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Orbital Motion Approximation with Constant Circumferential Acceleration

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Nomenclature

A, B, C	=	auxiliary dimensionless functions
$\mathcal{A}$	=	interceptor spacecraft
a	=	semimajor axis, [km]
a_T	=	propulsive acceleration magnitude, [mm/s^2]
$\mathcal{B}$	=	target spacecraft
e	=	eccentricity
F	=	thrust magnitude, [N]
f_1, f_2	=	auxiliary functions, see Eq. (24)
g_0	=	standard gravity, [m/s^2]
$\widetilde{h}$	=	dimensionless angular momentum
$I_{\rm sp}$	=	specific impulse, [s]
K	=	number of revolutions
m	=	spacecraft mass, [kg]
N	=	number of rectifications
q_1, q_2, q_3	=	dimensionless orbital parameters
r	=	radial distance, [km]
s	=	dimensionless auxiliary parameter
T	=	orbital period, [hours]
t	=	time, [hours]
u	=	radial component of velocity, [km/s]
v	=	circumferential component of velocity, [km/s]
ϵ	=	dimensionless propulsive acceleration magnitude
θ	=	polar angle, [deg]
μ	=	primary body's gravitational parameter, [km^3/s^2]
ν	=	true anomaly, [deg]
ρ	=	dimensionless radial error
τ	=	thrust direction parameter
ω	=	apse line rotation angle, [deg]

Subscripts

0	=	initial, parking orbit
$\mathcal{A}$	=	interceptor spacecraft
$\mathcal{B}$	=	target spacecraft
k	=	value at k-th rectification point
s	=	numerical integration

Superscripts

$\cdot$	=	time derivative

Introduction

The orbital motion of a spacecraft subjected to a constant and circumferential propulsive acceleration (i.e., perpendicular to the position vector direction) is a classical problem of orbital mechanics, which has been thoroughly investigated since the pioneering work of Tsien [1]. An interesting approximate solution is presented in Battin's textbook [2], obtained under the assumptions of a two-dimensional motion, a circular parking orbit, and a low propulsive acceleration magnitude. More recently, Ref. [3] proposes an analytical approximation of the spacecraft escape conditions and gives a simple and accurate formula for the escape distance, which only depends on the propulsive acceleration magnitude.

The aim of this Note is to analyze the two-dimensional trajectory of a spacecraft with a constant, low, and circumferential propulsive acceleration, using the procedure proposed by Bombardelli et al. [4]. The latter describes the spacecraft dynamics through a set of generalized (non-singular) orbital parameters, and uses a perturbative approach to get an

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analytical representation of the resulting orbit in case of tangential propulsive acceleration, that is, when the thrust and velocity vectors are parallel. This method [4] gives accurate results when the propulsive acceleration magnitude is significantly smaller than the local gravitational pull, and its effectiveness in trajectory approximation has been discussed in the noteworthy cases of a spacecraft propelled by either a solar sail [5], or an electric solar wind sail [6] with fixed attitude. In fact, the perturbative approach allows an approximation of the osculating orbit characteristics to be derived, for both circular and elliptic parking orbits, assuming either a constant propulsive acceleration or a constant thrust magnitude with a time-varying spacecraft mass. In the latter case a rectification procedure, useful for improving the method accuracy [4], may also be employed to model a time-varying propulsive acceleration and to estimate the propellant mass consumption. The set of equations discussed in this Note is useful to obtain an analytical approximation of the spacecraft propelled trajectory for a generic (closed) parking orbit. In particular, when a circular parking orbit and a constant propulsive acceleration magnitude are considered, the model also gives an analytical estimation of flight time and of propulsive performance necessary for a circle-to-circle low-thrust rendezvous maneuver [7].

Two-dimensional trajectory approximation

Consider a spacecraft subjected to a continuous propulsive acceleration of constant magnitude a_T, which initially covers a closed parking orbit of semimajor axis a_0 and eccentricity e_0, moving around a primary body with gravitational parameter μ. The propulsion system is switched-on at time $t_0 \triangleq 0$, when the spacecraft true anomaly is $\nu_0 \in [0, 2\,\pi]$ rad. In the special case of circular parking orbit ($e_0 = 0$), ν_0 represents the spacecraft initial angular position relative to a fixed direction.

Assume the thrust vector to lie on the parking orbit plane for $t \geq t_0$, and its direction to be perpendicular to the primary-spacecraft line (case of circumferential thrust). In this scenario, the spacecraft equations of motion can be conveniently written in a polar reference frame $\mathcal{T}(O; r, \theta)$ as

$$\ddot{r} - r\,\dot{\theta}^2 = -\frac{\mu}{r^2} \tag{1}$$

$$r\,\ddot{\theta} + 2\,\dot{r}\,\dot{\theta} = \tau\,a_T \tag{2}$$

where r is the primary-spacecraft distance, θ is the polar angle measured counterclockwise from the parking orbit apse line (with $\theta(t_0) = \nu_0$), and $\tau \in \{-1, 1\}$ is a dimensionless parameter that models either an orbit raising ($\tau = 1$) or an orbit lowering case ($\tau = -1$). Note that the situation described by Eqs. (1)-(2) is substantially different from that of Ref. [4], where the propulsive acceleration is parallel to the spacecraft velocity (tangential thrust case) and, therefore, has a radial component different from zero.

