Abelian varieties as automorphism groups of smooth projective varieties

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Abstract

We determine which complex abelian varieties can be realized as the automorphism group of a smooth projective variety.

1 Introduction

In this note we determine which complex abelian varieties A can be realized as the automorphism group of a complex smooth projective variety. Given an abelian variety A, we denote by $\mathrm{Aut}_{\mathrm{Grp}}(A)$ (respectively $\mathrm{Aut}(A)$) the automorphism group of A as an algebraic group (respectively as a projective variety). We prove that if $\mathrm{Aut}_{\mathrm{Grp}}(A)$ is infinite then A can never be realized as the automorphism group of a smooth projective variety (Theorem 2.1), whereas if $\mathrm{Aut}_{\mathrm{Grp}}(A)$ is finite there exists a smooth projective variety Y of dimension $2+\dim A$ such that $\mathrm{Aut}(Y) = A$ (Theorem 3.9).

Acknowledgments. This work was motivated by a more general question posed by Michel Brion in Oberwolfach; the specific case of abelian varieties, which we analyse in this paper, was raised by Corrado De Concini. We would like to thank Angelo Vistoli for some very useful correspondence regarding the comparison between étale and analytic cohomology. We also thank Enrico Arbarello and Michel Brion for useful comments on a preliminary version of this paper and for an improvement in the proof of Lemma 3.11.

2 Abelian varieties with infinite automorphism group

In this section we show that no abelian variety with infinite $\mathrm{Aut}_{\mathrm{Grp}}(A)$ can be realized as the automorphism group of a smooth projective variety:

Theorem 2.1. *Let A be an abelian variety such that $\mathrm{Aut}_{\mathrm{Grp}}(A)$, the automorphism group of A as an algebraic group, is infinite. Let X be a smooth projective variety on which A acts faithfully: then the index of A in the automorphism group of X is infinite.*

The proof relies on the following result of Nishi and Matsumura [Mat63], which we quote here in the version due to Brion [Bri10, page 2]:

Theorem 2.2. *Let X be a smooth projective variety on which an abelian variety A acts faithfully. There is a positive integer n and a $A[n]$-invariant closed subscheme Y of X such that there is an A-equivariant isomorphism*

$$X \cong Y \times^{A[n]} A.$$

Proof. (of Theorem 2.1) Let $\iota : A \hookrightarrow \mathrm{Aut}(X)$ be the given action of A on X and write $X \cong Y \times^{A[n]} A$ as in Theorem 2.2. We can represent $X \cong Y \times^{A[n]} A$ more explicitly as the quotient

$$X \cong \frac{Y \times A}{A[n]},$$

where $t \in A[n]$ acts on (y, a) as $t \cdot (y, a) = (\iota(t)(y), a - t)$. This quotient is well-behaved, because $A[n]$ is a finite group acting on $Y \times A$ with no fixed points. In particular, in order to give an (invertible) map $X \to X$ it is enough to give an (invertible) map $Y \times A \to Y \times A$ that is compatible with the action of $A[n]$. Notice that, since $A[n]$ is finite and stable under the action of $\mathrm{Aut}_{\mathrm{Grp}}(A)$, the group $K = \ker(\mathrm{Aut}_{\mathrm{Grp}}(A) \to \mathrm{Aut}(A[n]))$ is infinite. We claim that for any $\varphi \in K$ the automorphism ψ of $Y \times A$ given by $(y, a) \mapsto (y, \varphi(a))$ descends to an automorphism $\overline{\psi}$ of X. Indeed, it suffices to check that for every $t \in A[n]$ we have $\psi(t \cdot (y, a)) = t \cdot \psi((y, a))$, that is,

$$\psi((\iota(t)y, a - t)) = t \cdot (y, \varphi(a)) \iff (\iota(t)(y), \varphi(a - t)) = (\iota(t)(y), \varphi(a) - t);$$

this last equality holds since φ is a group homomorphism and t is in $A[n]$, which φ fixes pointwise. Finally one checks that the map $\varphi \mapsto \overline{\psi}$ is injective, and since $\overline{\psi}$ is not contained in the image of ι for any nontrivial φ this proves that $\mathrm{Aut}(X)$ contains $\iota(A)$ with infinite index. □

3 Abelian varieties with finite automorphism group

We will now prove that any abelian variety such that $\mathrm{Aut}_{\mathrm{Grp}}(A)$ is finite can be realized as the automorphism group of a smooth projective variety Y. We first make some remaks on the structure of abelian varieties with finite automorphism group.

