BRILL-NOETHER LOCI FOR DIVISORS ON IRREGULAR VARIETIES

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Abstract. We take up the study of the Brill-Noether loci $W^r(L,X) := \{\eta \in \mathrm{Pic}^0(X) \mid h^0(L\otimes\eta) \geq r+1\}$, where X is a smooth projective variety of dimension >1, $L \in \mathrm{Pic}(X)$, and $r \geq 0$ is an integer.

By studying the infinitesimal structure of these loci and the Petri map (defined in analogy with the case of curves), we obtain lower bounds for $h^0(K_D)$, where D is a divisor that moves linearly on a smooth projective variety X of maximal Albanese dimension. In this way we sharpen the results of [Xi] and we generalize them to dimension > 2.

In the 2-dimensional case we prove an existence theorem: we define a Brill-Noether number $\rho(C,r)$ for a curve C on a smooth surface X of maximal Albanese dimension and we prove, under some mild additional assumptions, that if $\rho(C,r) \geq 0$ then $W^r(C,X)$ is nonempty of dimension $\geq \rho(C,r)$.

Inequalities for the numerical invariants of curves that do not move linearly on a surface of maximal Albanese dimension are obtained as an application of the previous results.
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Contents

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1. Introduction

The classical Brill-Noether theory studies the loci

$$W_d^r(C) := \{L \in \mathrm{Pic}(C) |\deg L = d,\ h^0(L) \geq r+1\},$$

where C is a smooth projective curve of genus $g \geq 2$. We refer the reader to [ACGH] for a comprehensive treatment of this beautiful topic and to [ACG] for further information. We only recall here that all the theory revolves around the Brill-Noether number $\rho(g,r,d) = g-(r+1)(r+g-d)$: if $\rho(g,r,d) \geq 0$ then $W_d^r(C)$ is not empty, and if $\rho(g,r,d) > 0$ then $W_d^r(C)$ is connected of dimension $\geq \rho(g,r,d)$. In addition, if C is general in moduli then $\dim W_d^r(C) = \rho(g,r,d)$.

Several possible generalizations of this theory have been investigated in the past years, the most studied being the case in which divisors of fixed degree are replaced by stable vector bundles of fixed rank and degree on the smooth curve C, see [GT] for a recent survey. Generalizations to some varieties of dimension > 1 have been considered by several people (see for instance [DL], [H], [Le]).

Moreover, Brill-Noether type loci for higher dimensional varieties occur naturally in the theory of deformations, Hilbert schemes, Picard schemes and Fourier-Mukai transforms, but usually not as the main object of study. In [CM] the case of vector bundles on an arbitrary smooth projective variety is considered under the assumption that all the cohomology groups of degree > 1 vanish. In [Kl1] the Brill-Noether loci are defined in great generality for relative subschemes of any codimension of a family of projective schemes, but the theory is developed only in the case of linear series on smooth projective curves. Any concrete theory of special divisors, like for instance existence theorems, seems impossible in such generality.

Here we take up what seems to us the most straightforward generalization of the classical theory of linear series on curves, namely the case of line bundles on an arbitrary projective variety. In this setup, the Brill-Noether loci are a special instance of the cohomological support loci, whose study has been started in [GL1], [GL2], focusing on the case of topologically trivial line bundles, and has been extended and refined in the context of rank one local systems (see for instance [DPS]). However, our point of view and that of [GL1], [GL2] are different, since we look for lower bounds on the dimension of these loci rather than for upper bounds.

Let's now summarize the content of the paper. Given a projective variety X, a line bundle $L \in \mathrm{Pic}(X)$ and an integer $r \geq 0$, we set $W^r(L,X) := \{\eta \in \mathrm{Pic}^0(X) \mid h^0(L \otimes \eta) \geq r+1\}$. First we recall the natural scheme structure on $W^r(L,X)$ and, by analysing it, we show that, if X has maximal Albanese dimension (i.e, it has generically finite Albanese map) and D is an effective divisor contained in the fixed part of $|K_X|$, then $0 \in W(D,X)$ is an isolated point (Corollary 3.5).

Then we focus on two special cases: (a) singular projective curves and (b) smooth surfaces of maximal Albanese dimension. In case (a) we assume that X is a connected reduced projective curve and, since in general $\mathrm{Pic}^0(X)$ is not an abelian variety, we study the intersection of $W^r(L,X)$ with a compact subgroup $T \subseteq \mathrm{Pic}^0(X)$ of dimension t. We define the Brill-Noether number $\rho(t,r,d) := t-(r+1)(p_a(C)-d+r)$, where $d := \deg L$. In this set-up we prove the exact analogue of the existence theorem of Brill-Noether theory under a technical assumption on T (Theorem 5.1, cf. also Remark 5.2). We do not consider here the very important theory of the compactifications of the Brill-Noether loci of singular curves and the theory of limit linear series as treated by many authors (see, for instance, [Gi, EH, Al, EK, Cap, ACG], and see also [ACG] for a complete bibliography). Our reason is that we are interested in line bundles coming from a smooth complete variety via restriction, and these are naturally parametrized by a compact subgroup.

Theorem 5.1 is a key step in our approach to case (b): we combine it with the generic vanishing theorem of Green and Lazarsfeld to obtain an analogue of the existence theorem of Brill-Noether theory. Given a curve C on a surface S with irregularity $q > 1$, we consider the image T of the natural map $\mathrm{Pic}^0(S) \to \mathrm{Pic}^0(C)$. If this map has finite kernel and we take $L = \mathcal{O}_C(C)$, then the Brill-Noether number introduced above can be written as $\rho(C,r) := q-(r+1)(p_a(C)-C^2+r)$. (For $r=0$ we write simply $\rho(C)$).

For surfaces without irrational pencils of genus >1 and reduced curves C all of whose components have positive self-intersection, we prove that if $\rho(C,r) > 1$, then $W^r(C,S)$ is nonempty of dimension $\geq \min\{q, \rho(C,r)\}$, and that for $\rho(C,r) = 1$ the same statement holds under an additional assumption. In the specific case $r = 0$ and under the same hypotheses we are also able to show that C actually moves algebraically in a family of dimension $\geq \min\{q, \rho(C\})$. For the precise statement see Theorem 6.2. We remark that the assumptions on the Albanese map and on the structure of V^1 in these results are quite mild (see Remark 6.3 and Section 2).

A second theme of the paper, strictly interwoven with the analysis of the Brill-Noether loci, is the study of the restriction map $r_D\colon H^0(K_X) \to H^0(K_X|_D)$, where X is a smooth variety of maximal Albanese dimension and $D \subset X$ is an effective divisor whose image via the Albanese map $a\colon X \to \mathrm{Alb}(X)$ generates $\mathrm{Alb}(X)$. In Proposition 4.6 we establish a uniform lower bound for the rank of r_D under the only assumption that D is not contained in the ramification locus of the Albanese map (this is also one of the ingredients of the proof of the Brill-Noether type result for surfaces). Then we show how one can improve on this bound if the tangent space to $W^0(D,X)$ at 0 has positive dimension (Proposition 4.9); in order to do this we introduce and study, in analogy with the case of curves, the Petri map $H^0(D) \otimes H^0(K_X - D) \to H^0(K_X)$. If $h^0(D) > 1$, a lower bound for the rank of r_D gives immediately a lower bound for $h^0(K_D)$ (Corollaries 4.7 and 4.10): in this way we extend to arbitrary dimension the main result of [Xi], which treats the case in which X is a surface and D is a general fiber of a fibration $X \to \mathbb{P}^1$.

All the previous results are applied in §7 to the study of curves on a surface of general type S with $q(S) := h^0(\Omega^1_S) > 1$ that is not fibered onto a curve of genus > 1. More precisely, we give inequalities for the numerical invariants of a curve $C = \sum_i C_i$ of S such that $C_i^2 > 0$ for all i and $h^0(C) = 1$ (Corollary 7.4); in the special case in which $p_a(C) \leq 2q(S) - 2$ we obtain a stronger inequality and a lower bound on the codimension of $W^0(C, S)$ in $\mathrm{Pic}^0(S)$ (Corollary 7.6). Remark that, apart from the case when $p_a(C) = q(S)$ (classified in [MPP1]), the question of the existence on a surface of general type S of curves C with $C^2 > 0$ and $p_a(C) \leq 2q(S) - 2$ is, as far as we know, completely open. Finally, we prove a result (Proposition 7.7) that relates the fixed locus of the paracanonical system of S to the ramification divisor of the Albanese map.

In §8 we collect several examples in order to illustrate the phenomena that occur for Brill-Noether loci on surfaces and to clarify to what extent the results that we obtain are optimal. We also pose some questions: in our opinion, the most important of these is whether a statement analogous to Theorem 6.2 holds if one replaces the effective divisor C by, say, an ample line bundle, and whether a similar statement holds in arbitrary dimension. The main difficulty here is that, while in the case of curves the cohomology of a family of line bundles of fixed degree is computed by a complex with only 2 terms, in the case of a variety of dimension n one has to deal with a complex of length $n+1$. Hence the negativity of Picard sheaves, which has been established for projective varieties of any dimension (cf. [La], 6.3.C and 7.2.15), does not suffice alone to prove non emptyness results for the Brill-Noether loci.

In the surface case the method of restriction to curves and the use of the generic vanishing theorems overcome the cohomological problem. We are aware however that usually curves lying on surfaces are not general in the sense of the Brill-Noether theory, hence, although the existence theorem 6.2 is sharp, one cannot expect that the Brill-Noether number computes precisely the dimension of the Brill-Noether locus in most cases. In fact, in view of the complexity of the geometry and of the topology of irregular surfaces (even the geographical problem has not been solved yet, cf. [MP]), it is somewhat surprising that a single numerical invariant, such as the Brill-Noether number, can give a definite existence result for continuous families of effective divisors on surfaces. Our methods are also very useful to attack problems in classification theory and questions about on curves on surfaces, as we illustrated in [MPP1] and in section 7 of this paper.

In addition, the use of the generic vanishing combined with the infinitesimal analysis in sections 3 and 4 shows the importance of the Petri map in the higher dimensional case.

We are convinced that the methods of the present paper together with the use of some fine obstructions theory as in [MPP2] will give some striking new results on the theory of continuous families of divisor on irregular varieties, which is ultimately the Brill Noether theory.

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Notation and conventions. We work over the complex numbers. All varieties are assumed to be complete. We do not distinguish between divisors on smooth varieties and the corresponding line bundles and we denote linear equivalence by $\equiv$.