According to Bombardelli et al. [4], the osculating orbit semimajor axis a, eccentricity e, and apse line direction, can be written in terms of dimensionless non-singular modified orbital parameters $\{q_1, q_2, q_3\}$ as

$$q_1 \triangleq \frac{e}{\widetilde{h}}\cos\omega \quad , \quad q_2 \triangleq \frac{e}{\widetilde{h}}\sin\omega \quad , \quad q_3 \triangleq \frac{1}{\widetilde{h}} \tag{3}$$

where ω is the angle between the osculating orbit apse line and that of the parking orbit, and $\widetilde{h}$ is the dimensionless angular momentum magnitude, given by

$$\widetilde{h} = \sqrt{\frac{a\,(1-e^2)\,(1+e_0\,\cos\nu_0)}{a_0\,(1-e_0^2)}} \tag{4}$$

Note that a, e, and ω may be recovered from Eqs. (3) as [4]

$$a = \frac{a_0\,\left(1-e_0^2\right)}{(q_3^2 - q_1^2 - q_2^2)\,(1+e_0\,\cos\nu_0)} \quad , \quad e = \frac{\sqrt{q_1^2+q_2^2}}{q_3} \quad , \quad \omega = \arctan\left(\frac{q_2}{q_1}\right) \tag{5}$$

The primary-spacecraft distance r may be written as a function of $\{q_1, q_2, q_3\}$ and θ as

$$r = \frac{a_0\,\left(1-e_0^2\right)}{(1+e_0\,\cos\nu_0)\,(q_1\,q_3\,\cos\theta + q_2\,q_3\,\sin\theta + q_3^2)} \tag{6}$$

Finally, the radial (u) and the circumferential (v) component of the spacecraft velocity are

$$u = (q_1\,\sin\theta - q_2\,\cos\theta)\,\sqrt{\frac{\mu\,(1+e_0\,\cos\nu_0)}{a_0\,(1-e_0^2)}} \tag{7}$$

$$v = s\,\sqrt{\frac{\mu\,(1+e_0\,\cos\nu_0)}{a_0\,(1-e_0^2)}} \tag{8}$$

where $s \triangleq q_1\,\cos\theta + q_2\,\sin\theta + q_3$ is an auxiliary function, which coincides with the dimensionless circumferential component of the spacecraft velocity.

Using the angular coordinate θ as the independent variable, the variation of $\{q_1,\, q_2,\, q_3\}$ is described by the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \frac{\tau\,\epsilon}{q_3\, s^3}\begin{bmatrix} s\sin\theta & (s+q_3)\cos\theta \\ -s\cos\theta & (s+q_3)\sin\theta \\ 0 & -q_3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{9}$$

where

$$\epsilon \triangleq \frac{a_T}{\mu/r_0^2} \qquad \text{with} \qquad r_0 = \frac{a_0\,\left(1-e_0^2\right)}{1+e_0\,\cos\nu_0} \tag{10}$$

is the dimensionless propulsive acceleration, defined as the ratio of a_T to the initial gravitational acceleration μ/r_0^2 . Assume the propulsive acceleration magnitude to be small when compared to the initial gravitational acceleration, that is, $\epsilon \ll 1$. In this case, a_T plays the role of a perturbative term in the Keplerian motion [4]. Accordingly, the generic modified parameter q_i can be written with an asymptotic series expansion, viz.

$$q_i = q_{i_0} + \tau\,\epsilon\, q_{i_1} + O(\epsilon^2) \qquad \text{with} \quad i = \{1,\,2,\,3\} \tag{11}$$

where q_{i_0} are the unperturbed terms (Keplerian case), q_{i_1} are the first order perturbative terms, and $O(\epsilon^2)$ represent higher-order terms in ϵ, which will be neglected. Substituting Eqs. (11) into Eqs. (9) and equating the unperturbed terms, q_{i_0} turn out to be constant and may be obtained from the initial conditions, that is

$$q_{1_0} = \frac{e_0}{\widetilde{h}_0} \quad , \quad q_{2_0} = 0 \quad , \quad q_{3_0} = \frac{1}{\widetilde{h}_0} \tag{12}$$

where $\widetilde{h}_0 = \widetilde{h}(t_0) \equiv \sqrt{1+e_0\,\cos\nu_0}$ is the angular momentum along the parking orbit, see Eq. (4).

Since ϵ is a constant, the first order terms q_{i_1} are found substituting Eqs. (11)–(12) into (9) and solving the resulting set of differential equations. For a circular parking orbit ($e_0 = 0$), the solution is

$$q_1 = 2\,\tau\,\epsilon\,(\sin\theta - \sin\nu_0) \tag{13}$$

$$q_2 = -2\,\tau\,\epsilon\,(\cos\theta - \cos\nu_0) \tag{14}$$

$$q_3 = 1 - \tau\,\epsilon\,(\theta - \nu_0) \tag{15}$$

while, for an elliptic parking orbit

$$q_1 = \frac{e_0}{\widetilde{h}_0} + \tau\,\epsilon\,[A(\theta) - A(\nu_0)] \tag{16}$$

$$q_2 = \tau\,\epsilon\,[B(\theta) - B(\nu_0)] \tag{17}$$

$$q_3 = \frac{1}{\widetilde{h}_0} + \tau\,\epsilon\,[C(\theta) - C(\nu_0)] \tag{18}$$

where $A = A(\theta)$, $B = B(\theta)$, and $C = C(\theta)$ are auxiliary (dimensionless) functions defined in the appendix. Having obtained $\{q_1, q_2, q_3\}$, the orbital parameters of the osculating orbit may be calculated with Eqs. (5), whereas the polar form of the spacecraft propelled trajectory is given by Eq. (6).