Lemma 3.1. *Let A and B two isogenous abelian varieties then* $\mathrm{Aut}_{\mathrm{Grp}}(A)$ *is finite if and only if* $\mathrm{Aut}_{\mathrm{Grp}}(B)$ *is finite.*

Proof. Since being isogenous is a symmetric relation, it suffices to prove that if $A \to B$ is an isogeny and $\mathrm{Aut}_{\mathrm{Grp}}(A)$ is infinite, then so is $\mathrm{Aut}_{\mathrm{Grp}}(B)$. Write $B \cong A/H$, where H is a finite subgroup of A, and assume that $\mathrm{Aut}_{\mathrm{Grp}}(A)$ is infinite. Notice that every automorphism φ of A which leaves H stable induces an automorphism $\bar{\varphi}$ of B, and that $\bar{\varphi}$ is trivial if and only if φ is trivial. Let n be the order of H; in particular, we have $H \subset A[n]$. Any automorphism φ of A leaves $A[n]$ stable, so, since $\mathrm{Aut}_{\mathrm{Grp}}(A)$ is infinite and $A[n]$ is finite, the subgroup of automorphisms φ which fix $A[n]$ pointwise is infinite. Every such automorphism leaves H stable, hence it descends to an automorphism of B, and since the map $\varphi \mapsto \bar{\varphi}$ is injective we deduce that $\mathrm{Aut}_{\mathrm{Grp}}(B)$ is infinite. □

Lemma 3.2. *Let A be an abelian variety such that* $\mathrm{Aut}_{\mathrm{Grp}}(A)$ *is finite. Then any two simple abelian subvarieties A_1, A_2 of A are isogenous if and only if they coincide. Moreover, if A_1 is a simple abelian subvariety of A, then* $\mathrm{Aut}_{\mathrm{Grp}}(A_1)$ *is finite.*

Proof. Suppose by contradiction that we can find two distinct but isogenous simple abelian subvarieties A_1, A_2 of A. By Poincaré's reducibility theorem, there is an abelian subvariety C of A such that the multiplication map $A_1 \times A_2 \times C \to A$ is an isogeny. Let B be an abelian variety such that there exists isogenies $\varphi_i : B \longrightarrow A_i$ and define the isogeny

$$\varphi : B^2 \times C \to A \quad \text{by} \quad \varphi(b_1, b_2, c) = \varphi_1(b_1) + \varphi_2(b_2) + c.$$

Now notice that $\psi(b_1, b_2, c) = (b_1, b_1 + b_2, c)$ defines an automorphism of $B^2 \times C$ of infinite order, and by the previous Lemma we conclude that $\mathrm{Aut}_{\mathrm{Grp}}(A)$ is also infinite, contradiction. The proof that for any simple abelian subvariety A_1 of A the group $\mathrm{Aut}_{\mathrm{Grp}}(A_1)$ is finite is completely analogous. □

From now on we fix an abelian variety A with finite automorphism group $\mathrm{Aut}_{\mathrm{Grp}}(A)$. By the previous Lemma and Poincaré reducibility Theorem we know that there exist uniquely determined simple abelian subvarieties $A_1, \ldots, A_h$ of A such that the sum

$$\sigma : A_1 \times \cdots \times A_h \longrightarrow A \qquad \sigma(a_1, \ldots, a_h) = a_1 + \cdots + a_h$$

is an isogeny. We denote by Σ the finite kernel of this map and denote by N its order. By Lemma 3.2, A_i and A_j are not isogenous if $i \neq j$ and $\mathrm{Aut}_{\mathrm{Grp}}(A_i)$ is finite for all i. Finally, notice that any abelian variety constructed in this way has finite automorphism group.

3.1 Construction of the example

Let A be as above and choose a prime number $p \geq 7$ such that

(*) for $i = 1, \ldots, h$, for any subgroup H of A_i contained in $A[N]$, and for any nontrivial $\varphi \in \mathrm{Aut}_{\mathrm{Grp}}(A_i/H)$, p is larger than the order of $(A_i/H)^\varphi = \{x \in A_i/H : \varphi(x) = x\}$.

Notice that if φ is a nontrivial automorphism of a simple abelian variety then φ has only finitely many fixed points, so a prime number p with this property exists (notice that $\mathrm{Aut}_{\mathrm{Grp}}(A_i/H)$ is finite for every i and H thanks to Lemmas 3.1 and 3.2).

Let $S/\mathbb{C}$ be a smooth hypersurface of degree p in $\mathbb{P}^3$ with $\mathrm{Aut}(S) \cong \mathbb{Z}/p\mathbb{Z}$ and such that every automorphism of S acts on it without any fixed points; an explicit example of such a hypersurface is given in Theorem 3.12. Let $G = \mathrm{Aut}(S) \cong \mathbb{Z}/p\mathbb{Z}$ and set $X := S/G$. We now proceed to describe some basic properties of X (§3.1.1), construct a certain smooth projective variety Y of dimension $2 + \dim A$ (§3.1.2), and prove that Y has automorphism group isomorphic to A (Theorem 3.9 in §3.2).

