THE CLASSIFICATION OF MINIMAL IRREGULAR SURFACES OF GENERAL TYPE WITH $K^2 = 2p_g$

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ABSTRACT. Minimal irregular surfaces of general type satisfy $K^2 \geq 2p_g$ (see [13]). In this paper we classify those surfaces for which the equality $K^2 = 2p_g$ holds.

CONTENTS

1. INTRODUCTION

By a result of O. Debarre a minimal complex irregular surface S of general type satisfies $K^2 \geqslant 2p_g$ ([13, Théorème 6.1]). In this paper we complete the classification of the case $K^2 = 2p_g$, which can be summarized as follows:

Theorem 1.1. *Let S be a minimal complex surface of general type of irregularity $q > 0$ satisfying $K^2 = 2p_g$. Then $q \leq 4$.*

If $q = 1$, then the Albanese fibration is a genus 2 fibration with 2–connected fibres and S is a double cover of a $\mathbb{P}^1$-bundle over the elliptic curve $\mathrm{Alb}(S)$.

If $q \geqslant 2$ then $\chi = 1$ and:

(i) for $q = 2$, S is the minimal desingularization of a double cover of a principally polarized abelian surface (A, Θ) branched on an effective divisor of class 2Θ with at most non-essential singularities;

(ii) for $q = 3$, S is the symmetric product of a curve of genus 3;

(iii) for $q = 4$, S is the product of two curves of genus 2.

This classification is the result of the work of many authors.

Debarre, besides establishing the inequality $K^2 = 2p_g$ for minimal complex surfaces of general type of irregularity $q > 0$, has shown that if equality holds then $q \leqslant 4$ and that if $q = 4$ then S is the product of two curves of genus 2 ([13, Théorème 6.3]).

For $q = 1$, $K^2 = 2p_g$ is the same as $K^2 = 2\chi$. These surfaces had already been classified by Horikawa: their Albanese pencil has 2-connected genus 2 fibres, and the relative canonical map determines a 2–1 cover of a $\mathbb{P}^1$-bundle over the elliptic curve $\mathrm{Alb}(S)$ (see [19, Theorem 5.2] for a detailed description). We note that also Catanese in [8] studied the case $q = \chi = 1$.

Surfaces with $p_g = q = 3$ and $K^2 = 6$ have been shown to be the symmetric product of a curve of genus 3 by Catanese and the first two authors ([9, Prop. (3.22)]).

To obtain Theorem 1.1, we start by completing the classification of such irregular surfaces with $\chi = 1$ (see §3). As explained above the only yet unknown case was $p_g = q = 2$. In Theorem 3.2, which is the main result in this paper, we show that these surfaces are exactly the so called *Catanese surfaces*, described in (i) of Theorem 1.1. This had already been proved by Manetti in [20] under the extra assumption that K be ample and by the first two authors in [11] under the assumption that the bicanonical map is not birational, but the lack of a complete description of this case has been for some years an annoying gap in the classification of surfaces with small invariants.

Surfaces with $p_g = q = 2$ and $K^2 = 4$ are interesting also from the point of view of the Severi inequality $K^2 \geq 4\chi$ for minimal surfaces with Albanese dimension 2, proven by Manetti in [20] for the case where K is ample and then in general by the third author in [22]. Indeed, in [20] Manetti showed that for K ample the Severi inequality is an equality exactly when $q = 2$ and the Albanese map is a double cover of an abelian surface branched over a smooth curve B. Examples with K non ample and $K^2 = 4\chi$ can be obtained by letting the branch curve B acquire non-essential singularities and taking the minimal resolution of the double cover. It is natural to conjecture that these are the only surfaces with Albanese dimension 2 satisfying the equality in the Severi inequality. The present result proves the conjecture in the first case, i.e., for $\chi = 1$.

The new ingredient that allows us settle the case $p_g = q = 2$, in a way that is in our view more conceptually transparent, is the recent classification in [12] of curves C with the following property: a Brill–Noether locus $W_d^s(C)$, properly contained in the Jacobian $J(C)$ of C, contains a variety Z stable under translations by the elements of a positive dimensional abelian subvariety $A \subsetneq J(C)$ such that $\dim(Z) = d - \dim(A) - 2s$, i.e., the maximum possible dimension for such Z (see §2.1 and [12]). Besides this, the proof relies, as usual in these matters, on two main tools: the paracanonical system and the Albanese map (see §2.2). In fact we prove that the general paracanonical curve C is smooth of genus 5 and apply the results in [12] to the Abelian surface $A = \mathrm{Pic}^0(S)$, which is contained in the Theta divisor of $J(C)$, to deduce that the Albanese map of S has degree 2. The result then follows by "classical" arguments.