Flight time approximation for circular parking orbit

An analytical estimation of the flight time can be obtained, in principle, paralleling the procedure described in the appendix of Ref. [4]. However, that procedure is rather involved and requires the use of elliptic integrals. Here, a different approach is proposed, and an elegant, analytical approximation of the flight time is found in the special case of circular parking orbit with $r_0 \equiv a_0$. Assuming $e_0 = 0$ (circular orbit) and $\nu_0 = 0$, the radial distance r in Eq. (6) becomes

$$r = \frac{r_0}{(1-\tau\,\epsilon\,\theta)^2 + 2\,\epsilon\,\sin\theta\,\left(\tau - \epsilon\,\theta\right)} \tag{19}$$

whereas the radial and circumferential components of the spacecraft velocity, from Eqs. (7)–(8), are

$$u = 2\,\tau\,\epsilon\,\left(1-\cos\theta\right)\sqrt{\frac{\mu}{r_0}} \quad , \quad v = \left(1 - \tau\,\epsilon\,\theta + 2\,\tau\,\epsilon\,\sin\theta\right)\sqrt{\frac{\mu}{r_0}} \tag{20}$$

Note that the trajectory equation (19) is quite different from the original expression proposed by Battin [2], which was obtained by neglecting the radial acceleration $\dot{u}$, that is

$$r = \frac{r_0}{\sqrt{1-4\,\tau\,\epsilon\,\theta}} \tag{21}$$

Unlike Eq. (19), the relation (21) does not include any oscillatory term in θ, and is less accurate than Eq. (19) as long as $\epsilon\,\theta \ll 1$.

The flight time t may be estimated as a function of the polar angle θ by recalling that $\dot{\theta} = v/r$, where v and r are given by Eqs. (19) and (20), respectively. Therefore

$$\dot{\theta} = \sqrt{\frac{\mu}{r_0^3}}\,(1 - \tau\,\epsilon\,\theta)^3 \left(1 + 2\,\frac{\tau\,\epsilon\,\sin\theta}{1 - \tau\,\epsilon\,\theta}\right)^2 \tag{22}$$

Since q_3 is defined to be positive, see Eqs. (3), Eq. (15) implies that $\epsilon\,\theta < 1$. Recalling that $\epsilon \geq 0$ and $\tau = \{-1, 1\}$, the quadratic term in Eq. (22) is constrained by

$$f_1 \leq \left(1 + 2\,\frac{\tau\,\epsilon\,\sin\theta}{1 - \tau\,\epsilon\,\theta}\right)^2 \leq f_2 \tag{23}$$

where

$$f_1 \triangleq \left(1 - \frac{2\,\epsilon}{1 - \epsilon\,\theta}\right)^2 \quad , \quad f_2 \triangleq \left(1 + \frac{2\,\epsilon}{1 - \epsilon\,\theta}\right)^2 \tag{24}$$

Figure 1 shows the numerical values of the functions $f_1(\epsilon, \theta)$ and $f_2(\epsilon, \theta)$ defined in Eq. (24).

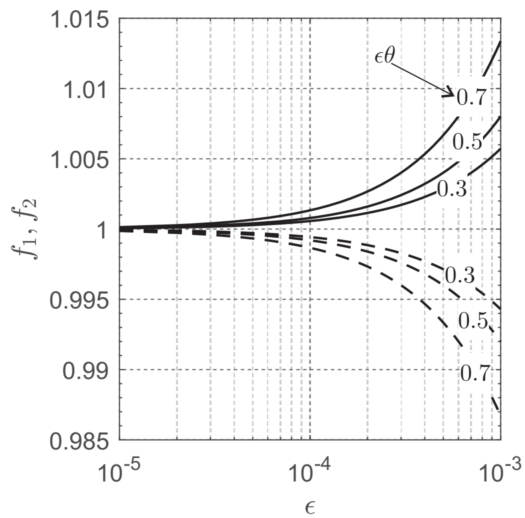


Figure 1 Numerical values of f_1 (solid line) and f_2 (dash line), see Eq. (24).

Both functions can be approximated with $f_1 \simeq f_2 \simeq 1$, with an error less than 1%, as long as $\epsilon\,\theta < 0.5$ and the dimensionless propulsive acceleration is $\epsilon \leq 10^{-3}$. Note that $\epsilon = 10^{-3}$ in a circular low-Earth orbit corresponds to an acceleration $a_T \simeq 9\,\mathrm{mm/s^2}$, a value well beyond any realistic case. Accordingly, Eq. (22) reduces to

$$\dot{\theta} \simeq \sqrt{\frac{\mu}{r_0^3}}\,(1 - \tau\,\epsilon\,\theta)^3 \tag{25}$$

and the approximate expression of the flight time as a function of the polar angle is

$$t \simeq \frac{\sqrt{r_0^3/\mu}}{2\,\tau\,\epsilon}\left[\frac{1}{(1 - \tau\,\epsilon\,\theta)^2} - 1\right] \tag{26}$$

Neglecting the oscillatory terms in Eq. (19), the time variation of the radial distance can be expressed as

$$r \simeq r_0\left(1 + \frac{2\,\tau\,\epsilon\,t}{\sqrt{r_0^3/\mu}}\right) \tag{27}$$

which coincides with the linearization of the analogue expression given by Battin [2] and by the recent model discussed in Ref. [3], when $\epsilon\,\theta \ll 1$ (or $\epsilon\,t \ll 1$).

Propellant consumption estimation for a constant-thrust case

The results given by Eqs. (16)–(18) (or Eqs. (13)–(15) for the circular case) are the application of the approximate model proposed by Bombardelli et al. [4] to the case of constant, low, and circumferential propulsive acceleration. In case of propellantless propulsion systems, Eqs. (16)–(18) (or Eqs. (13)–(15)) are sufficient to describe the spacecraft dynamics. A similar conclusion does not apply, however, to a conventional propulsion system, which provides a thrust of magnitude F with specific impulse I_{sp}. In fact, in that case the variation of the spacecraft mass m due to propellant consumption (and, therefore, the variation of $a_T = F/m$) must be taken into account in the analysis, especially when long flight times are considered. This is possible by introducing an orbit rectification procedure in the trajectory approximation [4], which essentially consists of updating the initial conditions of Eqs. (16)–(18) (or Eqs. (13)–(15)). Its effectiveness has been recently discussed for a propellantless propulsion system-based mission scenario [5,6]. Usually, this rectification procedure is used to improve the accuracy of the analytical model by preventing the angular coordinate θ to increase indefinitely. However, orbit rectifications may also be implemented to account for the instantaneous variation of the dimensionless propulsive acceleration ϵ in Eq. (10). In that case, the spacecraft propelled trajectory can be approximated assuming a piecewise constant propulsive acceleration $a_T = a_T(\theta)$ or $\epsilon = \epsilon(\theta)$.

