SHARP GAGLIARDO-NIRENBERG INEQUALITIES IN FRACTIONAL COULOMB-SOBOLEV SPACES

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Abstract. We prove scaling invariant Gagliardo-Nirenberg type inequalities of the form

$$\|\varphi\|_{L^p(\mathbb{R}^d)} \le C\|\varphi\|^{\beta}_{\dot{H}^s(\mathbb{R}^d)} \left(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \right)^{\gamma},$$

involving fractional Sobolev norms with $s > 0$ and Coulomb type energies with $0 < \alpha < d$ and $q \ge 1$. We establish optimal ranges of parameters for the validity of such inequalities and discuss the existence of the optimisers. In the special case $p = \frac{2d}{d-2s}$ our results include a new refinement of the fractional Sobolev inequality by a Coulomb term. We also prove that if the radial symmetry is taken into account, then the ranges of validity of the inequalities could be extended. More precisely, we show that such a radial improvement is possible if and only if $\alpha > 1$.

Contents

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1. Introduction and statement of results

1.1. **Introduction.** Given $d \in \mathbb{N}$, $s > 0$, $\alpha \in (0, d)$ and $q \in [1, \infty)$, we define the fractional Coulomb–Sobolev space by

$$\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d) = \Big\{\varphi : \mathbb{R}^d \to \mathbb{R} \,:\, \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y < \infty \text{ and } \int_{\mathbb{R}^d} \big||\xi|^s \widehat{\varphi}(\xi)\big|^2\,\mathrm{d}\xi < \infty\Big\}.$$

Since for every measurable function $\varphi : \mathbb{R}^d \to \mathbb{R}$

$$\Big(\int_{B_R(0)} |\varphi|^q \,\mathrm{d}x\Big)^2 \le C R^{d-\alpha} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y, \tag{1.1}$$

the boundedness of the double integral on the right-hand side of (1.1) ensures that φ is a tempered distribution and that its Fourier transform $\widehat{\varphi}$ is a well-defined tempered distribution. In particular $|\xi|^s\widehat{\varphi}$ is a well-defined distribution on $\mathbb{R}^d \setminus \{0\}$. The integrability condition in the definition of $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ means that this distribution can be represented by an L^2–function.

In the sequel we define the fractional Laplacian $(-\Delta)^{\frac{s}{2}}\varphi$ by

$$(\widehat{(-\Delta)^{\frac{s}{2}}\varphi})(\xi) = (2\pi|\xi|^2)^{\frac{s}{2}}\widehat{\varphi}(\xi).$$

We endow the space $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ with the norm

$$\|\varphi\|_{\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)} = \bigg(\big\|(-\Delta)^{\frac{s}{2}}\varphi\big\|^2_{L^2(\mathbb{R}^d)} + \Big(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{q}}\bigg)^{\frac{1}{2}}.$$

In particular, when $s < \frac{d}{2}$, a function φ is in the space $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ if and only if $\varphi \in \dot{H}^s(\mathbb{R}^d)$ and

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

The space $\mathcal{E}^{1,2,2}(\mathbb{R}^3)$ has been introduced and studied by P.-L. Lions [17, Lemma 4; 18, (55)] and D. Ruiz [24, section 2]. The space $\mathcal{E}^{s,\alpha,2}(\mathbb{R}^d)$ had been studied in [2], while $\mathcal{E}^{1,\alpha,q}(\mathbb{R}^d)$ had been studied in [21]. Following the arguments in [21, Section 2], it is not difficult to see that $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ is a Banach space (see Proposition 2.1 below).

The space $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ is the natural domain for the fractional Coulomb–Dirichlet type energy

$$\big\|(-\Delta)^{\frac{s}{2}}\varphi\big\|^2_{L^2(\mathbb{R}^d)} + \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y,$$

which appears in models of mathematical physics related to multi-particle systems, where the Coulomb term with $q = 2$ typically represents the electrostatic repulsion between the particles. Relevant models include Thomas–Fermi–Dirac–von Weizsäcker (TFDW) models of Density Functional theory [14]; or Schrödinger–Poisson–Slater approximation to Hartree–Fock theory [5]. The fractional case $d = 2$, $s = 1/2$ and $\alpha = 1$ appears in the recent TFDW–type model of charge screening in graphene [19], where relevant powers are $q = 2$ or $q = 1$. Interpolation inequalities (1.2) associated with the space $\mathcal{E}^{s,2s,2}(\mathbb{R}^d)$ are in some cases equivalent to the kinetic energy Lieb–Thirring type inequalities [20, Theorem 3], which are fundamental in the study of stability of non-relativistic ($s = 1$) and ultra-relativistic ($s = 1/2$) matter [16].

1.2. **Coulomb–Sobolev inequalities.** Our first main result in this paper is the continuous embedding

$$\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d) \hookrightarrow L^{\frac{2(2qs+\alpha)}{2s+\alpha}}(\mathbb{R}^d).$$

More specifically, we establish a family of scaling–invariant interpolation inequalities for the space $\mathcal{E}^{s,\alpha,p}(\mathbb{R}^d)$.

Theorem 1.1 (Coulomb–Sobolev inequalities)**.** *Let $d \in \mathbb{N}$, $s > 0$, $0 < \alpha < d$, $q, p \in [1, \infty)$ and $q(d - 2s) \neq d + \alpha$. There exists a constant $C = C(d, s, \alpha, q, p) > 0$ such that the scaling invariant inequality*

$$\|\varphi\|_p \leq C\|\varphi\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{p(d+\alpha)-2dq}{p(d+\alpha-q(d-2s))}} \Bigg(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\, |\varphi(y)|^q}{|x-y|^{d-\alpha}}\, \mathrm{d}x\, \mathrm{d}y \Bigg)^{\frac{2d-p(d-2s)}{2p(d+\alpha-q(d-2s))}} \tag{1.2}$$

holds for every function $\varphi \in \mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ if and only if

$$p \geq \frac{2(2qs+\alpha)}{2s+\alpha} \qquad \text{if } s \geq \frac{d}{2}, \tag{1.3}$$

$$p \in \Big[\frac{2(2qs+\alpha)}{2s+\alpha}, \frac{2d}{d-2s}\Big] \qquad \text{if } s < \frac{d}{2} \quad \text{and} \quad \frac{1}{q} > \frac{d-2s}{d+\alpha}, \tag{1.4}$$

$$p \in \Big[\frac{2d}{d-2s}, \frac{2(2qs+\alpha)}{2s+\alpha}\Big] \qquad \text{if } s < \frac{d}{2} \quad \text{and} \quad \frac{1}{q} < \frac{d-2s}{d+\alpha}. \tag{1.5}$$

Moreover, if p is not an end–point of the intervals (1.3)–(1.5), i.e. $p \neq \frac{2(2qs+\alpha)}{2s+\alpha}$ and $p \neq \frac{2d}{d-2s}$, then the best constant for (1.2) is achieved.

In the case $s = 1$ inequality (1.2) was known for $d = 3$, $\alpha = 2$ and $q = 2$ [18, (55); 24, Theorem 1.5]; and for $d \in \mathbb{N}$, $\alpha \in (0, N)$ and $q \geq 1$ [21, Theorem 1]. The fractional inequality (1.2) first appeared for $d = 3$, $s = 1/2$, $\alpha = 2$ and $q = 2$ in [4, Proposition 2.1]; and for $d \in \mathbb{N}$, $s > 0$, $\alpha \in (0, d)$ and $q = 2$ in [2, Proposition 2.1].

1.3. **Refined Sobolev inequalities.** The special case $q(d-2s) = d+\alpha$, which corresponds to $p = \frac{2d}{d-2s}$ and $q = \frac{d+\alpha}{d-2s}$, is not covered by the previous theorem and the exponents in (1.2) are meaningless. In this special case we obtain a refinement of the Sobolev embedding, extending the one observed for $s = 1$ [21, (1.7)] and for $q = 2$ [2, Proposition 2.1].

Theorem 1.2 (Endpoint refined Sobolev inequality)**.** *Let $d \in \mathbb{N}$, $0 < s < \frac{d}{2}$, $0 < \alpha < d$. Then there exists $C = C(d, s, \alpha) > 0$ such that the inequality*

$$\|\varphi\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \leq C\|\varphi\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{\alpha(d-2s)}{d(2s+\alpha)}} \left(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^{\frac{d+\alpha}{d-2s}}\, |\varphi(y)|^{\frac{d+\alpha}{d-2s}}}{|x-y|^{d-\alpha}}\, \mathrm{d}x\, \mathrm{d}y \right)^{\frac{s(d-2s)}{d(2s+\alpha)}} \tag{1.6}$$

holds for all $\varphi \in \mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$.

Remark 1.1. It is interesting to compare our refinement for Sobolev embedding with two other improvements. The Gérard–Meyer–Oru improvement [1, Theorem 1.43; 15] states that if $0 < s < \frac{d}{2}$ and $\theta \in \mathcal{S}(\mathbb{R}^d)$ is such that $\widehat{\theta}$ has compact support, has value 1 near the origin and satisfies $0 \le \widehat{\theta} \le 1$, then

$$\|\varphi\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \le C(d,s,\theta)\|\varphi\|_{\dot{H}^s(\mathbb{R}^d)}^{1-\frac{2s}{d}} \Bigl(\sup_{\lambda>0} \lambda^{\frac{d}{2}+s}\|\theta(\lambda\,\cdot) \star \varphi\|_\infty\Bigr)^{\frac{2s}{d}} \qquad \forall \varphi \in \dot{H}^s(\mathbb{R}^d). \tag{1.7}$$

The Palatucci–Pisante improvement [22, Theorem 1.1] (see also [27, (4.2)]) states that if $0 < s < \frac{d}{2}$, then

$$\|\varphi\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \le C(d,s)\|\varphi\|_{\dot{H}^s(\mathbb{R}^d)}^{1-\frac{2s}{d}}\|\varphi\|_{\mathcal{M}^{1,\frac{d}{2}-s}}^{\frac{2s}{d}} \qquad \forall \varphi \in \dot{H}^s(\mathbb{R}^d). \tag{1.8}$$

In the last inequality, the Morrey norm is defined as

$$\|\varphi\|_{\mathcal{M}^{r,\gamma}} := \sup_{R>0,\, x\in\mathbb{R}^d} R^\gamma \Bigl(\fint_{B_R(x)} |u|^r\Bigr)^{\frac{1}{r}};$$

one proof of (1.8) relies on (1.7) and on the observation that

$$\lambda^{\frac{d}{2}+s}\|\theta(\lambda\,\cdot) \star \varphi\|_\infty \le C\|\varphi\|_{\mathcal{M}^{1,\frac{d-2s}{2}}}.$$

In our case we have by Hölder's inequality and monotonicity of the integral

$$\Bigl(R^{\frac{d}{2}-s}\fint_{B_R(x)} |\varphi|\Bigr)^{\frac{d+\alpha}{d-2s}} \le R^{\frac{d+\alpha}{2}} \fint_{B_R(x)} |\varphi|^{\frac{d+\alpha}{d-2s}} \le C\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^{\frac{d+\alpha}{d-2s}}|\varphi(y)|^{\frac{d+\alpha}{d-2s}}}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{2}}$$

so that it is clear that Coulomb norm controls the Morrey norm $\mathcal{M}^{1,\frac{d}{2}-s}$. On the other hand, the exponent $\frac{\alpha(d-2s)}{d(2s+\alpha)} = (1-\frac{2s}{d})\frac{1}{1+2s/\alpha}$ for $\dot{H}^s$-norm in our improvement is always less than the exponent $1-\frac{2s}{d}$ for $\dot{H}^s$-norm in (1.7) and (1.8). This suggests that the inequality (1.6) cannot be derived directly from the already known ones.