Then we turn to the case $\chi > 1$, $q \geq 2$. Since $q \leq 4$ and the case $q = 4$ is already described by Debarre, to complete the proof of Theorem 1.1 one needs to rule out the existence of minimal irregular surfaces with $K^2 = 2p_g$, $3 \geqslant q \geqslant 2$, $\chi > 1$. This is done again using "classical" arguments in §4.

2. Preliminaries

2.1. **Abelian subvarieties of Theta divisors of Jacobians.** In [12], following [1, 14], we considered the following situation:

(*) C is a smooth, projective, complex curve of genus g, Z is an irreducible r–dimensional subvariety of a Brill–Noether locus $W_d^s(C) \subsetneq J^d(C)$, and Z is stable under translations by the elements of an abelian subvariety $A \subsetneq J(C)$ of dimension $a > 0$ (if so, we will say that Z is A–*stable*).

Here $J^d(C)$ is the set of equivalence classes of divisors of degree d on C and $J(C) := J^0(C)$ is the *Jacobian variety* of C. The Brill–Noether locus $W_d^s(C)$ is the subscheme of $J^d(C)$ corresponding to classes of divisors D such that $h^0(\mathcal{O}_C(D)) \geqslant s + 1$ and we set $W_d(C) := W_d^0(C)$ (see [2, Ch. IV]).

It was proved in [14] that if (*) holds, then $r + a + 2s \leqslant d$. In [12], improving on partial results in [1, 14], we give the full classification of the cases in which (*) holds and $r + a + 2s = d$. We

will not need here the full strength of the results in [12], but only the part concerning the case $(d, s) = (g - 1, 0)$ (see [12, Theorem 3.1]):

Theorem 2.1. *Let C be a curve of genus g. Let $A \subsetneq J(C)$ be an abelian variety of dimension $a > 0$ and $Z \subset W_{g-1}(C)$ an irreducible, A–stable variety of dimension $r = g - 1 - a$. Then there is a degree 2 morphism $\varphi\colon C \to C'$, with C' smooth of genus g', such that one of the following occurs:*
(a) $g' = a$, $A = \varphi^*(J(C'))$ and $Z = W_{g-1-2a}(C) + \varphi^*(J^a(C'))$;
(b) $g' = a + 1$, $r = a$, φ is étale, A is the Prym variety of φ and $Z \subset W_{g-1}(C)$ is the connected component of $\varphi_*^{-1}(K_{C'})$ consisting of divisor classes D with $h^0(C, \mathcal{O}_C(D))$ odd, where $\varphi_*\colon J^{g-1}(C) \to J^{g-1}(C')$ is the norm map.
  In particular, $Z \cong A$ is an abelian variety if and only if either we are in case (a) and $g = 2a+1$, or in case (b).

2.2. Some generalities on irregular surfaces.

Let S be a smooth projective irreducible surface over $\mathbb{C}$. We will use the standard notation $q(S) = h^1(S, \mathcal{O}_S)$, $p_g(S) = h^0(S, \Omega_S^2)$, $\chi(S) = \chi(\mathcal{O}_S) = p_g(S) - q(S) + 1$ for the *irregularity*, the *geometric genus*, and the *Euler characteristic* of the structure sheaf. We may often use the simplified notation p_g, q, χ. Note that, since we are working over the complex numbers, one has also $q = h^0(S, \Omega_S^1)$.

Numerical [resp. linear] equivalence will be denoted by $\sim$ [resp. by $\equiv$].

We will denote by $a\colon S \to \mathrm{Alb}(S)$ the *Albanese morphism* of S. The dimension of $a(S)$ is denoted by $\mathrm{Albdim}(S)$ and is called the *Albanese dimension* of S.

A *pencil of genus b* on S is a rational map $f\colon S \dashrightarrow B$, with connected fibres, with B a smooth curve of genus b. The indeterminacy points of f are the *base points* of the pencil. If $b > 0$, the pencil is said to be *irrational* and it has no base points.