3.1.1 Properties of X

Lemma 3.3. *X is a smooth projective variety.*

Proof. X is smooth since G acts on S without fixed points, and is projective since any quotient of a projective variety by a finite group of automorphisms is projective (see [Ser58, Remarque on page 51]). □

Lemma 3.4. *X does not admit any nontrivial automorphisms.*

Proof. Let $\varphi : X \to X$ be an automorphism. Composing with the natural projection $\pi : S \to X$, we obtain a map $\varphi \circ \pi : S \to X$ which, since S is simply connected, lifts to a map $\tilde{\varphi} : S \to S$. Clearly $\tilde{\varphi}$ is algebraic, and it is easily seen to be a covering map, so it is an isomorphism since S is connected and simply connected. It follows that $\tilde{\varphi} : S \to S$ is in G, hence (by passing to the quotient) it induces the identity on X. Since on the other hand $\tilde{\varphi}$ induces φ on X, we get $\varphi = \mathrm{id}_X$ as claimed. □

Lemma 3.5. *K_X is ample; in particular, X has Kodaira dimension 2.*

Proof. By adjunction, $K_S = O_{\mathbb{P}^3}(p-3-1)|_S$ is ample, so $\mathrm{kod}(S) = \dim(S) = 2$. Since $\pi : S \to X$ is finite étale, $K_X = \pi^*(K_S)$ is also ample, and $\mathrm{kod}(X) = \dim(X) = 2$. □

Lemma 3.6. *The Albanese variety of X is trivial, therefore there are no non-constant maps from X to any abelian variety.*

Proof. Clearly S is the universal cover of X, so $\pi_1(X)$ is isomorphic to $\mathrm{Aut}(S \to X) \cong \mathbb{Z}/p\mathbb{Z}$ and in particular is finite. Since the Albanese variety of X is dual to its Picard variety, one has $\dim \mathrm{Alb}(X) = \dim \mathrm{H}^1(X, \mathcal{O}_X) = h^{1,0}(X)$; on the other hand, the fact that $\pi_1(X)$ is finite implies that $H_1(X, \mathbb{Q})$ is trivial, so $h^{1,0}(X) \leq h^1(X) = \dim H^1(X, \mathbb{C}) = 0$, hence $\mathrm{Alb}(X)$ is trivial as claimed. □

3.1.2 A nontrivial A-torsor $Y \to X$

Definition 3.7. Fix an isomorphism $\chi : G \to \mathbb{Z}/p\mathbb{Z}$ and a point P such that

(**) P is a p-torsion point of A which is not contained in any proper abelian subvariety of A.

The abelian subvarieties of A are all of the form $A_{i_1} + \cdots + A_{i_k}$, so a point with this property exists. We let $\mathbb{Z}/p\mathbb{Z}$ act on the group generated by P in the obvious way (that is, for $n \in \mathbb{Z}$ the class of n in $\mathbb{Z}/p\mathbb{Z}$ sends P to nP). We set $Y = (S \times A)/G$, where the action of G on the product $S \times A$ is given by

$$g \cdot (s, a) = (g \cdot s, a + \chi(g)P).$$

As in the proof of Lemma 3.3, it is easy to see that Y is a smooth projective variety; moreover, Y has a natural structure of principal space under A. Indeed for each $b \in A$, the translation map

$$\begin{array}{ccc} S \times A & \to & S \times A \\ (s,a) & \mapsto & (s, a+b) \end{array}$$

commutes with the action of G, so it descends to an automorphism of $Y = (S \times A)/G$ that we denote by $y \mapsto b + y$ or by τ_b. This defines an action of A on Y which is free and transitive along the fibers of the map $Y \to X$. Moreover, $Y \to X$ is an A-torsor in the analytic (and in fact even étale) topology: indeed, S is an étale covering of X, and the pullback of Y to S is trivial.

Lemma 3.8. *The map $Y \to X$ does not admit a section (in the analytic topology).*

Proof. Notice that $Y \to X$ admits a section if and only if it is trivial as a torsor. Indeed if $Y \to X$ has a section s then the map $A \times X \to Y$ given by $(a, x) \mapsto a + s(x)$ is an isomorphism of torsors. Let $\mathcal{A}$ be the sheaf of holomorphic functions on X with values in A; A-torsors on X are classified by $H^1(X, \mathcal{A})$, where the cohomology is taken in the analytic category. For any fixed $n > 0$, consider the exact sequence of sheaves on X

$$0 \to \mathcal{A}[n] \to \mathcal{A} \xrightarrow{[n]} \mathcal{A} \to 0$$

and take cohomology to obtain the long exact sequence

$$0 \to H^0(X, \mathcal{A}[n]) \to H^0(X, \mathcal{A}) \xrightarrow{[n]} H^0(X, \mathcal{A}) \to H^1(X, \mathcal{A}[n]) \to H^1(X, \mathcal{A}).$$

