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GROUND STATES OF A TWO PHASE MODEL WITH CROSS AND SELF ATTRACTIVE INTERACTIONS*

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Abstract. We consider a variational model for two interacting species (or phases), subject to cross and self attractive forces. We show existence and several qualitative properties of minimizers. Depending on the strengths of the forces, different behaviors are possible: phase mixing or phase separation with nested or disjoint phases. In the case of Coulomb interaction forces, we characterize the ground state configurations.

Key words. nonlocal interactions, variational methods, Coulomb interactions, shape optimization

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Introduction. Models of two or more interacting species find applications in several fields of science, such as physics, chemistry, and biology. To cite a few examples one may think about the formation of bacterial colonies in biology [24], the self assembly of nano-particles in physical chemistry [25], and the problem of two species group consensus [14] as well as that of pedestrian dynamics [11]. The basic feature of all these models is the presence of competing forces aiming to drive two phases toward different shapes.

An interesting example of this phenomenon has been recently reported in [25]. There, it has been observed that, during the assembly process of two nano-scaled polyprotic macroions in a dilute aqueous solution, the system may be driven toward phase segregation as opposed to phase mixtures via a complex self recognition mechanism involving multiple scales optimization.

Far from thinking to propose realistic models for these complex mechanisms, we aim at reproducing such limit behaviors while keeping the number of parameters as small as possible. We propose and study a toy model for two interacting phases subject to self and cross attractive forces depending only on the distance between particles. Such a model may be introduced as follows. Two phases, represented by two subsets of $\mathbb{R}^N$, say, E_1 and E_2 with $E_1 \cap E_2 = \emptyset$, with masses m_1 and

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m_2, respectively, interact both with themselves and with the other phase trying to minimize an energy of the form

$$\mathcal{F}(E_1, E_2) = \sum_{i,j=1}^{2} J_{K_{ij}}(E_i, E_j). \tag{0.1}$$

Here

$$J_{K_{ij}}(E_i, E_j) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi_{E_i}(x)\, \chi_{E_j}(y)\, K_{ij}(x-y)\, \mathrm{d}x\, \mathrm{d}y \tag{0.2}$$

is a nonlocal interaction energy with interaction potential $K_{ij} : \mathbb{R}^N \to \mathbb{R}$. Energy functionals of this type have been considered by many authors in the context of nonlinear aggregation-diffusion problems, modeling biological swarming, and crowd congestion (see [33, 7, 9, 13, 27, 29] and the references therein).

In the present paper we initiate an analysis of the ground states of the energy functional $\mathcal{F}$ assuming that for $i, j \in \{1, 2\}$ the interaction forces, still having different intensities, obey the same nonlocal law. More precisely, we consider $K \in L^1_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R})$ a nonincreasing radially symmetric interaction potential and restrict our analysis to those $K_{ij} = c_{ij} K$. Our scope is studying the solutions of

$$\min_{\substack{E_1 \cap E_2 = \emptyset \\ |E_i| = m_i}} c_{11}\, J_K(E_1, E_1) + c_{22}\, J_K(E_2, E_2) + (c_{12} + c_{21})\, J_K(E_1, E_2). \tag{0.3}$$

Moreover, we assume that the interactions are attractive; more precisely, we deal with coefficients $c_{i,j} \le 0$. Without this assumption, different phenomena may appear, related to loss of mass at infinity. As a consequence, the minimization problem is in general ill-posed and requires specific care. One possibility would consist in adding some confinement conditions. In [3], the authors propose a different kind of problem: they focus on the case $c_{11} = c_{22} = 1$, $c_{12}+c_{21} = -2$, fix E_1, and study the minimization of (0.1) as a function of E_2. They prove that such a problem admits a solution if and only if $m_2 \le m_1$. Similar threshold phenomena appear in energetic models for di-block copolymers, where a confining perimeter term and a repulsive force compete [2, 12, 15, 17, 19, 20, 26] as well as in attractive/repulsive Lennard-Jones-type models (see, e.g., [4, 5, 8, 10, 21, 22, 32] and the references therein).

Let us go back to the case of attractive interactions $c_{ij} \le 0$ considered in this paper. We will see that, also in this case, the minimization problem in (0.3) is actually ill-posed. Indeed, in Proposition 2.8 and Theorem 3.9, we will show that if $|c_{11}|, |c_{22}|$ are small enough, any minimizing sequence mixes the two phases. We are then led to consider a relaxed version of the problem above where the notion of phase is weakened to allow local mixing. Now the phases are described in terms of their densities $f_1, f_2 \in L^1(\mathbb{R}^N; [0, 1])$, so that $\int_{\mathbb{R}^N} f_i(x)\, \mathrm{d}x = m_i$ and the functional becomes

$$\mathcal{E}_K(f_1, f_2) = c_{11}\, J_K(f_1, f_1) + c_{22}\, J_K(f_2, f_2) + (c_{12} + c_{21})\, J_K(f_1, f_2), \tag{0.4}$$

where $J_K(f_i, f_j)$ has the same form of (0.2) with K and f_i in place of K_{ij} and χ_{E_i}, respectively.

For all masses $m_i > 0$ and all $c_{ij} \le 0$, we prove existence of minimizers of $\mathcal{E}_K$ under the constraint $f_1 + f_2 \le 1$ (Theorem 1.9). Such a constraint is inherited by the original problem, naturally arising from the relaxation procedure, but it also has a clear physical meaning. Indeed, if we interpret the densities f_i as proportional to the number of particles per unit volume on a certain mesoscopic ball of a lattice gas model, the condition reflects the fact that two particles are not allowed to occupy the same elementary cell. Note that for a slightly different problem in the one-dimensional case, a similar existence result has appeared in [18].

In the case $c_{12} = c_{21} = 0$, our problem reduces to two independent one-phase problems given by

$$\min_{\substack{f_i \in L^1(\mathbb{R}^N;[0,1]) \\ \int_{\mathbb{R}^N} f_i(x)\,\mathrm{d}x = m_i}} c_{ii}\, J_K(f_i, f_i) \quad \text{for } i = 1, 2.$$

If $c_{ii} < 0$, it is well known that the minimizer above is (the characteristic function of) a ball having mass equal to m_i (see [31, 16] or Lemma 1.6). Therefore, we focus on the case $c_{12} + c_{21} < 0$. Clearly, by the scaling and symmetry properties of the energy, it is not restrictive to assume $c_{12} = c_{21} = -1$. With this interaction term in the energy, the geometry of the phases becomes a more delicate issue and it drastically depends on the strength of the interaction constants c_{11} and c_{22}. On one hand, if the cross interaction forces prevail, *phase mixing* occurs, that is, a new phase appears which is a combination of the two pure phases. On the other hand, if one of the two self interaction forces is sufficiently strong, *phase segregation* occurs, with the presence of two pure phases which can be nested or adjacent, depending on the strength of the other force. The latter behavior is in a certain sense reminiscent of clusters of two phases in an infinite ambient phase, minimizing an inhomogeneous perimeter functional with surface tension depending on the two touching phases [1]. In this case the mixing of phases is impossible but, depending on the strength of the surface tensions, minimizers may exhibit disjoint or nested phases [28].

Our analysis focuses also on qualitative properties of solutions. In some cases, we have determined the explicit geometry of the phases of the minimizers. Such an analysis is almost complete for the Coulomb interaction kernel.