An effective non-zero divisor D on a smooth surface is said to be *m-connected* if for every decomposition $D = A + B$ with A, B effective non-zero divisors one has $A \cdot B \geq m$.

A *non-essential singularity* of the branch locus B of a double cover $X \to Y$ of surfaces, with Y smooth and X normal is a double point or a triple point that has no infinitely near triple points (see [6, Ch.III, Section 7] where these are called *simple singularities*).

As usual, $|K_S|$ (or simply $|K|$) denotes the *canonical system* of S. If $\eta \in \mathrm{Pic}^0(S)$, the linear system $|K + \eta|$ is called a *twisted canonical system* of S. A curve $C_\eta \in |K + \eta|$ is a *paracanonical curve* on S.

Assume S is irregular of general type. We will denote by $\mathfrak{K}_S$ (or simply by $\mathfrak{K}$) the *paracanonical system* of S, i.e., the Hilbert scheme of paracanonical curves on S. Note the morphism $p\colon \mathfrak{K} \to \mathrm{Pic}^0(S)$ defined by $C \mapsto \mathcal{O}_S(C - K)$. As $\chi(\mathcal{O}_S) \geq 1$ there is a unique component $\mathcal{K}_S$ (or simply $\mathcal{K}$) of $\mathfrak{K}$ dominating $\mathrm{Pic}^0(S)$ via p. It is called the *main paracanonical system* of S.

If $\mathrm{Albdim}(S) = 2$ and $\eta \in \mathrm{Pic}^0(S)$ is general, one has $h^1(S, \eta) = 0$ hence $\dim(|K + \eta|) = \chi - 1$ (this is the so–called *generic vanishing theorem*, see [16, Theorem 1]). If $\eta \in \mathrm{Pic}^0(S)$ and $C \in |K + \eta|$ are general, then C corresponds to the general point of $\mathcal{K}$, which has dimension $q + \dim(|K + \eta|) = p_g$.

3. Surfaces with $K^2 = 4$ and $p_g = q = 2$

Let S be minimal, of general type, with $p_g = q = 2$. One has $K^2 \geq 4$ by [13, Théorème 6.1] and the equality is attained in the following example.

Example 3.1. Let (A, Θ) be a principally polarized abelian surface. Let $p\colon S \to A$ be the double cover branched on an effective, smooth divisor B in the class of 2Θ so that $p_*\mathcal{O}_S = \mathcal{O}_A \oplus \mathcal{O}_A(-\theta)$, with θ in the class of Θ and S is smooth and minimal. Then $K = p^*(\theta)$ and the invariants of S are $p_g = q = 2$, $K^2 = 4$. We can also allow non-essential singularities on the divisor B. In that case S will be the minimal resolution of the double cover of A branched on B. One has $\mathrm{Alb}(S) \cong A$.

A special case is when Θ is *reducible*, i.e. A is the product of elliptic curves, and $\Theta = E_1 + E_2$, with E_i elliptic curves such that $E_1 \cdot E_2 = 1$. In that case S has two elliptic pencils $\mathcal{M}_1, \mathcal{M}_2$ of curves of genus 2 (see [11, Example 7.1]; according to [25], this is the only case in which S has an irrational pencil of curves of genus 2: we will not use this result though).

Note that the curves in $\mathcal{K}_S$ are the proper transforms via p of the curves in the class of Θ. If Θ is irreducible, then the general curve in $\mathcal{K}_S$ is smooth, otherwise the general curve in $\mathcal{K}_S$ is the sum of a curve in $\mathcal{M}_1$ plus a curve in $\mathcal{M}_2$.

In this section we prove the following classification result.

Theorem 3.2. *A minimal surface of general type with $p_g = q = 2$, $K^2 = 4$ is as in Example 3.1.*

By [11, Proposition 2.3], minimal surfaces S of general type with $K^2 = 4$ and $p_g = q = 2$ have $\mathrm{Albdim}(S) = 2$, i.e., $a \colon S \to \mathrm{Alb}(S)$ is surjective. Surfaces of general type with $p_g = q = 2$ and with an irrational pencil have been studied in [23, 24, 25], but we will not use their results here.