By Serre's GAGA principle, all maps from X to A are algebraic, so by Lemma 3.6 we have $H^0(X, \mathcal{A}) = A$, and $H^0(X, \mathcal{A}) \xrightarrow{[n]} H^0(X, \mathcal{A})$ is just $A \xrightarrow{[n]} A$, which is surjective. It follows in particular that the natural arrow

$$H^1(X, \mathcal{A}[n]) \to H^1(X, \mathcal{A}) \tag{1}$$

is injective. Consider $Z := (S \times \langle P \rangle)/G \hookrightarrow Y$, where $\langle P \rangle$ denotes the order p subgroup of $A(\mathbb{C})$ generated by P. By the injectivity of (1) (with $n = p$), proving that Z is a nontrivial covering space of X suffices to show that $Y \to X$ is a nontrivial torsor. But this is clear, because the natural map $S \to S \times \langle P \rangle \to (S \times \langle P \rangle)/G$ is injective and surjective, hence (since S is compact) a homeomorphism. It follows that $Z \cong S$ is a nontrivial cover of X as desired. □

3.2 Determination of $\mathrm{Aut}(Y)$

In this section we show:

Theorem 3.9. *The automorphism group of Y is isomorphic to A.*

3.2.1 Preliminaries on simple abelian varieties

We shall need the following basic fact about simple abelian varieties.

Lemma 3.10. *Let T be a projective complex torus. Let A be the abelian variety obtained from T by fixing an arbitrary origin; notice that T is naturally a torsor under A. Finally let α be an automorphism of T (as a projective variety) and assume that A is simple. Then:*

1. *if α is translation by a point of A, then the determinant of $(1-\alpha)_* : H_1(T, \mathbb{Q}) \to H_1(T, \mathbb{Q})$ is 0;*
2. *if α is not translation by a point of A, then α has at least one fixed point and the determinant of $(1-\alpha)_* : H_1(T, \mathbb{Q}) \to H_1(T, \mathbb{Q})$ is the number of fixed points of α.*

Proof. The statement of (1) is obvious, because translations induce the identity on $H_1(T, \mathbb{Q})$. Assume now that α is not a translation and identify T with A by choosing a point $t_0 \in T$ as the origin. We prove first that α has at least one fixed point. Letting $a = \alpha(t_0) - t_0$ we have $\alpha(t) = \varphi(t) + a$, where $\varphi \in \mathrm{Aut}_{\mathrm{Grp}}(A)$ is different from the identity. Let $\psi = \varphi - \mathrm{id}_A : A \longrightarrow A$; it is an endomorphism of A, and since A is simple and φ is nontrivial the image of ψ is A itself. One checks that $b \in T$ is a fixed point of α if and only if $\psi(b) = -a$. As ψ is surjective, such b exist, and there are only finitely many of them because the set $\{b : \psi(b) = -a\}$ is naturally a torsor under the finite group $\ker \psi$. We can then choose the origin t_0 to be a fixed point of α, in which case α belongs to $\mathrm{Aut}_{\mathrm{Grp}}(A)$ and we have $\psi(t) = \alpha(t) - t$, so that A^{φ} is equal to the kernel of ψ and its order is the degree of ψ. The lemma follows from the fact that for a complex torus T the degree of ψ is equal to $\det(\psi_* : H_1(T, \mathbb{Q}) \longrightarrow H_1(T, \mathbb{Q}))$. □

3.2.2 Preliminaries on surfaces of Kodaira dimension 2

We shall need the following consequence of [DHP08].

Lemma 3.11. *Let X be a surface of Kodaira dimension 2 and A be an abelian variety. The image of any morphism $f : A \to X$ is either a point or a (possibly singular) irreducible curve of geometric genus at most one.*

Proof. Let Y be the image of A. By [DHP08, Lemma 2.1] there exists a subtorus B of A such that f factors through $C := A/B$ and the induced map $g : C \to Y$ is finite. Hence if Y is a curve its (geometric) genus must be equal to zero or one. If $Y = X$, letting R be the ramification divisor of g and using that g is finite we have

$$0 = K_C = g^* K_X + R.$$

Since K_X is ample and $R \geq 0$ we get a contradiction. □

3.2.3 Proof of Theorem 3.9

We already noticed that A injects into $\mathrm{Aut}(Y)$. For the other inclusion let φ be an automorphism of Y. We prove first that φ preserve the fibers of the map $\pi : Y \longrightarrow X$. For each $x \in X$, let Y_x be the fiber of π over x and let

$$\varphi_x : Y_x \hookrightarrow Y \xrightarrow{\varphi} Y \xrightarrow{\pi} X.$$

Suppose that for general x the image of Y_x is not reduced to a single point: then Lemma 3.11 implies that generically the image of φ_x is a (possibly singular) curve of genus at most 1. By [BHPVdV04, Proposition VII.2.1], a surface of Kodaira dimension 2 admits no algebraic system (of positive dimension) of effective divisors whose general member is a (possibly singular) rational or elliptic curve. By Lemma 3.5 we know that X is a surface of Kodaira dimension 2, so it follows that φ_x is constant for all $x \in X$. In particular,

$$Y \xrightarrow{\varphi} Y \to A \backslash Y = X$$

descends to a map $\varphi_X : X \to X$, which is easily seen to be biregular (its inverse being $(\varphi^{-1})_X$), and hence an automorphism. It follows from Lemma 3.4 that φ_X is the identity, which implies that the equality $\varphi(Y_x) = Y_x$ holds for all $x \in X$. Thus we see that for every $x \in X$ the automorphism φ of Y induces an automorphism $\varphi|_{Y_x}$ of Y_x.