Let X be a smooth projective variety. We denote as usual by $\chi(X)$ the Euler characteristic of $\mathcal{O}_X$, by $p_g(X)$ the *geometric genus* $h^0(X, K_X)$ and by $q(X)$ the *irregularity* $h^0(X, \Omega^1_X)$. We denote by $\operatorname{albdim}(X)$ the dimension of the image of the Albanese map $a\colon X \to \operatorname{Alb}(X)$. As usual, a *fibration of* X is a surjective morphism with connected fibers $X \to Y$, where Y is a variety with $\dim Y < \dim X$. We say that X has an *irrational pencil of genus* $g > 0$ if it admits a fibration $X \to B$ onto a smooth curve of genus $g > 0$.

If D is an effective divisor of a smooth variety X we denote by $p_a(D)$ the *arithmetic genus* $\chi(K_D) - 1$, where K_D is the canonical divisor of D. In particular, if $\dim X = 2$ and D is a nonzero effective divisor (a *curve*) then by the adjunction formula the arithmetic genus of D of S is $p_a(D) = (K_S D + D^2)/2 + 1$; the curve D is said to be *m-connected* if, given any decomposition $D = A + B$ of D with $A, B > 0$, one has $AB \geq m$.

Given a product of varieties $V_1 \times ... \times V_n$ we denote by pr_i the projection onto the $i-$th factor.

2. Preliminaries on irregular varieties

We recall some by now classical results on irregular varieties that are used repeatedly throughout the paper.

2.1. **Albanese dimension and irregular fibrations.** Let X be a smooth projective variety of dimension n. The *Albanese dimension* $\operatorname{albdim}(X)$ is defined as the dimension of the image of the Albanese map of X; in particular, X has *maximal Albanese dimension* if its Albanese map is generically finite onto its image and it is of *Albanese general type* if in addition $q(X) > n$. For a normal variety Y, we define the Albanese variety $\operatorname{Alb}(Y)$ and all the related notions by considering any smooth projective model of Y.

An *irregular fibration* $f\colon X \to Y$ is a morphism with positive dimensional connected fibers onto a normal variety Y with $\operatorname{albdim} Y = \dim Y > 0$; the map f is called an *Albanese general type fibration* if in addition Y is of Albanese general type. If $\dim Y = 1$, then Y is a smooth curve of genus $b > 0$; in that case, f is called an *irrational pencil of genus* b and it is an Albanese general type fibration if and only if $b > 1$.

Notice that if $q(X) \geq n$ and X has no Albanese general type fibration, then X has maximal Albanese dimension.

The so-called generalized Castelnuovo–de Franchis Theorem (see [Cat, Thm. 1.14] and Ran [Ra]) shows how the existence of Albanese general type fibrations is detected by the cohomology of X:

Theorem 2.1 (Catanese, Ran)**.** *The smooth projective variety X has an Albanese general type fibration $f\colon X \to Y$ with $\dim Y \leq k$ if and only if there exist independent 1-forms $\omega_0, \dots \omega_k \in H^0(\Omega^1_X)$ such that $\omega_0 \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0 \in H^0(\Omega^{k+1}_X)$.*

So in particular the existence of irrational pencils of genus > 1 is equivalent to the existence of two independent 1-forms $\alpha, \beta \in H^0(\Omega^1_X)$ such that $\alpha \wedge \beta = 0$.

2.2. **Generic vanishing.** Let X be a projective variety of dimension n and let $L \in \mathrm{Pic}(X)$; the generic vanishing loci, or Green-Lazarsfeld loci, are defined as $V^i(X) := \{\eta \mid h^i(\eta) > 0\} \subseteq \mathrm{Pic}^0(X)$, $i = 0, \dots n$. They have been object of intense study since the groundbreaking papers [GL1], [GL2] and their structure is very well understood.

We only summarize here for later use the properties of $V^1(X)$, established in [GL1], [GL2], [Be2], [Be3] and [Si]:

Theorem 2.2. *Let X be a smooth projective variety; then:*

(i) *if X has maximal Albanese dimension, then $V^1(X)$ is a proper closed subset of $\mathrm{Pic}^0(X)$ whose components are translates by torsion points of abelian subvarieties;*

(ii) *if X has no irrational pencil of genus > 1, then $\dim V^1(X) \leq 1$ and $0 \in V^1(X)$ is an isolated point.*

3. The Brill-Noether loci

In this section we recall the definition of the Brill-Noether loci and some general facts on their geometry. The scheme structure and the tangent space to the Brill-Noether loci have been described in several contexts, however for clarity's sake we choose to spell out and prove the properties we need in the sequel. We close the section by proving some properties of the ramification divisor of the Albanese map and of the fixed divisor of the canonical system of a variety of maximal Albanese dimension (Proposition 3.4 and Corollary 3.5).

Let X be a projective variety and let $L \in \mathrm{Pic}(X)$. For $r \geq 0$ we define the *Brill-Noether locus*

$$W^r(L, X) := \{\eta \in \mathrm{Pic}^0(X) \mid h^0(L \otimes \eta) \geq r + 1\}.$$

If $T \subseteq \mathrm{Pic}^0(X)$ is a subgroup, we set $W^r_T(L, X) := W^r(L, X) \cap T$. For $r = 0$ we write $W(L, X)$ instead of $W^0(L, X)$.

Remark 3.1. When X is a smooth curve, the Brill-Noether loci are a very classical object of study (cf. [ACGH], Ch. III, IV and V). The definition we give here is slightly different from the classical one, which consists in fixing a class $\lambda \in \mathrm{NS}(X)$ and defining the Brill-Noether locus as $W^r_\lambda(X) := \{M \in \mathrm{Pic}^\lambda(X) \mid h^0(M) \geq r+1\}$, where $\mathrm{Pic}^\lambda(X)$ denotes the preimage of λ via the natural map $\mathrm{Pic}(X) \to \mathrm{NS}(X)$. Of course, if λ is the class of L in $\mathrm{NS}(X)$, then $W^r(L,X)$ is mapped isomorphically onto $W^r_\lambda(X)$ by the translation by $L \in \mathrm{Pic}(X)$. Our choice of definition is motivated by technical reasons that become apparent, for instance, in the proof of Theorem 6.2.

By the semicontinuity theorem (cf. [Mu], p. 50) the Brill-Noether loci are closed in $\mathrm{Pic}^0(X)$. In fact they are a particular case of the cohomological support loci introduced in [GL1, §1].

The scheme structure of $W^r(L,X)$ is described by following the approach of [Kl1]. Our point of view differs slightly from [Kl1] in that we consider line bundles rather than subschemes.

We recall the following consequence of Grothendieck duality:

Lemma 3.2. *Let X be a projective variety of dimension n, let $L \in \mathrm{Pic}(X)$ and let $\mathcal{P}$ be a Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$. Then there exists a coherent sheaf Q on $\mathrm{Pic}^0(X)$, unique up to canonical isomorphism, such that:*

(i) *for every coherent sheaf M on $\mathrm{Pic}^0(X)$ there is a canonical isomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathrm{Pic}^0(X)}}(Q,M) \cong \mathrm{pr}_{2*}(\mathcal{P} \otimes \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* M)$;*

(ii) *if X is Gorenstein, then $Q \cong \mathrm{R}^n\mathrm{pr}_{2*}(\mathrm{pr}_1^*(K_X - L) \otimes \mathcal{P}^\vee)$.*

Proof. (i) Follows by applying [EGAIII2, Thm.7.7.6] to the morphism $\mathrm{pr}_2 \colon X \times \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ and to the sheaf $\mathcal{P} \otimes \mathrm{pr}_1^* L$.

(ii) By (i) it is enough to show that for every coherent sheaf M on $\mathrm{Pic}^0(X)$ there is a canonical isomorphism $\underline{\mathrm{Hom}}_{\mathcal{O}_{\mathrm{Pic}^0(X)}}(\mathrm{R}^n\mathrm{pr}_{2*}(\mathrm{pr}_1^*(K_X - L) \otimes \mathcal{P}^\vee, M) \cong \mathrm{pr}_{2*}(\mathcal{P} \otimes \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* M)$. If X is Gorenstein, then $\mathrm{pr}_1^*\omega_X$ is the relative dualizing sheaf for $\mathrm{pr}_2 \colon X \times \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ and, since X is Cohen-Macaulay, the required functorial isomorphism exists by [Kl2, Thm. 21]. $\square$

By Lemma 3.2 (i), a point $\eta \in \mathrm{Pic}^0(X)$ belongs to $W^r(L,X)$ iff $\dim_{\mathbb{C}}(Q \otimes \mathbb{C}(\eta)) \geq r+1$; hence we give $W^r(L,X)$ the r-th Fitting subscheme structure associated with the sheaf Q. Notice that, since $\mathcal{P}$ is determined up to tensoring with $\mathrm{pr}_2^* M$ for M a line bundle on $\mathrm{Pic}^0(X)$, Q is also determined up to tensoring with M; however Q and $Q \otimes M$ have the same Fitting subschemes, hence our definition is well posed.

Given $\eta \in \mathrm{Pic}^0(X)$, we identify as usual the tangent space to $\mathrm{Pic}^0(X)$ at the point η with $H^1(\mathcal{O}_X)$; then, generalizing the case when X is a curve, we have the following description of the Zariski tangent space to $W^r(L,X)$.

Proposition 3.3. *Let $r \geq 0$ be an integer, let X be a projective variety, let $L \in \mathrm{Pic}(X)$ and let $\eta \in W^r(L,X)$. Then:*

(i) *if* $\eta \in W^{r+1}(L, X)$, *then* $T_\eta W^r(L, X) = H^1(\mathcal{O}_X)$;
(ii) *if* $\eta \notin W^{r+1}(L, X)$, *then* $T_\eta W^r(L, X)$ *is the kernel of the linear map* $H^1(\mathcal{O}_X) \to \mathrm{Hom}(H^0(X, L+\eta), H^1(X, L+\eta))$ *induced by cup product.*

Proof. Let Q be the coherent sheaf of Lemma 3.2. As usual, we denote by $\mathbb{C}[\epsilon]$ the algebra of dual numbers. We regard an element $v \in H^1(\mathcal{O}_X)$ as a morphism $v\colon \operatorname{Spec}\mathbb{C}[\epsilon] \to \operatorname{Pic}^0(X)$ mapping the closed point of $\operatorname{Spec}\mathbb{C}[\epsilon]$ to η and we denote by Q_v the pull back of Q via v. By the functorial properties of Fitting ideals, v is in the tangent space to $W^r(L, X)$ iff the r-th Fitting ideal of Q_v as a $\mathbb{C}[\epsilon]$-module vanishes. Set $m := h^0(X, L+\eta)$. By the definition of Q (Lemma 3.2), there is an isomorphism $\mathrm{Hom}_{\mathbb{C}[\epsilon]}(Q_v, \mathbb{C}) \cong H^0(X, L+\eta)$; it is not hard to show that there is an isomorphism $Q_v \cong \mathbb{C}[\epsilon]^{m-l} \oplus \mathbb{C}^l$ for some $0 \le l \le m$. Hence Q_v has a presentation by a $m \times l$ matrix with ϵ on the diagonal and 0 elsewhere. A direct computation shows that the r-th Fitting ideal is 0 iff $m > r+1$ or $m = r+1$ and $l = 0$. In particular, this proves claim (i) and we may assume from now on that $m = r+1$.