Lemma 3.3. *Let S be a minimal surface of general type with $p_g = q = 2$, $K^2 = 4$. Then, either:*
(i) *the general curve in the main paracanonical system $\mathcal{K}$ is smooth, or*
(ii) *S is as in Example 3.1, with Θ reducible.*

Proof. Let $C \in \mathcal{K}$ be general. Write $C = F + M$ where F is the fixed divisor of $\mathcal{K}$. Since $\dim(\mathcal{K}) = 2$, the curve M moves in a 2-dimensional system $\mathcal{M}$ of curves. Suppose first M reducible. Then M must consist of two distinct irreducible components M_i each moving in a 1–dimensional family of curves $\mathcal{M}_i$, with $1 \leqslant i \leqslant 2$. The index theorem (see [6, Ch. IV, Corollary 2.16]) yields $K \cdot M_i \geqslant 2$, for $1 \leqslant i \leqslant 2$. On the other hand $4 = K^2 = K \cdot F + K \cdot M_1 + K \cdot M_2$, hence $K \cdot F = 0, K \cdot M_1 = K \cdot M_2 = 2$. Then $M_1^2 = M_2^2 = 0$ and $\mathcal{M}_i$ is a pencil of curves of genus 2, for $1 \leqslant i \leqslant 2$. Such a pencil is not rational by [4, Lemme on p. 345], hence it is of genus 1, because $q(S) = 2$ and $\mathrm{Albdim}(S) = 2$. This implies $M_1 \cdot F = M_2 \cdot F = 0$, hence $F = 0$ by the 2–connectedness of paracanonical divisors (see, e.g. [6, Ch.VII, Proposition 6.2]) and $M_1 \cdot M_2 = 2$. Let $f_i \colon S \to E_i$ be the elliptic pencils $\mathcal{M}_i$, for $1 \leqslant i \leqslant 2$. The morphism $f = f_1 \times f_2 \colon S \to E_1 \times E_2$ is a double cover and we are in case (ii).

Next we may assume M irreducible, and we prove that C is irreducible.

We claim that $M^2 \geq 3$. In fact, by [9, Lemma 0.14], $M^2 = 0$ would mean that $\mathcal{M}$ is a pencil and this is impossible because it has dimension 2. Now, by [9, Proposition 0.16], a curve M with $M^2 = 1$ moving in a 2-dimensional system of curves varies in a linear system of rational curves and this is impossible because S is of general type. Finally $M^2 = 2$ is also excluded because, by [9, Theorem 0.20], S would be birational to a product of curves, which is not the case.

Suppose that $F \neq 0$. Then by 2-connectedness of paracanonical divisors one has $K^2 \geqslant K \cdot M = F \cdot M + M^2 \geqslant 5$, a contradiction.

Finally we prove that C is smooth. Assume, to the contrary, C singular. Let $c \in \mathcal{K}$ be the point corresponding to C. The 2–dimensional tangent space $T_{\mathcal{K},c}$ is contained in $H^0(C, N_{C|S}) \cong H^0(C, \mathcal{O}_C(C))$, which is also 2–dimensional, so that equality holds. It therefore corresponds to a g_4^1 on C. Every section of $H^0(C, \mathcal{O}_C(C))$ in $T_{\mathcal{K},c}$ vanishes at each singular point of C. This implies that C has a unique singular point x_C. Consider the rational map $\mathrm{Pic}^0(S) \dashrightarrow S$ which associates to C its singular point x_C. Since this map cannot be dominant, it follows that for C general in $\mathcal{K}$ there is a 1–dimensional system $\mathcal{K}_C$ of curves in $\mathcal{K}$ having, as C, a singular point at x_C. Since the curves in $\mathcal{K}_C$ have no variable intersection off x_C, then $\mathcal{K}_C$ is a pencil, and it is rational, because it has the base point x_C. Thus the surface P parametrizing $\mathcal{K}$ would be ruled, a contradiction, since P is birational to $\mathrm{Pic}^0(S)$. $\square$

By Lemma 3.3 we may assume from now on that the general curve $C \in \mathcal{K}$ is smooth of genus 5. Consider the *restriction morphism* $\mathfrak{r} \colon \mathrm{Pic}^0(S) \to J^4(C)$ sending η to $\mathcal{O}_C(C + \eta)$. This map is injective by [10, Proposition 1.6]. By generic vanishing, if $\eta \in \mathrm{Pic}^0(S)$ is general, one has $h^0(C, O_C(C + \eta)) = h^0(C, O_S(C + \eta)) = 1$. Hence the image A of $\mathfrak{r}$ is an abelian surface contained in $W_4(C)$.

