Optimal Solar Sail Phasing Trajectories for Circular Orbit

Alessandro A. Quarta* and Giovanni Mengali†

University of Pisa, I-56122 Pisa, Italy

Nomenclature

$a_\oplus$	=	characteristic acceleration [mm/s^2]
A, B, C	=	constants of integration, see Eqs. (14)-(15)
e	=	error function, see Eq. (18)
$\mathcal{H}$	=	Hamiltonian
r	=	Sun-spacecraft distance ($r_\oplus \triangleq 1$ AU) [AU]
t	=	time [years]
T	=	parking orbit's orbital period [years]
α	=	sail pitch angle [deg]
β	=	sail lightness number
λ_i	=	adjoint to variable i
θ	=	polar angle [deg]
$\mu_\odot$	=	Sun's gravitational parameter
ω	=	parking orbit's angular velocity
ϕ	=	phase angle [deg]
ρ	=	relative radial distance [AU]

Subscripts

0	=	initial, parking orbit
f	=	final
max	=	maximum

*Research Assistant, Department of Aerospace Engineering, a.quarta@ing.unipi.it. Senior Member AIAA.
†Associate Professor, Department of Aerospace Engineering, g.mengali@ing.unipi.it. Senior Member AIAA.

Superscripts

$\cdot$	=	time derivative

Introduction

The concept of using a solar sail for the repositioning maneuver of a spacecraft on a heliocentric orbit was first proposed by McInnes [1]. In particular, assuming a circular orbit and a linearized dynamics model, Ref. [1] discusses the performance of a small (ideal) solar sail, subjected to a piecewise constant steering law, and compares its effectiveness with a situation in which a chemical (high-thrust) propulsion system is employed to perform the same maneuver. Subsequently, the study was extended by Mengali and Quarta [2], who solved the problem within an optimal framework, generalizing it to an elliptic orbit, a solar sail with an optical force model, and a scenario in which two spacecraft are simultaneously repositioned along the same starting orbit.

The analysis of Ref. [2] was conducted using a nonlinear dynamics model for the solar sail's heliocentric motion, which is, of course, fully general, but whose results can only be managed through numerical simulations. The aim of this Note is to show that, under the assumption of a linearized mathematical model [1], the optimal repositioning problem of an ideal, low-performance, solar sail can be solved within a semi-analytical framework. In particular, starting from a circular parking orbit, it is shown that the truly optimal steering law may be recovered in closed form.

Mathematical Model

Consider a spacecraft, whose primary propulsion system is a perfectly reflecting solar sail, moving along a circular heliocentric parking orbit of radius r_0 with a constant angular velocity $\omega = \sqrt{\mu_\odot / r_0^3}$, where $\mu_\odot$ is the Sun's gravitational parameter. The mission requirement is

to vary the spacecraft azimuthal position along the parking orbit of a prescribed angle ϕ_f. Note that a positive (negative) value of ϕ_f corresponds to an angular displacement in the same (opposite) direction with respect to how the spacecraft tracks its orbit in a Keplerian scenario. For solar sails with moderate performances, the repositioning trajectory takes place such that the Sun-spacecraft distance r remains close to r_0. Therefore, according to McInnes [1], the spacecraft dynamics may be conveniently described by means of the following linearized equations of motion

$$\dot{\rho} = u \tag{1}$$

$$\dot{\phi} = \frac{v}{r_0} \tag{2}$$

$$\dot{u} = 2\,\omega\,v + 3\,\omega^2\,\rho + \beta\,\frac{\mu_\odot}{r_0^2}\,\cos^3\alpha \tag{3}$$

$$\dot{v} = -2\,\omega\,u + \beta\,\frac{\mu_\odot}{r_0^2}\,\cos^2\alpha\,\sin\alpha \tag{4}$$

with $\rho \triangleq (r - r_0)$ and $\phi \triangleq (\theta - \omega\,t)$, where θ is the spacecraft (heliocentric) polar angle with $\theta(t_0) \triangleq 0$ and $t_0 \triangleq 0$. In Eqs. (3)-(4), β is the sail lightness number, that is, the ratio of the solar radiation force to the solar gravitational force acting on the solar sail [3], whereas $\alpha \in [-90,\, 90]$ deg is the sail pitch angle, defined as the angle between the Sun-spacecraft line and the normal to the sail nominal plane in the direction of the propulsive thrust. Recall that the spacecraft characteristic acceleration $a_\oplus$, that is, the maximum propulsive acceleration when the Sun-spacecraft distance is $r = r_\oplus \triangleq 1\,\mathrm{AU}$, is given by $a_\oplus \simeq 5.93\,\beta\,(r_\oplus/r_0)^2$, where $a_\oplus$ is in millimeters per second squared.