For example, assume the propulsion system to provide a constant magnitude thrust F. The spacecraft mass m_{k+1} (and, therefore, the propulsive acceleration magnitude) at the generic rectification time instant t_{k+1} is given by

$$m_{k+1} = m_k - \frac{F}{g_0\, I_{\rm sp}}\,(t_{k+1} - t_k) \quad \text{with} \quad k \in [0,\, N] \tag{28}$$

where g_0 is the standard gravity, and $N \geq 0$ is the (integer) total number of rectifications. The time interval $(t_{k+1} - t_k)$ between two consecutive rectifications can be calculated with the equations described in the appendix of Ref. [4] (not reported here for the sake of conciseness). From Eq. (28), the propulsive acceleration can be obtained as $a_{T_{k+1}} = F/m_{k+1}$, whereas the dimensionless parameter ϵ_{k+1} at time t_{k+1} is given by Eq. (10). In particular, the value of m_{k+1} when $k = N$ is an estimate of the total propellant consumption during the whole flight.

Model validation

The proposed model has been validated by comparing the results of the analytical approximation with the outputs of an orbit simulator in which the spacecraft equations of motion are integrated in double precision using a variable order Adams-Bashforth-Moulton solver scheme [8,9] with absolute and relative errors of 10^{-12}. The accuracy of the approximate results (see Eq. (6)) can be evaluated by analyzing, in a given mission scenario, the dimensionless error $\rho = \rho(\theta)$ defined as [10]

$$\rho = \frac{|r(\theta) - r_s(\theta)|}{r_s(\theta)} \tag{29}$$

where the subscript "s" refers to the output of the numerical integration.

For exemplary purposes, consider a geocentric orbit raising problem ($\mu = \mu_\oplus = 3.986 \times 10^5\,\mathrm{km^3/s^2}$, and $\tau = 1$) in which the parking orbit is circular with radius $r_0 \triangleq r(t_0) = 6640\,\mathrm{km}$. Assume the conventional propulsion system to provide a constant magnitude thrust $F = 0.1\,\mathrm{N}$ with a specific impulse $I_{\rm sp} = 3000\,\mathrm{s}$, a typical value of a Hall effect thruster. The variation of the dimensionless error $\rho = \rho(\theta)$ is shown in Fig. 2 for two values of the spacecraft initial mass $m_0 \in \{100,\, 1000\}\,\mathrm{kg}$, without any rectifications ($N = 0$). The error ρ of the analytical approximation increases (decreases) with the polar angle θ (initial spacecraft mass m_0), but its numerical value remains confined to a few percentage points, even when the spacecraft completes a large number (about 100) of revolutions around the Earth.

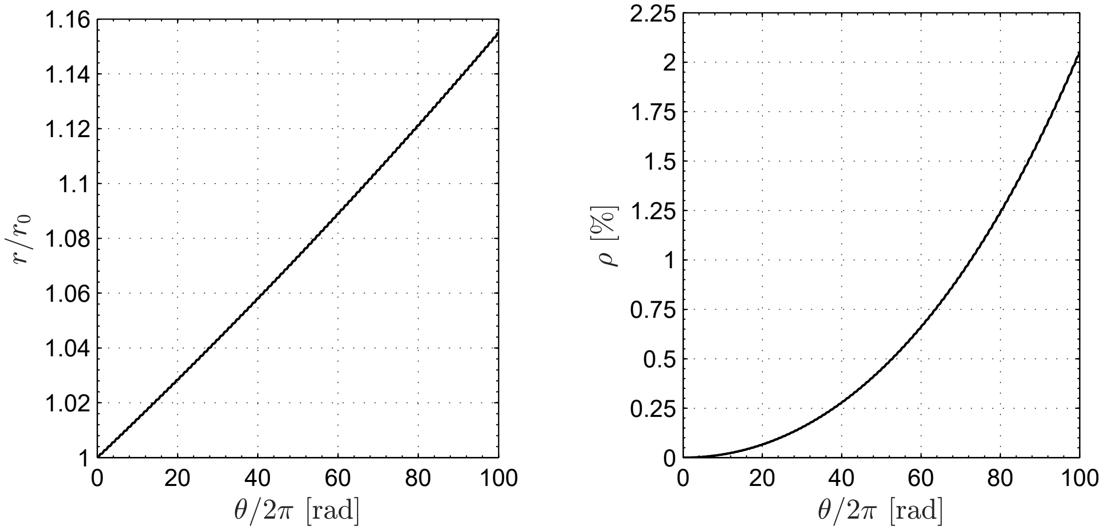
The accuracy of the analytical method also depends on the value of the parking orbit eccentricity e_0. For example, consider an elliptic parking orbit with perigee (apogee) radius equal to 6640 km (42168 km), which corresponds to a typical geosynchronous transfer orbit (GTO) with $e_0 \simeq 0.73$. In this case, the results (without rectifications) of an orbit raising problem, starting at perigee, are shown in Fig. 3. When the propulsive acceleration is sufficiently small (for example when the initial mass is $m_0 = 1000\,\mathrm{kg}$), the analytical model accurately approximates the actual propelled trajectory even for a large number of revolutions. This is shown in Fig. 3(b), where the dimensionless error is below 0.1% after thirty revolutions. When the propulsive acceleration is increased (for example, $m_0 = 100\,\mathrm{kg}$), the value of ρ is still acceptable (below 1%) when $\theta/2\pi < 10$, see Fig. 3(a). However, the dimensionless error ρ quickly grows when both a_T and θ increase, as is shown in Fig. 3(a), where $\rho \simeq 50\%$ when $m_0 = 100\,\mathrm{kg}$ and $\theta/(2\pi) = 30$. In the latter case, a more accurate approximation can be obtained with a rectification procedure. This is shown in Tab. 1, where the maximum dimensionless error ρ is reported as a function of the number of rectifications N. For example, assuming $m_0 = 100\,\mathrm{kg}$ and $N = 49$, ρ is less than 2% when $\theta/(2\pi) = 30$. Also, the propellant consumption, estimated to be 8.04 kg from Eq. (28), is close to its actual value of 7.84 kg.