Denote by L_v the pull back of $\mathcal{P} \otimes \mathrm{pr}_1^* L$ to $X_\epsilon := X \times_{\operatorname{Spec}\mathbb{C}} \operatorname{Spec}\mathbb{C}[\epsilon]$. The condition $l = 0$ is equivalent to the surjectivity of the map $\mathrm{Hom}_{\mathbb{C}[\epsilon]}(Q_v, \mathbb{C}[\epsilon]) \to \mathrm{Hom}_{\mathbb{C}[\epsilon]}(Q_v, \mathbb{C})$. By Lemma 3.2, we have canonical isomorphisms:

$$\mathrm{Hom}_{\mathbb{C}[\epsilon]}(Q_v, \mathbb{C}[\epsilon]) \cong H^0(X_\epsilon, L_v), \quad \mathrm{Hom}_{\mathbb{C}[\epsilon]}(Q_v, \mathbb{C}) \cong H^0(X, L+\eta)$$

So v is tangent to $W^r(L, X)$ at η iff the restriction map $H^0(X_\epsilon, L_v) \to H^0(X, L+\eta)$ is surjective. On the other hand, this map is part of the long cohomology sequence associated with the extension

$$0 \to L+\eta \xrightarrow{\epsilon} L_v \to L+\eta \to 0,$$

hence it is surjective iff the coboundary map $H^0(X, L+\eta) \to H^1(L+\eta)$ vanishes. Since it is well known that the latter map is given by cupping with v, statement (ii) follows. □

As an application of Proposition 3.3 we prove the following:

Proposition 3.4. *Let X be a smooth projective variety such that $n := \dim X = \operatorname{albdim} X$ and let R be the ramification divisor of the Albanese map of X; if $0 < Z \le R$ is a divisor and $s \in H^0(\mathcal{O}_X(Z))$ is a section that defines Z, then:*

(i) *the map* $H^1(\mathcal{O}_X) \xrightarrow{\cup s} H^1(\mathcal{O}_X(Z))$ *is injective;*
(ii) *if* $h^0(Z) = 1$, *then* $H^0(Z|_Z) = 0$ *and* $0 \in W(Z, X)$ *is an isolated point (with reduced structure).*

Proof. (i) Denote by Λ the image of the map $\bigwedge^n H^0(\Omega^1_X) \to H^0(K_X)$; the divisor R is the fixed part of the linear subsystem $|\Lambda| \subseteq |K_X|$.

Assume for contradiction that $v \in H^1(\mathcal{O}_X)$ is a nonzero vector such that $s \cup v = 0$; then, since $Z \le R$, we have $t \cup v = 0$ for every $t \in \Lambda$.

By Hodge theory there exists a nonzero $\beta \in H^0(\Omega^1_X)$ such that $v = \bar\beta$ and the condition $t \cup v = 0$ is equivalent to $t \wedge \bar\beta$ being an exact form:

$t \wedge \bar{\beta} = d\phi$. Let now $x \in X$ be a point such that $\beta(x) \neq 0$ and such that the differential of the Albanese map at x is injective. Then we can find $\alpha_1, \dots \alpha_{n-1} \in H^0(\Omega^1_X)$ such that $\alpha_1, \dots \alpha_{n-1}, \beta$ span the cotangent space $T^*_x X$. Hence the form $t := \alpha_1 \wedge \dots \wedge \alpha_{n-1} \wedge \beta$ is nonzero at x and therefore $t \wedge \bar{t} \neq 0$ and $(-i)^n \int_X t \wedge \bar{t} > 0$. On the other hand, we have

$$\int_X t \wedge \bar{t} = \pm \int_X (t \wedge \bar{\beta}) \wedge \bar{\alpha}_1 \wedge \dots \wedge \bar{\alpha}_{n-1} = \int_X d(\phi \wedge \bar{\alpha}_1 \wedge \dots \wedge \bar{\alpha}_{n-1}) = 0,$$

a contradiction. So $H^1(\mathcal{O}_X) \xrightarrow{\cup s} H^1(\mathcal{O}_X(R))$ is injective.

(ii) By (i) and by Proposition 3.3, the tangent space to $W(Z, X)$ at 0 is zero, hence $\{0\}$ with reduced structure is a component of $W(Z, X)$. The vanishing of $H^0(Z|_Z)$ also follows by (i) taking cohomology in the usual restriction sequence $0 \to \mathcal{O}_X \to \mathcal{O}_X(Z) \to \mathcal{O}_Z(Z) \to 0$. □

Corollary 3.5. *Let X be a smooth projective variety such that $n := \dim X = \operatorname{albdim} X$ and let $Z > 0$ be a divisor contained in the fixed part of $|K_X|$. Then $H^0(Z|_Z) = 0$ and $0 \in W(Z, X)$ is an isolated point.*

Proof. As usual, let R denote the ramification divisor of the Albanese map of X. Since Z is contained in the fixed part of $|K_X|$, we have $Z \subseteq R$ and $h^0(Z) = 1$. So the claim follows by Proposition 3.4. □

4. Restriction maps

In this section we consider a smooth projective variety X of maximal Albanese dimension and an effective divisor $D \subset X$ and we establish lower bounds for the rank of the restriction map

$$r_D \colon H^0(K_X) \to H^0(K_X|_D),$$

and for the corank of the residue map

$$\operatorname{res}_D \colon H^0(K_X + D) \to H^0(K_D).$$

Such bounds, besides being intrinsically interesting, can be used to give lower bounds for the arithmetic genus of divisors moving in a positive dimensional linear system (Corollaries 4.7 and 4.10).

More precisely, we give three inequalities. The first two (Proposition 4.6) are uniform bounds for the rank of r_D and the corank of res_D under the assumptions that D is irreducible, not contained in the ramification locus of the Albanese map $a \colon X \to \operatorname{Alb}(X)$, and $a(D)$ generates $\operatorname{Alb}(X)$. This is one of the ingredients in the proof of Theorem 6.2, which is our main result on the structure of Brill-Noether loci in the case of surfaces.

The third one (Proposition 4.9) is based on the infinitesimal analysis of the Brill-Noether locus $W^r(D, X)$ carried out in §3: the bound that we obtain is stronger than that of Proposition 4.6 but it requires further assumptions.

4.1. **Preliminary results.** The main goal of this section is to prove Proposition 4.5, which is the key result that enables us to obtain the inequalities of §4.2.

We start by listing some well known facts of linear algebra:

Lemma 4.1 (Hopf lemma)**.** *Let U, V and W be complex vector spaces of finite dimension and let $f\colon U \otimes V \to W$ be a linear map.*

If $\ker f$ *does not contain any nonzero simple tensor $u \otimes v$, then* $\operatorname{rk} f \geq \dim U + \dim V - 1$.

Lemma 4.2. *Let V, W be complex vector spaces of finite dimension and let $f\colon \bigwedge^k V \to W$ be a linear map.*

If $\ker f$ *does not contain any nonzero simple tensor $v_1 \wedge \dots \wedge v_k$, then* $\operatorname{rk} f \geq k(\dim V - k) + 1$.

Lemma 4.3 (ker/coker lemma)**.** *Let V, W be complex vector spaces of finite dimension and let $f, g\colon V \to W$ be linear maps.*

If $\operatorname{rk}(f + tg) \leq \operatorname{rk} f$ *for every $t \in \mathbb{C}$, then* $g(\ker f) \subseteq \operatorname{Im} f$.

The next result is possibly also known, but since it is less obvious we give a proof for completeness.

Lemma 4.4. *Let V, W be complex vectors spaces of finite dimension and set* $q := \dim V$. *Let $\phi : \bigwedge^2 V \to W$ be a linear map such that:*
(a) for every $0 \neq v \in V$, there exists $w \in V$ such that $\phi(v \wedge w) \neq 0$;
(b) if $\phi(v \wedge w) = \phi(v \wedge u) = 0$ and $v \neq 0$, then $\phi(u \wedge w) = 0$.
Then:

(i) $\dim \phi(V) \geq q - 1$.

(ii) *there exists $v \in V$ such that the restriction of ϕ to $v \wedge V$ is injective.*

Proof. We observe first of all that (i) follows from (ii), hence it is enough to prove (ii).

For every $v \in V$ we let $k_v\colon V \to W$ be the linear map defined by $x \mapsto \phi(v \wedge x)$, and we let $U(v)$ be the kernel and $S(v)$ be the image of k_v. Of course $v \in U(v)$, and for $v \neq 0$ the assumptions give:
(1) $U(v) \subsetneq V$;
(2) $U(v) = U(v') \iff v' \in U(v) \setminus \{0\}$.

Claim (ii) is equivalent to the existence of a vector $v \in V$ such that $U(v)$ is 1-dimensional. Choose v with $m := \dim U(v)$ minimal; notice that we have $0 < m < q$. For any vector $u \in V$ and any $t \in \mathbb{C}$ the map $k_v + tk_u = k_{v+tu}$ has rank $\leq q - m$; by Lemma 4.3 we have $k_u(U(v)) \subseteq S(v)$. Hence if $\phi(v' \wedge v) = 0$, then for any $u \in V$ there exists $h \in V$ such that

$$k_u(v') = \phi(u \wedge v') = \phi(v \wedge h) = k_v(h).$$

Since $k_u(v') = -k_{v'}(u)$, it follows that for every $v' \in U(v)$ we have $S(v') \subseteq S(v)$; since $U(v) = U(v')$ by (2), it follows $S(v) = S(v')$. Let now $L \subset V$ be a subspace such that $V = U(v) \oplus L$. Then for every $0 \neq v' \in U(v)$, $k_{v'}$

restricts to an isomorphism $h_{v'}\colon L \to S(v)$. Fixing bases for L and $S(v)$, this isomorphism is represented by an invertible matrix of order $q-m>0$, whose entries depend linearly on v'. Then taking determinants one obtains a homogeneous polynomial of degree $q-m>0$ that has no zeros on $\mathbb{P}(U(v))$. Since we are working over an algebraically closed field, this is possible only if $\dim U(v)=1$. □

Given a vector bundle E on a variety X and a finite dimensional subspace $V\subseteq H^0(X,E)$, for any integer $k\geq 0$ we denote by $\psi_k\colon \bigwedge^k V \to H^0(X,\bigwedge^k E)$ the natural map. Here is the main result of this section:

Proposition 4.5. *Let X be an irreducible variety, let E be a rank n vector bundle on X. Assume that there exists a subspace $V\subseteq H^0(X,E)$ of dimension q that generates E generically.*

Then the map $\psi_n\colon \bigwedge^n V\to H^0(X,\det E)$ has rank $\geq q-n+1$.