We first describe the case of general kernels (see Figure 1). First, consider the case $c_{11}+c_{22} > -2$, which we will call the *weakly attractive case*. In this case, we explicitly characterize the shape of minimizers only if K is positive definite, $-1 < c_{11} \le 0$, $-1 < c_{22} \le 0$ and $(c_{11}+1)\, m_1 = (c_{22}+1)\, m_2$. If this occurs, the unique minimizer is given by $(f_1, f_2) = (\frac{m_1}{m_1+m_2}\chi_B, \frac{m_2}{m_1+m_2}\chi_B)$, where B is a ball with $|B| = m_1 + m_2$ (Proposition 2.8). In all the remaining weakly attractive cases we cannot provide the explicit shape of the minimizers. In particular, if $(c_{11}+1)(c_{22}+1) \le 0$ and $(c_{11}, c_{22}) \ne (-1, -1)$ (notice that in these cases the condition $(c_{11}+1)\, m_1 = (c_{22}+1)\, m_2$ cannot be satisfied), we do not know the shape of the minimizers.

The *strongly attractive case* $c_{11} + c_{22} \le -2$ (Theorem 2.3) needs to be classified into the four subcases listed below. If $c_{11} = c_{22} = -1$, the problem is extremely degenerate, i.e., the minimizers are given by all the pairs (f_1, f_2), with $f_1 + f_2 = \chi_B$. If $c_{11} = -1$ and $c_{22} < -1$, then the minimizers of the problem are the pairs (f_1, f_2), where $f_1 + f_2 = \chi_B$ and f_2 is (the characteristic function of) a ball contained in B (not necessarily concentric). If $c_{22} < -1 < c_{11}$ (with $c_{11} + c_{22} \le -2$), then the minimizer is unique and it is given by a ball and a concentric annulus around it. Finally, for $c_{11}, c_{22} < -1$, the minimizer is fully characterized only in the one-dimensional case (Proposition 2.4) and it is given by the two tangent balls (namely, segments).

As for the Coulomb interactions (see Figure 2), we have fully characterized the minimizers also in the weakly attractive case.

We have proven (Theorem 3.9, Corollaries 3.5 and 3.12) that if $-1 < c_{11}, c_{22} < 0$ the minimizer is given by an interior ball in which f_1 and f_2 mix each other with specific volume fractions, according to their self attraction coefficients, and a concentric

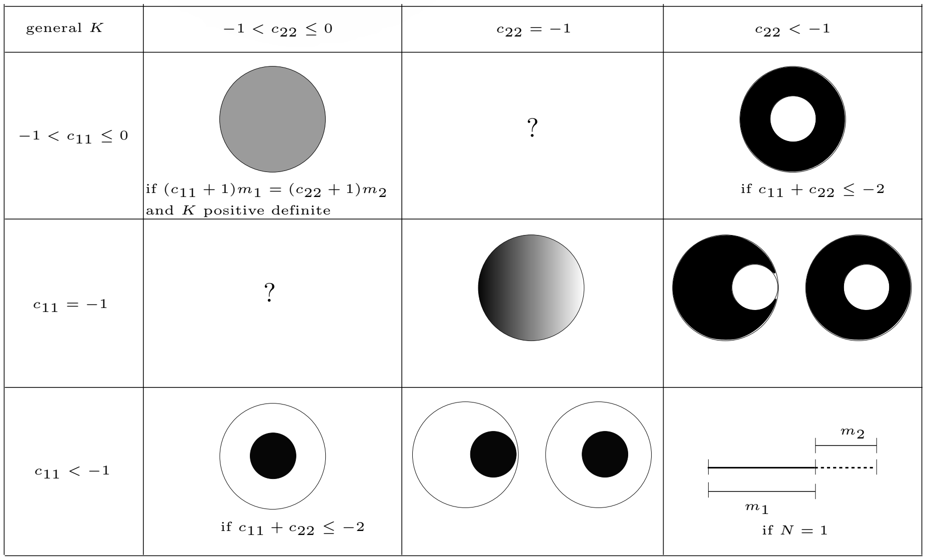


FIG. 1. *The phase f_1 is the black one, whereas the phase f_2 is white. The gray region represents the mixing of the two phases. The gradational shaded ball in the central box represents the extremely degenerate character of minimizers for $c_{11} = c_{22} = -1$.*

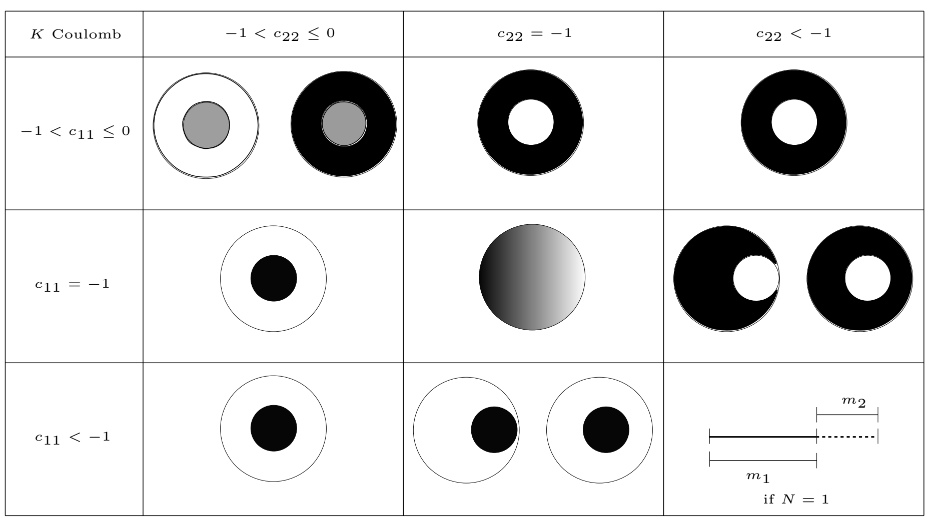


FIG. 2. *The phase f_1 is the black one, and the phase f_2 is white. The gray region represents the mixing of the two phases.*

annulus where only the remaining homogeneous phase is present. If $c_{22} \leq -1 < c_{11}$, then the minimizer is unique and it is given by a ball and a concentric annulus around it. In this respect, for $c_{22} \leq -1 < c_{11}$ the solution is the same in the weakly and strongly attractive cases.

Clearly, in the strongly attractive case the analysis done for general kernels applies in particular to the case of Coulomb interactions. For $c_{11}, c_{22} < -1$, we partially extend the one-dimensional result proven in the case of general kernels by showing that for Coulombic kernels in any dimension, the two phases do not mix each other (Proposition 3.2). The determination of their shapes seems to be a challenging problem that could be explored through numerical methods. Switching the roles of c_{11}, c_{22}, f_1, f_2 in the discussion above, the description of minimizers extends to all the other cases not explicitly mentioned.

We remark that the analysis for the Coulomb interaction kernel is much richer, since we can exploit methods and tools of potential theory such as maximum principles. The characterization of minimizers in the weakly attractive case reduces to the case $c_{11} = c_{22} = 0$, considered in Theorem 3.9. Even if the two phases interact only through a cross attractive force, this case turns out to be nontrivial. The strategy to tackle this problem is based on a rearrangement argument that resembles the Talenti inequality. This is the content of Lemma 3.8, which establishes that given a charge configuration f which generates a potential V, one can rearrange the masses on every superlevel of V, so that the new potential turns out to be greater than the radially symmetric rearrangement V^* of V.

The plan of the paper is the following. In section 1 we introduce the nonlocal model, we prove existence of minimizers, and we show that they have compact support. In section 2 we show some qualitative properties of minimizers and we characterize them explicitly in some strongly attractive cases. Eventually, in section 3 we study in detail the case of Coulomb interactions.

1. The variational problem.

In this section we state our variational problem, proving existence and some qualitative properties of the minimizers.

1.1. Description of the model.

We first introduce a functional modeling the interaction between two non-self-repulsive and mutually attractive species.