Thus, locally in the analytic topology, the automomorphism φ can be described as follows. For each $x \in X$ we can choose an open connected neighborhood $U \subset X$ of x such that $V = \pi^{-1}(U)$ can be identified with $U \times A$ (as an A-torsor) and $\varphi(u, a) = (u, \phi(u, a))$. Let $r : U \longrightarrow A$ be defined by $r(u) = \phi(u, 0) - 0$. Then $a \mapsto \phi(u, a) - r(u)$ is an automorphism of A as an algebraic group, and since $\mathrm{Aut}_{\mathrm{Grp}}(A)$ is finite it must be equal to an automorphism ϕ independent of u. Hence $\varphi(u, a) = (u, \phi(a) + r(u))$ and $\psi = \phi - id_A$ is an endomorphism of A as an algebraic group. Furthermore, since any two identifications of a fiber of $Y \to X$ with the trivial A-torsor differ only by a translation, we see that the endomorphism ψ thus obtained is independent of our choice of U and of the local trivialization $\pi^{-1}(U) \cong U \times A$.

We now prove the theorem by induction on h, the number of simple factors of A. Assume first that $h = 1$, so that A is simple. For $x \in X$ we define

$$n(x) = \det\left((1 - \varphi|_{Y_x})_* \;\middle|\; H_1(Y_x, \mathbb{Q})\right);$$

it is a continuous function on X. Since X is connected and $\mathbb{Z}$ is discrete, it follows that $n(x)$ is actually constant: let n be the common value of the various $n(x)$. We show that $n = 0$. Suppose by contradiction that $n > 0$. Let $\tilde{X} = Y^\varphi$ and let $\tilde{\pi}$ be the restriction of π to $\tilde{X}$. We prove that $\tilde{\pi}$ is an n-to-1 covering of X. The fact that it is n-to-1 follows from Lemma 3.10. The claim that it is a covering can be checked locally using the analytic topology: using the local description above we obtain $V^\varphi = \{(u, a) : \psi(a) = r(u)\}$, which is a covering of U.

If at least one of the connected components of $\tilde{X}$ is the trivial cover of X, then this gives a section of the projection map $\pi : Y \to X$, contradicting Lemma 3.8. Otherwise, take a connected subcover of $\tilde{X}$: this is a connected m-to-1 cover of X for some $m \le n$ which is smaller than p by our assumption (*) on p. This contradicts the fact that $\#\pi_1(X) = p$.

It follows that $n(x) = n = 0$ for all x, hence by Lemma 3.10 $\varphi|_{Y_x}$ is translation by a point $a(x) \in A$ (recall that Y_x is naturally a torsor under A, so it makes sense to identify translations of Y_x with elements of A). Now $x \mapsto a(x)$ gives a map $X \to A$, which is necessarily constant by Lemma 3.6, hence φ is globally a translation by a point of A.

We now prove the inductive step. Let $h > 1$. Since φ preserves the fibers Y_x, composing with a translation by an element of A we can assume that there exists $y_0 \in Y$ such that $\varphi(y_0) = y_0$. We want to prove that in this case φ is the identity. Let $\pi : A \to A' := A/A_1$ be the natural projection and set $A'_i := \pi(A_i)$ for $i = 2, \dots, h$. We let $P' = \pi(P)$ and write $\tilde{\pi} : A_1 \times \cdots \times A_h \to A'_2 \times \cdots \times A'_h$ for the homomorphism

$$\tilde{\pi}(a_1, \dots, a_h) = (\pi(a_2), \dots, \pi(a_h));$$

finally, we set $\Sigma' := \tilde{\pi}(\Sigma)$. One then checks that the sum $\sigma' : A'_2 \times \cdots \times A'_h \to A'$ is an isogeny with kernel Σ'.

Let $K = \ker(\Sigma \to \Sigma')$. For every $i = 2, \dots, h$, the intersection $A_1 \cap A_i$ embeds naturally into K, so $N' \cdot \#(A_1 \cap A_i) \mid N' \cdot \#K = N$. It follows that every quotient of $A'_i = A_i/(A_1 \cap A_i)$ by a subgroup of $A'_i[N']$ is a quotient of A_i by a subgroup of $A_i[N]$, so the analogue of condition (*) is satisfied by A' and the prime p. It is immediate to check that (**) also holds for A', p, and the point P'. In particular, by induction, the automorphism group of $Y' = S \times^G A'$ is equal to A'.