The optimal solar sail repositioning problem is to find the minimum time t_f required for the spacecraft to vary its azimuthal position of a prescribed angle ϕ_f. For convenience, a modified version of this problem will be considered. Indeed, an equivalent formulation

is to find the maximum (or the minimum) of $\phi(t_f)$ for a given flight time t_f. This latter formulation is simpler to address in mathematical terms and, as such, will be used in the following discussion. Note that the phase angle $\phi(t_f)$ is maximized (or minimized), according to whether the spacecraft is moved ahead (or behind) with respect to its position, along the circular parking orbit, at time t_f in a Keplerian scenario.

The equivalent optimal repositioning problem is solved by means of the calculus of variations, and the Hamiltonian function $\mathcal{H}$ is [see Eqs. (1)–(4)]

$$\mathcal{H} \triangleq \lambda_\rho\, u + \frac{\lambda_\phi\, v}{r_0} + \lambda_u \left(2\,\omega\, v + 3\,\omega^2\,\rho + \beta\,\frac{\mu_\odot}{r_0^2}\,\cos^3\alpha\right) + \lambda_v \left(\beta\,\frac{\mu_\odot}{r_0^2}\,\cos^2\alpha\,\sin\alpha - 2\,\omega\, u\right) \tag{5}$$

where λ_ρ, λ_ϕ, λ_u, and λ_v are the adjoint variables associated with the state variables ρ, ϕ, u, and v, respectively. The corresponding Euler-Lagrange equations are:

$$\dot{\lambda}_\rho \triangleq -\frac{\partial \mathcal{H}}{\partial \rho} = -3\,\omega^2\,\lambda_u \tag{6}$$

$$\dot{\lambda}_\phi \triangleq -\frac{\partial \mathcal{H}}{\partial \phi} = 0 \tag{7}$$

$$\dot{\lambda}_u \triangleq -\frac{\partial \mathcal{H}}{\partial u} = -\lambda_\rho + 2\,\omega\,\lambda_v \tag{8}$$

$$\dot{\lambda}_v \triangleq -\frac{\partial \mathcal{H}}{\partial v} = -\frac{\lambda_\phi}{r_0} - 2\,\omega\,\lambda_u \tag{9}$$

The optimal steering law involving the sail pitch angle α is well known [4], and is here repeated for convenience

$$\tan\alpha = \frac{-3\,\lambda_u + \sqrt{9\,\lambda_u^2 + 8\,\lambda_v^2}}{4\,\lambda_v} \tag{10}$$

Note that α is a function of adjoint variables λ_u and λ_v only, while its sign is the same of λ_v.

The linear differential system, constituted by the four Euler-Lagrange equations and the four equations of motion (1)–(4) is completed by eight boundary conditions. Of these, seven

depend on the state variables calculated at either initial or final time, that is

$$\rho(t_0) = 0 \quad , \quad \phi(t_0) = 0 \quad , \quad v(t_0) = 0 \quad , \quad u(t_0) = 0 \tag{11}$$

$$\rho(t_f) = 0 \quad , \quad v(t_f) = 0 \quad , \quad u(t_f) = 0 \tag{12}$$

while the eighth relationship comes from the transversality condition [5]

$$\lambda_\phi(t_f) = \begin{cases} 1 & \text{for phasing ahead} \\ -1 & \text{for phasing behind} \end{cases} \tag{13}$$

Because all of the boundary conditions involving the state variables are zero, see Eqs. (11)-(12), and since the propulsive acceleration depends linearly on the sail lightness number, it turns out that the optimal solution of the repositioning problem is linear in β. Therefore, for a given t_f, the problem may be solved for a single value of $\beta \neq 0$. In addition, it is useful to introduce a set of canonical units such that the parking orbit's radius r_0, and the Sun's gravitational parameter $\mu_\odot$ are both unitary [6]. As a result, the repositioning problem depends on a single design parameter, that is, the flight time t_f.