Let $N \in \mathbb{N}$ and let $K : \mathbb{R}^N \to \mathbb{R}$ be a nonincreasing radially symmetric interaction potential, with $K \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. For any pair of measurable sets (A, B) with finite measure, we set

$$J_K(A,B) := \int_A \int_B K(x-y)\,\mathrm{d}x\,\mathrm{d}y \tag{1.1}$$

and we notice that, by the assumptions on K, the functional J_K is well defined and takes values in $\mathbb{R} \cup \{-\infty\}$.

Given $c_{ij} \le 0$ for $i, j = 1, 2$, for any pair of measurable sets (E_1, E_2), we set

$$\mathcal{F}_K(E_1,E_2) := c_{11}\, J_K(E_1,E_1) + c_{22}\, J_K(E_2,E_2) + (c_{12}+c_{21})\, J_K(E_1,E_2).$$

Here E_1 and E_2 represent two species, c_{11}, c_{22} the self interaction and $c_{12} + c_{21}$ the cross interaction coefficients. For any fixed $m_1, m_2 > 0$, we are interested in studying the problem

$$\min_{\substack{E_1\cap E_2=\emptyset\\ |E_i|=m_i}} \mathcal{F}_K(E_1,E_2). \tag{1.2}$$

As mentioned in the introduction, for $c_{12} + c_{21} = 0$ the problem decouples into two independent minimization problems, one for each phase. These are of the form

$$\min\left\{-J_K(E,E) : |E| = m\right\}.$$

By the Riesz inequality [31] (see Lemma 1.6), such a one-phase problem is well known to be solved by a ball [16]. As a consequence we focus on the case $c_{12} + c_{21} < 0$

and furthermore, without loss of generality, we set $c_{12} + c_{21} = -2$. From Proposition 2.8 and Theorem 3.9, it will follow that if $|c_{11}|, |c_{22}|$ are small enough, the minimum problem in (1.2) does not admit in general a minimizer. Roughly speaking, the reason is that, in some cases, any minimizing sequence wants to mix the two phases. As a result, we are led to consider a *relaxed problem*. More precisely, according to (1.1), for any $f_1,\ f_2 \in L^1(\mathbb{R}^N;\mathbb{R}^+)$ we set

$$J_K(f_1, f_2) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f_1(x)\, f_2(y)\, K(x-y)\,\mathrm{d}x\,\mathrm{d}y.$$

Then, we consider the functional $\mathcal{E}_K^{c_{11},c_{22}} : L^1(\mathbb{R}^N;\mathbb{R}^+) \times L^1(\mathbb{R}^N;\mathbb{R}^+) \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) = c_{11}\, J_K(f_1, f_1) + c_{22}\, J_K(f_2, f_2) - 2\, J_K(f_1, f_2). \tag{1.3}$$

We introduce the class of admissible densities $\mathcal{A}_{m_1,m_2}$ defined by

$$\begin{aligned}\mathcal{A}_{m_1,m_2} := \Bigg\{&(f_1, f_2) \in L^1(\mathbb{R}^N;\mathbb{R}^+) \times L^1(\mathbb{R}^N;\mathbb{R}^+) :\\ &\int_{\mathbb{R}^N} f_i(x)\,\mathrm{d}x = m_i \text{ for } i = 1,2,\ f_1(x) + f_2(x) \le 1 \text{ for a.e. } x \in \mathbb{R}^N\Bigg\}.\end{aligned} \tag{1.4}$$

It is easy to see that for any $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$

$$\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) = \inf \liminf_{n\to\infty} \mathcal{F}_K(E_1^n, E_2^n),$$

where the infimum is taken among all sequences $\{E_i^n\}$ $(i = 1,2)$ with $|E_i^n| = m_i$ and such that $\chi_{E_i^n}$ converge tightly to f_i. We also observe that if the kernel K is bounded at infinity, then the energy is continuous with respect to tight convergence: if $f_i^n \overset{*}{\rightharpoonup} f_i$ and $\|f_i^n\|_1 \to \|f_i\|_1$ for $i = 1,2$, then $\mathcal{E}_K^{c_{11},c_{22}}(f_1^n, f_2^n) \to \mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2)$.

For $i = 1,2$, set $V_i := f_i * K$, so that we can write

$$\begin{aligned}\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) &= c_{11}\int_{\mathbb{R}^N} f_1(x)V_1(x)\,\mathrm{d}x + c_{22}\int_{\mathbb{R}^N} f_2(x)V_2(x)\,\mathrm{d}x\\ &\quad - 2\int_{\mathbb{R}^N} f_1(x)V_2(x)\,\mathrm{d}x\\ &= c_{11}\int_{\mathbb{R}^N} f_1(x)V_1(x)\,\mathrm{d}x + c_{22}\int_{\mathbb{R}^N} f_2(x)V_2(x)\,\mathrm{d}x\\ &\quad - 2\int_{\mathbb{R}^N} f_2(x)V_1(x)\,\mathrm{d}x.\end{aligned} \tag{1.5}$$

We now recall the definitions of the main classes of kernels we will focus on. We say that the kernel K is *positive definite* if

$$\begin{aligned}&J_K(\varphi, \varphi) \ge 0 \ \ \forall \varphi \in L^1(\mathbb{R}^N) \text{ and}\\ &J_K(\varphi, \varphi) = 0 \text{ if and only if } \varphi = 0 \text{ a.e. in } \mathbb{R}^N.\end{aligned} \tag{1.6}$$

We notice that if K is positive definite, locally integrable, radially symmetric, and nonincreasing, then it can be easily seen that $K \ge 0$ a.e.

We denote by K_{C_N} the Coulomb kernel in $\mathbb{R}^N$, defined by

$$K_{C_N}(x) := \begin{cases} -\dfrac{1}{2}\,|x| & \text{for } N=1, \\[2ex] -\dfrac{1}{2\pi}\log|x| & \text{for } N=2, \\[2ex] \dfrac{1}{(N-2)\,\omega_N}\,\dfrac{1}{|x|^{N-2}} & \text{for } N\geq 3, \end{cases} \tag{1.7}$$

where ω_N is the N-dimensional measure of the unitary ball in $\mathbb{R}^N$. By definition, $-\Delta K_{C_N} = \delta_0$ for any N so that $-\Delta V_i(x) = f_i(x)$. In the following remark we list some properties of the Coulomb kernels that will be useful in the rest of the paper.

Remark 1.1. By [23, Theorem 1.15] K_{C_N} is positive definite for $N \geq 3$ but not for $N = 1, 2$. Nevertheless, by [23, Theorem 1.16], for any $\varphi \in L^1(\mathbb{R}^2)$ with compact support and $\int_{\mathbb{R}^2} \varphi(x)\,\mathrm{d}x = 0$, we have

$$J_{K_{C_2}}(\varphi,\varphi) \geq 0,$$

where equality holds true if and only if $\varphi(x) = 0$ for a.e. $x \in \mathbb{R}^2$. Finally, it is easy to see that the same result holds true also for K_{C_1}.

1.2. First and second variations. For any given $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ set

$$G_i := \{x \in \mathbb{R}^N :\, 0 < f_i(x) < 1\}, \quad F_i := \{x \in \mathbb{R}^N :\, f_i(x) = 1\}, \quad i = 1,2. \tag{1.8}$$

Moreover, we set

$$S := \{x \in \mathbb{R}^N :\, f_1(x) + f_2(x) = 1\}. \tag{1.9}$$

LEMMA 1.2 (first variation). *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Let $i, j \in \{1,2\}$ with $i \neq j$. For any $\varphi_i, \psi \in L^1(\mathbb{R}^N;\mathbb{R}^+)$ with $\varphi_i = 0$ a.e. in $\mathbb{R}^N \setminus (G_i \cup F_i)$, $\psi = 0$ a.e. in S, and $\int_{\mathbb{R}^N} \varphi_i(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \psi(x)\,\mathrm{d}x$, we have*

$$\int_{\mathbb{R}^N} (\psi(x) - \varphi_i(x))(c_{ii}V_i(x) - V_j(x))\,\mathrm{d}x \geq 0. \tag{1.10}$$

As a consequence,

$$c_{ii}V_i - V_j = \gamma_i \text{ a.e. in } G_i \setminus S \tag{1.11}$$

for some constant $\gamma_i \in \mathbb{R}$.