The projection map $S \times A \longrightarrow S \times A'$ is G-equivariant, so it induces a map $q : Y \longrightarrow Y'$ which we prove to be a categorical quotient by the action of A_1. Indeed let $f : Y \longrightarrow Z$ be a A_1-invariant map. It induces a $G \times A_1$-invariant map $f_1 : S \times A \longrightarrow Z$ and therefore a map $f_2 : S \times (A_1 \times \cdots \times A_h) \longrightarrow Z$ which is invariant by the action of both A_1 and Σ on the second factor. Since the quotient of $A_1 \times \cdots \times A_h$ by the subgroup generated by A_1 and Σ is A', the map f_2 induces a regular map $g_2 : S \times A' \longrightarrow Z$ such that $f_2 = g_2 \circ (id_S \times \pi')$, where $\pi' := \pi \circ \sigma$ is the natural map $A_1 \times \cdots \times A_h \to A'$. Since furthermore f_2 is G-invariant, g_2 is also G-invariant, hence it induces a map $g : Y' \longrightarrow Z$ such that $f = g \circ q$. Moreover, as q is surjective, the map g is unique.

We can now prove that φ is the identity. For $a \in A$ denote by τ_a the translation by a in Y. Notice that for each a and for each $x \in X$ there exists $\phi_x \in \mathrm{Aut}_{\mathrm{Grp}}(A)$ such that

$$\varphi \circ \tau_a \circ \varphi^{-1} = \tau_{\phi_x(a)} : Y_x \longrightarrow Y_x.$$

In particular, if $a \in A_1$, then $\phi_x(a) \in A_1$. Being Y' a categorical quotient of Y by the action of A_1, we have that φ induces a map $\varphi' : Y' \longrightarrow Y'$, which is an automorphism since $(\varphi^{-1})'$ is its inverse. Moreover, the image of y_0 in Y' is fixed by φ', so φ' is equal to the identity.

Hence $\varphi(y) - y \in A_1$ for all $y \in Y$. Arguing in the same way, but using A_2 instead of A_1, we obtain $\varphi(y) - y \in A_2$ for all y. So $\varphi(y) - y \in A_1 \cap A_2$ for all $y \in Y$, and since $A_1 \cap A_2$ is finite and $\varphi(y_0) = y_0$ we obtain $\varphi(y) = y$ for all y. □

3.3 A hypersurface in $\mathbb{P}^3$ with automorphism group $\mathbb{Z}/p\mathbb{Z}$

In this section we explicitly construct, for every prime $p \geq 7$, an algebraic surface in $\mathbb{P}^3$ of degree p whose automorphism group is cyclic of order p:

Theorem 3.12. *Let $p \geq 7$ be a prime number, and for $\lambda \in \mathbb{C}$ let S_λ be the algebraic surface over $\mathbb{C}$ given by the zero locus in $\mathbb{P}^3$ of the homogeneous polynomial*

$$f_\lambda(x_1, x_2, x_3, x_4) := x_1^p + x_2^p + x_3^p + x_4^p + \lambda(x_1^2 x_2^{p-4} x_3^2 + x_1^4 x_2^{p-6} x_4^2).$$

The surface S_λ is smooth for all but finitely many $\lambda \in \mathbb{C}$; if $\lambda \neq 0$, the automorphism group of S_λ is cyclic of order p, generated by $[x_1 : x_2 : x_3 : x_4] \mapsto [x_1 : \zeta_p x_2 : \zeta_p^2 x_3 : \zeta_p^3 x_4]$, where ζ_p is a primitive p-th root of unity. In particular, each nontrivial element of $\mathrm{Aut}(S_\lambda)$ acts on S_λ without any fixed points.

We start by noticing that for $\lambda = 0$ the surface S_0 is smooth. Since being smooth is a Zariski-open condition in the defining polynomial, this shows that S_λ is smooth away from a proper Zariski-closed subset of $\mathbb{C}$, that is, S_λ is smooth for all but finitely many values of λ. From now on fix a nonzero value of λ such that S_λ is smooth, and to simplify the notation write S for S_λ and $f(x_1, x_2, x_3, x_4)$ for $f_\lambda(x_1, x_2, x_3, x_4)$.