Lemma 3.4. *In the above set up there exists a smooth genus* 2 *curve* C' *and a degree* 2 *morphism* $\varphi \colon C \to C'$ *such that* $A = \varphi^*(J^2(C'))$.

Proof. We can apply Theorem 2.1 with $g = 5$ and $r = a = 2$. Hence to prove the assertion it suffices to show that case (b) of Theorem 2.1 does not occur. Suppose we are in case (b). Then, by Theorem 2.1 (cf. [7, §12.6]), for all $\eta \in \text{Pic}^0(S)$, one has $h^0(C, L_\eta) \equiv 1 \pmod 2$. On the other hand $L_0 = \mathcal{O}_C(C)$ and, by generic vanishing, $h^0(C, \mathcal{O}_C(C)) = 2$, a contradiction. $\square$

End of proof of Theorem 3.2. By Lemma 3.3, we may assume $C \in \mathcal{K}$ general to be smooth. By Lemma 3.4, there is a degree 2 morphism $\varphi \colon C \to C'$, with C' of genus 2, such that $A = \varphi^*(J^2(C'))$. Since φ is ramified at four points, $A \cong J(C')$ is a principally polarized abelian surface. So also $\text{Pic}^0(S)$ is principally polarized and we may identify both $\text{Pic}^0(S)$ and $\text{Alb}(S)$ with A.

Next we prove that there is an involution $\mathfrak{i}$ of S such that $\mathfrak{i}(C) = C$ and $\mathfrak{i}_{|C}$ coincides with the involution ι determined by the double cover $\varphi \colon C \to C'$ (then $\mathfrak{i}$ will be biregular, since S is minimal of general type). To define $\mathfrak{i}$ we argue as follows.

The curve C is smooth by Lemma 3.3; let $x \in C$ be general and let $\mathcal{K}_x$ be the family of curves in $\mathcal{K}$ containing x. We claim that the set $\{x, \iota(x)\}$ coincides with the base locus B_x of $\mathcal{K}_x$, and therefore it is independent of C. Indeed, by the generality of x, there exists a $D \in \mathcal{K}_x$ such that $h^0(C, \mathcal{O}_C(D)) = 1$ and the divisor $D|_C$ is reduced and it is equal to $x + \iota(x) + y + \iota(y)$ for some $y \in C$. Since B_x is invariant under ι by construction, we have two possibilities: (a) $B_x = \{x, \iota(x), y, \iota(y)\}$ or (b) $B_x = \{x, \iota(x)\}$. In case (a) all the curves of $\mathcal{K}_x$ would cut on C the same divisor: since the map $\text{Pic}^0(C) \to \text{Pic}^0(S)$ is injective, this would mean that all the curves of $\mathcal{K}_x$ are linearly equivalent, against the assumption that C be general.

Consider the minimal desingularization X of the quotient $S/\mathfrak{i}$. It contains a 2-dimensional family of (generically smooth and irreducible) curves Γ of genus 2, with $\Gamma^2 = 2$, i.e. the genus 2 quotients of the curves in $\mathcal{K}_S$ under the involution $\mathfrak{i}$. By [9, Theorem 0.20], all curves Γ are isomorphic and X is birational to the symmetric product $\Gamma(2)$, which is birational to $J(\Gamma)$. So we have a degree two map $S \dashrightarrow J(\Gamma)$, which is in fact a morphism, since $J(\Gamma)$ is an abelian variety and S is smooth. The universal property of the Albanese map implies immediately that $A = J(\Gamma)$ and $S \to J(\Gamma)$ is the Albanese morphism $a \colon S \to A$. Note that via $X \dashrightarrow A$ the curves Γ are mapped to Θ-divisors, hence the paracanonical curves are mapped 2-to-1 to Θ-divisors by a.