$N+1$	$\max(\rho)$ [%]
1	49.01
10	27.63
20	16.35
30	8.12
40	2.79
50	1.77

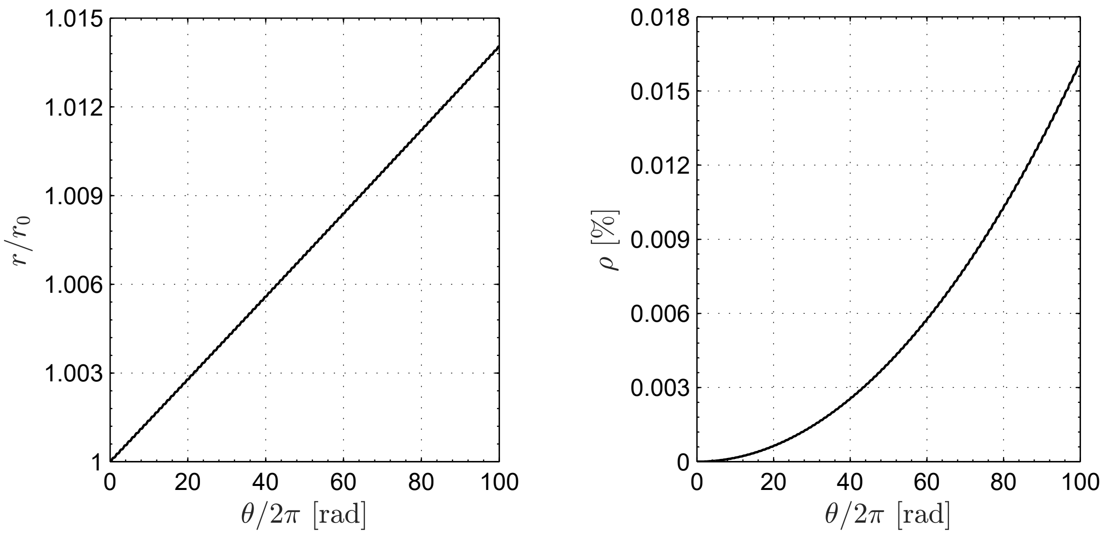
Table 1 Orbit raising from a GTO with $T/m_0 = 1\,\mathrm{mm/s^2}$: $\max(\rho)$ vs. N.

Mission application

The previously discussed approximate method is now used to analyze a classical mission scenario involving the rendezvous of two spacecraft, the interceptor $\mathcal{A}$ and the target $\mathcal{B}$. These spacecraft initially cover two circular coplanar orbits with radius $r_\mathcal{A}$ and $r_\mathcal{B}$, respectively. Assume that $\mathcal{A}$, equipped with a propulsion system providing a circumferential acceleration with constant (small) magnitude, must perform a rendezvous with $\mathcal{B}$. The problem consists of estimating the propulsive acceleration (i.e., the value of ϵ) and the initial angular separation between $\mathcal{A}$ and $\mathcal{B}$ necessary for the



a) $T/m_0 = 1\,\mathrm{mm/s^2}$.



b) $T/m_0 = 0.1\,\mathrm{mm/s^2}$.

Figure 2 Results for an orbit raising from a circular LEO ($r_0 = 6640\,\mathrm{km}$).

rendezvous maneuver. The latter starts at $t_0 \triangleq 0$, when the angular coordinate of $\mathcal{A}$ and $\mathcal{B}$ are $\theta_{\mathcal{A}}(t_0) \equiv \nu_{\mathcal{A}_0} \triangleq 0$ and $\theta_{\mathcal{B}}(t_0) \triangleq \nu_{\mathcal{B}_0} \in [0,\, 2\pi]$ rad, respectively, see Fig. 4. Since the angular position of $\mathcal{A}$ must coincide with that of $\mathcal{B}$ at the end of the rendezvous maneuver (that is, at time $t = t_f$, where t_f is the flight time) the final polar angle of the target, $\theta_{\mathcal{B}_f} = \theta_{\mathcal{B}}(t_f)$, can be written as

$$\theta_{\mathcal{B}_f} = \nu_{\mathcal{B}_0} + t_f \sqrt{\frac{\mu}{r_{\mathcal{B}}^3}} = \theta_{\mathcal{A}_f} \tag{30}$$