Remark 1.2. The refinement of the Sobolev inequality in Theorem 1.2 is sharp. Indeed, by scaling it can be proved that if a scaling invariant inequality of the following form holds

$$\|\varphi\|_{L^{\frac{2d}{d-2s}}(\mathbb{R}^d)} \le C(d,s,\alpha)\|\varphi\|_{\dot{H}^s(\mathbb{R}^d)}^{\beta}\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^{\frac{d+\alpha}{d-2s}}|\varphi(y)|^{\frac{d+\alpha}{d-2s}}}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\gamma}, \tag{1.9}$$

then the exponents γ and β are related by the equation

$$\frac{d-2s}{2} = \Bigl(\frac{d}{2}-s\Bigr)\beta + (d+\alpha)\gamma$$

On the other hand, estimates (3.7)–(3.9) in the proof of Theorem 1.1 below imply that

$$\frac{d-2s}{2d} \le \frac{\beta}{2} + \gamma.$$

We conclude that $\beta \ge \frac{\alpha(d-2s)}{d(2s+\alpha)}$ is necessary for (1.9) to hold.

Interpolating between the refined and classical Sobolev inequalities, we obtain a new family of interpolation inequalities, for which the best constant is achieved.

Theorem 1.3 (Non-endpoint refined Sobolev inequalities)**.** *Let $d \in \mathbb{N}$, $0 < s < \frac{d}{2}$, $0 < \alpha < d$ and $0 < \varepsilon < \frac{s(d-2s)}{d(2s+\alpha)}$. Then there exists $C = C(d,s,\alpha,\varepsilon) > 0$ such that the inequality*

$$||\varphi||_{\frac{2d}{d-2s}} \le C\|\varphi\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{\alpha(d-2s)}{2sd+\alpha d}+\varepsilon\frac{2(\alpha+d)}{d-2s}} \left(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^{\frac{d+\alpha}{d-2s}}|\varphi(y)|^{\frac{d+\alpha}{d-2s}}}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{s(d-2s)}{d(2s+\alpha)}-\varepsilon} \tag{1.10}$$

holds for all $\varphi \in \mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$. Moreover, the best constant for (1.10) *is achieved.*

When $\varepsilon = \frac{s(d-2s)}{d(2s+\alpha)}$ the inequality (1.10) is the classical Sobolev inequality.

The existence of optimizers for the non-endpoint inequality (1.6) provides a partial answer towards the question raised in the case $s = 1$ in [21, Section 1.5.5]. The existence of optimizers for the endpoint inequality (1.6) remains open.

1.4. **Radial improvements.** We now consider the question of embeddings for *radial functions.* Since the symmetric decreasing rearrangement *increases* the nonlinear nonlocal Coulomb energy term, the situation might be more favorable for radial functions. Our next result shows that for the subspace of radially symmetric functions in the Coulomb–Sobolev space $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ the intervals (1.3)–(1.5) of the validity of the Coulomb–Sobolev inequality (1.2) can be extended provided that $\alpha > 1$.

Theorem 1.4 (Sharp Improvement in the radial case for $\alpha > 1$)**.** *Let $d \ge 2$, $s > 0$, $1 < \alpha < d$, $q, p \in [1,\infty)$, $q(d-2s) \ne d+\alpha$ and*

$$p_{\mathrm{rad}} := q + \frac{((2s-1)q+2)(d-\alpha)}{2s(d+\alpha-2)+d-\alpha}.$$

There exists a constant $C_{\mathrm{rad}} = C_{\mathrm{rad}}(d,s,\alpha,q,p) > 0$ such that the scaling invariant inequality

$$\|\varphi\|_{L^p(\mathbb{R}^d)} \le C_{\mathrm{rad}}\|\varphi\|_{\dot{H}^s(\mathbb{R}^d)}^{\frac{p(d+\alpha)-2dq}{p(d+\alpha-q(d-2s))}} \left(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{2d-p(d-2s)}{2p(d+\alpha-q(d-2s))}} \tag{1.11}$$

hold for all radially symmetric functions $\varphi \in \mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ if and only if

$$p > p_{\mathrm{rad}} \qquad \text{if } s \ge \frac{d}{2}, \tag{1.12}$$

$$p \in \Big(p_{\mathrm{rad}}, \frac{2d}{d-2s}\Big] \qquad \text{if } s < \frac{d}{2} \text{ and } \frac{1}{q} > \frac{d-2s}{d+\alpha}, \tag{1.13}$$

$$p \in \Big[\frac{2d}{d-2s}, p_{\mathrm{rad}}\Big) \qquad \text{if } s < \frac{d}{2} \text{ and } \frac{1}{q} < \frac{d-2s}{d+\alpha},\ \frac{1}{q} \ne \frac{1-2s}{2}, \tag{1.14}$$

$$p \in \Big[\frac{2d}{d-2s}, q\Big] \qquad \text{if } s < \frac{1}{2} \text{ and } \frac{1}{q} = \frac{1-2s}{2}. \tag{1.15}$$

If $0 < \alpha \le 1$ then inequality (1.11) *holds on $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ if and only if* (1.2) *holds on $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$.*

In the important special case $s = 1/2$ we have the simplified expression $p_{\mathrm{rad}} = q + \frac{d-\alpha}{d-1}$, while for $s = 0$ we formally obtain $p_{\mathrm{rad}} = 2$.

In the special case $d = 3$, $s = 1$, $\alpha = 2$ and $q = 2$ the improved radial inequality (1.11) was first established in [24, Theorem 1.2]. For $d \in \mathbb{N}$, $s = 1$, $\alpha \in (0,d)$ and $q \ge 1$ the improved radial inequalities (1.2) were studied in [21, Theorem 4]. The fractional case $d = 3$, $1/2 < s < 3/2$, $\alpha = 2$, $q = 2$ was considered in [3].

We shall emphasise that the radial improvement is possible for any $s > 0$ but if and only if $\alpha > 1$. The *universality* of the threshold $\alpha = 1$ which does not depend on any other parameter in the problem is quite interesting.

Another new and purely fractional phenomenon is the special role of the exponent $q = \frac{2}{1-2s}$ in the case $s < 1/2$. Observe that for $s \geq 1/2$ we always have $p_{\mathrm{rad}} > q$, while $p_{\mathrm{rad}} < q$ if $s < 1/2$ and $q > \frac{2}{1-2s}$, the latter requires $q > \frac{d+\alpha}{d-2s}$. If $s < 1/2$ and $q = \frac{2}{1-2s}$ then $p_{\mathrm{rad}} = q$ and this is the only case when the endpoint embedding $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d) \hookrightarrow L^{p_{\mathrm{rad}}}(\mathbb{R}^d)$ is valid.

Finally, we prove that the embedding $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ is compact provided that p is not an endpoint of the embedding intervals.

Theorem 1.5 (Compact embeddings for radial functions)**.** *Let $d \geq 2$, $s > 0$, and $q \in [1, \infty)$. Moreover we assume that p is away from the endpoints of the intervals in (1.3)–(1.5) when $0 < \alpha \leq 1$ and in (1.12)–(1.15) when $1 < \alpha < d$. Then, the embedding $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ is compact.*

Compactness of the radial embedding implies in a standard way the existence of radial optimizers associated to the inequalities (1.11), cf. [21, Section 7] where the case $s = 1$ was considered.

1.5. **Outline.** The rest of the paper is organised as follows. Section 2 contains a short proof of the completeness of the Coulomb–Sobolev spaces. In Section 3 we discuss the spaces $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ in the nonradial context and show that interpolation inequalities of Theorems 1.1 and 1.2 can be deduced from the standard fractional Gagliardo–Nirenberg inequality (3.3) using a fractional chain rule. We also discuss the existence of the optimisers and prove Theorem 1.3. In Section 4 we derive the radial improvement of Theorem 1.4 as a consequence of Ruiz's inequality for Coulomb energy (see Theorem 4.1) and either de Napoli's interpolation inequality (see Theorem 4.2), which is a fractional extension of the classical pointwise Strauss type bounds valid only for $s > 1/2$; or, in case $s \leq 1/2$ a consequence of Rubin's inequality (Theorem 4.3), which is refinement for radial functions of the classical Stein–Weiss inequality. In Section 5 we construct special families of functions which are used to prove the optimality of the radial embeddings, while in Section 6 we prove the compactness of the radial embedding.

1.6. **Asymptotic notation.** For real valued functions $f(t), g(t) \geq 0$, we write:

$f(t) \lesssim g(t)$ if there exists $C > 0$ independent of t such that $f(t) \leq Cg(t)$;

$f(t) \simeq g(t)$ if $f(t) \lesssim g(t)$ and $g(t) \lesssim f(t)$.

As usual, C, c, c_1, etc., denote generic positive constants independent of t.

2. Completeness of the fractional Coulomb–Sobolev space

As in [21, Section 2], it is not difficult to see that the space $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ is a normed space.

Proposition 2.1. *For every $d \in \mathbb{N}$, $s > 0$, $0 < \alpha < d$ and $q \in [1, \infty)$, the normed space $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ is complete.*

Proof. If $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$, then $((-\Delta)^{\frac{s}{2}}u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{R}^d)$ and there exists thus $f \in L^2(\mathbb{R}^d)$ such that $((-\Delta)^{\frac{s}{2}}u_n)_{n\in\mathbb{N}}$ converges strongly to f in $L^2(\mathbb{R}^d)$. On the other hand, by (1.1) we have for every $R > 0$,

$$\lim_{m,n\to\infty} \int_{B_R(0)} |u_n - u_m|^q = 0.$$

There exists thus a measurable function $u : \mathbb{R}^d \to \mathbb{R}$ such that $(u_n)_{n\in\mathbb{N}}$ converges to u in $L^q_{\mathrm{loc}}(\mathbb{R}^d)$. By Fatou's lemma, we have

$$\lim_{n\to\infty} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u_n(x) - u(x)|^q\,|u_n(y) - u_n(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\ \le \lim_{n\to\infty} \liminf_{m\to\infty} \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u_n(x) - u_m(x)|^q\,|u_n(y) - u_m(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y.$$

It remains now to prove that $(-\Delta)^{\frac{s}{2}}u = f$. We observe that by (1.1),

$$\lim_{n\to\infty} \sup_{R>0} \frac{1}{R^{\frac{d-\alpha}{2}}} \int_{B_R(0)} |u_n - u|^q = 0.$$

Therefore $(u_n)_{n\in\mathbb{N}}$ converges to u as tempered distributions $\mathbb{R}^d$, and thus the sequence $(\widehat{u}_n)_{n\in\mathbb{N}}$ converges to $\widehat{u}$ as tempered distributions on $\mathbb{R}^d$. It follows that $((2\pi)^{s/2}|\xi|^s\widehat{u_n})_{n\in\mathbb{N}}$ converges to $2\pi^{s/2}|\xi|^s|\xi|^s\widehat{u}$ as distributions on $\mathbb{R}^d$. Since on the other hand, $((2\pi)^{s/2}|\xi|^s\widehat{u_n})_{n\in\mathbb{N}}$ converges to $\widehat{f}$ it follows that $(-\Delta)^{\frac{s}{2}}u = f$. □

3. Gagliardo–Nirenberg inequalities: Proof of Theorems 1.1, 1.2 and 1.3

We first establish the endpoint inequality.