Proof. The proof is by induction on the rank n of E, the case $n=1$ being trivial.

Up to restricting to a Zariski open set, we may assume that X is affine and that V generates E.

Consider first the map $\psi_2\colon \bigwedge^2 V\to W:=H^0(X,\bigwedge^2 E)$. Since ψ_2 satisfies the assumptions of Lemma 4.4, there exist a section $s\in V$ such that $\psi_2(s\wedge t)=0$ if and only if $t=\lambda s$ for some $\lambda\in\mathbb{C}$. Up to replacing X by an open subset, we may assume that s vanishes nowhere on X, hence there is a short exact sequence

$$0\to\mathcal{O}_X\xrightarrow{s}E\to E'\to 0, \tag{4.1}$$

with E' a rank $n-1$ vector bundle. We denote by $V'\subseteq H^0(X,E')$ the image of V; the subspace V' has dimension $q-1$ and generates E' on X, hence by the inductive assumption the map $\psi'_{n-1}\colon \bigwedge^{n-1}V'\to H^0(X,\det E')$ has rank $\geq (q-1)-(n-1)+1=q-n+1$.

The sequence (4.1) induces an isomorphism $\det E\to\det E'$ and the induced map $H^0(X,\det E)\to H^0(X,\det E')$ maps $\operatorname{Im}\psi_n$ to a subspace containing $\operatorname{Im}\psi'_{n-1}$. Hence $\operatorname{rk}\psi_n\geq\operatorname{rk}\psi'_{n-1}\geq q-n+1$. □

4.2. **Uniform bounds.** Here we use the results of §4.1 to bound the rank of the map $r_D\colon H^0(K_X)\to H^0(K_X|_D)$ and the corank of the residue map $\operatorname{res}_D\colon H^0(K_X+D)\to H^0(K_D)$, where D is an effective divisor of an irregular variety X.

Proposition 4.6. *Let X be a smooth projective variety with* $\operatorname{albdim}X=\dim X=n$ *and let $D>0$ be an irreducible divisor of X such that the image of D via the Albanese map $a\colon X\to\operatorname{Alb}(X)$ generates* $\operatorname{Alb}(X)$. *Assume that D is not contained in the ramification divisor of a. Then, letting $q:=q(X)$:*

(i) *the rank of $r_D\colon H^0(K_X)\to H^0(K_X|_D)$ is $\geq q-n+1$;*
(ii) *the corank of* $\operatorname{res}_D\colon H^0(K_X+D)\to H^0(K_D)$ *is $\geq q-n+2$.*

Proof. (i) The inequality follows by Proposition 4.5 taking $E = \Omega^1_X|_D$ and $V = i^*H^0(X, \Omega^1_X)$, where $i\colon D \to X$ is the inclusion.

(ii) Consider the short exact sequence $0 \to K_X \to K_X + D \to K_D \to 0$. Taking cohomology, we see that the corank of res_D is equal to the rank of the coboundary map $\partial\colon H^0(K_D) \to H^1(K_X)$ or, taking duals, to the rank of ${}^t\partial\colon H^{n-1}(\mathcal{O}_X) \to H^{n-1}(\mathcal{O}_D)$.

Now let (X', D') be an embedded resolution of (X, D); then there is a commutative diagram:

$$\begin{array}{ccc} H^{n-1}(\mathcal{O}_X) & \longrightarrow & H^{n-1}(\mathcal{O}_{X'}) \\ {\scriptstyle {}^t\partial}\big\downarrow & & {\scriptstyle {}^t\partial}\big\downarrow \\ H^{n-1}(\mathcal{O}_D) & \longrightarrow & H^{n-1}(\mathcal{O}_{D'}) \end{array}$$

where the top horizontal map is an isomorphism. Hence we may assume without loss of generality that D is smooth.

Then, by Hodge theory, the map ${}^t\partial$ is the complex conjugate of the natural map $\rho\colon H^0(\Omega^{n-1}_X) \to H^0(K_D)$. Here we set $E = \Omega^1_D$ and $V = i^*H^0(\Omega^1_X) \subseteq H^0(\Omega^1_D)$, where $i\colon D \to X$ is the inclusion; then the image of ρ contains the image of $\psi_{n-1}\colon \bigwedge^{n-1} V \to H^0(K_D)$. The required inequality now follows by Proposition 4.5, since V has dimension q by the assumption that $a(D)$ generates $\mathrm{Alb}(X)$. $\square$

Statement (i) of Proposition 4.6 has been proven in the case of surfaces fibered over $\mathbb{P}^1$ by Xiao Gang in [Xi]. The following corollary generalizes to arbitrary dimension the main result of [Xi].

Corollary 4.7. *Let X be a smooth projective variety with* $\mathrm{albdim}\, X = \dim X = n$ *and let $D > 0$ be an irreducible divisor of X. If $h^0(D) = r + 1 \geq 2$, then:*

$$h^0(K_D) \geq 2(q + 1 - n) + r.$$

Proof. By the semicontinuity of $h^0(K_{D'})$ as $D' \in |D|$ varies, we may replace D by a general element of $|D|$, and assume that D is not contained in the ramification locus of the Albanese map $a\colon X \to \mathrm{Alb}(X)$. Observe also that $a(D)$ generates $\mathrm{Alb}(X)$, since D moves linearly.

Consider the map $\mathrm{res}_D\colon H^0(K_X + D) \longrightarrow H^0(K_D)$. By Proposition 4.6, we have $h^0(K_D) \geq \mathrm{rk}\, \mathrm{res}_D + q - n + 2$.

To give a bound on the rank of res_D, we observe that the image of res_D contains the image of the multiplication map

$$(\mathrm{Im}\, r_D) \otimes H^0(D)/ < s > \longrightarrow H^0(K_D),$$

where $s \in H^0(D)$ is a section defining D. By Proposition 4.6, $\mathrm{rk}\, r_D \geq q - n + 1$ and so applying Lemma 4.1 we obtain $\mathrm{rk}\, \mathrm{res}_D \geq q - n + r$. Hence $h^0(K_D) \geq (q - n + r) + q - n + 2 = 2(q + 1 - n) + r$. $\square$

For future reference, we observe the following:

Corollary 4.8. *Let S be a smooth complex surface with* $\mathrm{albdim}(S) = 2$*, let* $a\colon S \to \mathrm{Alb}(S)$ *be the Albanese map and let $C \subset S$ be a 1-connected curve having a component C_1 not contained in the ramification locus of a and such that $C_1^2 > 0$. Then:*

(i) $h^0(K_S - C) \leq \chi(S)$*;*
(ii) $h^0(C|_C) \geq q + C^2 - p_a(C)$.

Proof. (i) Note that C_1 is nef and big; therefore $h^1(\mathcal{O}_S(-C_1)) = 0$, the map $H^1(\mathcal{O}_S) \to H^1(\mathcal{O}_{C_1})$ is an injection and $a(C_1)$ generates $\mathrm{Alb}(S)$. Since, by Proposition 4.6 we have $\mathrm{rk}\, r_{C_1} \geq q - 1$, we obtain $\mathrm{rk}\, r_C \geq q - 1$ and so $h^0(K_S - C) \leq \chi(S)$.

(ii) Since C is 1-connected, Riemann–Roch on C gives:

$$h^0(C|_C) = C^2 + h^0(K_S|_C) + 1 - p_a(C) \geq C^2 + q - p_a(C).$$

□

4.3. **The Petri map.** Let X be a smooth projective variety and let D be an effective divisor on X.

As a tool for studying the rank or r_D we introduce the *Petri map*, which, in analogy with the case of curves, is the map

$$\beta_D\colon H^0(K_X - D) \otimes H^0(D) \to H^0(K_X),$$

induced by cup product.

The Petri map is strictly related to the infinitesimal structure of the Brill-Noether loci, as follows. Let $r = h^0(D) - 1$, let T be the tangent space to $W^r(D, X)$ at 0 and let $\alpha\colon H^1(\mathcal{O}_X) \otimes H^{n-1}(\mathcal{O}_X) \to H^n(\mathcal{O}_X)$ be the map induced by cup product. Then by Proposition 3.3 for all $\sigma \in T \otimes H^{n-1}(\mathcal{O}_X)$ and $\psi \in H^0(D) \otimes H^0(K_X - D)$ we have

$$\alpha(\sigma) \cup \beta_D(\psi) = 0,$$

namely $V := \alpha(T \otimes H^{n-1}(\mathcal{O}_X)) \subseteq H^n(\mathcal{O}_X)$ is orthogonal to $\mathrm{Im}\, \beta_D \subseteq H^0(K_X)$.

Proposition 4.9. *Let X be a smooth projective variety of dimension n and irregularity $q \geq n$, let $D > 0$ be a divisor of X, let $r := h^0(D) - 1$ and let T be the tangent space to $W^r(D, X)$ at the point* 0*. Assume that* $\dim T > 0$ *and that X has no fibration $f\colon X \to Z$, with Z normal of Albanese general type and* $0 < \dim Z < n$*; then:*

(i) *if* $h^0(K_X - D) = 0$*, then* $\mathrm{rk}\, r_D \geq n(q - n) + 1$*;*
(ii) *if* $h^0(K_X - D) > 0$*, then:*

$$\mathrm{rk}\, r_D \geq (n-1)(q-n) + \dim T + r, \quad \textit{if } \dim T \leq q - n,$$
$$\mathrm{rk}\, r_D \geq n(q-n) + 1 + r, \qquad \textit{if } \dim T \geq q + 1 - n.$$

Proof. (i) If $h^0(K_X - D) = 0$ then $\mathrm{rk}\, r_D = h^0(K_X)$. By Theorem 2.1, under our assumptions the map $\bigwedge^n H^0(\Omega^1_X) \to H^0(K_X)$ does not map any simple tensor to 0, hence Lemma 4.2 gives $h^0(K_X) \geq n(q - n) + 1$.

(ii) Let $\alpha\colon H^1(\mathcal{O}_X)\otimes H^{n-1}(\mathcal{O}_X)\to H^n(\mathcal{O}_X)$ be the map induced by cup product. As we have remarked at the beginning of the section, $V := \alpha(T\otimes H^{n-1}(\mathcal{O}_X))\subseteq H^n(\mathcal{O}_X)$ is orthogonal to $\operatorname{Im}\beta_D\subseteq H^0(K_X)$, where β_D is the Petri map. Let $\mathbb{G}_T\subseteq\mathbb{G}(n,H^1(\mathcal{O}_X))$ be the subset consisting of the subspaces that have non trivial intersection with T. Since, as we remarked in (i), the map $\bigwedge^n H^0(\Omega^1_X)\to H^0(K_X)$ does not map any simple tensor to 0, the complex conjugate map $\bigwedge^n H^1(\mathcal{O}_X)\to H^n(\mathcal{O}_X)$ induces a morphism $\mathbb{G}_T\to\mathbb{P}(V)$ which is finite onto its image. It follows that $\dim V\geq\dim\mathbb{G}_T+1$. Since, as noticed above, the space V is orthogonal to $\operatorname{Im}\beta_D$, the codimension of $\operatorname{Im}\beta_D$ is $\geq\dim\mathbb{G}_T+1$.