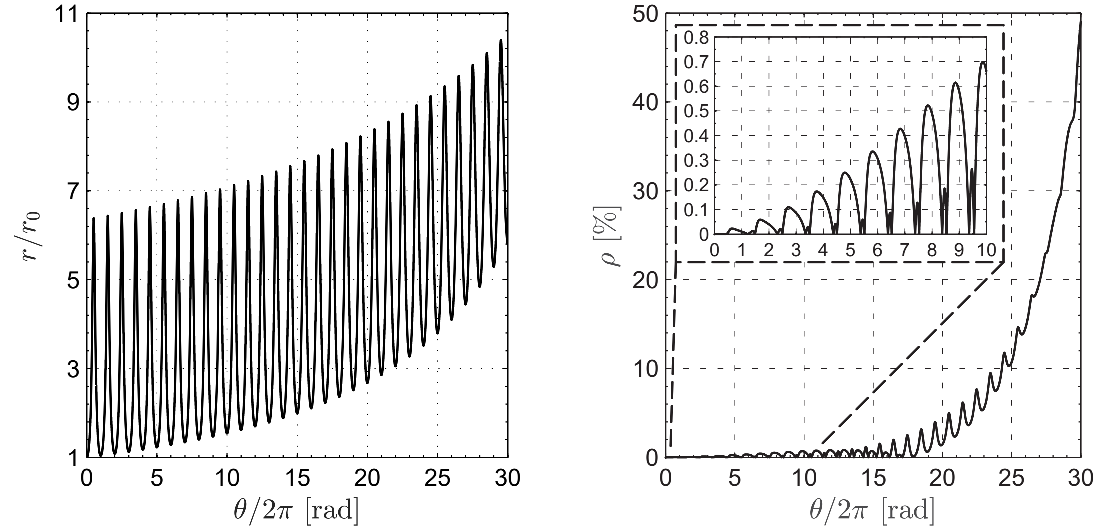
where $\theta_{\mathcal{A}_f} = \theta_{\mathcal{A}}(t_f)$ is the total angle swept by $\mathcal{A}$ during the transfer (propelled) trajectory.

The value of $\theta_{\mathcal{A}_f}$ may be found by taking into account that, at the end of the maneuver, the radial velocity of $\mathcal{A}$ must be $u_{\mathcal{A}}(t_f) = 0$. From the first of Eqs. (20), the result is

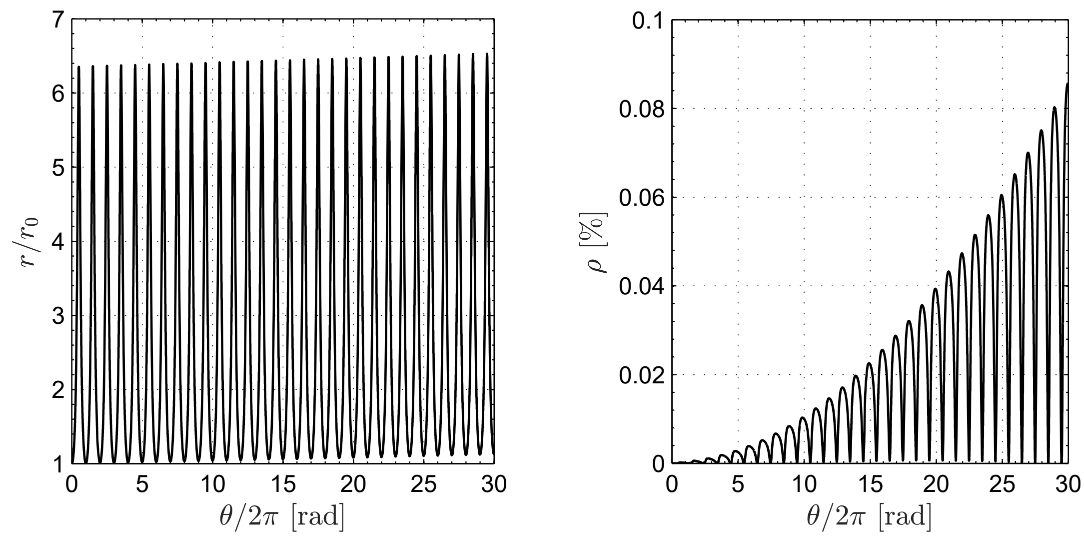
$$\theta_{\mathcal{A}_f} = 2\,K\,\pi \tag{31}$$

where $K \geq 1$ is the integer number of revolutions around the primary body completed by $\mathcal{A}$ during the transfer. Note that, substituting Eq. (31) into Eq. (19) and the second of Eqs. (20), the circumferential velocity of the interceptor is $v = \sqrt{\mu/r}$, that is, it matches the target orbital velocity at the end of the rendezvous maneuver. The latter unusual result is related to the approximate nature of the model.

The required value of the dimensionless propulsive acceleration ϵ can be written as a function of K by enforcing the initial condition $r_0 = r_{\mathcal{A}}$ and the rendezvous condition $r(\theta_{\mathcal{A}_f}) = r_{\mathcal{B}}$ into Eq. (19), where $\theta_{\mathcal{A}_f}$ is given by Eq. (31). The



a) $T/m_0 = 1\,\mathrm{mm/s^2}$.



b) $T/m_0 = 0.1\,\mathrm{mm/s^2}$.

Figure 3 Results for an orbit raising from a GTO ($e_0 \simeq 0.73$) starting at perigee ($r_0 = 6640\,\mathrm{km}$).

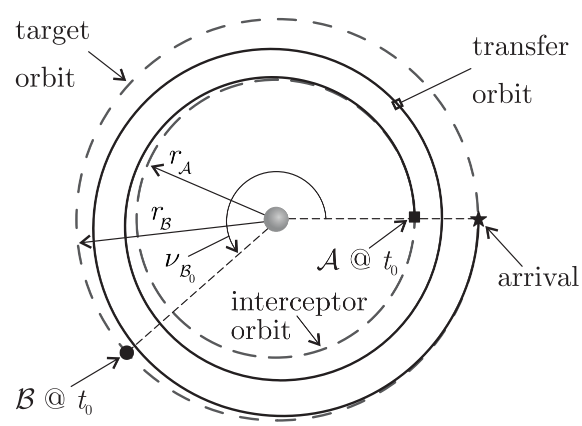


Figure 4 Conceptual scheme of the circle-to-circle rendezvous.

