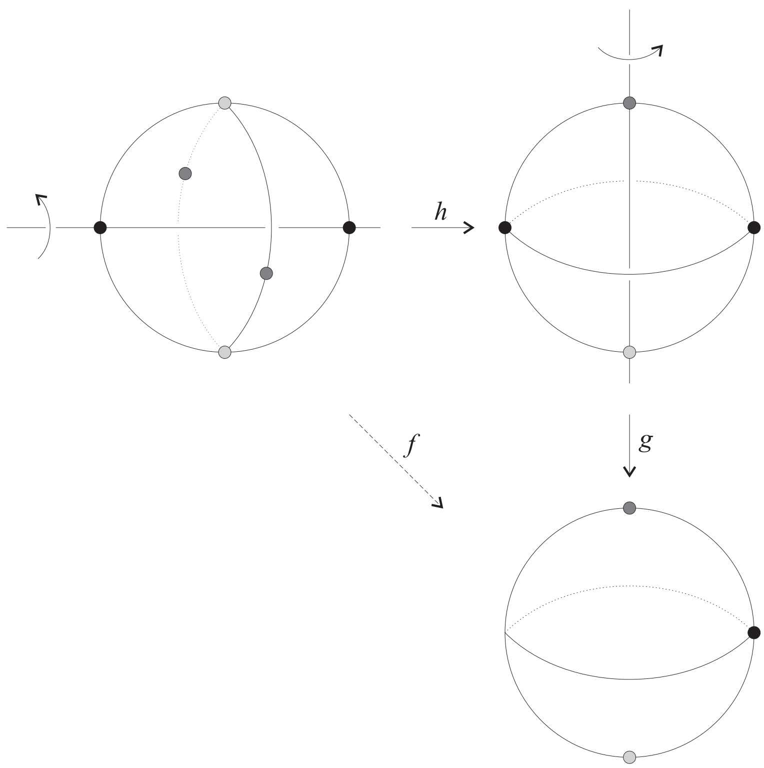

Figure 5: Stepwise construction of a degree-4 cover

General technique and results The missing portion of the previous argument is deduced from the following result established in [8], together with its main consequence stated soon after.

Proposition 5.1. *Let a candidate surface branched cover* $S^2 \overset{d:1}{\underset{\Pi_1,\ldots,\Pi_n}{\dashrightarrow}} S^2$ *be realized by a map* f *and suppose that all the entries of* Π_1 *and* Π_2 *are even, so* d *also is. Then* f *can be expressed as* $f = g \circ h$ *where* g *is the realizable cover* $S^2 \overset{2:1}{\underset{(2),(2)}{\longrightarrow}} S^2$ *and* h *realizes a candidate of the form*

$$S^2 \overset{d/2:1}{\underset{\frac{1}{2}\Pi_1,\frac{1}{2}\Pi_2,\Pi_3',\Pi_3'',\ldots,\Pi_n',\Pi_n''}{\dashrightarrow}} S^2$$

where Π_i *is the juxtaposition of* Π_i' *and* Π_i'' *for* $i \geqslant 3$.

Theorem 5.2. *If* d *and all the entries of* Π_1 *and* Π_2 *are even and the candidate surface branched cover*

$$S^2 \overset{d:1}{\underset{\Pi_1,\ldots,\Pi_n}{\dashrightarrow}} S^2$$

is realizable then Π_i *is the juxtaposition of two partitions of* $d/2$ *for* $i \geqslant 3$.

On the realizability side, the following easy but useful fact was also established in [8] exploiting the idea of factorization.

Theorem 5.3. *Consider a candidate surface branched cover*

$$\widetilde{\Sigma} \overset{d:1}{\underset{\Pi_1,\Pi_2,\Pi_3}{\dashrightarrow}} S^2$$

and suppose there exists an odd number $p \geqslant 3$ *dividing each entry in each* Π_i*. Then the candidate is realizable.*

6 Geometric 2-orbifolds

In this last section, before giving yet another proof of the exceptionality of $S^2 \overset{4:1}{\underset{(2,2),(2,2),(3,1)}{\dashrightarrow}} S^2$, we will have to review some general notions.