Let B be the branch divisor of $a \colon S \to A$. Since the curves in $\mathcal{K}$ have genus 5, we have $B \cdot \Theta = 4$. By the index theorem we have $B^2 \leqslant 8$. On the other hand B is divisible by 2 in the Neron-Severi group $NS(A)$, hence $B^2 \geqslant 8$. In conclusion $B^2 = 8$ and the index theorem implies that the class of B is 2Θ and the assertion follows. $\square$

Remark 3.5. Let S be as in Example 3.1 with B smooth. Then the ramification curve R is a canonical curve isomorphic to B, hence smooth. So the general curve $C \in |K|$ is smooth and, as above, we can consider the restriction map $\text{Pic}^0(S) \to W_4(C)$, whose image is an abelian surface $A \subset W_4(C)$. We can use Theorem 2.1, but now the argument of Lemma 3.4 does not apply because C is not a paracanonical curve. In this situation the Prym case (b), and not case (a), of Theorem 2.1 occurs. Indeed, the bicanonical morphism is not birational for S: it is in fact composed with an involution σ such that $\Sigma = S/\sigma$ is a surface with 20 nodes (over which the double cover $\phi \colon S \to \Sigma$ is ramified). If X is the minimal desingularization of Σ, one has $p_g(X) = 2, q(X) = 0, K_X^2 = 2$ (see [11, Proposition 6.5]). So the general curve $C \in |K_S|$ is the étale double cover of the general curve $C' \in |K_X|$. One has the following commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\phi} & \Sigma \\
p \downarrow & & \downarrow q \\
A & \xrightarrow{\kappa} & Y
\end{array}
$$

where $Y \subset \mathbb{P}^3$ is the 16–nodal Kummer surface of A (κ is the obvious double cover), and $q\colon \Sigma \to Y$ is a double cover branched along a smooth plane section H of Y (which pulls back via κ to the branch divisor of a), plus six nodes lying on a conic $\Gamma \subset Y$.

By [5, Proposition 4], there is a unique curve C_0 in the intersection of $|K_S|$ with $\mathcal{K}$, i.e., the proper transform of Γ on S. It is interesting to notice that for C_0 both cases (a) and (b) of Theorem 2.1 occur at the same time (see [12, Remark 3.2]).

We finally notice that the same idea of proof of Theorem 3.2 can be applied to recover the classification of minimal surfaces S with $p_g = q = 3$ (see [9, Proposition (3.22)]). In this case q is odd, the analogue of Lemma 3.4 does not hold, and the Prym case (b) of Theorem 2.1 occurs. We do not dwell on this here.

4. PROOF OF THEOREM 1.1

As explained in the introduction, once we have Theorem 3.2, to prove Theorem 1.1, it suffices to show that:

Proposition 4.1. *There is no minimal complex surface of general type S with $2 \le q \le 3$, $\chi \ge 2$ and $K^2 = 2p_g$.*

Proof. Suppose by contradiction that such a surface S exists.

Claim 4.2. *One has* $\mathrm{Albdim}(S) = 2$, $q = 3$, $p_g = 4$.

Proof of the Claim. Suppose, by contradiction, that $\mathrm{Albdim}(S) = 1$ and let $a\colon S \to B$ be the Albanese pencil, with B of genus q. We denote by g the genus of the fibres of a and one has $g \geqslant 2$ because S is of general type,

By [19, Theorem 3.1], if $K^2 < 3\chi$, the general fibre of a is hyperelliptic of genus $g = 2$ or 3. Since B has genus q, applying the slope inequality for hyperelliptic fibrations ([19, Theorem 2.1]) one has $K^2 \geqslant 4\frac{g-1}{g}(\chi+(g+1)(q-1))$. From $K^2 = 2p_g$, one concludes that $q = 1$, a contradiction.

So $K^2 \geqslant 3\chi$. Then $2p_g = K^2 \geqslant 3p_g - 3q + 3$ yields $p_g \leqslant 6$ for $q = 3$ and $p_g \leqslant 3$ for $q = 2$. Since $K^2 = 2p_g$, one has $K^2 \leqslant 12$, if $q = 3$ and $K^2 \leqslant 6$ if $q = 2$. But, by Arakelov's theorem (see [4, Corollaire on p. 344]), we have $K^2 \geqslant 8(g-1)(q-1) \geqslant 8(q-1)$, a contradiction.

So $\mathrm{Albdim}(S) = 2$. Then the Severi inequality proved in [22] gives $2p_g = K^2 \geqslant 4\chi$, i.e., $p_g \le 2q - 2$. Since $\chi \ge 2$ and $q \le 3$, the only possibility is $p_g = 4$, $q = 3$. $\square$

Claim 4.3. *S has no irrational pencil $f\colon S \to B$ with B of genus $b > 1$. Therefore:*
(i) the map $\Phi\colon \bigwedge^2 H^0(S, \Omega_S^1) \to H^0(S, \mathcal{O}_S(K_S))$ is injective;
(ii) the Albanese image Ξ of S is not covered by elliptic curves.