Proof. To simplify notation we prove the claim for $i = 1$ and $j = 2$. The proof of the other case can be obtained by switching f_1 with f_2 and c_{11} with c_{22}. Without loss of generality, we assume $\varphi_1, \psi \in L^\infty(\mathbb{R}^N;\mathbb{R}^+)$. For any $\varepsilon > 0$, we set

$$A^\varepsilon := \{x \in G_1 \cup F_1 :\, \varepsilon < f_1(x) \leq 1\}, \quad B^\varepsilon := \{x \in \mathbb{R}^N :\, f_1(x) + f_2(x) < 1 - \varepsilon\}.$$

It is easy to see that $A^\varepsilon \nearrow (G_1 \cup F_1)$, $B^\varepsilon \nearrow (\mathbb{R}^N \setminus S)$ as $\varepsilon \searrow 0$. Set

$$\varphi_1^\varepsilon := \frac{\int_{\mathbb{R}^N} \varphi_1(x)\,\mathrm{d}x}{\int_{A^\varepsilon} \varphi_1(x)\,\mathrm{d}x} \cdot \varphi_1 \llcorner A^\varepsilon, \quad \psi^\varepsilon := \frac{\int_{\mathbb{R}^N} \psi(x)\,\mathrm{d}x}{\int_{B^\varepsilon} \psi(x)\,\mathrm{d}x} \cdot \psi \llcorner B^\varepsilon;$$

then $\|\varphi_1^\varepsilon - \varphi_1\|_{L^1} \to 0$ and $\|\psi^\varepsilon - \psi\|_{L^1} \to 0$. For $t > 0$ small enough, $(f_1 + t(\psi^\varepsilon - \varphi_1^\varepsilon), f_2) \in \mathcal{A}_{m_1,m_2}$ and, since (f_1, f_2) is a minimizer for $\mathcal{E}_K^{c_{11},c_{22}}$, we have

$$0 \le \lim_{t\to 0} \frac{\mathcal{E}_K^{c_{11},c_{22}}(f_1 + t(\psi^\varepsilon - \varphi_1^\varepsilon), f_2) - \mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2)}{t}$$
$$= \int_{\mathbb{R}^N} 2(\psi^\varepsilon(x) - \varphi_1^\varepsilon(x))\,(c_{11}V_1(x) - V_2(x))\,\mathrm{d}x.$$

As $\varepsilon \to 0$, we get the claim.

Finally, taking $\varphi_1 = \psi \equiv 0$ in $\mathbb{R}^N \setminus (G_1 \setminus S) = S \setminus G_1$ we are allowed to switch the roles of ψ and φ_1 in (1.10), obtaining (1.11). ◻

From now on, given any subset E of $\mathbb{R}^N$, we will always assume that E coincides with the set of the Lebesgue points of its characteristic function. In this way, ∂E will be well defined and will always refer to this precise representative of E.

COROLLARY 1.3. *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Then, for any $\varphi_1, \varphi_2 \in L^1(\mathbb{R}^N;\mathbb{R}^+)$ with $\varphi_i = 0$ a.e. in $\mathbb{R}^N \setminus (G_i \cup F_i)$ for $i = 1, 2$, and $\int_{\mathbb{R}^N} \varphi_1(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \varphi_2(x)\,\mathrm{d}x$, we have*

$$\int_{\mathbb{R}^N} (\varphi_2(x) - \varphi_1(x))((c_{11} + 1)V_1(x) - (c_{22} + 1)V_2(x))\,\mathrm{d}x \ge 0. \tag{1.12}$$

In particular, for any $x_1 \in \overline{G_1 \cup F_1}$ and $x_2 \in \overline{G_2 \cup F_2}$, we have

$$(c_{11} + 1)V_1(x_1) - (c_{22} + 1)V_2(x_1) \le (c_{11} + 1)V_1(x_2) - (c_{22} + 1)V_2(x_2). \tag{1.13}$$

Moreover,

$$(c_{11} + 1)V_1 - (c_{22} + 1)V_2 = \gamma \qquad \text{a.e. in } G_1 \cap G_2 \tag{1.14}$$

for some constant $\gamma \in \mathbb{R}$.

Proof. The proof of (1.12) follows along the lines of that of Lemma 1.2. For the sake of completeness we include here the details. Without loss of generality, we assume φ_1, $\varphi_2 \in L^\infty(\mathbb{R}^N;\mathbb{R}^+)$. For any $\varepsilon > 0$, we set

$$A_1^\varepsilon := \{x \in G_1 \cup F_1 : \varepsilon < f_1(x) \le 1\},\ \ A_2^\varepsilon := \{x \in G_2 \cup F_2 : \varepsilon < f_2(x) \le 1\}.$$

It is easy to see that $A_i^\varepsilon \nearrow (G_i \cup F_i)$ as $\varepsilon \searrow 0$ (for any $i = 1, 2$). Set

$$\varphi_i^\varepsilon := \frac{\int_{\mathbb{R}^N} \varphi_i(x)\,\mathrm{d}x}{\int_{A_i^\varepsilon} \varphi_i(x)\,\mathrm{d}x} \cdot \varphi_i \llcorner A_i^\varepsilon \ \ (i = 1, 2);$$

then $\|\varphi_i^\varepsilon - \varphi_i\|_{L^1} \to 0$ for $i = 1, 2$. For $t > 0$ small enough, $(f_1 + t(\varphi_2^\varepsilon - \varphi_1^\varepsilon), f_2 - t(\varphi_2^\varepsilon - \varphi_1^\varepsilon)) \in \mathcal{A}_{m_1,m_2}$ and, since (f_1, f_2) is a minimizer for $\mathcal{E}_K^{c_{11},c_{22}}$, we have

$$0 \le \lim_{t\to 0} \frac{\mathcal{E}_K^{c_{11},c_{22}}(f_1 + t(\varphi_2^\varepsilon - \varphi_1^\varepsilon), f_2 - t(\varphi_2^\varepsilon - \varphi_1^\varepsilon)) - \mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2)}{t}$$
$$= \int_{\mathbb{R}^N} 2(\varphi_2^\varepsilon(x) - \varphi_1^\varepsilon(x))\,((c_{11} + 1)V_1(x) - (c_{22} + 1)V_2(x))\,\mathrm{d}x.$$

As $\varepsilon \to 0$, we get (1.12). Finally, taking $\varphi_1, \varphi_2 \in L_c^\infty(\mathbb{R}^N;\mathbb{R}^+)$, with $\varphi_1 = \varphi_2 = 0$ a.e. in $\mathbb{R}^N \setminus (G_1 \cap G_2)$ and $\int_{\mathbb{R}^N} \varphi_1(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \varphi_2(x)\,\mathrm{d}x$, we have that (1.12) holds true also switching φ_1 with φ_2, whence we get (1.14). ◻

Using Lemma 1.2 and Corollary 1.3, we prove the following stationarity equations for the boundaries of the two phases (see also [30, equations (1.2)–(1.4)] for similar conditions in a related model for triblock copolymers).

Corollary 1.4. *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ and assume that $f_i = \chi_{E_i}$ for some sets $E_i \subset \mathbb{R}^N$. Then, the following equalities hold:*

$$c_{11}V_1 - V_2 = c_1 \qquad \text{on } \partial E_1 \setminus \partial E_2, \tag{1.15}$$

$$c_{22}V_2 - V_1 = c_2 \qquad \text{on } \partial E_2 \setminus \partial E_1, \tag{1.16}$$

$$(c_{11}+1)V_1 - (c_{22}+1)V_2 = c_1 - c_2 \qquad \text{on } \partial E_1 \cap \partial E_2 \tag{1.17}$$

for some c_1, $c_2 \in \mathbb{R}$.