Note that the Euler-Lagrange equations (6)–(9) are uncoupled from the equations of motion (1)–(4) (recall that ω is constant). In particular, the linear differential equations for the adjoint variables λ_u and λ_v, which univocally describe the optimal steering law [see Eq. (10)], may be solved to give

$$\lambda_u = -\frac{2\,\lambda_\phi}{\omega\, r_0} + A\,\cos\left(\omega\, t + B\right) \tag{14}$$

$$\lambda_v = \frac{3\,\lambda_\phi\, t}{r_0} - 2\,A\,\sin\left(\omega\, t + B\right) + C \tag{15}$$

where λ_ϕ is given by Eq. (13), whereas A, B, and C are three constants of integration whose value is obtained by solving the two-point boundary value problem (TPBVP) associated to the optimal problem, that is, by enforcing the three boundary conditions (12).

It may be verified by simulation, by solving a set of TPBVPs for different values of t_f, that the optimal sail steering law $\alpha = \alpha(t - t_f/2)$ is an odd function of time and, therefore, $\alpha(t_0) = -\alpha(t_f)$. This behaviour agrees with the results of a non-linear analysis [2], when a parking orbit of low eccentricity (for example, the Earth's heliocentric orbit) is considered in the problem. From Eq. (10) the symmetry property of α implies that $\lambda_u(t_0) = \lambda_u(t_f)$ and $\lambda_v(t_0) = -\lambda_v(t_f)$, independent of t_f. Substituting these last two conditions into Eqs. (14)-(15), the expressions for B and C are obtained to be

$$B = \pi - \frac{\omega\, t_f}{2} \qquad , \qquad C = -\frac{3\,\lambda_\phi\, t_f}{2\, r_0} \tag{16}$$

As a result, the adjoint variables λ_u and λ_v may be written as a function of the constant A alone, viz.

$$\lambda_u = -\frac{2\,\lambda_\phi}{\omega\, r_0} - A\,\cos\left[\omega\,(t - t_f/2)\right] \quad , \quad \lambda_v = \frac{3\,\lambda_\phi}{r_0}\,(t - t_f/2) + 2\,A\,\sin\left[\omega\,(t - t_f/2)\right] \tag{17}$$

The optimal solar sail repositioning problem is now considerably simplified, because the associated TPBVP can be solved by simply looking for the roots of a single-variable error function depending on A and defined as

$$e = \sqrt{\left(\frac{\rho(t_f)}{r_0}\right)^2 + \left(\frac{u(t_f)}{\omega\, r_0}\right)^2 + \left(\frac{v(t_f)}{\omega\, r_0}\right)^2} \tag{18}$$

The values $\rho(t_f)$, $u(t_f)$, and $v(t_f)$ in the previous equation are obtained, for a given value of A, by forward integrating the equations of motions (with zero initial conditions) and using

the optimal steering law defined by Eqs. (10) and (17). Intensive numerical simulations have shown that the function $e = e(A)$ always shows a single root, whose value is easily calculated with a standard numerical algorithm such as a bisection method.

t_f/T	$\phi(t_f)/\beta$ [deg]	A	$\lvert\rho\rvert_{\max}/(\beta\, r_0)$
0.25	2.1736×10^{-3}	-1.5894	1.0136×10^{-4}
0.5	1.0614	-1.7757	2.0530×10^{-2}
0.75	2.1406×10^{1}	-1.9447	2.4374×10^{-1}
1	1.2469×10^{2}	-2.0494	9.4800×10^{-1}
1.25	3.8928×10^{2}	-2.0661	2.0926
1.5	8.4094×10^{2}	-1.9385	3.3922
1.75	1.4141×10^{3}	-1.3401	4.5147
2	1.9831×10^{3}	1.2469×10^{-1}	4.9214
2.25	2.5791×10^{3}	6.3829×10^{-1}	4.8448
2.5	3.2550×10^{3}	-7.4824×10^{-1}	4.9831
2.75	3.9160×10^{3}	-1.7282	5.0332
3	4.6389×10^{3}	-1.9885	5.6842
3.25	5.5370×10^{3}	-2.0149	6.9225
3.5	6.6330×10^{3}	-1.8897	8.2625
3.75	7.8504×10^{3}	-1.2345	9.3859
4	9.0657×10^{3}	8.9855×10^{-2}	9.7674

Table 1: Optimal performances for $\lambda_\phi = 1$ (phasing ahead).