On the other hand, by Lemma 4.1 the dimension of $\operatorname{Im}\beta_D$ is at least $h^0(K_X-D)+h^0(D)-1$. Since $h^0(K_X-D)=p_g-\operatorname{rk}r_D$ and $h^0(D)=r+1$, one obtains that the codimension of $\operatorname{Im}\beta_D$ is $\leq\operatorname{rk}r_D-r$. So $\operatorname{rk}r_D\geq\dim\mathbb{G}_T+r+1$, which is precisely the statement. $\square$

Arguing as in the proof of Corollary 4.7, one obtains the following:

Corollary 4.10. *Let X be a smooth projective variety of dimension n and irregularity $q\geq n$ that has no fibration $f\colon X\to Z$, with Z normal of Albanese general type and $0<\dim Z<n$. Let $D>0$ be a divisor of X, let $r:=h^0(D)-1$ and let T be the tangent space to $W^r(D,X)$ at the point 0. Assume that $r>0$ and $\dim T>0$; then*

(i) *if $h^0(K_X-D)=0$, then $h^0(K_D)\geq(n+1)(q-n)+r+2$;*
(ii) *if $h^0(K_X-D)>0$, then:*

$$h^0(K_D)\geq n(q-n)+\dim T+2r+1,\qquad \textit{if } \dim T\leq q-n,$$
$$h^0(K_D)\geq(n+1)(q-n)+2r+2,\qquad \textit{if } \dim T\geq q+1-n.$$

5. Brill-Noether theory for singular curves

Here we prove a generalization to the case of a compact subgroup of the Jacobian of a reduced connected curve of the classical results on the Brill-Noether loci of smooth curves. The results of this section are used in section 6 to prove the analogous result for smooth irregular surfaces (Thm. 6.2).

Assume that C is a reduced connected projective curve with irreducible components $C_1,\dots C_k$; for every i, denote by $\nu_i\colon C_i^\nu\to C_i$ the normalization map. We refer the reader to [BLR, §9.2, 9.3] for a detailed description of the Jacobian $\operatorname{Pic}^0(C)$. We just recall here that $\operatorname{Pic}^0(C)$ is a smooth algebraic group and that there is an exact sequence:

$$0\to G\to\operatorname{Pic}^0(C)\xrightarrow{f}\operatorname{Pic}^0(C_1^\nu)\times\cdots\times\operatorname{Pic}^0(C_k^\nu)\to 0,$$

where G is a smooth connected linear algebraic group and $f(\eta)=(\nu_1^*\eta,\dots,\nu_k^*\eta)$. Notice that if $T\subseteq\operatorname{Pic}^0(C)$ is a complete subgroup, then $G\cap T$ is a finite group, and therefore the induced map $T\to\operatorname{Pic}^0(C_1^\nu)\times\cdots\times\operatorname{Pic}^0(C_k^\nu)$ has finite kernel.

Fix $L \in \mathrm{Pic}(C)$, an integer $r \geq 0$ and a complete connected subgroup $T \subseteq \mathrm{Pic}^0(C)$, and consider the Brill-Noether locus $W^r_T(L, C) := \{\eta \in T \mid h^0(L \otimes \eta) \geq r + 1\}$.

As in the case of a smooth curve C, we define the Brill-Noether number $\rho(t, r, d) := t - (r + 1)(p_a(C) - d + r)$, where d is the total degree of L and $t = \dim T$. In complete analogy with the classical situation, we prove:

Theorem 5.1. *Let $r \geq 0$ be an integer, let C be a reduced connected projective curve and let L be a line bundle on C of total degree d. If $T \subseteq \mathrm{Pic}^0(C)$ is a complete connected subgroup of dimension t such that for every component C_i of C the map $T \to \mathrm{Pic}^0(C_i^\nu)$ has finite kernel, then:*

- (i) *if $\rho(t, r, d) \geq 0$, then $W^r_T(L, C)$ is nonempty;*
- (ii) *if $\rho(t, r, d) > 0$, then $W^r_T(L, C)$ is connected, it generates T and each of its components has dimension $\geq \min\{\rho(t, r, d), t\}$.*

Proof. The proof follows closely the proof given by Fulton and Lazarsfeld in the case of a smooth curve (cf. [FL], [La, §6.3.B, 7.2]).

Denote by $\mathcal{P}$ the restriction to $C \times T$ of a normalized Poincaré line bundle on $C \times \mathrm{Pic}^0(C)$. Let H be a sufficiently high multiple of an ample line bundle of C and let $M := L \otimes H$. Recall that for any product of varieties we denote by pr_i the projection onto the i-th factor; we define:

$$E := \mathrm{pr}_{2*}(\mathrm{pr}_1^* M \otimes \mathcal{P}).$$

By the choice of H, E is a vector bundle of rank $d + \deg H + 1 - p_a(C)$ on T and for every $\eta \in T$ the natural map $E \otimes \mathbb{C}(\eta) \to H^0(M \otimes \eta)$ is an isomorphism and $M \otimes \eta$ is generated by global sections.

We let $Z = x_1 + \cdots + x_m \in |H|$ be a general divisor and we set $F := \mathrm{pr}_{2*}(\mathrm{pr}_1^* M|_Z \otimes \mathcal{P})$. The sheaf F is a vector bundle of rank $m = \deg H$ on $\mathrm{Pic}^0(C)$ and the evaluation map $\mathrm{pr}_1^* M \otimes \mathcal{P} \to \mathrm{pr}_1^* M|_Z \otimes \mathcal{P}$ induces a sheaf map $E \to F$. The locus where this map drops rank by $r + 1$ is $W^r_T(L, C)$.

By Theorem II and Remark 1.9 of [FL], to prove the theorem it suffices to show that $\mathrm{Hom}(E, F)$ is an ample vector bundle. We have $F = \oplus_i \mathcal{P}_{x_i}$, where $\mathcal{P}_{x_i}$ is (isomorphic to) the restriction of $\mathcal{P}$ to $\{x_i\} \times T$. Since $\mathcal{P}$ is the restriction of a normalized Poincaré line bundle, $\mathcal{P}_{x_i}$ is algebraically equivalent to $\mathcal{O}_T$. Hence $\mathrm{Hom}(E, F) = \oplus_{i=1}^n (E^\vee \otimes \mathcal{P}_{x_i})$ is ample if and only if $E^\vee$ is ample.

To show the ampleness of $E^\vee$ we adapt the proof of [La, Thm. 6.3.48]. Denote by ξ the linear equivalence class of the tautological line bundle of $\mathbb{P}(E^\vee)$; we are going to show that for any irreducible positive dimensional subvariety V of $\mathbb{P}(E^\vee)$ the cycle $V \cap \xi$ is represented, up to numerical equivalence, by a proper nonempty subvariety of V.

Given a point $x \in C$, the evaluation map $E \to \mathcal{P}_x$ is surjective, since $M \otimes \eta$ is globally generated for every $\eta \in T$, hence it defines an effective divisor I_x algebraically equivalent to ξ. Denote by $p\colon \mathbb{P}(E^\vee) \to T$ the natural projection. A point $v \in \mathbb{P}(E^\vee)$ is determined by a section $s_v \in H^0(M \otimes p(v))$, and $v \in I_x$ if and only if $s_v(x) = 0$. Let C_i be a component of C such that the general element of V does not vanish identically on C_i. If the support of the zero locus of s_v on C_i varies, then for a general $x \in C_i$ the set $V \cap I_x$ is a proper nonempty subvariety algebraically equivalent to $V \cap \xi$ and we are done. So assume that for general $v \in V$ the support of the zero locus of v on C_i is constant: then, pulling back via $\nu_i \colon C_i^\nu \to C_i$, we see that the line bundle $\nu_i^*(M \otimes p(v))$ stays constant as $v \in V$ varies. Since the map $T \to \mathrm{Pic}^0(C_i^\nu)$ has finite kernel by assumption, $p(V)$ is a point $\eta_V \in T$ and $V \subseteq \mathbb{P}(H^0(M \otimes \eta_V))$. Since $\dim V > 0$ and ξ restricts to the class of a hyperplane of $\mathbb{P}(H^0(M \otimes \eta_V))$, the cycle $V \cap \xi$ is represented by a proper nonempty subvariety of V also in this case. This completes the proof. $\square$

Remark 5.2. The proof of Theorem 5.1 does not extend to the case of a complete subgroup $T \subseteq \mathrm{Pic}^0(C)$ such that the the map $T \to \mathrm{Pic}^0(C_i)$ does not have finite kernel for some component C_i of C. Indeed, take $C = C_1 \cup C_2$, with C_i smooth curves of genus $g_i > 0$ meeting transversely at only one point P, and $T = \mathrm{Pic}^0(C_1) \subset \mathrm{Pic}^0(C) = \mathrm{Pic}^0(C_1) \times \mathrm{Pic}^0(C_2)$. Twisting by $H \otimes \eta$, $\eta \in T$, the exact sequence $0 \to \mathcal{O}_{C_2}(-P) \to \mathcal{O}_C \to \mathcal{O}_{C_1} \to 0$ and taking global sections, one gets inclusions

$$H^0(\mathcal{O}_{C_2}(H-P)) = H^0(\mathcal{O}_{C_2}(H-P) \otimes \eta) \hookrightarrow H^0(\mathcal{O}_C(H) \otimes \eta)$$

that sheafify to a vector bundle map $\mathcal{O}_T \otimes H^0(\mathcal{O}_{C_2}(H-P)) \to E$. So the bundle $E^\vee$ is not ample.

We do not know whether the statement of Theorem 5.1 still holds without this assumption on T.

6. Brill-Noether theory for curves on irregular surfaces

Our approach to the study of the Brill-Noether loci $W^r(D, X)$ for an effective divisor D in an n-dimensional variety X of maximal Albanese dimension consists in comparing it with a suitable Brill-Noether locus on the $(n-1)$-dimensional variety D. Let $i^* \colon \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(D)$ be the map induced by the inclusion $i \colon D \to X$ and denote by T the image of i^*. The key observation is the following:

Proposition 6.1. *Let X be a variety of dimension $n > 1$ with* $\mathrm{albdim}\, X = n$ *and without irrational pencils of genus > 1 and let $D > 0$ be a divisor of X. Let Y be a positive dimensional irreducible component of $W_T^r(D|_D, D)$; if $\dim Y \geq 2$ or $0 \in Y$, then ${i^*}^{-1}Y$ is a component of $W^r(D, X)$.*

Proof. Let $V^1(X) \subset \mathrm{Pic}^0(X)$ the first Green-Lazarsfeld locus, namely $V^1(X) = \{\eta \in \mathrm{Pic}^0(X) | h^1(\eta) > 0\}$ (see §2.2).