Orbifolds and orbifold covers A 2-orbifold $X = \Sigma(p_1, \ldots, p_n)$ is a closed orientable surface Σ with n cone points of orders $p_i \geqslant 2$, at which X has a singular differentiable structure given by the quotient $\mathbb{C}/_{\langle \mathrm{rot}(2\pi/p_i)\rangle}$. Thurston [10] defined an orbifold cover $f : \widetilde{X} \xrightarrow{d:1} X$ of degree d as a map such that generic points have d preimages, and f is locally modelled on functions of the form

$$\mathbb{C}/_{\langle \mathrm{rot}(2\pi/\widetilde{p})\rangle} \xrightarrow{k:1} \mathbb{C}/_{\langle \mathrm{rot}(2\pi/p)\rangle}$$

induced by the identity of $\mathbb{C}$, where $p = k \cdot \widetilde{p}$. He also introduced the notion of orbifold Euler characteristic

$$\chi^{\mathrm{orb}}\big(\Sigma(p_1, \ldots, p_n)\big) = \chi(\Sigma) - \sum_{i=1}^{n} \left(1 - \frac{1}{p_i}\right),$$

designed so that if $f : \widetilde{X} \xrightarrow{d:1} X$ is an orbifold cover then $\chi^{\mathrm{orb}}(\widetilde{X}) = d \cdot \chi^{\mathrm{orb}}(X)$. He then proved that 2-orbifolds are almost always *geometric*, that is:

- If $\chi^{\mathrm{orb}}(X) > 0$ then X is either *bad* (not orbifold-covered by a surface) or *spherical*, namely the quotient of the metric 2-sphere $\mathbb{S}^2$ under a finite isometric action;

- If $\chi^{\mathrm{orb}}(X) = 0$ (respectively, $\chi^{\mathrm{orb}}(X) < 0$) then X is *Euclidean* (respectively, *hyperbolic*), namely the quotient of the Euclidean plane $\mathbb{E}^2$ (respectively, the hyperbolic plane $\mathbb{H}^2$) under a discrete isometric action.

Finally, he showed that any orbifold has an *orbifold universal cover*, which easily implies the following:

Lemma 6.1. *If $\widetilde{X}$ is bad and X is good then there cannot exist any orbifold cover $\widetilde{X} \to X$.*

Induced orbifold covers The local model described above for an orbifold cover $f : \widetilde{X} \to X$ can be viewed as the map $\mathbb{C} \ni z \mapsto z^k \in \mathbb{C}$, therefore f is also a branched cover $\widetilde{\Sigma} \to \Sigma$ between the surfaces underlying $\widetilde{X}$ and X. However different $\widetilde{X} \to X$ can give the same $\widetilde{\Sigma} \to \Sigma$, because in the local model one can arbitrarily replace p by some $h \cdot p$ and $\widetilde{p}$ by $h \cdot \widetilde{p}$. On the other hand, any $\widetilde{\Sigma} \to \Sigma$ has an associated "easiest" $\widetilde{X} \to X$, where the cone orders are chosen as small as possible. This carries over to *candidate* covers, as we will now spell out. Consider a candidate surface branched cover

$$\widetilde{\Sigma} \xrightarrow[(d_{11},\ldots,d_{1m_1}),\ldots,(d_{n1},\ldots,d_{nm_n})]{d:1} \Sigma$$

and define

$$\begin{aligned} p_i &= \text{l.c.m.}\{d_{ij}:\ j=1,\ldots,m_i\}, & p_{ij} &= p_i/d_{ij}, \\ X &= \Sigma(p_1,\ldots,p_n), & \widetilde{X} &= \widetilde{\Sigma}\big((p_{ij})_{i=1,\ldots,n}^{j=1,\ldots,m_i}\big) \end{aligned}$$

where "l.c.m." stands for "least common multiple." Then we have an induced candidate 2-orbifold cover $\widetilde{X} \xrightarrow{d:1} X$ satisfying the condition $\chi^{\text{orb}}(\widetilde{X}) = d \cdot \chi^{\text{orb}}(X)$, which is easily deduced from the Riemann-Hurwitz condition.