Figure 1 shows how the constant of integration A varies with the flight time t_f. The simulation results for $t_f \leq 4\,T$ have been summarized in Tables 1 and 2, where $T \triangleq 2\,\pi/\omega$ is the orbital period of the circular parking orbit (recall that A corresponds to unitary values of both $\mu_\odot$ and r_0). In all of the simulations, the differential equations have been integrated in double precision using a variable order Adams-Bashforth-Moulton solver [7] with absolute and relative errors of 10^{-12}. The numerical root of the error function (18) was found with a tolerance of 10^{-8}.

In Tables 1 and 2 the value of $\phi(t_f)$ is expressed in percentage terms of β. The same result holds for the dimensionless maximum displacement $\lvert\rho\rvert_{\max}/r_0$, which defines the maximum spacecraft radial displacement from the circular parking orbit during the phasing maneuver.

t_f/T	$\phi(t_f)/\beta$ [deg]	A	$\lvert\rho\rvert_{\max}/(\beta\, r_0)$
0.25	-1.8060×10^1	2.5051	6.5772×10^{-2}
0.5	-1.2859×10^2	2.3793	5.5616×10^{-1}
0.75	-4.0346×10^2	2.3126	1.6389
1	-8.8100×10^2	2.0740	3.0877
1.25	-1.5113×10^3	1.7773	4.2969
1.5	-2.1786×10^3	5.7584×10^{-1}	5.0309
1.75	-2.7966×10^3	-1.6467	5.0106
2	-3.5335×10^3	-1.6620×10^{-1}	5.0410
2.25	-4.2642×10^3	1.5257	5.1831
2.5	-4.9960×10^3	1.9684	5.3315
2.75	-5.8711×10^3	2.0854	6.4319
3	-6.9596×10^3	2.0060	7.8518
3.25	-8.2207×10^3	1.7035	9.1212
3.5	-9.5277×10^3	5.6017×10^{-1}	9.8420
3.75	-1.0807×10^4	-9.1596×10^{-1}	9.8545
4	-1.2183×10^4	-1.1807×10^{-1}	9.8629

Table 2: Optimal performances for $\lambda_\phi = -1$ (phasing behind).

In fact, because the equations of motion have been approximated by a linear expansion, it is important to verify that ρ remains sufficiently small for all $t \in [t_0,\, t_f]$. For this reason the last column of Tables 1 and 2 is useful for checking the soundness of results, as is now better explained with a numerical example.

Case study

Consider the same application scenario used by McInnes in Ref. [1], in which the problem is to change the azimuthal position of an ideal solar sail that initially tracks a circular heliocentric orbit of radius $r_0 = r_\oplus = 1\,\mathrm{AU}$ ($T = 1\,\mathrm{year}$ and $\omega = 2\,\pi\,\mathrm{rad/year}$). Using the simplified steering law of Ref. [1], a low-performance solar sail with a lightness number $\beta = 8.6191 \times 10^{-3}$ (corresponding to a characteristic acceleration of $a_\oplus \simeq 0.051\,\mathrm{mm/s^2}$) is able to complete a two years repositioning maneuver with a phase angle $\phi_f = -30\,\mathrm{deg}$.