result is

$$\epsilon = \frac{1 - \sqrt{r_{\mathcal{A}}/r_{\mathcal{B}}}}{2\,\tau\,K\,\pi} \tag{32}$$

where

$$\tau = \operatorname{sign}\left(\frac{r_{\mathcal{B}}}{r_{\mathcal{A}}} - 1\right) \tag{33}$$

and sign $(\square)$ is the signum function. The flight time t_f is given by Eqs. (26) and (32) as

$$\frac{t_f}{T_{\mathcal{A}}} = \frac{K\,(r_{\mathcal{B}}/r_{\mathcal{A}} - 1)}{2\left(1 - \sqrt{r_{\mathcal{A}}/r_{\mathcal{B}}}\right)} \tag{34}$$

where $T_{\mathcal{A}} = 2\,\pi\,\sqrt{r_{\mathcal{A}}^3/\mu}$ is the initial orbital period of the interceptor spacecraft. The required propulsive acceleration and flight time are shown in Fig. 5. Finally, substituting Eqs. (31)–(34) into Eq. (30), the initial angular position of the

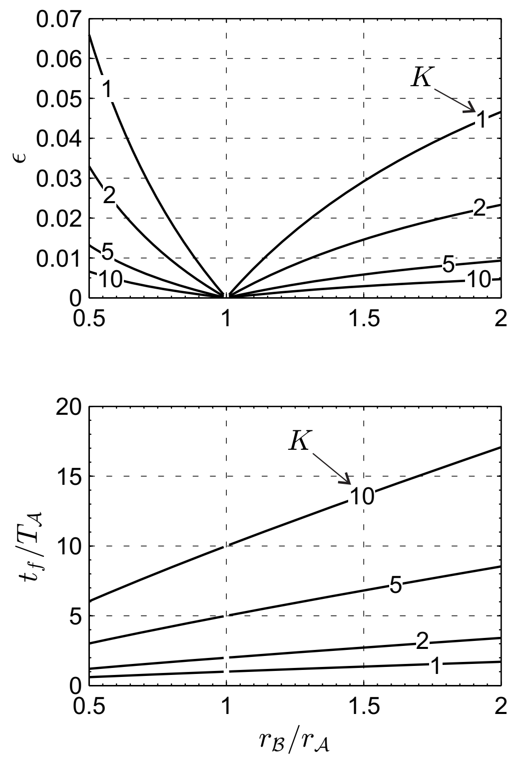


Figure 5 Dimensionless propulsive acceleration and flight time for a circle-to-circle orbit transfer.

target spacecraft $\nu_{\mathcal{B}_0}$, necessary to accomplish the rendezvous maneuver, is

$$\nu_{\mathcal{B}_0} = K\,\pi\left(2 - \sqrt{\frac{r_{\mathcal{A}}}{r_{\mathcal{B}}}} - \frac{r_{\mathcal{A}}}{r_{\mathcal{B}}}\right) \tag{35}$$

Case study

Equations (32)–(35) are now used to analyze a low-thrust rendezvous between an interceptor in a circular low-Earth orbit of radius $r_{\mathcal{A}} = 6640\,\mathrm{km}$ (with $\mu_\oplus/r_{\mathcal{A}}^2 \simeq 9\,\mathrm{m/s^2}$), and a target spacecraft that covers a coplanar (close) circular orbit of radius $r_{\mathcal{B}} = 6740\,\mathrm{km}$. Since the analysis is based on the assumption that $\nu_{\mathcal{A}}(t_0) = 0$, the value of $\nu_{\mathcal{B}_0}$ in Eq. (35) represents the initial angular separation between the interceptor and the target. In this case $r_{\mathcal{B}}/r_{\mathcal{A}} \simeq 1.015$, $\tau = 1$, and the values of $\{\epsilon,\, t_f,\, \nu_{\mathcal{B}_0}\}$, given by Eqs. (32)–(35), are summarized in Fig. 6 for different values of $K \le 10$.

For example, assuming $K = 10$, the required dimensionless propulsive acceleration is $\epsilon \simeq 1.2\times10^{-4}$ (about $1.08\,\mathrm{mm/s^2}$), the flight time is $t_f \simeq 10.1\,T_{\mathcal{A}} \simeq 15\,\mathrm{hours}$, and the initial angular separation is $\nu_{\mathcal{B}_0} \simeq 40\,\mathrm{deg}$, see Fig. 6. Note that, according to the approximate method, the polar angle of the interceptor at rendezvous is $\theta_{\mathcal{A}}(t_f) = 20\,\pi = 3600\,\mathrm{deg}$. In this case, the actual (numerical) values of the radial and angular position of spacecraft $\mathcal{A}$ at time $t_f \simeq 15\,\mathrm{hours}$ are $r_{\mathcal{A}}(t_f) = 6741\,\mathrm{km}$ and $\theta_{\mathcal{A}}(t_f) \simeq 3599.7\,\mathrm{deg}$, respectively. Hence, the actual $\mathcal{A}$-$\mathcal{B}$ final distance, at time $t = t_f$, is less than $36\,\mathrm{km}$. A more accurate result can be obtained by considering a smaller value of K and, therefore, a smaller value of the final polar angle $\theta_{\mathcal{A}_f}$. Indeed, assuming $K = 2$, Fig. 6 gives $\epsilon \simeq 5.93\times10^{-4}$, $t_f \simeq 2\,T_{\mathcal{A}} \simeq 3\,\mathrm{hours}$, $\nu_{\mathcal{B}_0} \simeq 8\,\mathrm{deg}$. In this case, $a_T \simeq 5.3\,\mathrm{mm/s^2}$ and the actual $\mathcal{A}$-$\mathcal{B}$ distance at time $t = t_f$ is reduced to about 6.7 km.