Proof of the Claim. Let g be the genus of the general fibre of f. Again by Arakelov's theorem (see [4, Corollaire on p. 344]) we have $8 = K^2 \geqslant 8(g-1)(b-1)$, hence $g = b = 2$. By [18, Theorem 3], one has $K^2 \geqslant 2\chi - 6 + 6b = 10$, a contradiction. This proves the first assertion and (i) follows by the Castelnuovo–de Franchis' theorem ([15], see [3, Proposition X.9] for a modern reference) and the fact that, since $q = 3$, all 2-vectors are decomposable.

To prove (ii), suppose Ξ is covered by elliptic curves. Then there is an elliptic curve $E \subset \mathrm{Alb}(S)$ such that the image of S via the composition of the Albanese morphism and the morphism $\mathrm{Alb}(S) \to \mathrm{Alb}(S)/E$ is a curve B. Since B has to span $\mathrm{Alb}(S)/E$, which has dimension 2, then B has genus $b \geqslant 2$, contradicting the first assertion. $\square$

Claim 4.4. *Let F be the general fibre of a pencil $f\colon S \dashrightarrow B$ with B of genus b (possibly $b = 0$ and the pencil is linear, with base points), and let F be irreducible of geometric genus g. Then:*
(i) $K \cdot F \geqslant 4$ and $g \geqslant 3$;
(ii) if $K \cdot F = 4$, then either $F^2 = 0, g = 3, b = 1$ or $F^2 = 2, b = 0, g = 4$ and $K \sim 2F$.
In particular S has no pencil of curves of genus 2 and its bicanonical map is a birational morphism.

Proof of the Claim. By Claim 4.3, we have $b \leqslant 1$. By blowing up if necessary, we can assume the pencil has no base points and F is smooth of genus g. Then, by [4, Lemme on p. 345], one has $3 = q \leqslant b + g$ with equality only if S is birational to a product of curves of genus b and g. Since S is of general type and $b \leqslant 1$, equality cannot hold and thus either $b = 1$ and $g \geqslant 3$ or $b = 0$ and $g \geqslant 4$.

Consider now again the original surface S.

If $K \cdot F < 4$, by the index theorem and the fact that, by the adjunction formula, $K \cdot F$ and $F^2 = 0$ have the same parity, one either has $F^2 = 0, K \cdot F = 2$ or $F^2 = 1, K \cdot F = 3$. The former case implies $g = 2$, a contradiction. The latter case gives $g = 3$ and $b = 0$ because the pencil has a base point, a contradiction again. Hence $K \cdot F \geqslant 4$. If equality holds, again by the index theorem and parity either $F^2 = 0$ or $F^2 = 2$ and $K_S \sim 2F$. In the former case $g = 3$ and thus $b = 1$.

The bicanonical map is a morphism, as it always is for a minimal surface of general type with $p_g \geq 1$ (see [6, Ch. VII, Theorem 7.4]). Since we just proved that S has no genus 2 pencil S, by [9, Theorem A] the bicanonical morphism of S is birational. □

Finally we prove Proposition 4.1 by showing that surfaces S as in Claim 4.2 do not exist. Suppose otherwise. Consider the map $\phi \colon S \dashrightarrow \mathbb{P}^2$ associated to the linear system $|\operatorname{Im}(\Phi)|$, (see Claim 4.3, (i)). A standard local computation shows that this is the composition of the Albanese map $a \colon S \to \Xi$ with the *Gauss map* $\gamma \colon \Xi \dashrightarrow \mathbb{P}^2$ which maps a general point $x \in \Xi$ to the point of $\mathbb{P}(T_{\mathrm{Alb}(S),0})^* \cong \mathbb{P}^2$ corresponding to the line $\mathbb{P}(T_{\Xi,x})$ translated to the origin of $\mathrm{Alb}(S)$. The map γ is dominant. Otherwise the line bundle determined by Ξ would not be positive definite (see [7, Proposition 4.4.2]) and therefore Ξ would be covered by elliptic curves, which is not the case. Hence the image of the canonical map $\varphi \colon S \dashrightarrow \mathbb{P}^3$ is a surface Σ.