Proof. As mentioned above, we assume that the sets E_i coincide with the sets of the Lebesgue points of their characteristic functions. We start by proving (1.15). Let ξ, $\eta \in \partial E_1 \setminus \partial E_2$ and let r_ξ, $r_\eta > 0$ be such that

$$|B_{r_\xi}(\xi) \cap E_2| = |B_{r_\eta}(\eta) \cap E_2| = 0 \text{ and } |E_1 \cap B_{r_\xi}(\xi)| = |B_{r_\eta}(\eta) \setminus E_1|.$$

Set $\varphi_1 := \chi_{E_1 \cap B_{r_\xi}(\xi)}$ and $\psi := \chi_{B_{r_\eta}(\eta)\setminus E_1}$. It is easy to see that φ_1 and ψ satisfy all the assumptions of Lemma 1.2, and by (1.10) we have immediately

$$\int_{B_{r_\eta}(\eta)\setminus E_1} (c_{11}V_1(x) - V_2(x))\,\mathrm{d}x \geq \int_{B_{r_\xi}(\xi)\cap E_1} (c_{11}V_1(x) - V_2(x))\,\mathrm{d}x.$$

Since V_1 and V_2 are continuous, taking the limit as $r_\xi, r_\eta \to 0$ we get

$$c_{11}V_1(\eta) - V_2(\eta) \geq c_{11}V_1(\xi) - V_2(\xi); \tag{1.18}$$

switching the roles of ξ and η, we get the equality in (1.18) and hence (1.15) holds true. The proof of (1.16) is fully analogous and is left to the reader, so it remains to prove only (1.17). Let ξ, $\eta \in \partial E_1 \cap \partial E_2$ and let r_ξ, $r_\eta > 0$ be such that

$$|B_{r_\xi}(\xi) \cap E_1| = |B_{r_\eta}(\eta) \cap E_2|.$$

Set $\varphi_1 := \chi_{B_{r_\xi}(\xi)\cap E_1}$ and $\varphi_2 := \chi_{B_{r_\eta}(\eta)\cap E_2}$. It is easy to see that φ_1 and φ_2 satisfy the assumptions of Corollary 1.3, so by (1.12) we have immediately that

$$\int_{B_{r_\eta}(\eta)\cap E_2} ((c_{11}+1)V_1(x) - (c_{22}+1)V_2(x))\,\mathrm{d}x \\ \geq \int_{B_{r_\xi}(\xi)\cap E_1} ((c_{11}+1)V_1(x) - (c_{22}+1)V_2(x))\,\mathrm{d}x.$$

Taking the limit as $r_\xi, r_\eta \to 0$, we have

$$(c_{11}+1)V_1(\eta) - (c_{22}+1)V_2(\eta) \geq (c_{11}+1)V_1(\xi) - (c_{22}+1)V_2(\xi). \tag{1.19}$$

By switching the roles of ξ and η, we get the equality in (1.19) and hence

$$(c_{11}+1)V_1 - (c_{22}+1)V_2 = c \quad \text{on } \partial E_1 \cap \partial E_2$$

for some $c \in \mathbb{R}$. Finally, since V_1 and V_2 are continuous we obtain that $c = c_1 - c_2$ and hence (1.17). ∎

LEMMA 1.5 (second variation). *Let (f_1, f_2) be a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Then for any $\varphi \in L^1(\mathbb{R}^N;\mathbb{R})$ with $\varphi = 0$ in $\mathbb{R}^N \setminus (G_1 \cap G_2)$ and $\int_{\mathbb{R}^N} \varphi = 0$, we have*

$$(c_{11} + c_{22} + 2) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x - y)\, \varphi(x)\, \varphi(y)\, \mathrm{d}x\, \mathrm{d}y \geq 0. \tag{1.20}$$

Proof. Without loss of generality assume that $\varphi \in L^\infty(\mathbb{R}^N;\mathbb{R})$. It is easy to see that

$$\begin{aligned}\mathcal{E}_K^{c_{11},c_{22}}(f_1 + t\varphi, f_2 - t\varphi) = \mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) \\ + 2\,t \int_{\mathbb{R}^N} \varphi(x)\, ((c_{11} + 1)V_1(x) - (c_{22} + 1)V_2(x))\, \mathrm{d}x \\ + t^2(c_{11} + c_{22} + 2) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x - y)\, \varphi(x)\, \varphi(y)\, \mathrm{d}x\, \mathrm{d}y.\end{aligned}$$

Since $(f_1 + t\varphi, f_2 - t\varphi) \in \mathcal{A}_{m_1,m_2}$ for t small enough and (f_1, f_2) is a minimizer, the last term in the sum above is nonnegative and hence (1.20) holds true. ☐

1.3. Existence of minimizers. Here we prove that for every c_{11}, $c_{22} \leq 0$, the functional $\mathcal{E}_K^{c_{11},c_{22}}$ defined in (1.3) admits a minimizer in $\mathcal{A}_{m_1,m_2}$.

First, we recall the classical Riesz inequality [31]. To this purpose, for any $m > 0$ and $x_0 \in \mathbb{R}^N$, we denote by $B^m(x_0)$ the ball centered in x_0 with $|B^m(x_0)| = m$ (B^m if $x_0 = 0$). With a little abuse of notation, for any $x_0 \in \mathbb{R}^N$ and for any $f \in L^1(\mathbb{R}^N)$, we set $B^f(x_0) := B^{\|f\|_{L^1}}(x_0)$ ($B^f := B^{\|f\|_{L^1}}$ if $x_0 = 0$). Moreover, for every function $u \in L^1(\mathbb{R}^N;\mathbb{R}^+)$ we denote by u^* the spherical symmetric nonincreasing rearrangement of u, satisfying

$$\{u^* > t\} = B^{m_t}, \text{ where } m_t := |\{u > t\}| \qquad \text{for all } t > 0. \tag{1.21}$$

LEMMA 1.6 (Riesz inequality). *Let f, $g \in L^1(\mathbb{R}^N;[0,1])$ with $\|f\|_{L^1}, \|g\|_{L^1} > 0$. Then,*

$$\begin{aligned}\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f(x)\, g(y)\, K(x - y)\, \mathrm{d}x\, \mathrm{d}y &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} f^*(x)\, g^*(y)\, K(x - y)\, \mathrm{d}x\, \mathrm{d}y \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi_{B^f}(x)\, \chi_{B^g}(y)\, K(x - y)\, \mathrm{d}x\, \mathrm{d}y,\end{aligned}$$

where the first inequality is in fact an equality if and only if $f(\cdot) = f^*(\cdot - x_0)$ and $g(\cdot) = g^*(\cdot - x_0)$ for some $x_0 \in \mathbb{R}^N$, whereas the second inequality holds with the equality if and only if $f^* = \chi_{B^f}$ and $g^* = \chi_{B^g}$.

Moreover, for any $m_1, m_2 > 0$, we set

$$I_{m_1,m_2}^{c_{11},c_{22}} := \inf_{(f_1,f_2) \in \mathcal{A}_{m_1,m_2}} \mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2)$$

and we extend this definition to the case of possibly null masses by setting

$$I_{m_1,m_2}^{c_{11},c_{22}} := \begin{cases} \min\limits_{\substack{f_i \in L^1(\mathbb{R}^N;[0,1]) \\ \int_{\mathbb{R}^N} f_i(x)\, \mathrm{d}x = m_i}} c_{ii}\, J_K(f_i, f_i) & \text{if } m_i > 0 \text{ and } m_j = 0, \\ 0 & \text{if } m_1 = m_2 = 0. \end{cases}$$

The following two lemmas state monotonicity and subadditivity properties of the energy with respect to the masses m_1, m_2 for nonnegative kernels. Their proofs can be easily obtained exploiting the fact that the two phases attract each other: adding masses or moving back masses going to infinity decreases the energy. The details of the proofs are left to the reader.