Using the optimal approach described in this Note, from Table 2 the value corresponding

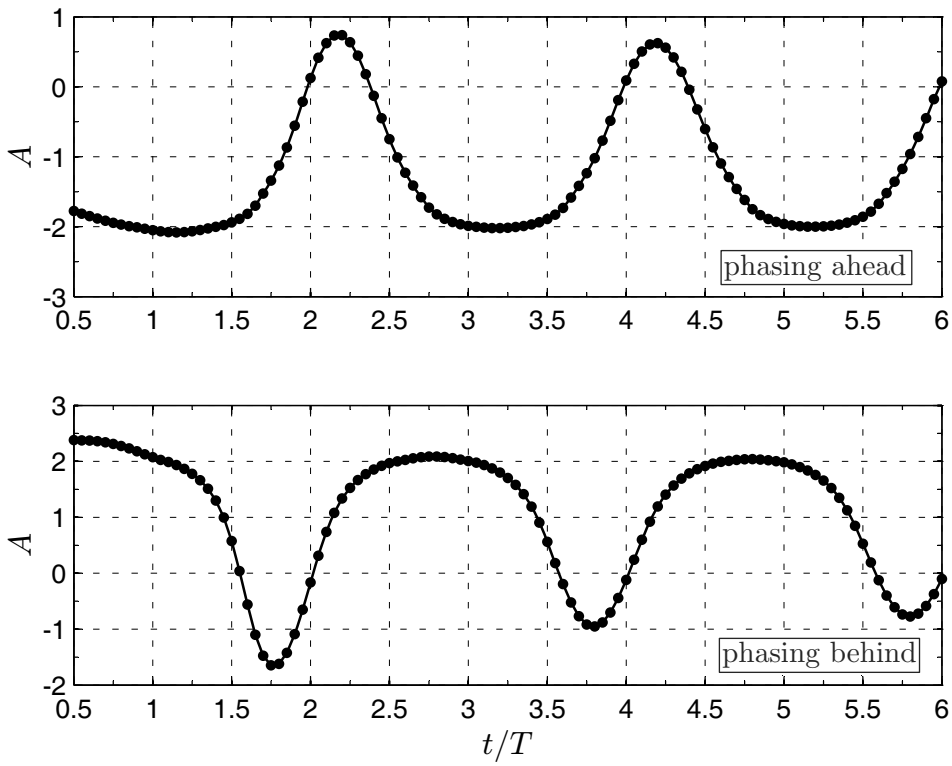

Figure 1: Constant of integration A as a function of flight time t_f (large dots correspond to simulated points).

to $t_f/T = 2$ is $\phi(t_f)/\beta = -3.5335 \times 10^3$ deg. Therefore, the maximum attainable phase angle is $\phi(t_f) = -8.6191 \times 3.5335 \simeq -30.46$ deg, whose modulus is, as expected, greater than that obtained in Ref. [1], because now the steering law is optimal.

Such an optimal steering law is available in closed form substituting $A = -0.1662$ (see Table 2) into Eqs. (10) and (17). The simulation results are shown in Fig. 2. Note the symmetry of the state variables with respect to one half of the mission time, that is, $t_f/2$. In particular, ϕ and u are odd functions, while ρ and v are even functions.

From the upper time history of Fig. 2, the maximum distance of the spacecraft from the circular parking orbit is about $|\rho|_{\max} = 5.0410\, r_\oplus \times 8.6191 \times 10^{-3} \simeq 0.04345$ AU. Such a value is less than 5% of r_0, which guarantees the correctness of a liner simulation.

Assume now, instead, that the same final phase angle (that is, -30.46 deg) is obtained