Theorem 3.1. *Let $d \in \mathbb{N}$, $s > 0$, $0 < \alpha < d$ and $q \in [1,\infty)$. Then the following inequality holds*

$$\|\varphi\|_{L^{\frac{2(2qs+\alpha)}{2s+\alpha}}(\mathbb{R}^d)} \lesssim \|(-\Delta)^{\frac{s}{2}}\varphi\|_{L^2(\mathbb{R}^d)}^{\frac{\alpha}{2qs+\alpha}} \Bigg(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \Bigg)^{\frac{s}{2qs+\alpha}} \qquad \forall \varphi \in \mathcal{E}^{s,\alpha,q}(\mathbb{R}^d).$$

In particular, $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d) \hookrightarrow L^{\frac{2(2qs+\alpha)}{2s+\alpha}}(\mathbb{R}^d)$ continuously.

The above inequality in the particular case $q = 1$ implies that $\mathcal{E}^{s,\alpha,1}(\mathbb{R}^d)$ embeds continuously into $H^s(\mathbb{R}^d)$.

Proof of Theorem 3.1. Recall that for all $\phi \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ such that

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\phi(x)\phi(y)}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y < \infty \tag{3.1}$$

there holds

$$\|(-\Delta)^{-\frac{\alpha}{4}}\phi\|_{L^2(\mathbb{R}^d)}^2 = c \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\phi(x)\phi(y)}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.2}$$

Moreover we recall the endpoint Gagliardo-Nirenberg inequality (see for example [1, Theorem 2.44])

$$\|(-\Delta)^{\frac{\alpha}{4}}\psi\|_{L^p(\mathbb{R}^d)} \le C\|\psi\|_{L^2(\mathbb{R}^d)}^{\frac{2s}{\alpha+2s}}\|(-\Delta)^{\frac{\alpha}{4}+\frac{s}{2}}\psi\|_{L^r(\mathbb{R}^d)}^{\frac{\alpha}{\alpha+2s}} \tag{3.3}$$

where

$$\frac{1}{p} = \frac{1}{2}\Big(\frac{2s}{\alpha+2s}\Big) + \frac{1}{r}\Big(\frac{\alpha}{\alpha+2s}\Big).$$

When $q=1$ by (3.1) and (3.2) there holds

$$\|(-\Delta)^{-\frac{\alpha}{4}}\varphi\|_{L^2(\mathbb{R}^d)}^2 = c\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\varphi(x)\varphi(y)}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le c\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|\varphi(x)|\,|\varphi(y)|}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y. \tag{3.4}$$

Setting $\psi = (-\Delta)^{-\frac{\alpha}{4}}\varphi$ and $p=r=2$, (3.3) together with (3.4) yields the inequality for $q=1$.

Let $q>1$. Setting $\psi = (-\Delta)^{-\frac{\alpha}{4}}|\varphi|^q$ in (3.3), we get

$$\||\varphi|^q\|_{L^p(\mathbb{R}^d)} \le C\|(-\Delta)^{-\frac{\alpha}{4}}|\varphi|^q\|_{L^2(\mathbb{R}^d)}^{\frac{2s}{\alpha+2s}}\|(-\Delta)^{\frac{s}{2}}|\varphi|^q\|_{L^r(\mathbb{R}^d)}^{\frac{\alpha}{\alpha+2s}}$$

which implies

$$\|\varphi\|_{L^{qp}(\mathbb{R}^d)}^q \le C\|(-\Delta)^{-\frac{\alpha}{4}}|\varphi|^q\|_{L^2(\mathbb{R}^d)}^{\frac{2s}{\alpha+2s}}\|(-\Delta)^{\frac{s}{2}}\varphi\|_{L^2(\mathbb{R}^d)}^{\frac{\alpha}{\alpha+2s}}\|\varphi\|_{L^{(q-1)l}(\mathbb{R}^d)}^{(q-1)\frac{\alpha}{\alpha+2s}} \tag{3.5}$$

by the fractional chain rule where $\frac{1}{r} = \frac{1}{2}+\frac{1}{l}$ [12, Corollary of Theorem 5]. Now choosing l such that $(q-1)l = qp$, i.e. such that

$$\frac{1}{l} = \frac{q-1}{qp}$$

we conclude that $p = \frac{2\alpha+4qs}{q(2s+\alpha)}$. By (3.5) and setting $\phi = |\varphi|^q$ in (3.2), this concludes the proof. □

Proof of Theorem 1.1 and Theorem 1.2. The exponents for the refined Sobolev inequality given by Theorem 1.2 are derived directly from the endpoint Gagliardo-Nirenberg inequality of Theorem 3.1.

The scaling-invariant inequalities of Theorem 1.1 follows from the fact that by interpolation between Theorem 3.1 and the classical fractional Sobolev embedding, $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ continuously for

$$p \in \Big(\frac{2(2qs+\alpha)}{2s+\alpha}, \frac{2d}{d-2s}\Big] \qquad \text{if } 1 < q < \frac{d+\alpha}{d-2s}.$$
$$p \in \Big[\frac{2d}{d-2s}, \frac{2(2qs+\alpha)}{2s+\alpha}\Big) \qquad \text{if } q > \frac{d+\alpha}{d-2s}$$

Indeed, let us consider the scaling $u_\lambda(x) = \lambda^{\frac{d}{p}}u(\lambda x)$ such that $\|u_\lambda\|_{L^p(\mathbb{R}^d)} = ||u||_{L^p(\mathbb{R}^d)}$. From the embedding we get

$$\|u_\lambda\|_{L^p(\mathbb{R}^d)}^2 \lesssim \|(-\Delta)^{\frac{s}{2}}u_\lambda\|_{\dot{L}^2(\mathbb{R}^d)}^2 + \Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|u_\lambda(x)|^q\,|u_\lambda(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{q}},$$

which gives by scaling

$$\|u\|^2_{L^p(\mathbb{R}^d)} \lesssim \lambda^{\frac{2d}{p}-d+2s}\|(-\Delta)^{\frac{s}{2}}u_\lambda\|^2_{L^2(\mathbb{R}^d)} + \lambda^{\frac{2d}{p}-\frac{(d+\alpha)}{q}}\Bigg(\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u_\lambda(x)|^q\,|u_\lambda(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y\Bigg)^{\frac{1}{q}}. \tag{3.6}$$

Notice that when $q=\frac{d+\alpha}{d-2s}$ and $p=\frac{2d}{d-2s}$ we obtain as expected $\frac{2d}{p}-d+2s=0$, $\frac{2d}{p}-\frac{(d+\alpha)}{q}=0$. Minimizing the right-hand side of (3.6) with respect to λ we get the scaling invariant inequality given by Theorem 1.1. The same computation of course works in the radial case.

Optimality of the embedding intervals. Given a nonnegative function $\eta \in C_c^\infty(\mathbb{R}^d)\setminus\{0\}$ and a vector $a\in\mathbb{R}^d\setminus\{0\}$, for $k\in\mathbb{N}$ set

$$u_{a,k}(x)=\eta(x+ka).$$

Following [24, Section 5], we define the functions $v_{a,m}\in C_c^\infty(\mathbb{R}^N)$ by

$$v_{a,m}=\sum_{k=1}^m u_{a,k}.$$

Then for $|a|\to\infty$ we obtain

$$\|v_{a,m}\|^p_{L^p(\mathbb{R}^d)} \simeq m, \tag{3.7}$$

$$\|v_{a,m}\|^2_{\dot H^s(\mathbb{R}^d)} \lesssim m, \tag{3.8}$$

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|v_{a,m}(x)|^q\,|v_{a,m}(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \lesssim m. \tag{3.9}$$

To deduce (3.8), choose $k\in\mathbb{N}$ such that $k\geq s$. Interpolating between homogeneous L^2 and $\dot H^k$ norms (cf. [1, Proposition 1.32]), for $|a|\to\infty$ we conclude that

$$\|v_{a,m}\|^2_{\dot H^s(\mathbb{R}^d)} \leq \|v_{a,m}\|^{\frac{2s}{k}}_{\dot H^k(\mathbb{R}^d)}\|v_{a,m}\|^{2\left(1-\frac{s}{k}\right)}_{L^2(\mathbb{R}^d)} \lesssim \Big(m\|\eta\|^2_{\dot H^k(\mathbb{R}^d)}\Big)^{\frac{s}{k}}\Big(m\|\eta\|^2_{L^2(\mathbb{R}^d)}\Big)^{1-\frac{s}{k}} \lesssim m.$$

Using the diagonal argument, from (1.2) we deduce that for all sufficiently large $m\in\mathbb{N}$ it must hold

$$m \lesssim m^{\frac{p(d+\alpha)-2dq}{2(d+\alpha-q(d-2s))}}\, m^{\frac{2d-p(d-2s)}{2(d+\alpha-q(d-2s))}},$$

which implies the optimality of the embedding intervals (1.3)–(1.5).

Existence of the optimizers. The existence of optimizers follows almost identically to the proof of [2, Theorem 2.2], see also [3, proof of Corollary 0.1]. We only sketch the argument.

Fix p inside one of the intervals (1.3)–(1.5). By homogeneity and scaling we can assume that an optimizing sequence $(\varphi_n)_{n\in\mathbb{N}}$ in $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$ satisfies

$$\|\varphi_n\|_{\dot H^s(\mathbb{R}^d)} = \iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi_n(x)|^q|\varphi_n(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y = 1,$$

and $\|\varphi_n\|_{L^p(\mathbb{R}^d)} = C(d,s,\alpha)+o(1)$.

Since p is not an end-point of the intervals (1.3)–(1.5), we can use interpolation inequality (1.2) to find a uniform upper bound on $\|\varphi_n\|_{L^{p_1}(\mathbb{R}^d)}$ and $\|\varphi_n\|_{L^{p_2}(\mathbb{R}^d)}$, for some $p_1<p<p_2$. Therefore, via the *pqr*-lemma [11, Lemma 2.1 p.258] and Lieb's compactness lemma in $\dot H^s(\mathbb{R}^d)$ [2, Lemma 2.1], we conclude that $\varphi_n \rightharpoonup \bar\varphi \neq 0$ in $H^s(\mathbb{R}^d)$. Finally, using the non-local Brezis–Lieb splitting lemma for the Coulomb term [21, Proposition 4.8], the existence of a

maximizer could be proved similarly to the arguments in [2, pp.661–662] (see also the proof of Theorem 1.3 below for similar estimates). □

Proof of Theorem 1.3. Inequality (1.10) is obtained directly by interpolation between the classical Sobolev inequality and endpoint refined Sobolev inequality (1.6)

To prove that the best constant $C(d, s, \alpha, \varepsilon)$ in (1.10) is achieved, we will use the following result.