LEMMA 1.7. *Assume that $K(x) \geq 0$ for all $x \in \mathbb{R}^N$. For any $m_1 \geq \tilde{m}_1 \geq 0$ and $m_2 \geq \tilde{m}_2 \geq 0$ we have*

$$I^{c_{11},c_{22}}_{m_1,m_2} \leq I^{c_{11},c_{22}}_{\tilde{m}_1,\tilde{m}_2}.$$

Moreover, if m_1, $m_2 > 0$, equality holds true if and only if $m_i = \tilde{m}_i$ for $i = 1, 2$.

LEMMA 1.8. *Assume that $K(x) \geq 0$ for all $x \in \mathbb{R}^N$. Let $\{m^l_1\}$, $\{m^l_2\}$ be two nonnegative sequences such that $0 \leq \tilde{m}_i := \sum_{l\in\mathbb{N}} m^l_i < +\infty$ for $i = 1, 2$. Then*

$$\sum_{l\in\mathbb{N}} I^{c_{11},c_{22}}_{m^l_1,m^l_2} \geq I^{c_{11},c_{22}}_{\tilde{m}_1,\tilde{m}_2}. \tag{1.22}$$

Moreover, if $\tilde{m}_1$, $\tilde{m}_2 > 0$, then equality holds true if and only if $\tilde{m}^l_i \equiv 0$ for any $l \neq \bar{l}$, for some $\bar{l} \in \mathbb{N}$, and for $i = 1, 2$.

We are now in a position to prove the existence of minimizers of the energy $\mathcal{E}^{c_{11},c_{22}}_K$ in $\mathcal{A}_{m_1,m_2}$.

THEOREM 1.9. *Let c_{11}, $c_{22} \leq 0$. Then, the functional $\mathcal{E}^{c_{11},c_{22}}_K$ defined in* (1.3) *admits a minimizer in $\mathcal{A}_{m_1,m_2}$. More precisely, let $\{(f_{1,n}, f_{2,n})\}$ be a minimizing sequence. Then, there exists a sequence of translations $\{\tau_n\} \subset \mathbb{R}^N$ such that (up to a subsequence) $f_{i,n}(\cdot - \tau_n) \to f_i$ tightly for some $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ which minimizes $\mathcal{E}^{c_{11},c_{22}}_K$.*

Proof. We distinguish between two cases.

First case: $\lim_{|x|\to+\infty} K(x) = -\infty$. For every $\varepsilon > 0$ and for every pair of sets $A_{1,n}$, $A_{2,n} \subset \mathbb{R}^N$ such that

$$\int_{A_{i,n}} f_{i,n}(x)\, \mathrm{d}x \geq \varepsilon,$$

we have $\mathrm{dist}(A_{1,n}, A_{2,n}) \leq C$ for some C independent of n; otherwise, we would clearly have $-J_K(f_{1,n}, f_{2,n}) \to +\infty$. As a consequence, by the triangular inequality we deduce that for every pair of sets $A_{i,n}$, $B_{i,n} \subset \mathbb{R}^N$ such that

$$\int_{A_{i,n}} f_{i,n}(x)\, \mathrm{d}x \geq \varepsilon, \qquad \int_{B_{i,n}} f_{i,n}(x)\, \mathrm{d}x \geq \varepsilon,$$

we have $\mathrm{dist}(A_{i,n}, B_{i,n}) \leq C$ for some C independent of n. As a result there exists $\{\tau_n\} \subset \mathbb{R}^N$ such that, up to a subsequence, $f_{i,n}(\cdot - \tau_n)$ tightly converge to some f_i in L^1. By the lower semicontinuity of $\mathcal{E}^{c_{11},c_{22}}_K$ with respect to the tight convergence, we conclude that (f_1, f_2) is a minimizer of $\mathcal{E}^{c_{11},c_{22}}_K$ in $\mathcal{A}_{m_1,m_2}$.

Second case: $\lim_{|x|\to+\infty} K(x) + C = 0$ *for some $C \in \mathbb{R}$*. For simplicity, we assume that $C = 0$, since additive constants in the kernel bring only an additive constant in the total energy. Set $Q_0 := [0,1]^N$, and for every $z \in \mathbb{Z}^N$, let $Q^z := z + Q_0$ and $m^z_{i,n} := \int_{Q^z} f_{i,n}(x)\,\mathrm{d}x$. For any given $\varepsilon > 0$, we set

$$\mathcal{I}_{\varepsilon,n} := \{z \in \mathbb{Z}^N : m^z_{i,n} \le \varepsilon,\, i = 1,2\}, \mathcal{J}_{\varepsilon,n} := \{z \in \mathbb{Z}^N : \max_i m^z_{i,n} > \varepsilon\},$$
$$A^\varepsilon_n := \bigcup_{z\in\mathcal{I}_{\varepsilon,n}} Q^z, g^\varepsilon_{i,n} := f_{i,n}\,\chi_{A^\varepsilon_n},$$
$$E^\varepsilon_n := \bigcup_{z\in\mathcal{J}_{\varepsilon,n}} Q^z, f^\varepsilon_{i,n} := f_{i,n}\,\chi_{E^\varepsilon_n}.$$

We first prove that

$$J_K(g^\varepsilon_{1,n}, f_{1,n}) + J_K(g^\varepsilon_{2,n}, f_{2,n}) + J_K(g^\varepsilon_{1,n}, f_{2,n}) + J_K(f_{1,n}, g^\varepsilon_{2,n}) \le r(\varepsilon), \tag{1.23}$$

where $r(\varepsilon) \to 0$ as $\varepsilon \to 0$. We show only that $J_K(g^\varepsilon_{1,n}, f_{2,n}) < r(\varepsilon)$ (the other cases being analogous). For every fixed $R \in \mathbb{N}$ we have

$$\begin{aligned} J_K(g^\varepsilon_{1,n}, f_{2,n}) &= \sum_{z\in\mathcal{I}_{\varepsilon,n}} \sum_{w\in\mathbb{Z}^N} J_K(f_{1,n}\chi_{Q^z}, f_{2,n}\chi_{Q^w}) \\ &= \sum_{z\in\mathcal{I}_{\varepsilon,n}, w\in\mathbb{Z}^N : |z-w|\le R} J_K(f_{1,n}\chi_{Q^z}, f_{2,n}\chi_{Q^w}) \\ &\quad + \sum_{z\in\mathcal{I}_{\varepsilon,n}, w\in\mathbb{Z}^N : |z-w|> R} J_K(f_{1,n}\chi_{Q^z}, f_{2,n}\chi_{Q^w}). \end{aligned} \tag{1.24}$$

Set $h(t) := \int_{B^t} K(\xi)\,\mathrm{d}\xi$; using the Riesz inequality (see Lemma 1.6), it is easy to see that

$$J_K(f_{1,n}\chi_{Q^z}, f_{2,n}\chi_{Q^w}) \le \int_{B^{m^w_{2,n}}} \mathrm{d}x \int_{B^{m^z_{1,n}}(x)} K(\xi)\,\mathrm{d}\xi \le h(m^z_{1,n})\, m^w_{2,n},$$

where the last inequality is a consequence of the fact that K is nonincreasing radially symmetric. We deduce that the first addendum in (1.24) tends to zero as $\varepsilon \to 0$ (for R fixed). Moreover, the second addendum is bounded (uniformly with respect to ε) from above by a function $\omega(R)$ such that $\omega(R) \to 0$ as $R \to \infty$. This completes the proof of (1.23).