By [MM64, Theorem 2] we know that all the automorphisms of S are induced by (linear) automorphisms of $\mathbb{P}^3$, so we only need to consider these. Let $L : \mathbb{P}^3 \to \mathbb{P}^3$ be a linear transformation that satisfies $L(S) = S$. We identify L to the class $[M] \in \mathrm{PGL}_4(\mathbb{C})$ of a matrix $M = (M_{ij}) \in \mathrm{GL}_4(\mathbb{C})$. Furthermore, we let $e_1, \ldots, e_4$ be the canonical basis of $\mathbb{C}^4$ and denote by $\langle e_i \rangle$ the 1-dimensional $\mathbb{C}$-vector subspace of $\mathbb{C}^4$ generated by e_i. We shall show Theorem 3.12 in three steps: first we shall prove that M either fixes or permutes the lines generated by e_3 and e_4; then we shall show that the same statement holds for the lines generated by e_1 and e_2; finally, we shall deduce from this that M needs to be a diagonal matrix, at which point a direct computation concludes the proof. This approach is inspired by [Poo05].

3.3.1 Step 1: M permutes $\langle e_3 \rangle$ and $\langle e_4 \rangle$

The condition that $L(S) = S$ translates into the polynomial equality

$$f \circ M(x_1, \ldots, x_4) = \alpha f(x_1, \ldots, x_4) \tag{2}$$

for some $\alpha \in \mathbb{C}^\times$. Applying $\frac{\partial^2}{\partial x_i \partial x_j}$ to the two members of this equation and setting

$$H_{ij}(x_1, \ldots, x_4) := \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, \ldots, x_4)$$

we find

$$\sum_k \sum_m M_{kj} M_{mi} H_{mk}(M(x_1, \ldots, x_4)) = \alpha H_{ij}(x_1, \ldots, x_4).$$

Let u, v be two vectors in $\mathbb{C}^4$. Multiplying the previous identity by $u_i v_j$ and summing over i and j we get

$$\sum_{k,m} (Mv)_k (Mu)_m H_{mk}(M(x_1, \ldots, x_4)) = \alpha \sum_{i,j} H_{ij}(x_1, \ldots, x_4) u_i v_j. \tag{3}$$

We now define a bilinear pairing

$$\begin{array}{rccc} \langle \cdot, \cdot \rangle : & \mathbb{C}^4 \times \mathbb{C}^4 & \to & \mathbb{C}[x_1, \ldots, x_4] \\ & (u, v) & \mapsto & \sum_{i,j} H_{i,j}(x_1, \ldots, x_4) u_i v_j, \end{array}$$

so that Equation 3 reads

$$\langle Mu, Mv \rangle (M(x_1, \ldots, x_4)) = \alpha \langle u, v \rangle.$$

In particular, since M is invertible we obtain:

Proposition 3.13. *Let u, v be vectors in $\mathbb{C}^4$. The equalities $\langle u, v\rangle = 0$ and $\langle Mu, Mv\rangle = 0$ are equivalent.*

Lemma 3.14. *Let $a, b \in \mathbb{C}^4$ be two nonzero vectors such that $\langle a, b\rangle = 0$. Then there exist $\lambda, \mu \in \mathbb{C}^\times$ such that either $a = \lambda e_3, b = \mu e_4$, or $a = \lambda e_4, b = \mu e_3$ hold.*

Proof. Write $a = (a_1, a_2, a_3, a_4)$ and $b = (b_1, b_2, b_3, b_4)$. By direct inspection, one checks that, for $i = 1, 2, 3, 4$, the only second derivative of f involving the monomial x_i^{p-2} is H_{ii}. This immediately implies that $a_i b_i = 0$ for $i = 1, \dots, 4$, and by symmetry we can assume $a_1 = 0$. The coefficients of the monomials $x_1 x_2^{p-5} x_3^2$, $x_1 x_2^{p-4} x_3$ and $x_1^3 x_2^{p-6} x_4$ in $\langle a, b\rangle$ are given by $2\lambda(p-4)(a_2 b_1 + a_1 b_2), 4\lambda(a_3 b_1 + a_1 b_3)$ and $8\lambda(a_4 b_1 + a_1 b_4)$ respectively, so under our assumptions $\langle a, b\rangle = 0$, $\lambda \neq 0$ and $a_1 = 0$ we obtain $b_1 a_2 = b_1 a_3 = b_1 a_4 = 0$. If we had $b_1 \neq 0$, this would imply $a = (0, 0, 0, 0)$, contradicting our assumptions, so we must have $b_1 = 0$ as well. The situation is now again symmetric in a, b, so we might assume $a_2 = 0$. Arguing as before (but looking at the monomials $x_1^2 x_2^{p-5} x_3$ and $x_1^4 x_2^{p-7} x_4$) one finds $a_3 b_2 = a_4 b_2 = 0$, so that $b_2 = 0$ as well. The conclusion now follows easily from the equalities $a_3 b_3 = a_4 b_4 = 0$. □

Corollary 3.15. *One of the following holds:*

- *$M\langle e_3\rangle = \langle e_3\rangle$ and $M\langle e_4\rangle = \langle e_4\rangle$;*
- *$M\langle e_3\rangle = \langle e_4\rangle$ and $M\langle e_4\rangle = \langle e_3\rangle$.*