Denote by U the complement of $V^1(X)$ in $\mathrm{Pic}^0(X)$; for $\eta \in U$, the short exact sequence:

$$0 \to \eta \to \mathcal{O}_X(D + \eta) \to (D + \eta)|_D \to 0$$

induces an isomorphism $H^0(\mathcal{O}_X(D + \eta)) \cong H^0((D + \eta)|_D)$. Hence $U \cap W^r(D, X) = U \cap {i^*}^{-1} W_T^r(D|_D, D)$ and to prove the claim it is enough to show that $i^{*-1}Y \not\subset V^1(X)$. By Theorem 2.2, if $\dim Y \geq 2$ this follows by the fact that $\dim V^1(X) \leq 1$ and if $0 \in Y$ this follows by the fact that 0 is an isolated point of $V^1(X)$. □

In the case of surfaces, Proposition 6.1 can be made effective.

Let S be a surface with $q(S) = q$ and let $C \subset S$ be a curve; we define the Brill-Noether number $\rho(C, r) := q - (r+1)(p_a(C) - C^2 + r)$. For $r = 0$ we write simply $\rho(C)$ for $\rho(C, 0) = q + C^2 - p_a(C)$. Recall that by the adjunction formula $q + C^2 - p_a(C) = q - 1 + \frac{C^2 - K_S C}{2}$.

Theorem 6.2. *Let $r \geq 0$ be an integer. Let S be a surface with irregularity $q > 1$ that has no irrational pencil of genus > 1 and let $C \subset S$ be a reduced curve such that $C_i^2 > 0$ for every irreducible component C_i of C.*

(i) *If $\rho(C, r) > 1$ or $\rho(C, r) = 1$ and $V^1(S) = \{\eta \in \mathrm{Pic}^0(S) | h^1(\eta) > 0\}$ does not generate* $\mathrm{Pic}^0(S)$*, then $W^r(C, S)$ is nonempty of dimension $\geq \min\{q, \rho(C, r)\}$.*

(ii) *If $\rho(C) > 1$, or $\rho(C) = 1$ and C is not contained in the ramification locus of the Albanese map, or $\rho(C) = 1$ and $V^1(S)$ does not generate* $\mathrm{Pic}^0(S)$*, then $W(C, S)$ has an irreducible component of dimension $\geq \min\{q, \rho(C\})$ containing* 0.

Proof. We start by observing that by the Hodge index theorem any two irreducible components of C intersect, hence in particular C is connected.

Let C_i be a component of C and denote by C_i^ν its normalization; since $C_i^2 > 0$, by [CFM, Prop. 1.6] the map $\mathrm{Pic}^0(S) \to \mathrm{Pic}^0(C_i)$ is an injection. Since $\mathrm{Pic}^0(S)$ is projective and the kernel of $\mathrm{Pic}^0(C_i) \to \mathrm{Pic}^0(C_i^\nu)$ is an affine algebraic group, it follows that the map $\mathrm{Pic}^0(S) \to \mathrm{Pic}^0(C_i^\nu)$ has finite kernel and we may apply Theorem 5.1.

By Theorem 5.1, if $\rho(C, r) > 0$ then $W^r_{\mathrm{Pic}^0(S)}(C|_C, C)$ is nonempty, it generates $\mathrm{Pic}^0(S)$ and all its components have dimension $\geq \min\{q, \rho(C, r)\}$. Claim (i) follows directly by Proposition 6.1 if $\rho(C, r) > 1$. If $\rho(C, r) = 1$ and $V^1(S)$ does not generate $\mathrm{Pic}^0(S)$, then there exist a positive dimensional component Y of $W^r_{\mathrm{Pic}^0(S)}(C|_C, C)$ not contained in $V^1(S)$ and arguing as in the proof of Proposition 6.1 one shows that Y is a component of $W^r(C, S)$.

By Proposition 6.1 to prove claim (ii) it is enough to show that $0 \in W_{\mathrm{Pic}^0(S)}(C|_C, C)$, namely that $h^0(C|_C) > 0$.

If $\rho(C) = 1$ and C is not contained in the ramification locus of the Albanese map of S, this follows by Corollary 4.8.

Otherwise assume that $\rho(C) > 1$ or $\rho(C) = 1$ and $V^1(S)$ does not generate $\mathrm{Pic}^0(S)$. Then by claim (i), $(-1)^* W(C, S)$ has dimension $\geq \min\{q, \rho(C)\}$. As previously we conclude that the hypotheses $\rho(C) > 1$ or $\rho(C) = 1$ and $V^1(S)$ does not generate $\mathrm{Pic}^0(S)$ also imply that $(-1)^* W(C, S)$ is not contained in $V^1(S)$.

Assume for contradiction that $h^0(C|_C) = 0$. Then the Riemann-Roch theorem on C gives $h^0(K_S|_C) = p_a(C) - C^2 - 1$. Since $p_a(C) - C^2 - 1 = q - 1 - \rho(C)$, one obtains $h^0(K_S|_C) < q - 1$ and thus $h^0(K_S - C) > \chi(S)$.

For every $\eta \in W(C, S)$ we have $h^0(K_S + \eta) \geq h^0(K_S + \eta - (C + \eta)) = h^0(K_S - C) > \chi(S)$, hence $-\eta \in V^1(S)$, a contradiction. This completes the proof. □

Remark 6.3. There are plenty of irregular surfaces without irrational pencils, for instance complete intersections in abelian varieties and symmetric products of curves (cf. [MP, §2]); indeed such surfaces can be regarded in some sense as "the general case".

Note that if S has no irrational pencil of genus > 1 and $\mathrm{Alb}(S)$ is not isogenous to a product of elliptic curves, then the assumption that $V^1(S)$ does not generate $\mathrm{Pic}^0(S)$ is verified, since by Theorem 2.2 the positive dimensional components of $V^1(S)$ are elliptic curves. In Example 8.5 we describe a surface without irrational pencils of genus > 1 such that $V^1(S)$ generates $\mathrm{Pic}^0(S)$.

Furthermore the inequalities of Theorem 6.2 are sharp: see Example 8.1.

7. Applications to curves on surfaces of maximal Albanese dimension

7.1. **Curves that do not move in a linear series.** Here we apply the results of the previous sections to curves C with $h^0(C) = 1$ on a surface of general type S.

The cohomology sequence associated to the restriction sequence for such a curve C gives an exact sequence:

$$0 \to H^0(C|_C) \longrightarrow H^1(\mathcal{O}_S) \xrightarrow{\cup s} H^1(\mathcal{O}_S(C)),$$

where $s \in H^0(\mathcal{O}_S(C))$ is a nonzero section vanishing on C. Hence by Proposition 3.3, the space $H^0(C|_C)$ is naturally isomorphic to the tangent space to $W(C, S)$. This remark, together with Proposition 4.9, gives the following:

Lemma 7.1. *Let S be a surface of general type with irregularity $q > 0$ that has no irrational pencil of genus > 1 and let $C \subset S$ be a 1-connected curve with $h^0(C) = 1$. Then one of the following occurs:*

(i) *$0 \in W(C, S)$ is an isolated point (with reduced structure);*
(ii) *$0 < h^0(C|_C) < q$ and $C^2 + 2q - 4 \leq K_S C$;*
(iii) *$h^0(C|_C) = q$ and $C^2 + 2q - 6 \leq K_S C$.*

Proof. As we observed above, the tangent space to $W(C, S)$ has dimension equal to $h^0(C|_C)$, therefore case (i) occurs for $h^0(C|_C) = 0$. If $h^0(C|_C) > 0$, then we can apply Proposition 4.9, which gives $h^0(K_S|_C) \geq q - 2 + h^0(C|_C)$ if $h^0(C|_C) < q$ and $h^0(K_S|_C) \geq 2q - 3$ if $h^0(C|_C) = q$. By Riemann-Roch and by the adjunction formula, we have:

$$h^0(K_S|_C) = h^0(C|_C) + K_S C + 1 - p_a(C) = h^0(C|_C) + \frac{K_S C - C^2}{2},$$

and statements (ii) and (iii) follow immediately by plugging this expression in the above inequalities. □

Remark 7.2. The inequality (ii) of Lemma 7.1 is sharp (cf. Example 8.1). Using the adjunction formula it can be rewritten as:

$$C^2 \le (p_a(C) - q) + 1,$$

or, equivalently, $\rho(C) \le 1$.

In the situation of Lemma 7.1 (i) we can also find a lower bound for $K_S C$ using the results of Section 6.

Proposition 7.3. *Let S be a surface of general type with irregularity $q > 1$ that has no irrational pencil of genus > 1 and let $C \subset S$ be a curve with $h^0(C) = 1$ and $h^0(C|_C) = 0$. Assume that C is connected and reduced and that every irreducible component C_i of C satisfies $C_i^2 > 0$; then:*

$$C^2 + 2q - 4 \le K_S C,$$

or, equivalently, $C^2 \le (p_a(C) - q) + 1$.

Furthermore if equality occurs then $V^1(S)$ generates $\mathrm{Pic}^0(S)$ and C is contained in the ramification locus of the Albanese map.

Proof. Since $h^0(C) = 1$ and $h^0(C|_C) = 0$, $0 \in W(C, S)$ is an isolated point. So by Theorem 6.2 (ii), $\rho(C) \le 1$, i.e. $q + C^2 - p_a(C) \le 1$ and this last inequality can be written $C^2 + 2q - 4 \le K_S C$.

The last assertion is also an immediate consequence of Theorem 6.2 (ii). □

As immediate consequences of the two above propositions we obtain:

Corollary 7.4. *Let S be a surface of general type with irregularity $q > 1$ that has no irrational pencil of genus > 1 and let $C \subset S$ be a curve with $h^0(C) = 1$. Assume that C is connected and reduced and that every irreducible component $C_i \subseteq C$ satisfies $C_i^2 > 0$; then:*

$$C^2 + 2q - 6 \le K_S C,$$

or, equivalently, $C^2 \le (p_a(C) - q) + 2$.

Furthermore, if equality holds then $h^0(C|_C) = q$.