Theorem 3.2 (Gerard–Meyer–Oru)**.** *Let $0 < s < d/2$ and let $\theta \in \mathcal{S}(\mathbb{R}^d)$ be such that $\hat\theta$ has compact support, has value 1 near the origin and satisfies $0 \le \hat\theta \le 1$. Then there is a constant $C = C_{s,d}(\theta)$ such that for all $u \in \dot H^s(\mathbb{R}^d)$,*

$$\|u\|_{\frac{2d}{d-2s}} \le C\|u\|_{\dot H^s}^{\frac{d-2s}{d}} \left(\sup_{A>0} A^{d/2+s}\|\theta(A\,\cdot) \star u\|_\infty\right)^{\frac{2s}{d}}.$$

Consider a maximizing sequence $(\varphi_n)_{n\in\mathbb{N}}$ for (1.10) such that $\|\varphi_n\|_{\dot H^s(\mathbb{R}^d)} = 1$ and

$$||\varphi_n||_{\frac{2d}{d-2s}} = (C(d, s, \alpha, \varepsilon) + o(1))\,(D(\varphi_n))^{\frac{s(d-2s)}{d(2s+\alpha)}-\varepsilon},$$

where for brevity, we denoted

$$D(\varphi) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^{\frac{d+\alpha}{d-2s}}|\varphi(y)|^{\frac{d+\alpha}{d-2s}}}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y.$$

Using the endpoint refined Sobolev inequality we infer that

$$(D(\varphi_n))^{\frac{s(d-2s)}{d(2s+\alpha)}-\varepsilon} \lesssim ||\varphi_n||_{\frac{2d}{d-2s}} \lesssim (D(\varphi_n))^{\frac{s(d-2s)}{d(2s+\alpha)}}.$$

This implies that

$$1 \lesssim D(\varphi_n)$$

and hence,

$$1 \lesssim ||\varphi_n||_{\frac{2d}{d-2s}}. \tag{3.10}$$

Let $\bar\varphi$ denotes the weak limit of (φ_n) in $\dot H^s(\mathbb{R}^d)$. Recall that our inequality (1.10) is critical, i.e. it is both scaling and translation invariant. From Theorem 3.2 together with (3.10) there exists sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ of translations and $(A_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^+$ of dilations such that

$$\inf_n A_n^{\frac{d}{2}+s}\int_{\mathbb{R}^d} \theta(A_n(x_n-y))\varphi_n(y)dy > 0.$$

This fact implies by the change of variable that

$$A_n^{s-\frac{d}{2}}\varphi_n\Big(\frac{x-x_n}{A_n}\Big) \rightharpoonup \bar\varphi \neq 0.$$

The fact that $\bar\varphi$ is an optimizer is now standard. By the Brezis–Lieb type splitting properties of the three terms in (1.10) (for the splitting of nonlocal term D see [21, Proposition 4.7]), we obtain

$$C(d, s, \alpha, \varepsilon)^{-\frac{2d}{d-2s}}\left(||\bar\varphi||_{\frac{2d}{d-2s}}^{\frac{2d}{d-2s}} + ||\varphi_n - \bar\varphi||_{\frac{2d}{d-2s}}^{\frac{2d}{d-2s}} + o(1)\right)$$

$$\geq \Big(\|\bar{\varphi}\|^2_{\dot{H}^s(\mathbb{R}^d)} + \|\varphi_n - \bar{\varphi}\|^2_{\dot{H}^s(\mathbb{R}^d)} + o(1)\Big)^{\frac{d\alpha}{d(2s+\alpha)}+\varepsilon\frac{2d(\alpha+d)}{(d-2s)^2}} \big(D(\bar{\varphi}) + D(\varphi_n - \bar{\varphi})\big)^{\frac{2ds}{d(2s+\alpha)}-\varepsilon\frac{2d}{d-2s}}.$$

Since

$$\Big(\frac{d\alpha}{d(2s+\alpha)} + \varepsilon\frac{2d(\alpha+d)}{(d-2s)^2}\Big) + \Big(\frac{2ds}{d(2s+\alpha)} - \varepsilon\frac{2d}{d-2s}\Big) = 1 + \varepsilon\frac{2d(\alpha+2s)}{(d-2s)^2} > 1,$$

As a consequence of the discrete Hölder inequality we have

$$a^{\frac{2d\alpha}{d(2s+\alpha)}+\varepsilon\frac{4d(\alpha+d)}{(d-2s)^2}} c^{\frac{2ds}{d(2s+\alpha)}-\varepsilon\frac{2d}{d-2s}} + b^{\frac{2d\alpha}{d(2s+\alpha)}+\varepsilon\frac{4d(\alpha+d)}{(d-2s)^2}} e^{\frac{2ds}{d(2s+\alpha)}-\varepsilon\frac{2d}{d-2s}} \leq \Big(a^2+b^2\Big)^{\frac{d\alpha}{d(2s+\alpha)}+\varepsilon\frac{2d(\alpha+d)}{(d-2s)^2}} (c+e)^{\frac{2ds}{d(2s+\alpha)}-\varepsilon\frac{2d}{d-2s}}$$

for all $a, b, c, e \geq 0$. Hence

$$\begin{aligned} C(d,s,\alpha,\varepsilon)^{-\frac{2d}{d-2s}} &\Big(||\bar{\varphi}||^{\frac{2d}{d-2s}}_{\frac{2d}{d-2s}} + ||\varphi_n - \bar{\varphi}||^{\frac{2d}{d-2s}}_{\frac{2d}{d-2s}} + o(1)\Big) \\ &\geq \|\bar{\varphi}\|^{\frac{2d\alpha}{d(2s+\alpha)}+\varepsilon\frac{4d(\alpha+d)}{(d-2s)^2}}_{\dot{H}^s(\mathbb{R}^d)} D(\bar{\varphi})^{\frac{2ds}{d(2s+\alpha)}-\varepsilon\frac{2d}{d-2s}} \\ &\quad + \|\varphi_n - \bar{\varphi}\|^{\frac{2d\alpha}{d(2s+\alpha)}+\varepsilon\frac{4d(\alpha+d)}{(d-2s)^2}}_{\dot{H}^s(\mathbb{R}^d)} D(\varphi_n - \bar{\varphi})^{\frac{2ds}{d(2s+\alpha)}-\varepsilon\frac{2d}{d-2s}} + o(1). \end{aligned}$$

Therefore we can conclude that

$$C(d,s,\alpha,\varepsilon)^{-\frac{2d}{d-2s}} ||\bar{\varphi}||^{\frac{2d}{d-2s}}_{\frac{2d}{d-2s}} \geq \|\bar{\varphi}\|^{\frac{2d\alpha}{d(2s+\alpha)}+\varepsilon\frac{4d(\alpha+d)}{(d-2s)^2}}_{\dot{H}^s(\mathbb{R}^d)} D(\bar{\varphi})^{\frac{2ds}{d(2s+\alpha)}-\varepsilon\frac{2d}{d-2s}} + o(1),$$

which implies that φ is an optimizer. □

4. Sharp improvement in the radial case

In order to establish the radial inequality (1.11) we will use a version of the weighted estimate involving the Coulomb term which was originally established by Ruiz [24].

Theorem 4.1 (Ruiz [24, Theorem 1.1], see also [21, Proposition 3.8])**.** *Let $d \in \mathbb{N}$, $0 < \alpha < d$, $q \in [1, \infty)$. Then for every $\varepsilon > 0$ and $R > 0$ there exists $C = C(d, \alpha, q, \varepsilon) > 0$ such that for all $\varphi \in L^{\frac{2dq}{d+\alpha}}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d \setminus B_R(0)} \frac{|\varphi(x)|^q}{|x|^{\frac{d-\alpha}{2}+\varepsilon}} \,\mathrm{d}x \leq \frac{C}{R^\varepsilon}\Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}, \tag{4.1}$$

$$\int_{B_R(0)} \frac{|\varphi(x)|^q}{|x|^{\frac{d-\alpha}{2}-\varepsilon}} \,\mathrm{d}x \leq CR^\varepsilon\Big(\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{2}}. \tag{4.2}$$

We will also employ two different estimate on the functions in $\dot{H}^s_{\mathrm{rad}}(\mathbb{R}^d)$. In the case $s > 1/2$ our proof of (1.11) relies on the following interpolation result.

Theorem 4.2 (De Nápoli [7, Theorem 3.1])**.** *Let $d \geq 2$, $s > \frac{1}{2}$, $r > 1$ and*

$$-(d-1) \leq a < d(r-1). \tag{4.3}$$

Then

$$|\varphi(x)| \le C(d,s,r,a)|x|^{-\sigma}\|(-\Delta)^{\frac{s}{2}}\varphi\|^{\theta}_{L^2(\mathbb{R}^d)}\|\varphi\|^{1-\theta}_{L^r_a(\mathbb{R}^d)} \qquad \forall \varphi \in \dot H^s_{\mathrm{rad}}(\mathbb{R}^d)\cap L^r_a(\mathbb{R}^d), \tag{4.4}$$

where $\sigma = \frac{2s(d-1)+(2s-1)a}{(2s-1)r+2}$, $\theta = \frac{2}{(2s-1)r+2}$ *and* $L^r_a(\mathbb{R}^d)$ *is the weighted Lebesgue space with the norm*

$$\|u\|_{L^r_a(\mathbb{R}^d)} = \Big(\int_{\mathbb{R}^d} |x|^a |u(x)|^r \,\mathrm{d}x\Big)^{\frac{1}{r}}.$$

Remark 4.1. The inequality (4.2) has important special cases:

i) When $r = \frac{2d}{d-2s}$ and $a = 0$ we obtain Cho–Ozawa's inequality [6]:

$$\sup_{|x|>0} |\varphi(x)| \lesssim |x|^{-\frac{d-2s}{2}}\|\varphi\|_{\dot H^s(\mathbb{R}^d)} \qquad \forall \varphi \in \dot H^s_{\mathrm{rad}}(\mathbb{R}^d), \tag{4.5}$$

ii) When $r = 2$ and $a = 0$ we obtain Ni type inequality

$$\sup_{|x|>0} |\varphi(x)| \lesssim |x|^{-\frac{d-1}{2}}\|\varphi\|^{\frac{1}{2s}}_{\dot H^s(\mathbb{R}^d)}\|\varphi\|^{1-\frac{1}{2s}}_{L^2(\mathbb{R}^d)} \qquad \forall \varphi \in \dot H^s_{\mathrm{rad}}(\mathbb{R}^d).$$

In the case $s \le 1/2$ pointwise estimates on functions in $\dot H^s_{\mathrm{rad}}(\mathbb{R}^d)$ are no longer available. Instead, our proof of (1.11) relies on the radial version of the classical Stein–Weiss estimate [25].

Theorem 4.3 (Rubin [8; 9, Theorem 1.2; 23])**.** *Let* $d \ge 2$ *and* $0 < s < d/2$. *Then*

$$\Big(\int_{\mathbb{R}^d} |\varphi(x)|^r |x|^{-\beta r}\,\mathrm{d}x\Big)^{\frac{1}{r}} \le C(d,s,r,\beta)\|\varphi\|_{\dot H^s(\mathbb{R}^d)} \qquad \forall \varphi \in \dot H^s_{\mathrm{rad}}(\mathbb{R}^d), \tag{4.6}$$

where $r \ge 2$ *and*

$$-(d-1)\Big(\frac{1}{2}-\frac{1}{r}\Big) \le \beta < \frac{d}{r}, \tag{4.7}$$

$$\frac{1}{r} = \frac{1}{2} + \frac{\beta - s}{d}. \tag{4.8}$$

Remark 4.2. The difference with the classical (non-radial) Stein–Weiss estimate [25] is only in the extended range (4.7) for β (in the non-radial case we must have $0 \le \beta < \frac{d}{r}$). Note special cases of (4.6):

i) When $\beta = s$ and $s < \frac{d}{2}$ we obtain $r = 2$ which gives the Hardy inequality:

$$\Big(\int_{\mathbb{R}^d} |\varphi(x)|^2 |x|^{-2s}\,\mathrm{d}x\Big)^{\frac{1}{2}} \lesssim \|\varphi\|_{\dot H^s(\mathbb{R}^d)} \qquad \forall \varphi \in \dot H^s_{\mathrm{rad}}(\mathbb{R}^d),$$

ii) When $\beta = 0$ and $s < \frac{d}{2}$ we obtain $r = \frac{2d}{d-2s}$ which gives the Sobolev estimate:

$$\Big(\int_{\mathbb{R}^d} |\varphi|^{\frac{2d}{d-2s}}\Big)^{\frac{1}{2}-\frac{s}{d}} \lesssim \|\varphi\|_{\dot H^s(\mathbb{R}^d)} \qquad \forall \varphi \in \dot H^s_{\mathrm{rad}}(\mathbb{R}^d),$$

iii) When $\beta = -(d-1)(\frac{1}{2}-\frac{1}{r})$ and $s < \frac{1}{2}$ we see from (4.8) that $r = \frac{2}{1-2s}$ and hence $\beta = -(d-1)s$, so we obtain a "limiting" inequality

$$\Big(\int_{\mathbb{R}^d} |\varphi|^{\frac{2}{1-2s}}|x|^{\frac{2s(d-1)}{1-2s}}\,\mathrm{d}x\Big)^{\frac{1}{2}-s} \lesssim \|\varphi\|_{\dot H^s(\mathbb{R}^d)} \qquad \forall \varphi \in \dot H^s_{\mathrm{rad}}(\mathbb{R}^d).$$

A corollary of Rubin's inequality is an integral replacement of the Cho–Ozawa bound (4.5).