Consider the multiplication map

$$\mu \colon \operatorname{Sym}^2(H^0(S, \mathcal{O}_S(K))) \to H^0(S, \mathcal{O}_S(2K)).$$

Since both spaces have dimension 10, we have two possibilities:
(i) μ is an isomorphism: since the bicanonical map of S is a birational morphism (see Claim 4.4), then φ is also a birational morphism;
(ii) $\dim(\ker(\mu)) = 1$: in this case Σ is a quadric.

Since q is odd and $\mathrm{Albdim}(S) = 2$, the canonical system is contained in the main paracanonical system $\mathcal{K}$ (see [21, Theorem 1.3]). Hence if the general canonical curve is irreducible (smooth) then the general curve in $\mathcal{K}$ is also irreducible (smooth).

Assume we are in case (i). Since φ is a morphism, the general canonical curve is smooth. Let $C \in \mathcal{K}$ be general. Then by the above remark also C is smooth. For any paracanonical curve $D \in |2K - C|$ there exists a quadric Q_D of $\mathbb{P}^3$ such that $D + C$ is the divisor of $\varphi^*(Q_D)$. Since $h^0(S, \mathcal{O}_S(D)) = \chi = 2$, there exist at least two distinct quadrics Q_1 and Q_2 of $\mathbb{P}^3$ containing $\varphi(C)$. Since $\varphi(C)$ is irreducible and non degenerate of degree 8, this is a contradiction. So case (i) does not occur.

Assume now we are in case (ii) and suppose first that Σ is non–singular. The two line rulings $|L_1|$, $|L_2|$ of Σ determine two pencils on S with general fibres F_1, F_2. For $1 \leqslant i \leqslant 2$ the strict transform of a general element of $|L_i|$ is numerically equivalent to $r_i F_i$, with $r_i \geqslant 1$, and $K - (r_1 F_1 + r_2 F_2)$ is numerically equivalent to an effective divisor. Hence we have $8 = K^2 \geq r_1 K \cdot F_1 + r_2 K \cdot F_2 \geq 4(r_1 + r_2)$ (the last inequality follows by Claim 4.4). So $r_1 = r_2 = 1$, $K \cdot F_1 = K \cdot F_2 = 4$ and, by Claim 4.4, $|F_1|$ and $|F_2|$ are distinct linear pencils with $F_1^2 = F_2^2 = F_1 \cdot F_2 = 2$ (in particular $F_1 \sim F_2$ and $K \sim 2F_1 \sim 2F_2$). The pencil $|F_i|$ has a base scheme β_i of length 2, hence F_i is smooth of genus 4, for $1 \leqslant i \leqslant 2$.

Set $F = F_i$ (with $i = 1$ or 2). Since $h^1(S, \mathcal{O}_S(K_S + F)) = 0$ by Ramanujam's vanishing theorem (see [6, Ch. IV, Theorem 12.5]), the restriction sequence

$$0 \to \mathcal{O}_S(K_S) \to O_S(K_S + F) \to O_F(K_F) \to 0$$

yields the long exact cohomology sequence

$$0 \to H^0(S, \mathcal{O}_S(K_S)) \to H^0(S, \mathcal{O}_S(K_S + F)) \xrightarrow{r} H^0(F, \mathcal{O}_F(K_F)) \to H^1(S, \mathcal{O}_S(K_S)) \to 0$$

This implies that $\dim(\mathrm{Im}(r)) = 1$, i.e., the rational map determined by $|K_S + F|$ contracts the general curve in $|F|$ to a point. This is a contradiction, because the canonical image of S is a surface.

Finally, assume Σ is a quadric cone. The same arguments as before show that the line ruling of Σ determines a pencil $|F|$ on S with $K \cdot F = 4$, $F^2 = 2$. Then, as above, the rational map determined by $|K_S + F|$ contracts the general curve of $|F|$ to a point, again a contradiction, which shows that case (ii) cannot occur either. This finishes the proof of the Proposition.   □

Remark 4.5. In order to exclude the existence of surfaces S as in Claim 4.2 we could have argued, in principle, as in the proof of Theorem 3.2, i.e. we could have used the results in [12]. To do so, one should first prove that the general curve in $\mathcal{K}_S$ is smooth. This is not shorter, actually it is a bit more involved, than the proof presented here.

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