Proof. Apply Proposition 3.13 to $u = e_3$ and $v = e_4$: since $\langle e_3, e_4\rangle = H_{34} = 0$ we obtain $\langle Me_3, Me_4\rangle = 0$. The claim then follows from the previous lemma. □

3.3.2 Step 2: M permutes $\langle e_1\rangle$ and $\langle e_2\rangle$

Arguing as in the previous section, it is easily seen that if we let $A : (\mathbb{C}^4)^p \to \mathbb{C}$ denote the multilinear form

$$A : (u_1, \dots, u_p) \mapsto \sum_{i_1, \dots, i_p} \frac{\partial^p f}{\partial x_{i_p} \cdots \partial x_{i_p}} (u_1)_{i_1} \cdots (u_p)_{i_p},$$

where $(u_i)_j$ is the j-th coordinate of u_i, we have $A(Mu_1, \dots, Mu_p) = \beta A(u_1, \dots, u_p)$ for some $\beta \in \mathbb{C}^\times$; notice that here we do not need to compose with M on the left hand side, because p-th derivatives of f are just scalars. Suppose that $M\langle e_3\rangle = \langle e_3\rangle$ and $M\langle e_4\rangle = \langle e_4\rangle$; the case $M\langle e_3\rangle = \langle e_4\rangle$ and $M\langle e_4\rangle = \langle e_3\rangle$ is completely analogous. Rescaling M if necessary (which we can do, since we are only interested in its projective class) we can assume $Me_3 = e_3$. Choosing $u_1 = \cdots = u_{p-1} = e_3$ and $u_p = e_1$ we have

$$\beta A(e_3, \dots, e_3, e_1) = \beta \frac{\partial^p f}{\partial x_3^{p-1} \partial x_1} = 0,$$

from which we deduce

$$0 = A(Me_3, \dots, Me_3, Me_1) = A(e_3, \dots, e_3, Me_1) = \sum_{i_p} \frac{\partial^p f}{\partial x_3^{p-1} \partial x_{i_p}} (Me_1)_{i_p};$$

since the only nonvanishing partial derivative of the form $\frac{\partial^p f}{\partial x_3^{p-1} \partial x_{i_p}}$ is $\frac{\partial^p f}{\partial x_3^p}$, this implies $M_{31} = 0$. Similary, the choice $(e_4, \dots, e_4, e_1)$ shows $M_{41} = 0$, while the choices $(e_3, \dots, e_3, e_2)$ and $(e_4, \dots, e_4, e_2)$ give $M_{32} = M_{42} = 0$. It follows that M sends the 2-plane $\{x_3 = x_4 = 0\}$ to itself; in particular, M induces an automorphism of the finite set of points in $\mathbb{P}^3$ defined by the equations

$$f(x_1, x_2, x_3, x_4) = 0, \quad x_3 = x_4 = 0 \quad \Longleftrightarrow \quad x_3 = x_4 = 0, \quad x_1^p + x_2^p = 0.$$

From this it is immediate to deduce:

Corollary 3.16. *One of the following holds:*

- *$M\langle e_1\rangle = \langle e_1\rangle$ and $M\langle e_2\rangle = \langle e_2\rangle$;*
- *$M\langle e_1\rangle = \langle e_2\rangle$ and $M\langle e_2\rangle = \langle e_1\rangle$.*

3.3.3 Step 3: determination of $\mathrm{Aut}(S)$

Corollaries 3.15 and 3.16 tell us that M either fixes or permutes the lines $\langle e_1\rangle, \langle e_2\rangle$, and that the same holds for the lines $\langle e_3\rangle, \langle e_4\rangle$. One checks easily that if M exchanges $\langle e_1\rangle$ with $\langle e_2\rangle$, and/or it exchanges $\langle e_3\rangle$ with $\langle e_4\rangle$, then $f \circ M$ is not a scalar multiple of f, so that M needs to be a diagonal matrix. Normalize M so that $M_{11} = 1$ and write $M = \mathrm{diag}(1, \mu_2, \mu_3, \mu_4)$: replacing in Equation (2) and comparing the coefficients of x_1^p on the two sides we find $\alpha = 1$. Comparing the coefficients of x_i^p for $i = 2, 3, 4$ we then obtain $\mu_i^p = 1$ for $i = 2, 3, 4$, so that μ_2, μ_3, μ_4 are p-th roots of unity. It is now immediate to check that the only automorphisms of S are represented by the powers of the (order p) matrix $\begin{pmatrix} 1 & & & \\ & \zeta_p & & \\ & & \zeta_p^2 & \\ & & & \zeta_p^3 \end{pmatrix}$, where ζ_p is a primitive p-th root of unity. The fixed points (in $\mathbb{P}^3$) for the action of this matrix (or any of its powers, with the exception of the identity) are $[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1]$, none of which lies on the hypersurface $f(x_1, x_2, x_3, x_4) = 0$. This concludes the proof of Theorem 3.12.

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