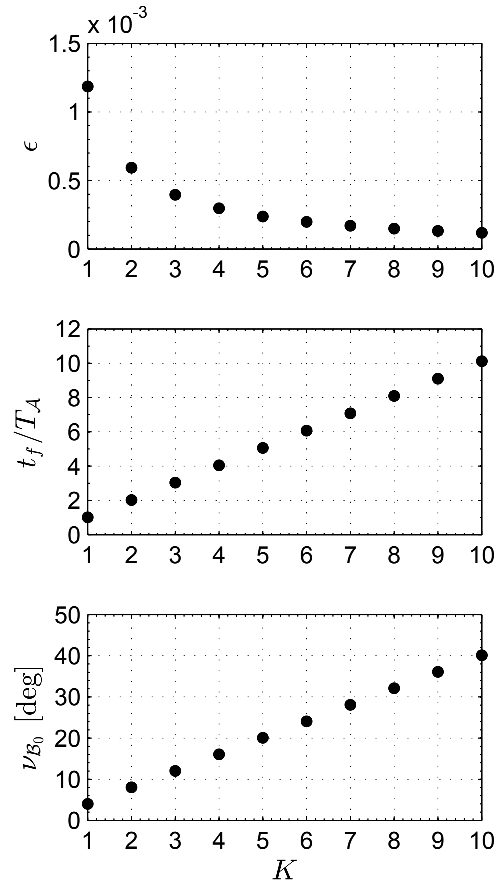


Figure 6 Results for a low-thrust rendezvous between circular orbits with $r_\mathcal{A} = 6640\,\mathrm{km}$ and $r_\mathcal{B} = 6740\,\mathrm{km}$.

Conclusions

The two-dimensional dynamics of a spacecraft, subjected to a small and constant circumferential propulsive acceleration, has been approximated using a perturbative approach. A comparison with the outputs of a numerical integration of the equations of motion shows that the approximate method is accurate when the parking orbit is circular, and the propulsive acceleration magnitude is sufficiently small. A rectification procedure may effectively be used for estimating the propellant consumption and improving the model accuracy, particularly in the case of an elliptic parking orbit, when a large polar angle (or propulsive acceleration) is considered. The proposed model is an useful tool for the analysis of a circle-to-circle orbit raising, and gives a set of analytical relationships for the preliminary phase of mission design.

Appendix: Auxiliary functions A, B, and C

Equations (16)–(18) represent the compact form of the modified parameters $\{q_1,\, q_2,\, q_3\}$ for an elliptic parking orbit, where the terms $\{A,\, B,\, C\}$ are three auxiliary functions of the polar angle θ. The function $A = A(\theta)$ in Eq. (16) is defined as

$$A(\theta) \triangleq A_1(\theta) + A_2(\theta) + A_3(\theta) + A_4(\theta) + A_5(\theta) + A_6(\theta) \tag{36}$$

where

$$A_1(\theta) \triangleq \frac{2\,\widetilde{h}_0^3\,\sin\theta}{(1-e_0^2)^2\,(1+e_0\,\cos\theta)^2} \tag{37}$$

$$A_2(\theta) \triangleq -\frac{\widetilde{h}_0^3}{e_0\,(1-e_0^2)^{5/2}}\,\phi(\theta) \tag{38}$$

$$A_3(\theta) \triangleq \frac{\widetilde{h}_0^3}{e_0\,\sqrt{1-e_0^2}}\,\phi(\theta) \tag{39}$$

$$A_4(\theta) \triangleq -\frac{e_0\,\widetilde{h}_0^3}{2\,(1-e_0^2)^{5/2}}\,\phi(\theta) \tag{40}$$

$$A_5(\theta) \triangleq -\frac{e_0^2\,\widetilde{h}_0^3\,\sin\theta}{2\,(1-e_0^2)^2\,(1+e_0\,\cos\theta)^2} \tag{41}$$

$$A_6(\theta) \triangleq \frac{3\,e_0\,\widetilde{h}_0^3\,\sin\theta\,\cos\theta}{2\,(1-e_0^2)^2\,(1+e_0\,\cos\theta)^2} \tag{42}$$

with

$$\phi(\theta) \triangleq \arctan\left(\frac{\sqrt{1-e_0^2}\,\sin\theta}{e_0+\cos\theta}\right) \tag{43}$$

The function $B = B(\theta)$ in Eq. (17) is defined as

$$B(\theta) \triangleq \frac{(2\,e_0\,\cos\theta+3)\,\widetilde{h}_0^3}{2\,e_0\,(1+e_0\,\cos\theta)^2} \tag{44}$$

Finally, the function $C = C(\theta)$ in Eq. (18) is given by

$$C(\theta) \triangleq C_1(\theta) + C_2(\theta) + C_3(\theta) + C_4(\theta) + C_5(\theta) \tag{45}$$

where

$$C_1 = -\frac{\widetilde{h}_0^3}{(1-e_0^2)^{5/2}}\,\phi(\theta) \tag{46}$$

$$C_2 = -\frac{\widetilde{h}_0^3\,e_0^2}{2\,(1-e_0^2)^{5/2}}\,\phi(\theta) \tag{47}$$

$$C_3 = \frac{2\,e_0\,\widetilde{h}_0^3\,\sin\theta}{(1-e_0^2)^2\,(1+e_0\cos\theta)^2} \tag{48}$$

$$C_4 = -\frac{e_0^3\,\widetilde{h}_0^3\,\sin\theta}{2\,(1-e_0^2)^2\,(1+e_0\cos\theta)^2} \tag{49}$$

$$C_5 = \frac{3\,e_0^2\,\widetilde{h}_0^3\,\sin\theta\,\cos\theta}{2\,(1-e_0^2)^2\,(1+e_0\cos\theta)^2} \tag{50}$$

where ϕ is given by Eq. (43). Note that $A(\nu_0)$, $B(\nu_0)$, and $C(\nu_0)$ in Eqs. (16)–(18) refer to the value of the functions A, B, and C at the initial time t_0, that is, when $\theta = \nu_0$.

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