Sample application If the candidate $S^2 \xrightarrow[(2,2),(2,2),(3,1)]{4:1} S^2$ were realizable, then the induced candidate orbifold cover

$$S^2(3) \dashrightarrow S^2(2,2,3)$$

would also be realizable, which is impossible by Lemma 6.1 because $S^2(3)$ is bad and $S^2(2,2,3)$ is good.

General technique If a candidate orbifold cover $\widetilde{X} \xrightarrow{d:1} X$ is complemented with the instructions of which cone points of $\widetilde{X}$ should be mapped to which cone points of X, one can reconstruct a unique candidate surface branched cover $\widetilde{\Sigma} \xrightarrow[\Pi]{d:1} \Sigma$, so one can fully switch to the viewpoint of candidate orbifold covers. The X's such that $\chi^{\text{orb}}(X) \geqslant 0$, namely the bad, spherical or Euclidean X's, are easily listed, and in addition on most of them the geometric structure (if any) is *rigid*. With these facts in mind the following program appears to be natural:

- Determine all the candidate surface branched covers inducing candidate orbifold covers $\widetilde{X} \dashrightarrow X$ with $\chi^{\text{orb}}(X) \geqslant 0$;

- For any such a candidate, analyze its realizability using geometric methods, namely Lemma 6.1 when $\widetilde{X}$ is bad, or the fact that a map realizing the cover can be lifted to an isometry $\mathbb{S}^2 \to \mathbb{S}^2$ (or $\mathbb{E}^2 \to \mathbb{E}^2$) when $\widetilde{X}$ and X are spherical (or Euclidean; it turns out that X is never bad).

The program has been fully carried out in [7], leading to the following main results:

Theorem 6.2. *Let a candidate surface branched cover* $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ *induce a candidate 2-orbifold cover* $\widetilde{X} \overset{d:1}{\dashrightarrow} X$ *with* $\chi^{\mathrm{orb}}(X) > 0$. *Then* $\widetilde{\Sigma} \overset{d:1}{\underset{\Pi}{\dashrightarrow}} \Sigma$ *is exceptional if and only if* $\widetilde{X}$ *is bad and* X *is spherical. All exceptions occur with non-prime degree.*

Theorem 6.3. *Suppose* $d = 4k + 1$ *for* $k \in \mathbb{N}$. *Then*

$$S \overset{d:1}{\underset{(\underbrace{2,\ldots,2}_{2k},1),(\underbrace{4,\ldots,4}_{k},1),(\underbrace{4,\ldots,4}_{k},1)}{\dashrightarrow}} S$$

is a candidate surface branched cover, and it is realizable if and only if d *can be expressed as* $x^2 + y^2$ *for some* $x, y \in \mathbb{N}$.

Theorem 6.4. *Suppose* $d = 6k + 1$ *for* $k \in \mathbb{N}$. *Then*

$$S \overset{d:1}{\underset{(\underbrace{2,\ldots,2}_{3k},1),(\underbrace{3,\ldots,3}_{2k},1),(\underbrace{6,\ldots,6}_{k},1)}{\dashrightarrow}} S$$

is a candidate surface branched cover and it is realizable if and only if d *can be expressed as* $x^2 + xy + y^2$ *for some* $x, y \in \mathbb{N}$.

We conclude by noting that the last two results provide strong supporting evidence for Conjecture 1.5, because of the following facts:

- A prime number of the form $4k+1$ can always be expressed as $x^2 + y^2$ for $x, y \in \mathbb{N}$ (Fermat);
- A prime number of the form $6k + 1$ can always be expressed as $x^2 + xy + y^2$ for $x, y \in \mathbb{N}$ (Gauss);
- The integers that can be expressed as $x^2 + y^2$ or as $x^2 + xy + y^2$ with $x, y \in \mathbb{N}$ have asymptotically zero density in $\mathbb{N}$.

This means that a candidate cover in any of these two statements is "exceptional with probability 1," even though it is realizable when its degree is prime. Note also that it was shown in [2] that establishing Conjecture 1.5 in the special case of three branching points would imply the general case.

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