Corollary 7.5. *Let S be a surface of general type with with irregularity $q > 1$ that has no irrational pencil of genus > 1 and let C be an irreducible component of the fixed part of $|K_S|$ such that $C^2 > 0$. Then:*

$$CK_S \ge C^2 + 2q - 4.$$

Proof. We have $h^0(C) = 1$ by assumption and $h^0(C|_C) = 0$ by Corollary 3.5. Hence the required inequality follows by Proposition 7.3. □

In [MPP1] we have characterized surfaces S of irregularity $q > 1$ containing a curve C such that $C^2 > 0$ and $p_a(C) = q$ (i.e., the smallest possible value). By [Xi] (cf. also Corollary 4.7), any irreducible curve with $h^0(C) \geq 2$ must satisfy $p_a(C) \geq 2q - 1$. We know no example of a curve with $C^2 > 0$ and $q < p_a(C) < 2q - 1$. The next result gives some information on this case:

Corollary 7.6. *Let S be a surface of general type with irregularity $q \geq 3$ that has no irregular pencil of genus > 1 and let $C \subset S$ be an irreducible curve such that $C^2 > 0$ and $p_a(C) \leq 2q - 2$. Then:*

(i) $C^2 \leq (p_a(C) - q) + 1$;

(ii) *the codimension of the tangent space at 0 to $W(C, S)$ is $\geq (3q - p_a(C) - 3)/2 \geq (q - 1)/2$.*

Proof. Since by Corollary 4.7 (cf. also [Xi]) we have $h^0(C) = 1$, by Proposition 3.3 the tangent space to $W(C, S)$ at 0 has dimension $w := h^0(C|_C)$. Note that by Lemma 4.1 we have $h^0(K_S|_C) + h^0(C|_C) \leq p_a(C) + 1 < 2q$.

Now observe that $w < q$. In fact, if $w = q$ then, by Proposition 4.9, one has $h^0(K_S|_C) \geq 2q - 3$. Since $p_a(C) \geq h^0(K_S|_C) + h^0(C|_C) - 1$ we obtain $p_a(C) \geq 3q - 4$, against the assumptions $p_a(C) \leq 2q - 2$ and $q \geq 3$. So (i) follows from Corollary 7.4.

Now Clifford's theorem gives $2w - 2 \leq C^2$. Since $p_a(C) \leq 2q - 2$, from (i) we obtain $w \leq (p_a(C) - q + 3)/2 \leq (q + 1)/2$. Statement (ii) then follows since w is the dimension of the tangent space to $W(C, S)$ at 0. □

7.2. **The fixed part of the paracanonical system.** Let S be a smooth surface of general type of irregularity $q \geq 2$ such that $\operatorname{albdim} S = 2$. Recall (cf. [Be2], §3) that the *paracanonical system* $\{K_S\}$ of S is the connected component of the Hilbert scheme of S containing a canonical curve. There is a natural morphism $c\colon \{K_S\} \to \operatorname{Pic}^0(S)$ defined by $[C] \mapsto \mathcal{O}_S(C - K_S)$ and the fiber of c over $\eta \in \operatorname{Pic}^0(S)$ is the linear system $|K_S + \eta|$, hence there is precisely one irreducible component $\mathcal{K}_{main}$ of $\{K_S\}$ (the so-called *main paracanonical system*) that dominates $\operatorname{Pic}^0(S)$. By the generic vanishing theorem of Green and Lazarsfeld, one has $\dim |K_S + \eta| = \chi(S) - 1$ for $\eta \in \operatorname{Pic}^0(S)$ general, and so the main paracanonical system $\mathcal{K}_{main}$ has dimension $q + \chi(S) - 1 = p_g(S)$. It is known ([Be2, Prop.4]) that if q is even and S has no irrational pencil of genus $> q/2$, then the canonical system $|K_S|$ is an irreducible component of $\{K_S\}$.

The relationship between the fixed part of $\mathcal{K}_{main}$ and the fixed part of $\{K_S\}$ does not seem to have been studied in general. Here we relate the fixed part of $\mathcal{K}_{main}$ to the ramification locus of the Albanese map.

Proposition 7.7. *Let S be a smooth surface of general type of irregularity $q \geq 2$ that has no irrational pencil of genus $> \frac{q}{2}$ and let $C \subset S$ be an irreducible curve with $C^2 > 0$ that is contained in the fixed part of the main paracanonical system $\mathcal{K}_{main}$.*

Then C is contained in the ramification locus of the Albanese map of S.

Proof. By the semi-continuity of the map $\eta \mapsto h^0(K_S - C + \eta)$, $\eta \in \mathrm{Pic}^0(S)$, we have $h^0(K_S - C) \geq \chi(S)$. Assume for contradiction that C is not contained in the ramification divisor of the Albanese map: then by Corollary 4.8 (i) we have $h^0(K_S - C) = \chi(S)$. By Proposition 3.3 it follows that the bilinear map $H^1(\mathcal{O}_S) \otimes H^0(K_S - C) \to H^1(K_S - C)$ given by cup product is zero. Hence for every section $s \in H^0(K_S)$ that vanishes along C and for every $v \in H^1(\mathcal{O}_S)$ we have $s \cup v = 0$. Therefore, by the proof of Proposition 3.4, it follows that if $\alpha, \beta \in H^0(\Omega^1_S)$ are such that $\alpha \wedge \beta \neq 0$, then $\alpha \wedge \beta$ does not vanish along C.

Consider the Grassmannian $\mathbb{G} := \mathbb{G}(2, H^0(\Omega^1_S)) \subseteq \mathbb{P}(\bigwedge^2 H^0(\Omega^1_S))$ and the projectivization $T \subset \mathbb{P}(\bigwedge^2 H^0(\Omega^1_S))$ of the kernel of $\bigwedge^2 H^0(\Omega^1_S) \to H^0(K_S)$. By the Theorem 2.1 the intersection $T \cap \mathbb{G}$ is the union of a finite number of Grassmannians $\mathbb{G}(2, V) \subset \mathbb{P}(\bigwedge^2 V)$ where $V \subset H^0(\Omega^1_S)$ is a subspace of the form $p^*H^0(\omega_B)$ for $p\colon S \to B$ an irrational pencil of genus > 1. Since by assumption S has no irrational pencil of genus $> \frac{q}{2}$, if $\mathbb{G}_0 \subset \mathbb{G}$ is a general codimension $q-3$ hyperplane section then $\mathbb{G}_0 \cap T = \emptyset$. Hence the image of $\mathbb{G}_0$ in $|K_S|$ is a closed subvariety Z of dimension $q-1$. Hence $(C + |K_S - C|) \cap Z$ is nonempty, namely there exist $\alpha, \beta \in H^0(\Omega^1_S)$ such that $\alpha \wedge \beta \neq 0$ and $\alpha \wedge \beta$ vanishes on C, a contradiction. $\square$

8. Examples and open questions

We collect here some examples to illustrate the phenomena that one encounters in studying the Brill-Noether loci of curves on irregular surfaces. We also give an example (Example 8.5) that shows that the hypothesis that $V^1(S)$ does not generate $\mathrm{Pic}^0(S)$ in Theorem 6.2 is not empty, i.e. that surfaces S of maximal Albanese dimension without irrational pencils of genus > 1 such that $V^1(S)$ generates $\mathrm{Pic}^0(S)$ do exist. We conclude the section by posing some questions.

Example 8.1 (Symmetric products)**.** Let C be a smooth curve of genus $q \geq 3$ and let $X := S^2C$ be the second symmetric product of C. The surface X is minimal of general type with irregularity q (cf. [MP, §2.4] for a detailed description of X).

For any $P \in C$, the curve $C_P = \{P + x | x \in C\} \subset X$ is a smooth curve isomorphic to C, in particular it has genus q. It satisfies $C_P^2 = 1$, $h^0(C_P) = 1$ and $\rho(C_P) = 1$. If we fix $P_0 \in C$, then it is easy to check that the map $C \to W(C_{P_0}, X)$ defined by $P \mapsto C_P - C_{P_0}$ is an isomorphism, hence Theorem 6.2 is sharp in this case.

Notice also that for every $P \in C$ we have $h^0(K_X|_{C_P}) = q-1$, hence both Proposition 4.6 and Proposition 4.9 are sharp in this case.

Example 8.2 (Étale double covers of symmetric products)**.** As in Example 8.1, take C a smooth curve of genus $q \geq 3$, let $X := S^2C$ be the second symmetric product and for $P \in C$ let $C_P = \{P + x | x \in C\} \subset X$. Let $f\colon C' \to C$ be an étale double cover and let $X' := S^2C'$. The involution

σ of C' associated to the covering $C' \to C$ induces an involution τ of X' defined by $\tau(A+B) = \sigma(A) + \sigma(B)$. The fixed locus of τ is the smooth curve $\Gamma = \{A + \sigma(A) | A \in C'\}$, hence $Y := X'/\tau$ is a smooth surface. It is easy to check that $q(Y) = q$ and that f descends to a degree 2 étale cover $Y \to X$.

Denote by D_P the inverse image of C_P in Y. The map $D_P \to C_P$ is a connected étale double cover, hence D_P is smooth (isomorphic to C') with $D_P^2 = 2$ and $g(D_P) = 2q-1$. In this case $\rho(D_P) = 3 - q \leq 0$ although D_P moves in a 1-dimensional algebraic family.

Next we study $h^0(D_P)$. The standard restriction sequence for D_P on Y gives $0 \to H^0(\mathcal{O}_Y) \to H^0(D_P) \to H^0(D_P|_{D_P}) \to H^1(\mathcal{O}_Y)$ and by Proposition 3.3 the last map in the sequence is nonzero for every P since D_P moves algebraically. Hence if $H^0(D_P|_{D_P}) = 1$ (e.g., if C' is not hyperelliptic) then $h^0(D_P) = 1$. Consider now a special case: take C hyperelliptic, $A, B \in C$ two Weierstrass points and $C' \to C$ the double cover given by the 2-torsion element $A - B$, so that the corresponding étale double cover $Y \to X$ is given by the equivalence relation $2(C_A - C_B) \equiv 0$. Then the curves D_A and D_B are linearly equivalent on Y and we have $h^0(D_A) = 2$.

With a little extra work it is possible to show that this is the only instance in which $h^0(D_P) > 1$.

Example 8.3 (The Fano surface of the cubic threefold)**.** This example deals with a well-known 2-dimensional family of curves of genus 11 on a surface of irregularity $q = 5$.

Let $V = \{f(x) = 0\} \subset \mathbb{P}^4$ be a smooth cubic 3-fold. Let $\mathbb{G} := \mathbb{G}(2,5)$ be the Grassmannian of lines of $\mathbb{P}^4$. The Fano surface (see [CG] and [Ty]) is the subset $F = F(V) \subset \mathbb{G}$ of lines contained in V. We recall that F is a smooth surface with irregularity $q = 5$. In fact, let $J(V)$ be the intermediate Jacobian of V: the Abel-Jacobi map $F \to J(V)$ induces an isomorphism $\mathrm{Alb}(F) \to J(V)$. Moreover one has: $H^2(F, \mathbb{C}) \cong \bigwedge^2 H^1(F, \mathbb{C})$, $K_F^2 = 45$, $\chi(\mathcal{O}_F) = 6$, $p_g(F) = 10$ and $c_2(F) = 27$.