Lemma 4.1 (Weak Ni's inequality)**.** *Let $d \geq 2$, $0 < s \leq 1/2$ and $\frac{1}{2} - s \leq \frac{1}{p} \leq \frac{1}{2} - \frac{s}{d}$. Then for $R > 0$,*

$$\int_{\mathbb{R}^d \setminus B_R(0)} |\varphi|^p \leq C(d,s,p) R^{d-p\left(\frac{d}{2}-s\right)} \|\varphi\|^p_{\dot{H}^s_{rad}(\mathbb{R}^d)} \qquad \forall \varphi \in \dot{H}^s_{\mathrm{rad}}(\mathbb{R}^d). \tag{4.9}$$

Proof. Follows from Rubin's inequality (4.6) by setting $r = p$ and $\beta = \frac{2d - p(d-2s)}{2p}$. □

Using (4.1), (4.4) and (4.6) in the exterior and the classical Sobolev inequality in the interior of a ball we deduce the following.

Proposition 4.1. *Let $d \geq 2$, $s > 0$, $1 < \alpha < d$ and $\left(\frac{d-2s}{d+\alpha}\right)_+ < \frac{1}{q} \leq 1$. Then the space $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ is continuously embedded into $L^p(\mathbb{R}^d)$ for*

$$p \in \Big(p_{\mathrm{rad}}, \frac{2d}{d-2s}\Big] \quad \textit{and} \quad s < \frac{d}{2}, \tag{4.10}$$

$$p > p_{\mathrm{rad}} \quad \textit{and} \quad s \geq \frac{d}{2}. \tag{4.11}$$

Proof. It is sufficient to establish continuous embedding $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ only for p in a small *right* neighbourhood of p_{rad}, the remaining values of p are then covered by interpolation via Theorem 3.1. Given $R > 0$, we shall estimate the L^p–norm of a function $\varphi \in \mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ separately in the interior and exterior of the ball $B_R(0)$. Since $p < \frac{2d}{d-2s}$, in the interior of the ball $B_R(0)$ we estimate by Sobolev inequality

$$\int_{B_R(0)} |\varphi|^p \leq C R^{1-p\left(\frac{1}{2}-\frac{s}{d}\right)} \|\varphi\|^p_{\dot{H}^s(\mathbb{R}^d)}.$$

The estimate in the exterior of the ball $B_R(0)$ will be split into the cases $s > 1/2$ and $s \leq 1/2$. Observe that $p > p_{\mathrm{rad}} > q$, since $q < \frac{d+\alpha}{d-2s}$. For a small $\varepsilon > 0$, denote

$$\gamma := \frac{d-\alpha}{2} + \varepsilon.$$

Case $s > 1/2$. Using successively the inequalities (4.4), (4.1) and (4.5), we estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^d \setminus B_R(0)} |\varphi|^p \\
&\qquad \le \sup_{|x|>R} \Bigl(|\varphi(x)|\,|x|^{\frac{\gamma}{p-q}}\Bigr)^{p-q} \int_{\mathbb{R}^d \setminus B_R(0)} \frac{|\varphi(x)|^q}{|x|^\gamma} \,\mathrm{d}x \\
&\qquad \lesssim \|\varphi\|_{\dot H^s(\mathbb{R}^d)}^{\theta(p-q)} \Bigl(\int_{\mathbb{R}^d} \frac{|\varphi(x)|^q}{|x|^\gamma} \,\mathrm{d}x\Bigr)^{\frac{(1-\theta)(p-q)}{q}} \int_{\mathbb{R}^d \setminus B_R(0)} \frac{|\varphi(x)|^q}{|x|^\gamma} \,\mathrm{d}x \\
&\qquad \lesssim \|\varphi\|_{\dot H^s(\mathbb{R}^d)}^{\theta(p-q)} \Bigl(\frac{1}{R^{2\varepsilon}} \iint_{\mathbb{R}^N} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x\,\mathrm{d}y\Bigr)^{\frac{1}{2}+\frac{(1-\theta)(p-q)}{2q}} \\
&\qquad\qquad + \|\varphi\|_{\dot H^s(\mathbb{R}^d)}^{p-q} \Bigl(\frac{1}{R^{2\varepsilon}} \iint_{\mathbb{R}^N} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x\,\mathrm{d}y\Bigr)^{\frac{1}{2}} \Bigl(\int_{B_R(0)} |x|^{-\frac{d-2s}{2}q-\gamma} \,\mathrm{d}x\Bigr)^{\frac{(1-\theta)(p-q)}{q}},
\end{aligned}
\tag{4.12}
$$

where $\theta = \frac{2}{(2s-1)q+2}$. The application of (4.4) requires that

$$
\frac{\gamma}{p-q} \le \sigma = \frac{2s(d-1-\gamma)+\gamma}{(2s-1)q+2}, \tag{4.13}
$$

which is fulfilled for a sufficiently small $\varepsilon > 0$ if $p > p_{\mathrm{rad}}$. The last integral in (4.12) is finite when

$$
-\frac{d-2s}{2}q - \gamma < -d; \tag{4.14}
$$

this is the case for a sufficiently small $\varepsilon > 0$ when $q < \frac{d+\alpha}{d-2s}$.

Case $s \le 1/2$. Let $r > p > q$ and $\theta \in [0,1]$ be such that $\frac{\theta}{q} + \frac{1-\theta}{r} = \frac{1}{p}$, i.e. $\theta = \frac{q}{p}\frac{r-p}{r-q}$. By the Hölder inequality together with (4.1) and (4.6), we estimate

$$
\begin{aligned}
\int_{\mathbb{R}^d \setminus B_R(0)} |\varphi|^p &\le \Bigl(\int_{\mathbb{R}^d \setminus B_R(0)} |\varphi(x)|^r |x|^{\gamma\frac{r-p}{p-q}} \,\mathrm{d}x\Bigr)^{\frac{p-q}{r-q}} \Bigl(\int_{\mathbb{R}^d \setminus B_R(0)} \frac{|\varphi(x)|^q}{|x|^\gamma} \,\mathrm{d}x\Bigr)^{\frac{r-p}{r-q}} \\
&\lesssim \Bigl(\int_{\mathbb{R}^d \setminus B_R(0)} |\varphi(x)|^r |x|^{-r\beta} \,\mathrm{d}x\Bigr)^{\frac{p-q}{r-q}} \Bigl(\frac{1}{R^{2\varepsilon}} \iint_{\mathbb{R}^N} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x\,\mathrm{d}y\Bigr)^{\frac{1}{2}\frac{r-p}{r-q}} \\
&\lesssim \|\varphi\|_{\dot H^s(\mathbb{R}^d)}^{r\frac{p-q}{r-q}} \Bigl(\frac{1}{R^{2\varepsilon}} \iint_{\mathbb{R}^N} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x\,\mathrm{d}y\Bigr)^{\frac{1}{2}\frac{r-p}{r-q}},
\end{aligned}
\tag{4.15}
$$

where in view of (4.8) we must express r and β as

$$
r = \frac{2(\gamma p - d(p-q))}{2\gamma - (d-2s)(p-q)}, \qquad \beta = \frac{1}{2}\frac{\gamma(2d - p(d-2s))}{\gamma p - d(p-q)}.
$$

Note that $\beta < 0$ for sufficiently small $\varepsilon > 0$, since $q < \frac{d+\alpha}{d-2s}$ and $p < \frac{2d}{d-2s}$. Hence (4.7) requires

$$
\beta \ge -\frac{d-1}{2}\frac{\gamma(p-2) - 2s(p-q)}{\gamma p - d(p-q)}.
$$

The latter is satisfied provided that

$$p \geq p_\varepsilon := 2\frac{qs(d-1)+\gamma}{2s(d-1)+\gamma(1-2s)}, \tag{4.16}$$

where $p_\varepsilon \searrow p_{\mathrm{rad}}$ as $\varepsilon \to 0$. In addition, observe that $r \nearrow \frac{2}{1-2s}$ as $p = p_\varepsilon$ and $\varepsilon \to 0$, which in particular, ensures that we can choose $r > p$ and $r > 2$ in (4.6). We conclude that (4.15) holds for $p > p_{\mathrm{rad}}$, provided that $\varepsilon > 0$ is sufficiently small. □

Proposition 4.2. *Let $d \geq 2$, $0 < s < \frac{d}{2}$, $1 < \alpha < d$ and $\frac{d+\alpha}{d-2s} < q < \infty$. Then the space $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ is continuously embedded into $L^p(\mathbb{R}^d)$ for*

$$p \in \Big[\frac{2d}{d-2s}, p_{\mathrm{rad}}\Big) \quad \textit{and} \quad \frac{1}{q} \neq \frac{1}{2} - s, \tag{4.17}$$

$$p \in \Big[\frac{2d}{d-2s}, p_{\mathrm{rad}}\Big] \quad \textit{and} \quad \frac{1}{q} = \frac{1}{2} - s. \tag{4.18}$$

Proof. Note that for $\frac{1}{q} \neq \frac{1}{2} - s$ it is sufficient to establish continuous embedding $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$ only for p in a small *left* neighbourhood of p_{rad}, the remaining values of p are then covered by interpolation via Theorem 3.1.

Given $R > 0$, we shall estimate the L^p–norm of a function $\varphi \in \mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ separately in the interior and exterior of the ball $B_R(0)$. The proof will be splitted into a number of separate cases, which we outline in Table 1.

s	q	$B_R(0)$	$\mathbb{R}^d \setminus B_R(0)$
$s > 1/2$	$q > \frac{d+\alpha}{d-2s}$	De Napoli + Ruiz as in (4.12)	Sobolev + Cho-Ozawa (4.5)
$s \leq 1/2$	$\frac{d+\alpha}{d-2s} < q < \frac{2}{1-2s}$	Rubin + Ruiz as in (4.15)	Weak Ni (4.9)
	$q = \frac{2}{1-2s}$	L^q-estimate (1.1)	Weak Ni (4.9)
	$q > \frac{2}{1-2s}$	L^q-estimate (1.1)	Rubin + Ruiz as in (4.15)

Table 1. Different cases in the proof of Proposition 4.2

Case $s > 1/2$. In the exterior of the ball $B_R(0)$, for any $p > \frac{2d}{d-2s}$ we can estimate

$$\int_{\mathbb{R}^d \setminus B_R(0)} |\varphi|^p \leq C R^{d-p\left(\frac{d}{2}-s\right)} \|\varphi\|^p_{\dot H^s(\mathbb{R}^d)}, \tag{4.19}$$

using the classical Sobolev inequality and Cho–Ozawa's inequality (4.5). To obtain an estimate in the interior of the ball $B_R(0)$, we observe that for $s > 1/2$ we have $q < p_{\mathrm{rad}}$ and hence we can assume that $q < p < p_{\mathrm{rad}}$. For a small $\varepsilon > 0$, set $\gamma := \frac{d-\alpha}{2} - \varepsilon$. Then the estimate on $\int_{B_R(0)} |\varphi|^p$ is identical to the argument in (4.12), but carried out in the interior of the ball $B_R(0)$, which reverses the inequalities in (4.13) and (4.14).