By the mass constraints on f_i we have that $\sharp\mathcal{J}_{\varepsilon,n} \le \frac{m_1+m_2}{\varepsilon}$. Therefore, up to a subsequence, we can always write $E^\varepsilon_n = \cup_{l=1}^{H_\varepsilon} J^l_{\varepsilon,n}$ for some $H_\varepsilon \le \frac{m_1+m_2}{\varepsilon}$, where the sets $J^l_{\varepsilon,n}$ are pairwise disjoint and satisfy

(1) for every l, $\mathrm{diam}(J^l_{\varepsilon,n}) \le M_\varepsilon$ for some $M_\varepsilon \in \mathbb{R}$ independent of n;

(2) for every $l_1 \ne l_2$, $\mathrm{dist}(J^{l_1}_{\varepsilon,n}, J^{l_2}_{\varepsilon,n}) \to \infty$ as $n \to \infty$.

Notice that by (1.23) we deduce that, for ε small enough, $E^\varepsilon_n \ne \emptyset$ and $H_\varepsilon \ge 1$ (otherwise $I^{c_{11},c_{22}}_{m_1,m_2}$ would be zero). Set $f^{\varepsilon,l}_{i,n} := f^\varepsilon_{i,n} \llcorner \bigcup_{z\in J^l_{\varepsilon,n}} Q^z$ for $i = 1,2$ and for every $l = 1,\dots,H_\varepsilon$. There exists a translation $\tau_{l,n}$ such that, up to a subsequence, $f^{\varepsilon,l}_{i,n}(\cdot - \tau_{l,n})$ converge tightly to some $f^{\varepsilon,l}_i$. By (1.23), recalling that $\lim_{|x|\to+\infty} K(x) = 0$ and using the continuity of the energy with respect to the tight convergence, we have

$$\begin{aligned} \lim_n \mathcal{E}^{c_{11},c_{22}}_K(f_{1,n}, f_{2,n}) &\ge \limsup_n \mathcal{E}^{c_{11},c_{22}}_K(f^\varepsilon_{1,n}, f^\varepsilon_{2,n}) - r(\varepsilon) \\ &= \limsup_n \sum_{l=1}^{H_\varepsilon} \mathcal{E}^{c_{11},c_{22}}_K(f^{\varepsilon,l}_{1,n}, f^{\varepsilon,l}_{2,n}) - r(\varepsilon) \ge \sum_{l=1}^{H_\varepsilon} \mathcal{E}^{c_{11},c_{22}}_K(f^{\varepsilon,l}_1, f^{\varepsilon,l}_2) - r(\varepsilon). \end{aligned} \tag{1.25}$$

Let now $\{\varepsilon_k\}$ be a decreasing sequence converging to zero as $k \to \infty$. We notice that H_{ε_k} is nondecreasing with respect to k and then $H_{\varepsilon_k} \to H \in \mathbb{N} \cup \infty$. We can always choose the labels in such a way that the sequences $\{f_{i,n}^{\varepsilon_k,l}\}$, and so their limits $f_i^{\varepsilon_k,l}$, are monotone with respect to k. As a consequence, it is not restrictive to assume that the translation vectors $\tau_{l,n}$ are independent of ε. By monotonicity, $f_i^{\varepsilon_k,l}$ converge strongly in L^1 to some f_i^l for any $1 \le l \le H$ and $i = 1, 2$. By (1.25) and the continuity of $\mathcal{E}_K^{c_{11},c_{22}}$ with respect to the tight convergence, it follows that

$$I_{m_1,m_2}^{c_{11},c_{22}} = \lim_n \mathcal{E}_K^{c_{11},c_{22}}(f_{1,n}, f_{2,n}) \ge \sum_{l=1}^{H} \mathcal{E}_K^{c_{11},c_{22}}(f_1^l, f_2^l). \tag{1.26}$$

Let $m_i^l := \int_{\mathbb{R}^N} f_i^l(x)\,\mathrm{d}x$; then $\tilde{m}_i := \sum_{l=1}^{H} m_i^l \le m_i$ for $i = 1, 2$.

By (1.26) and Lemmas 1.8 and 1.7, we get

$$I_{m_1,m_2}^{c_{11},c_{22}} \ge \sum_{l=1}^{H} \mathcal{E}_K^{c_{11},c_{22}}(f_1^l, f_2^l) \ge \sum_{l=1}^{H} I_{m_1^l,m_2^l}^{c_{11},c_{22}} \ge I_{\tilde{m}_1,\tilde{m}_2}^{c_{11},c_{22}} \ge I_{m_1,m_2}^{c_{11},c_{22}};$$

it follows that all the inequalities above are in fact equalities, $H = 1$ and $\tilde{m}_i = m_i$, which concludes the proof. ∎

Remark 1.10. As already explained in the introduction, the constraint $f_1+f_2 \le 1$ represents a noninterpenetration condition. One might wonder how relaxing this constraint affects the (existence result and) shape of the minimizer. Replacing the constraint $f_1 + f_2 \le 1$ with the weaker one $f_1, f_2 \le 1$, the Riesz inequality would immediately yield that f_i are characteristic functions of concentric balls. Finally, since all the forces are attractive, prescribing only a mass constraint on f_i yields to concentration in a single point, i.e., the solution becomes a measure given by $f_i = m_i\delta_x$ for some $x \in \mathbb{R}^N$.

Remark 1.11. The problem considered in this paper could be generalized to the case of more than two phases, with mutual and self attractive interactions. We notice that, with minor changes, the existence of a solution for this generalized problem would follow along the lines of the proof of Theorem 1.9.

Remark 1.12. Notice that in the case of $c_{11}, c_{22} > 0$ the functional $\mathcal{E}_K^{c_{11},c_{22}}$ does not admit in general a minimizer in $\mathcal{A}_{m_1,m_2}$. For instance, if $c_{11} > 0$, then it is easy to see that, for m_1 large enough, any minimizing sequence $f_{1,n}$ for the first phase tends to lose mass at infinity. As a consequence, $\mathcal{E}_K^{c_{11},c_{22}}$ does not admit a minimizer in $\mathcal{A}_{m_1,m_2}$ for m_1 large enough.

Moreover, assume that K is a positive definite kernel as in (1.6), and let $c_{11}, c_{22} \ge 1$. Then, for any $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$, we have

$$\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) = (c_{11} - 1)\, J_K(f_1, f_1) + (c_{22} - 1)\; J_K(f_2, f_2) + J_K(f_1 - f_2, f_1 - f_2) \ge 0.$$

Up to adding a constant to the kernel (and hence a constant to the energy functional), we can always assume that K vanishes at infinity. In this case, it is easy to see that the infimum of $\mathcal{E}_K^{c_{11},c_{22}}$ is zero. It follows that (f_1, f_2) is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $m_1 = m_2$, $c_{11} = c_{22} = 1$, and $f_i = f \in L^1(\mathbb{R}^N; [0, \frac{1}{2}])$ with $\int_{\mathbb{R}^N} f(x)\,\mathrm{d}x = m_1 = m_2$.