Following Fano, for any $r \in F$ we consider the curve $C_r \subset F$

$$C_r = \{s \in F : s \cap r \neq \emptyset\}.$$

We have: $h^0(C_r) = 1$, $C_r^2 = 5$, $p_a(C_r) = 11$ and $K_F \sim_{alg} 3C_r$; in addition, the general $C = C_r$ is smooth (see [CG] and [Ty]).

We remark that $W(C, F)$ contains a 2-dimensional variety isomorphic to F, while one would expect it to be empty, since $\rho(C) = -1$. On the other hand, since the family has dimension 2, we have $h^0(\mathcal{O}_C(C)) = 2$, hence $W_5^1(C) \neq \emptyset$ and C is not Brill-Noether general. In fact the corresponding Brill-Noether number is $\rho(11, 5, 1) = 11 - 2(11 - 5 + 1) = -3$. Moreover there is a degree 2 étale map $C \to D$, where D is a smooth plane quintic, and $\mathrm{Alb}(F)$ is isomorphic to the Prym variety $P(C, D)$ of the covering, thus the curve C has very special moduli.

Nevertheless we will see that the family $\{C_r\}_{r \in F}$ has a good infinitesimal behavior. Firstly we recall that $h^0(K_F - C_r) = 3$ by [Ty, Cor. 2.2]. Let

T be the tangent space to $W(C_r, F)$ at 0; since $\dim T \geq 2$, the image $V \subseteq H^2(\mathcal{O}_F) \cong \wedge^2 H^1(\mathcal{O}_F)$ of $T \otimes H^1(\mathcal{O}_F)$ has dimension ≥ 7. On the other hand, V is orthogonal to the image in $H^0(K_S)$ of the Petri map $\beta_{C_r} \colon H^0(C_r) \otimes H^0(K_F - C_r) \to H^0(K_F)$, therefore $\dim V \leq p_g(F) - h^0(K_F - C_r) = 7$. So we have $\dim V = 7$, $\dim T = 2$ and in this case the dimension of the family is predicted by the Petri map.

Example 8.4 (Ramified double covers)**.** Let X be a smooth surface of irregularity q such that the Albanese map $X \to \mathrm{Alb}(X)$ is an embedding (for instance take X a complete intersection in an abelian variety), and let $\pi \colon S \to X$ be the double cover given by a relation $2L \equiv B$ with B a smooth ample curve. Write $\pi^* B = 2R$; the induced map $\mathrm{Alb}(S) \to \mathrm{Alb}(X)$ is an isomorphism (cf. [MP, §2.4]), hence R is the ramification divisor of the Albanese map of S. We have $R \in |\pi^* L|$ and, by the projection formula for double covers, for every $\eta \in \mathrm{Pic}^0(S) = \mathrm{Pic}^0(X)$ we have $H^0(R + \eta) = H^0(L + \eta) \oplus H^0(\eta)$, where the first summand is the space of invariant sections and the second one is the space of anti-invariant sections. Hence for $\eta \neq 0$ all sections are invariant, while for $\eta = 0$ the curve R is the zero locus of the only (up to scalars) anti-invariant section. Hence R moves only linearly on S, as predicted by Proposition 3.4.

This construction can be used also to produce examples of surfaces of fixed irregularity q that contain smooth curves C with $C^2 > 0$, $h^0(C) = 1$ and unbounded genus. Assume that X contains a smooth curve D such that $D^2 > 0$ and $h^0(D) = 1$ (for instance, take X a symmetric product as in Example 8.1). Set $C := \pi^{-1}(D)$: if B meets D transversally, then C is smooth of genus $2g(D) - 1 + LD$, hence $g(C)$ can be arbitrarily large. Again by the projection formulae, for every $\eta \in \mathrm{Pic}^0(S) = \mathrm{Pic}^0(X)$ we have $h^0(C + \eta) = h^0(D + \eta) + h^0(D + \eta - L)$. Hence if $L - D > 0$, we have $h^0(C + \eta) = h^0(D + \eta) = 1$ and $W(C, S) = W(D, X)$.

Example 8.5. An example of surface S without pencils of genus > 1 such that $V^1(S)$ generates $\mathrm{Pic}^0(S)$ can be constructed as follows.

Let E be an elliptic curve, let $C \to E$ and $E' \to E$ be double covers with C a curve of genus 2 and E' an elliptic curve, and set $B := C \times_E E'$. The map $B \to E$ is a $\mathbb{Z}_2^2$-cover and B has genus 3. We denote by α the element of the Galois group of $B \to E$ such that $B/ < \alpha >= E'$ and by β, γ the remaining nonzero elements. The curves $B/ < \beta >$ and $B/ < \gamma >$ have genus 2.

Now choose elliptic curves E_1, E_2, E_3 and for $i = 1, 2, 3$ let $B_i \to E_i$ and $\alpha_i, \beta_i, \gamma_i$ be as above. Let $X := B_1 \times B_2 \times B_3$ and let G be the subgroup of $\mathrm{Aut}(X)$ generated by $g_1 = (\alpha_1, \beta_2, \gamma_3)$ and $g_2 = (\beta_1, \gamma_2, \alpha_3)$; note that G acts freely on X. Let $S' \subset X$ be a smooth ample divisor which is invariant under the G-action and let $S := S'/G$; we denote by $f_i \colon S \to E_i$ the induced map, $i = 1, 2, 3$. The surfaces S' and S are minimal of general type. By the Lefschetz Theorem, $\mathrm{Alb}(S') = \mathrm{Alb}(X) = J(B_1) \times J(B_2) \times J(B_3)$. It is immediate to check that $q(S) = 3$ and that the map $S \to A = E_1 \times E_2 \times E_3$

induces an isogeny $\mathrm{Alb}(S) \to A$. Consider the étale cover $S_1 = S'/< g_1 > \to S$; the map $S' \to B_1$ induces a map $S_1 \to E'_1 = B_1/< \alpha_1 >$ which is equivariant for the action of $G/< g_1 >$. The group $G/< g_1 >$ acts freely on E'_1, hence the cover $S_1 \to S$ is obtained from $E'_1 \to E_1$ by base change. It follows that the 2-torsion element $\eta_1 \in \mathrm{Pic}(S)$ associated to this double cover is a pull back from E_1. Thus η_1 belongs to $\mathrm{Pic}^0(S)$.

The map $S' \to B_2$ induces a fibration $S_1 \to C_2 = B_2/ < \beta_2 >$. There is a commutative diagram

$$\begin{array}{ccc} S_1 & \longrightarrow & S \\ \big\downarrow & & \big\downarrow{\scriptstyle f_2} \\ C_2 & \longrightarrow & E_2, \end{array}$$

where the map $S_1 \to S$ is obtained from $C_2 \to E_2$ by base change and normalization. This means that the fibration $S_1 \to C_2$ has two double fibres $2F_1$ and $2F_2$, occurring at the ramification points of f_2 and that $\eta_1 = F_1 - F_2 + \alpha$ for some $\alpha \in \mathrm{Pic}^0(E_2)$, hence η_1 restricts to 0 on the general fiber of f_2. By [Be4, Thm.2.2], this implies that $\eta_1 + f_2^* \mathrm{Pic}^0(E_2)$ is a component of $V^1(S)$. A similar argument shows that $V^1(S)$ contains translates of $f_1^* \mathrm{Pic}^0(E_1)$ and $f_3^* \mathrm{Pic}^0(E_3)$. Since $\mathrm{Alb}(S)$ is isogenous to $E_1 \times E_2 \times E_3$, it follows that $V^1(S)$ generates $\mathrm{Pic}^0(S)$.

To conclude, we show that if the curves E_i are general, then S has no irrational pencil of genus > 1. In this case $\mathrm{Hom}(E_i, E_j) = 0$ if $i \neq j$, hence $\mathrm{End}(A) = \mathbb{Z}^3$. Assume for contradiction that $S \to B$ is an irrational pencil of genus $b > 1$. Since the map $S \to A$ is generically finite by construction and $q(S) = 3$, $b = 2$ is the only possibility. Then we have a map with finite kernel $J(B) \to A$. Let W be the image of $J(B)$ in A. Consider the endomorphism ϕ of A defined as $A \to A/W = (A/W)^\vee \to A^\vee = A$ (both A and A/W are principally polarized). The connected component of $0 \in \ker\phi$ is W, hence W is a product of two of the E_i. So the map $S \to W$ has finite fibres, while $S \to J(B)$ is composed with a pencil, and we have a contradiction.

Question 8.6. Let S be a surface of general type of irregularity q and of maximal Albanese dimension. A curve $C \subset S$ with $C^2 > 0$ satisfies $p_a(C) \geq q$ and in [MPP1] we have proven that if S contains a 1-connected curve C with $C^2 > 0$ and $p_a(C) = q$, then S is birationally a product of curves or the symmetric square of a curve. On the other hand, the curves D_P in Example 8.2 have $D_P^2 > 0$ and genus equal to $2q-1$. We do not know any surface S containing a curve C with $C^2 > 0$ and arithmetic genus in the intermediate range $q < p_a(C) < 2q - 1$, so it is natural to ask whether such an example exists. Notice that, by Corollary 4.7 (cf. also [Xi]), a curve C with $C^2 > 0$ and $p_a(C) < 2q - 1$ cannot move linearly and that some further restrictions are given in Corollary 7.6. In addition, the image C' of such a curve C via the Albanese map generates $\mathrm{Alb}(C)$ and therefore by the Hurwitz formula C' is birational to C. Hence this question is also related to the question of existence of curves of genus $q < p_a(C) < 2q-1$ that generate an abelian variety of dimension q (see [Pi] for related questions).

Question 8.7. On a variety X with $\operatorname{albdim} X = \dim X$ there are three intrinsically defined effective divisors:

(a) the fixed part of $|K_X|$;
(b) the ramification divisor R of the Albanese map;
(c) the fixed part of the main paracanonical system $\mathcal{K}_{main}$ (cf. §7.2).

Clearly the fixed part of $|K_X|$ is a subdivisor of R. In the case of surfaces, in Proposition 7.7 it is shown that the components C of the fixed part of $\mathcal{K}_{main}$ with $C^2 > 0$ are contained in R if the surface has no irrational pencil of genus $> \frac{q}{2}$.

It would be interesting to know more precisely in arbitrary dimension how these three divisors are related.[1]

Question 8.8. In §6 we study the Brill-Noether loci for curves C on a surface S, namely we always assume that $0 \in W(C,S) \neq \emptyset$. It would be very interesting to find numerical conditions on a line bundle $L \in \operatorname{Pic}(X)$, X a smooth projective variety, that ensure that $W(L,X)$ is not empty.

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