Case $s \leq 1/2$ and $\frac{d+\alpha}{d-2s} < q < \frac{2}{1-2s}$. In the exterior of the ball $B_R(0)$ the estimate (4.19) follows directly from the weak Ni's inequality (4.9). To obtain an estimate in the interior of the ball $B_R(0)$, observe that for $q < \frac{2}{1-2s}$ we have $q < p_{\mathrm{rad}}$ and hence we can assume that

$q < p < p_{\text{rad}}$. For a small $\varepsilon > 0$, set $\gamma := \frac{d-\alpha}{2} - \varepsilon$. Then the estimate on $\int_{B_R(0)} |\varphi|^p$ is identical to the argument in (4.15), but carried out in the interior of the ball $B_R(0)$ with $q < p < r$. The only difference is that for $q > \frac{d+\alpha}{d-2s}$ the inequality in (4.16) reverses and that $p_\varepsilon \nearrow p_{\text{rad}}$ as $\varepsilon \to 0$, since $\gamma < \frac{d-\alpha}{2}$. Note that for $0 < s < 1/2$ and $q \geq \frac{2}{1-2s}$ we have $q \geq p_{\text{rad}}$ and a Hölder inequality estimate of type (4.15) on $\int_{B_R(0)} |\varphi|^p$ is no longer possible.

Case $s < 1/2$ and $q = \frac{2}{1-2s}$. Observe that in this case we have $p_{\text{rad}} = q$. In the exterior of the ball $B_R(0)$ the estimate

$$\int_{\mathbb{R}^d \setminus B_R(0)} |\varphi|^q \leq C R^{d - q(\frac{d}{2} - s)} \|\varphi\|^q_{\dot{H}^s(\mathbb{R}^d)}, \tag{4.20}$$

follows directly from the weak Ni's inequality (4.9), which is valid for $q = \frac{2}{1-2s}$. To estimate $\int_{B_R(0)} |\varphi|^q$, we can use the L^q–estimate (1.1), i.e.

$$\int_{B_R(0)} |\varphi|^q \leq C R^{\frac{d-\alpha}{2}} \Bigg(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\varphi(x)|^q \, |\varphi(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x \,\mathrm{d}y \Bigg)^{\frac{1}{2}}. \tag{4.21}$$

Combining (4.20) and (4.21) together we conclude that $\mathcal{E}^{s,\alpha,q}_{\text{rad}}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d)$, the remaining range of p follows by interpolation.

Case $s < 1/2$ and $q > \frac{2}{1-2s}$. Observe that in this case $p < p_{\text{rad}} < q$. To estimate $\int_{B_R(0)} |\varphi|^p$, we use the L^q–estimate (1.1) to obtain

$$\int_{B_R(0)} |\varphi|^p \leq C R^{\left(1 - \frac{p}{q}\right) \frac{d-\alpha}{2}} \Bigg(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\varphi(x)|^q \, |\varphi(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x \,\mathrm{d}y \Bigg)^{\frac{p}{2q}}.$$

To obtain an estimate in the exteriour of the ball $B_R(0)$, we will use Hölder, Rubin and Ruiz's inequalities similarly to (4.15), with $\gamma = \frac{d-\alpha}{2} + \varepsilon$ and $r < p < q$, which could be carried out for $p < p_{\text{rad}}$ provided that $\varepsilon > 0$ is sufficiently small, because $p_{\text{rad}} > \frac{2}{1-2s}$. ☐

Proof of Theorem 1.4. The scaling invariant inequalities of Theorem 1.4 follow from Propositions 4.1 and 4.2 by by the same scaling consideration as in the proof of Theorem 1.1. ☐

The estimates of Propositions 4.1 and 4.2 improve upon the estimate of Theorem 3.1 only when $\alpha > 1$. In the next section we show that the intervals of Propositions 4.1 and 4.2 are optimal and that for $\alpha \leq 1$ there is no improvement for the radial embedding.

5. Optimality of the radial embeddings

The optimality of the intervals in Theorems 1.1 and 1.4 for $s \leq 1$ is a consequence of the following.

Theorem 5.1. *Let $d \geq 2$, $1 < \alpha < d$, $0 < s < 1/2$ and $q = \frac{2}{1-2s}$. Then the space $\mathcal{E}^{s,\alpha,q}_{\text{rad}}(\mathbb{R}^d)$ is not continuously embedded into $L^p(\mathbb{R}^d)$ for $p > q = p_{\text{rad}}$.*

Theorem 5.2. *Let $d \geq 2$, $1 < \alpha < d$, $0 < s \leq 1$ and $p, q \in [1, +\infty)$. Then the space $\mathcal{E}^{s,\alpha,q}_{\text{rad}}(\mathbb{R}^d)$ is not continuously embedded in $L^p(\mathbb{R}^d)$ for*

$$p \leq p_{\text{rad}} \quad \text{and} \quad \frac{1}{q} > \frac{d-2s}{d+\alpha}, \tag{5.1}$$

$$p \geq p_{\mathrm{rad}} \quad \text{and} \quad \frac{1}{q} < \frac{d-2s}{d+\alpha},\ \frac{1}{q} \neq \frac{1-2s}{2}. \tag{5.2}$$

Theorem 5.3. *Let $d \geq 2$, $0 < \alpha \leq 1$, $0 < s \leq 1$ and $p, q \in [1, +\infty)$. Then the space $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ is not continuously embedded in $L^p(\mathbb{R}^d)$ for*

$$p < \frac{2(2qs+\alpha)}{2s+\alpha} \quad \text{and} \quad \frac{1}{q} > \frac{d-2s}{d+\alpha}, \tag{5.3}$$

$$p > \frac{2(2qs+\alpha)}{2s+\alpha} \quad \text{and} \quad \frac{1}{q} < \frac{d-2s}{d+\alpha}. \tag{5.4}$$

The proof of Theorems 5.2 and 5.3 is obtained by constructing counterexamples, i.e a family of functions u such that for a suitable p it holds

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^d)} \simeq 1$$
$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^q\,|u(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \simeq 1$$
$$||u||^p_{L^p(\mathbb{R}^d)} \to +\infty.$$

Given a nonnegative function $\eta \in C^\infty(\mathbb{R}) \setminus \{0\}$ such that $\operatorname{supp}\eta \subset [-1,1]$, we consider the family of functions

$$u_{\lambda,R,S}(x) = \lambda\,\eta\Big(\frac{|x|-R}{S}\Big), \tag{5.5}$$

where $R > S > 0$ and $\lambda > 0$ will be specified in the sequel.

By elementary computation we obtain

$$\|u_{\lambda,R,S}\|^p_p \simeq \lambda^p R^{d-1} S. \tag{5.6}$$

We also claim that

$$\|u_{\lambda,R,S}\|^2_{\dot{H}^s(\mathbb{R}^d)} \simeq \lambda^2 R^{d-1} S^{1-2s}, \tag{5.7}$$

and

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u_{\lambda,R,S}(x)|^q\,|u_{\lambda,R,S}(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \lesssim \begin{cases} \lambda^{2q} R^{d+\alpha-2} S^2 & \text{if } 1 < \alpha < d, \\ \lambda^{2q} R^{d-1} S^2 \log(R/S) & \text{if } \alpha = 1, \\ \lambda^{2q} R^{d-1} S^{1+\alpha} & \text{if } 0 < \alpha < 1. \end{cases} \tag{5.8}$$

The estimate (5.8) is proved in Appendix A below.

To prove (5.7), for any $s > 0$ choose $k \in \mathbb{N}$ such that $2k \geq s$. Taking into account that $S < R$, by the change of variables and scaling we compute

$$\begin{aligned} \|u_{\lambda,R,S}\|^2_{\dot{H}^{2k}(\mathbb{R}^d)} &\simeq \int_{\mathbb{R}^d} |\Delta^k u_{\lambda,R,S}|^2\,\mathrm{d}x \simeq \int_0^\infty \left|\Big\{\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r}\Big\}^k u_{\lambda,R,S}(r)\right|^2 r^{d-1}\,\mathrm{d}r \\ &= \int_0^\infty \left|\Big\{\frac{\partial^{2k}}{\partial r^{2k}} + \frac{a_1}{r}\frac{\partial^{2k-1}}{\partial r^{2k-1}} + \cdots + \frac{a_k}{r^k}\frac{\partial^k}{\partial r^k}\Big\} u_{\lambda,R,S}(r)\right|^2 r^{d-1} dr \\ &\leq \lambda^2 d\Big(\int_0^\infty \Big|\eta^{(2k)}\Big(\tfrac{r-R}{S}\Big)\Big|^2 r^{d-1} dr + |a_1| \int_0^\infty \eta^{(2k-1)}\Big|\Big(\tfrac{r-R}{S}\Big)\Big|^2 r^{d-3} dr \end{aligned} \tag{5.9}$$

$$+ \dots + |a_d| \int_0^\infty \eta^{(k)} \Big|\Big(\tfrac{r-R}{S}\Big)\Big|^2 r^{d-1-2k} dr\Big)$$
$$\lesssim \lambda^2 \Big(S^{1-4k} R^{d-1} + S^{1-2(2k-1)} R^{d-3} + \dots + S^{1-2k} R^{d-1-k}\Big)$$
$$\lesssim \lambda^2 S^{1-4k} R^{d-1}.$$

Interpolating between the L^2 and $\dot H^{2k}$–norm of $u_{\lambda,R,S}$ (cf. [1, Proposition 1.32]), we conclude from (5.6) and (5.9) that

$$\|u_{\lambda,R,S}\|^2_{\dot H^s(\mathbb{R}^d)} \le \|u_{\lambda,R,S}\|^{\frac{s}{k}}_{\dot H^{2k}(\mathbb{R}^d)} \|u_{\lambda,R,S}\|^{2-\frac{s}{k}}_{L^2(\mathbb{R}^d)} \lesssim \lambda^2 R^{d-1} S^{1-2s}.$$

Proof of Theorem 5.1. Let $u_S := u_{\lambda,R,S}$ be the function in (5.5), where we fix $R > 0$ and for $S < R$ set

$$\lambda = S^{-\frac{1}{q}}.$$

Then, since by our assumption $1 < \alpha < d$,

$$\|u_S\|^2_{\dot H^s(\mathbb{R}^d)} \lesssim R^{d-1}, \tag{5.10}$$

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u_S(x)|^q\,|u_S(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \lesssim R^{d+\alpha-2}, \tag{5.11}$$

$$\|u_S\|^p_{L^p(\mathbb{R}^d)} \simeq \lambda^p S R^{d-1} \simeq \lambda^{p-q} R^{d-1} \simeq S^{1-\frac{p}{q}} R^{d-1}, \tag{5.12}$$

Since R is fixed, we conclude that $\|u_S\|_{L^p(\mathbb{R}^d)} \to \infty$ for $p > q$ when $S \to 0$. □

Proof of Theorem 5.2. Let $u_R :=_{\lambda,R,S}$ be the function in (5.5), where we set

$$\lambda = R^\beta \quad \text{and} \quad S = (\lambda^2 R^{d-1})^{\frac{1}{2s-1}} = R^\gamma,$$

with

$$\beta = -\frac{2(d-1) + (d+\alpha-2)(2s-1)}{2q(2s-1)+4}, \qquad \gamma = \frac{q(d-1) - (d+\alpha-2)}{q(2s-1)+2}. \tag{5.13}$$

Then we compute

$$\|u_R\|^2_{\dot H^s(\mathbb{R}^d)} \lesssim 1, \tag{5.14}$$

$$\iint\limits_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u_R(x)|^q\,|u_R(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \lesssim 1, \tag{5.15}$$

$$\|u_R\|^p_{L^p(\mathbb{R}^d)} \simeq \lambda^p R^{d-1} S \simeq R^{\beta(p-p_{\mathrm{rad}})}, \tag{5.16}$$

provided that $R > S$, that is, either $R > 1$ and $\gamma < 1$ or $R < 1$ and $\gamma > 1$. To complete the proof Theorem 5.2 for $p \ne p_{\mathrm{rad}}$ we select R according to Table 2.