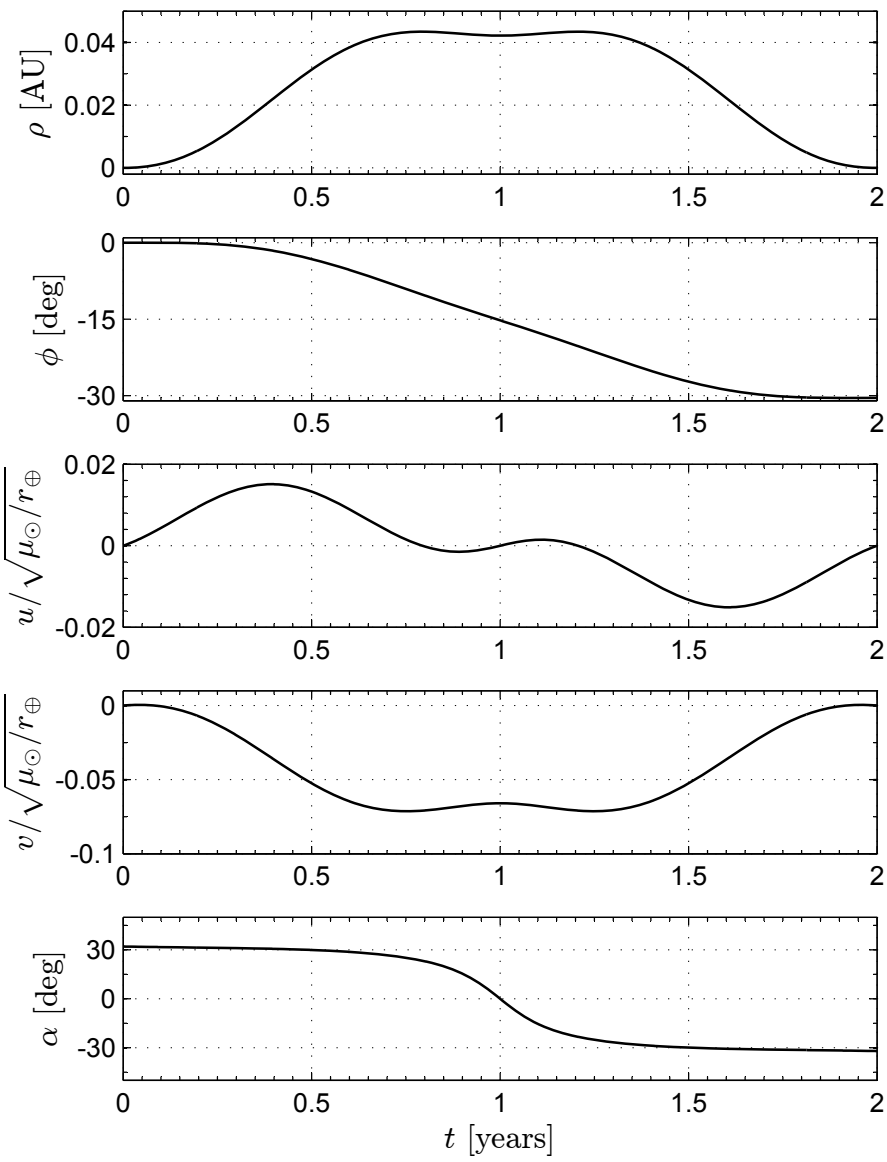

Figure 2: Simulation of a two-years mission with $\lambda_\phi = -1$, $r_0 = 1\,\mathrm{AU}$, and $\beta = 8.6191 \times 10^{-3}$.

using one half of the previous mission time, or $t_f = 1\,\mathrm{year}$. Table 2 shows that the required sail lightness number is $\beta \simeq 3.4574 \times 10^{-2}$ (corresponding to a characteristic acceleration $a_\oplus \simeq 0.205\,\mathrm{mm/s^2}$), and the maximum spacecraft radial distance from the circular parking orbit reaches $|\rho|_{\max} = 3.0877\, r_\oplus \times 3.4574 \times 10^{-2} \simeq 0.107\,\mathrm{AU}$. The latter value exceeds 10% of r_0 and is probably beyond the limits of a linear analysis.

Of course, the optimal problem could be restated by introducing a constraint on the maximum admissible value of $|\rho|_{\max}$. However the resulting problem would become much more involved and the convenience of using a linearized model for describing the equations of motion would be lost. The better choice in such a situation is therefore to resort to a nonlinear analysis. A detailed discussion of such an approach is described in Ref. [2] to which the interested reader is referred.

Conclusions

The two-dimensional repositioning problem of an ideal solar sail has been discussed in an optimal framework. Starting from a circular heliocentric parking orbit, an analytical expression for the optimal steering law was derived using an indirect approach, under the assumption that the spacecraft equations of motion may be described through a linear approximation. The available results have been collected within suitable tables to obtain a simple and accurate estimate of the obtainable performances that link the flight time and the phase angle for a given value of the sail lightness number. The procedure discussed in this Note may be easily extended to a solar sail with an optical force model, in which the sail thrust performances depend on the optical characteristics of the reflecting film.

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