Next we prove that $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d) \not\subset L^{p_{\mathrm{rad}}}(\mathbb{R}^d)$ when $\frac{1}{q} \ne \frac{1-2s}{2}$. Similarly to [21, Lemma 6.4], we consider the "multibump" sequence

$$v_{R,m} = \sum_{k=1}^m u_{R^k},$$

q	β	γ	Choice of R	Conclusion
$\frac{1}{q} > \frac{d-2s}{\alpha+d}$	$\beta < 0$	$0 < \gamma < 1$	$R \to \infty$	$\Vert u_R\Vert^p_{L^p(\mathbb{R}^d)} \to \infty$ for $p < p_{\mathrm{rad}}$
$\frac{1}{q} \in \left(\left(\frac{1-2s}{2}\right)_+, \frac{d-2s}{\alpha+d}\right)$	$\beta < 0$	$\gamma > 1$	$R \to 0$	$\Vert u_R\Vert^p_{L^p(\mathbb{R}^d)} \to \infty$ for $p > p_{\mathrm{rad}}$
$s < 1/2$ and $\frac{1}{q} < \frac{1-2s}{2}$	$\beta > 0$	$\gamma < 0$	$R \to \infty$	$\Vert u_R\Vert^p_{L^p(\mathbb{R}^d)} \to \infty$ for $p > p_{\mathrm{rad}}$

Table 2. Choice of R which ensures $R > S$ and $\Vert u_R\Vert^p_{L^p(\mathbb{R}^d)} \to \infty$ for $\alpha > 1$.

where the functions u_{R^k} are as in (5.5) with $R = R^k$, $\lambda = R^{k\beta}$, $S = R^{k\frac{2\beta+d-1}{2s-1}}$ and where β is given in (5.13). Note that for $R \neq 1$ and sufficiently large quotient R/S the functions u_{R^k} $(k = 1, \dots, m)$ have mutually disjoint supports.

If $\frac{1}{q} > \frac{d-2s}{\alpha+d}$, or $s < 1/2$ and $\frac{1}{q} < \frac{1-2s}{2}$ then we let $R \to \infty$. We obtain

$$\Vert v_{R,m}\Vert^p_{L^p(\mathbb{R}^d)} \simeq m, \tag{5.17}$$

$$\Vert v_{R,m}\Vert^2_{\dot H^s(\mathbb{R}^d)} \lesssim m, \tag{5.18}$$

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|v_{R,m}(x)|^q\,|v_{R,m}(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \lesssim m. \tag{5.19}$$

For derivation of (5.19) see [21, proof of Lemma 6.4]. To obtain (5.18), we observe that

$$\Vert v_{R,m}\Vert^2_{\dot H^s(\mathbb{R}^d)} = \sum_{k=1}^m \Vert u_{R^k}\Vert^2_{\dot H^s(\mathbb{R}^d)} + 2\sum_{i,j=1, i>j}^m (u_{R^i}, u_{R^j})_{\dot H^s(\mathbb{R}^d)}. \tag{5.20}$$

If s is an integer the second term vanishes, or if $s < 1$ then the second term is negative. Otherwise, $s = \ell + \sigma$, with $\ell \in \mathbb{N}$ and $\sigma \in (0, 1)$. Thus by the Gagliardo seminorm characterization of $\dot H^s(\mathbb{R}^d)$, if u_{R^i} and u_{R^j} have disjoint supports,

$$\begin{aligned}(u_{R^i}, u_{R^j})_{\dot H^s(\mathbb{R}^d)} &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(\nabla^\ell u_{R^i}(x) - \nabla^\ell u_{R^i}(y))\cdot(\nabla^\ell u_{R^j}(x) - \nabla^\ell u_{R^j}(y)}{|x-y|^{d+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y \\ &= -2C \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\nabla^\ell u_{R^i}(x)\cdot\nabla^\ell u_{R^j}(y)}{|x-y|^{d+2\sigma}}\,\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{5.21}$$

Similarly to (5.9), we deduce that $\Vert D^\ell u_{\lambda,R,S}\Vert_{L^1(\mathbb{R}^d)} \lesssim \lambda R^{d-1} S^{1-\ell}$ and hence

$$\Vert D^\ell u_{R^k}\Vert_{L^1(\mathbb{R}^d)} \lesssim R^{k(\beta+d-1+\gamma(1-\ell))}. \tag{5.22}$$

If $\frac{1}{q} > \frac{d-2s}{\alpha+d}$ then $\beta < 0$ and $0 < \gamma < 1$. For $i > j$ and if $R^i \gg R^j$ we estimate (5.21) as follows,

$$\begin{aligned}(u_{R^i}, u_{R^j})_{\dot H^s(\mathbb{R}^d)} &\lesssim \frac{\Vert D^\ell u_{R^i}\Vert_{L^1(\mathbb{R}^d)}\Vert D^\ell u_{R^j}\Vert_{L^1(\mathbb{R}^d)}}{(R^i - R^j)^{d+2\sigma}} \\ &\lesssim R^{-i(d+2\sigma)} R^{(i+j)(\beta+d-1+\gamma(1-\ell))} \\ &\lesssim R^{-i(d+2\sigma)} R^{i(2(\gamma s-\beta)+2\sigma\gamma)} \lesssim R^{-i(2\sigma(1-\gamma))},\end{aligned} \tag{5.23}$$

since we note that $2(\gamma s - \beta) < d$, provided that $q < \frac{d+\alpha}{d-2s}$. Then in (5.20) for all sufficiently large R we have

$$\|v_{R,m}\|^2_{\dot{H}^s(\mathbb{R}^d)} \lesssim m + \sum_{i,j=1, i>j}^{m} R^{-i(2\sigma(1-\gamma))} \lesssim m. \tag{5.24}$$

The case $\frac{1}{q} \in \Big(\big(\frac{1-2s}{2}\big)_+, \frac{d-2s}{\alpha+d}\Big)$ is similar, but letting $R \to 0$ and observing that $\gamma < 0$.

Now, set

$$w_{R,m}(x) = m^\theta v_{R,m}\Big(\frac{x}{m^\sigma}\Big).$$

Then by the standard scaling we have

$$\|w_{R,m}\|^p_{L^p(\mathbb{R}^d)} \simeq m^{p\theta + \sigma d + 1}, \tag{5.25}$$

$$\|w_{R,m}\|^2_{\dot{H}^s(\mathbb{R}^d)} \lesssim m^{2\theta + \sigma(d-2s)+1}, \tag{5.26}$$

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|w_{R,m}(x)|^q\, |w_{R,m}(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x\,\mathrm{d}y \lesssim m^{2q\theta + \sigma(d+\alpha)+1}. \tag{5.27}$$

If we set

$$\sigma = \frac{q-1}{d+\alpha - q(d-2s)}, \qquad \theta = -\frac{2s+\alpha}{2(d+\alpha-q(d-2s))},$$

then for $R \to \infty$ and $m \to \infty$ we obtain

$$\|w_{R,m}\|^2_{\dot{H}^s(\mathbb{R}^d)} \lesssim 1, \tag{5.28}$$

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|w_{R,m}(x)|^q\, |w_{R,m}(y)|^q}{|x-y|^{d-\alpha}} \,\mathrm{d}x\,\mathrm{d}y \lesssim 1, \tag{5.29}$$

$$\|w_{R,m}\|^p_{L^p(\mathbb{R}^d)} \simeq m^{p\theta+\sigma d+1} \simeq m^{\frac{2s(\alpha-1)}{2s(d+\alpha-2)+d-\alpha}} \to \infty, \tag{5.30}$$

since $\alpha > 1$ and $d \geq 2$.

The case $\frac{1}{q} \in \Big(\big(\frac{1-2s}{2}\big)_+, \frac{d-2s}{\alpha+d}\Big)$ is similar, by letting $R \to 0$. □

Proof of Theorem 5.3. The strategy in the case $0 < \alpha < 1$ and $\frac{1}{q} \neq \frac{1-2s}{1+\alpha}$ is the same as in the first part of the proof of Theorem 5.2. Let $u_R := u_{\lambda,R,S}$ be the function in (5.5) and we choose

$$\lambda = R^\beta, \qquad S = \big(\lambda^2 R^{d-1}\big)^{\frac{1}{2s-1}} = R^\gamma,$$

where

$$\beta = -\frac{(d-1)(2s+\alpha)}{2(q(2s-1)+1+\alpha)}, \qquad \gamma = \frac{(d-1)(q-1)}{q(2s-1)+1+\alpha}.$$

Then (5.14) and (5.15) hold, and

$$\|u_R\|^p_{L^p(\mathbb{R}^d)} \simeq \lambda^p R^{d-1} S \simeq R^{\beta(p - \frac{2(2qs+\alpha)}{2s+\alpha})},$$

provided that $R > S$. Then to construct the required counterexamples, we select R according to Table 3.

In the case $0 < \alpha < 1$, $s < 1/2$ and $q = \frac{1+\alpha}{1-2s}$ we note that $\frac{2(2qs+\alpha)}{2s+\alpha} = \frac{2}{1-2s} > q$. Similarly to the proof of Theorem 5.1, for $u_S := u_{\lambda,R,S}$ with a fixed $R > 0$ and for $S < R$ we set

$$\lambda = S^{-\frac{\alpha+2s}{2(q-1)}} = S^{\frac{2s-1}{2}}.$$

q	β	γ	Choice of R	Conclusion
$\frac{1}{q} > \frac{d-2s}{\alpha+d}$	$\beta < 0$	$0 < \gamma < 1$	$R \to \infty$	$\\|u_R\\|^p_{L^p(\mathbb{R}^d)} \to \infty$ for $p < \frac{2(2qs+\alpha)}{2s+\alpha}$
$\frac{1}{q} \in \left((\frac{1-2s}{1+\alpha})_+, \frac{d-2s}{\alpha+d} \right)$	$\beta < 0$	$\gamma > 1$	$R \to 0$	$\\|u_R\\|^p_{L^p(\mathbb{R}^d)} \to \infty$ for $p > \frac{2(2qs+\alpha)}{2s+\alpha}$
$s < 1/2$ and $\frac{1}{q} < \frac{1-2s}{1+\alpha}$	$\beta > 0$	$\gamma < 0$	$R \to \infty$	$\\|u_R\\|^p_{L^p(\mathbb{R}^d)} \to \infty$ for $p > \frac{2(2qs+\alpha)}{2s+\alpha}$

TABLE 3. Choice of R which ensures $R > S$ and $\|u_R\|^p_{L^p(\mathbb{R}^d)} \to \infty$ for $\alpha \leq 1$.

Then

$$\|u_S\|^2_{\dot{H}^s(\mathbb{R}^d)} \simeq R^{d-1}, \tag{5.31}$$

$$\iint\limits_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u_S(x)|^q \, |u_S(y)|^q}{|x-y|^{d-\alpha}} \, \mathrm{d}x \, \mathrm{d}y \lesssim R^{d-1}, \tag{5.32}$$

$$\|u_S\|^p_{L^p(\mathbb{R}^d)} \simeq \lambda^p S R^{d-1} \simeq \lambda^{p-\frac{2}{2s-1}} R^{d-1} \simeq S^{1-\frac{p(1-2s)}{2}} R^{d-1}. \tag{5.33}$$

Since R is fixed, we conclude that $\|u_S\|_{L^p(\mathbb{R}^d)} \to \infty$ for $p > \frac{2(2qs+\alpha)}{2s+\alpha} = \frac{2}{1-2s}$ when $S \to 0$.

The case $\alpha = 1$ is similar, but takes into account the logarithmic correction in (5.8). We omit the details. □

6. Radial compactness: Proof of Theorem 1.5

We need the following preliminary local compactness result.

Lemma 6.1 (Local compactness). *Let $d \in \mathbb{N}$, $s > 0$, $\alpha \in (0, d)$ and $q \in [1, \infty)$. Then the embedding $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d) \hookrightarrow L^1_{loc}(\mathbb{R}^d)$ is compact.*

Proof. Multiplication by $\theta \in \mathcal{S}(\mathbb{R}^d)$ is a continuous mapping $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d) \to \dot{H}^s(\mathbb{R}^d)$. Indeed by the fractional Leibniz rule, see e.g. [13, Theorem 1.4], we obtain

$$\|(-\Delta)^{\frac{s}{2}} \theta u\|_{L^2(\mathbb{R}^d)} \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^d)} \|\theta\|_{L^\infty(\mathbb{R}^d)} + \|(-\Delta)^{\frac{s}{2}} \theta\|_{L^r(\mathbb{R}^d)} \|u\|_{L^{\frac{2(2qs+\alpha)}{2s+\alpha}}(\mathbb{R}^d)},$$

with r such that $\frac{2s+\alpha}{2(2qs+\alpha)} + \frac{1}{r} = \frac{1}{2}$. For $q = 1$, we set $r = \infty$. Hence by Theorem 3.1,

$$\|\theta u\|_{\dot{H}^s(\mathbb{R}^d)} \leq C(\theta) \|u\|_{\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)}.$$

For every $\rho > 0$, we choose $\theta \in C^\infty(\mathbb{R}^d)$ such that $\theta = 1$ on B_ρ and $\theta = 0$ in $\mathbb{R}^d \setminus B_{2\rho}$. Let $(u_n)_{n \in \mathbb{N}}$ be a bounded sequence in $\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)$. Setting $v_n = \theta u_n$, theorem 3.1 implies that $(v_n)_{n \in \mathbb{N}}$ is also bounded in $H^s(\mathbb{R}^d)$. We can assume that v_n converges weakly to some v in $L^2(\mathbb{R}^d)$. By testing against suitable test functions, it follows that v is also supported in $B_{2\rho}$ and thus $\hat{v} \in L^\infty(\mathbb{R}^d)$. By Plancharel's identity we have

$$\|v_n - v\|^2_{L^2(\mathbb{R}^d)} = \int_{|\xi| \leq R} |\widehat{v}_n(\xi) - \widehat{v}(\xi)|^2 \, \mathrm{d}\xi + \int_{|\xi| > R} |\widehat{v}_n(\xi) - \widehat{v}(\xi)|^2 \, \mathrm{d}\xi.$$

By showing that the right hand side goes to zero we will infer by Hölder's inequality that $\|u_n - v\|_{L^1(B_\rho)} \to 0$. We have

$$\int_{|\xi|>R} |\widehat{v}_n(\xi) - \widehat{v}(\xi)|^2 \,\mathrm{d}\xi \le \frac{1}{(1+R^2)^s} \int_{\mathbb{R}^d} (1+|\xi|^2)^s |\widehat{v}_n(\xi) - \widehat{v}(\xi)|^2 \,\mathrm{d}\xi \le \frac{C}{(1+R^2)^s}.$$

Since $e^{ix\cdot\xi} \in L^2_x(B_{2\rho})$, by weak convergence in $L^2(B_{2\rho})$ we have $\widehat{v}_n(\xi) \to \widehat{v}(\xi)$ almost everywhere. To conclude it suffices to show that

$$\int_{|\xi|\le R} |\widehat{v}_n(\xi) - \widehat{v}(\xi)|^2 \,\mathrm{d}\xi = o(1). \tag{6.1}$$

Notice that $\|\widehat{v}_n\|_\infty \le \|v_n\|_{L^1(B_{2\rho})} \le \mu(B_{2\rho})^{\frac{1}{2}} \|v_n\|_{L^2(B_{2\rho})} \le \mu(B_{2\rho})^{\frac{1}{2}} \|v_n\|_{H^s(\mathbb{R}^d)}$ and hence $|\widehat{v}_n(\xi) - \widehat{v}(\xi)|^2$ is estimated by a uniform constant so that by Lebesgue's dominated convergence theorem (6.1) holds. This concludes the proof. □

Proof of theorem 1.5 . We sketch the proof only in the most interesting case $\alpha > 1$, $s < 1/2$, and $q \ge \frac{2}{1-2s}$, namely when $p_{\mathrm{rad}} \le q$. Notice that for all $R > 0$, by (1.1) and Lemma 6.1, interpolation between q and $p' = 1$ yields the compact embedding $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d) \hookrightarrow L^p_{loc}(\mathbb{R}^d)$ for all $1 \le p < q$. Thus it suffices to show that for any bounded sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{E}^{s,\alpha,q}_{\mathrm{rad}}(\mathbb{R}^d)$ there holds

$$\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d\setminus B_R(0)} |u_n|^p \to 0, \quad R \to \infty.$$

When $p \le \frac{2}{1-2s}$, we use Lemma 4.1 which yields

$$\int_{\mathbb{R}^d\setminus B_R(0)} |u_n|^p \le o(1) \|u_n\|^p_{\mathcal{E}^{s,\alpha,q}(\mathbb{R}^d)}, \qquad R \to \infty.$$

When $p > \frac{2}{1-2s}$ the same conclusion holds by arguing as in the proof of (4.15) and using the strict inequality $p < p_{\mathrm{rad}}$. This is enough to prove the theorem for $\alpha > 1$, $s < 1/2$, and $q \ge \frac{2}{1-2s}$.

The other cases are similar, estimating the various integrals as in Proposition 4.1 for $q < \frac{d+\alpha}{d-2s}$ and according to Table 1 for $q > \frac{d+\alpha}{d-2s}$. This concludes the proof. □

Appendix A. Proof of claim (5.8)

Proof of (5.8). We use an estimate for radial functions from [21]. Similar estimates were previously obtained in [10, 23, 26].

Lemma A.1 ([21, Lemma 6.3]). *Let $d \ge 2$ and $\alpha \in (0, d)$, then for every measurable function $f : [0,\infty) \to [0,\infty)$*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f(|x|)f(|y|)}{|x-y|^{d-\alpha}} \,\mathrm{d}x \,\mathrm{d}y = \int_0^\infty \int_0^\infty f(r) K^R_{\alpha,d}(r,s) f(s) r^{d-1} s^{d-1} \,\mathrm{d}r \,\mathrm{d}s$$

where the kernel $K^R_{\alpha,d} : [0,\infty) \times [0,\infty) \to \infty$ is defined for $r, s \in [0,\infty) \times [0,\infty)$ by

$$K^R_{\alpha,d}(r,s) = C_d \int_0^1 \frac{z^{\frac{d-3}{2}}(1-z)^{\frac{d-3}{2}}}{((s+r)^2 - 4srz)^{\frac{d-\alpha}{2}}} \,\mathrm{d}z.$$

Moreover, there exists $M > 0$ such that

$$K^R_{\alpha,d}(r,s) \le M \begin{cases} (\frac{1}{rs})^{\frac{d-1}{2}} \frac{1}{|r-s|^{1-\alpha}} & \text{if } \alpha < 1, \\ (\frac{1}{rs})^{\frac{d-1}{2}} \ln \frac{2|r+s|}{|r-s|} & \text{if } \alpha = 1, \\ (\frac{1}{rs})^{\frac{d-\alpha}{2}} & \text{if } \alpha > 1. \end{cases} \tag{A.1}$$

Case $\alpha > 1$. From (A.1) we obtain for radially symmetric functions that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\int_0^\infty\int_0^\infty \frac{|\varphi(r)|^q\,|\varphi(s)\,|^q r^{d-1}s^{d-1}}{(rs)^{\frac{d-\alpha}{2}}}\,\mathrm{d}r\,\mathrm{d}s,$$

and hence that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\left(\int_0^\infty |\varphi(r)|^q\, r^{\frac{d}{2}+\frac{\alpha}{2}-1}\,\mathrm{d}r\right)^2.$$

Let $u = u_{\lambda,R,S}$ be defined in (5.5). Then

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^q\,|u(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\lambda^{2q}\left(\int_{R-S}^{R+S}\left(\frac{S-|r-R|}{S}\right)^q r^{\frac{d}{2}+\frac{\alpha}{2}-1}\,\mathrm{d}r\right)^2.$$

Using the trivial estimate $\frac{S-\left|r-R\right|}{S} < 1$ it follows that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^q\,|u(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\lambda^{2q}\left((R+S)^{\frac{d}{2}+\frac{\alpha}{2}} - (R-S)^{\frac{d}{2}+\frac{\alpha}{2}}\right)^2$$

and we get the desired estimate.

Case $\alpha = 1$. From (A.1) we obtain for radially symmetric functions that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\int_0^\infty\int_0^\infty \frac{|\varphi(r)|^q|\varphi(s)|^q r^{d-1}s^{d-1}}{(rs)^{\frac{d-1}{2}}}\ln\frac{2|r+s|}{|r-s|}\,\mathrm{d}r\,\mathrm{d}s,$$

and hence that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|\varphi(x)|^q\,|\varphi(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\int_0^\infty\int_0^\infty |\varphi(r)|^q|\varphi(s)|^q r^{\frac{d}{2}-\frac{1}{2}}s^{\frac{d}{2}-\frac{1}{2}}\ln\frac{2|r+s|}{|r-s|}\,\mathrm{d}r\,\mathrm{d}s.$$

Let $u = u_{\lambda,R,S}$ be defined in (5.5). Using the estimates $\frac{S-\left|r-R\right|}{S} < 1$ and $r \le R+S$, $s \le R+S$ we have

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^q\,|u(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\lambda^{2q}(R+S)^{d-1}\int_{R-S}^{R+S}\int_{R-S}^{R+S}\ln\frac{2|r+s|}{|r-s|}\,\mathrm{d}r\,\mathrm{d}s$$

and we can conclude that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^q\,|u(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\lambda^{2q}R^{d-1}\int_{R-S}^{R+S}\int_{R-S}^{R+S}\ln\frac{2|r+s|}{|r-s|}\,\mathrm{d}r\,\mathrm{d}s$$

i.e.

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^q\,|u(y)|^q}{|x-y|^{d-\alpha}}\,\mathrm{d}x\,\mathrm{d}y \le C\lambda^{2q}R^{d-1}S^2(\ln R - \ln S + 1) = O(\lambda^{2q}R^{d-1+\beta}S^2).$$

Case $0 < \alpha < 1$. This case is similar to $\alpha = 1$, we omit the details. □

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