1.4. Compactness of minimizers. Here we prove the compactness property of minimizers.

PROPOSITION 1.13. *Every minimizer* (f_1, f_2) *of* $\mathcal{E}_K^{c_{11},c_{22}}$ *in* $\mathcal{A}_{m_1,m_2}$ *has compact support.*

Proof. Assume by contradiction that f_1 does not have compact support. Recalling the definition of S in (1.9), we set $r := (2\frac{m_1+m_2}{\omega_N})^{1/N}$ so that $|B_r \setminus S| > 0$. For $R > 0$ we now set $\varphi_1^R := f_1\chi_{(\mathbb{R}^N\setminus B_R)}$ and observe that for R large enough we can find $\psi^R \in L^1(\mathbb{R}^N;\mathbb{R}^+)$ such that $\psi^R \equiv 0$ in $S \cup (\mathbb{R}^N \setminus B_r)$ and at the same time $\int_{\mathbb{R}^N} \varphi_1^R(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} \psi^R(x)\,\mathrm{d}x > 0$. Hence by (1.10) we have

$$\int_{B_r} \psi^R(x)(c_{11}V_1(x) - V_2(x))\,\mathrm{d}x \ge \int_{\mathbb{R}^N\setminus B_R} \varphi_1^R(x)(c_{11}V_1(x) - V_2(x))\,\mathrm{d}x,$$

or, equivalently,

$$\int_{B_r} \psi^R(x)(|c_{11}|V_1(x) + V_2(x))\ \mathrm{d}x \ \le\ \int_{\mathbb{R}^N\setminus B_R} \varphi_1^R(x)(|c_{11}|V_1(x) + V_2(x))\ \mathrm{d}x.$$

Since $\int_{\mathbb{R}^N\setminus B_R} \varphi_1^R(x)\,\mathrm{d}x = \int_{B_r} \psi^R(x)\,\mathrm{d}x$, the previous inequality implies that

$$\inf_{B_r}(|c_{11}|V_1 + V_2) \le \sup_{\mathbb{R}^N\setminus B_R}(|c_{11}|V_1 + V_2),$$

which gives a contradiction for R large enough. ∎

2. Qualitative properties of minimizers and some explicit solutions.

In this section we discuss some qualitative properties of the minimizers of $\mathcal{E}_K^{c_{11},c_{22}}$, and we determine the explicit solutions for some specific choices of the coefficients c_{11}, c_{22}.

2.1. Some preliminary results.

The following lemma states that, for $c_{11} = 0$, there exists a minimizer $(\tilde f_1, f_2)$ such that $\tilde f_1 + f_2 = 1$ on the support of $\tilde f_1$.

LEMMA 2.1 (superlevels). *Let* (f_1, f_2) *be a minimizer of* $\mathcal{E}_K^{0,c_{22}}$ *in* $\mathcal{A}_{m_1,m_2}$. *Set*

$$t := \inf\left\{ s \in \mathbb{R} : \int_{\{V_2>s\}} (1 - f_2(x))\,\mathrm{d}x \le m_1 \right\}.$$

Then, $t \in \mathbb{R}$ *and the pair* $(\tilde f_1, f_2)$ *is still a minimizer of* $\mathcal{E}_K^{0,c_{22}}$ *in* $\mathcal{A}_{m_1,m_2}$ *if and only if* $\tilde f_1 \in L^1(\mathbb{R}^N;[0,1])$ *satisfies* (i), (ii), *and* (iii) *below:*

(i) $\int_{\mathbb{R}^N} \tilde f_1(x)\,\mathrm{d}x = m_1$;

(ii) $\tilde f_1(x) = 1 - f_2(x)$ *if* $V_2(x) > t$;

(iii) $\tilde f_1(x) = 0$ *if* $V_2(x) < t$.

Moreover, if $|\{V_2 = t\}| = 0$, *then* $\tilde f_1$ *is uniquely determined, and clearly* $f_1 = \tilde f_1$.

A similar statement holds true for the case $c_{22} = 0$.

Proof. Since V_2 is bounded from above, $\mathbb{R}^N = \cup_{s\in\mathbb{R}}\{V_2 > s\}$ and $|\{V_2 > t\}| \le m_1 + m_2$, we have $-\infty < t < +\infty$.

Let $\tilde f_1$ satisfy properties (i), (ii), and (iii) above. We show that $(\tilde f_1, f_2)$ is a minimizer of $\mathcal{E}_K^{0,c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Indeed, using that $\tilde f_1 \ge f_1$ on $\{V_2 > t\}$, we have

$$
\begin{aligned}
(2.1)\qquad \mathcal{E}_K^{0,c_{22}}(f_1,f_2) - \mathcal{E}_K^{0,c_{22}}(\tilde f_1,f_2) &= 2\int_{\mathbb{R}^N} V_2(x)(\tilde f_1(x) - f_1(x))\,\mathrm{d}x \\
&= -2\int_{\{V_2<t\}} V_2(x) f_1(x)\,\mathrm{d}x + 2\,t\int_{\{V_2=t\}} (\tilde f_1(x) - f_1(x))\,\mathrm{d}x \\
&\quad + 2\int_{\{V_2>t\}} V_2(x)(\tilde f_1(x) - f_1(x))\,\mathrm{d}x \\
\ge 2\,t\left(-\int_{\{V_2<t\}} f_1(x)\,\mathrm{d}x + \int_{\{V_2=t\}} (\tilde f_1(x) - f_1(x))\,\mathrm{d}x + \int_{\{V_2>t\}} (\tilde f_1(x) - f_1(x))\,\mathrm{d}x\right)& \\
&= 2\,t\int_{\mathbb{R}^N} (\tilde f_1(x) - f_1(x))\,\mathrm{d}x = 0,
\end{aligned}
$$

where the last equality is a direct consequence of (i).

Assume now that $(\hat f_1, f_2)$ is a minimizer of $\mathcal{E}_K^{0,c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. We trivially have that $\hat f_1$ satisfies (i). Notice that the inequality in (2.1) is an equality if and only if $f_1 = 0$ a.e. in $\{V_2 < t\}$ and $\tilde f_1 = f_1$ a.e. in $\{V_2 > t\}$. By replacing f_1 with $\hat f_1$ in (2.1), we have immediately that $\hat f_1$ should satisfy (ii) and (iii). □

We recall that the sets G_i are defined in (1.8).

COROLLARY 2.2. *Let (f_1, f_2) be a minimizer for $\mathcal{E}_K^{0,c_{22}}$ in $\mathcal{A}_{m_1,m_2}$. Then, for any measurable set $E_1 \subset G_1 \setminus G_2$ with $|E_1| = \int_{G_1\setminus G_2} f_1(x)\,\mathrm{d}x$, the function*

$$\tilde f_1(x) := \begin{cases} \chi_{E_1} & \text{if } x \in G_1 \setminus G_2, \\ f_1(x) & \text{otherwise in } \mathbb{R}^N \end{cases}$$

satisfies

$$\mathcal{E}_K^{0,c_{22}}(\tilde f_1, f_2) = \mathcal{E}_K^{0,c_{22}}(f_1, f_2).$$

A similar statement holds true in the case $c_{22} = 0$.

Proof. By Lemma 2.1, there exists $t \in \mathbb{R}$ so that $f_1 = 1 - f_2$ on $\{V_2 > t\}$ and $f_1 = 0$ on $\{V_2 < t\}$. It follows that $G_1 \setminus G_2 \subset \{V_2 = t\}$, and hence

$$
\begin{aligned}
\mathcal{E}_K^{0,c_{22}}(f_1,f_2) - \mathcal{E}_K^{0,c_{22}}(\tilde f_1,f_2) &= 2\int_{G_1\setminus G_2} V_2(x)(\tilde f_1(x) - f_1(x))\,\mathrm{d}x \\
&= 2\,t\int_{G_1\setminus G_2} (\tilde f_1(x) - f_1(x))\,\mathrm{d}x = 0.
\end{aligned}
$$
□

2.2. The strongly attractive case $c_{11} + c_{22} \le -2$. In the following theorem we characterize the minimizers for every c_{11}, c_{22} such that $c_{11} + c_{22} \le -2$ and $\max\{c_{11}, c_{22}\} \ge -1$ (see Figure 3).

THEOREM 2.3. *Let $c_{11} + c_{22} \le -2$. The following statements hold true:*

(i) *if $c_{11} = c_{22} = -1$, then (f_1, f_2) is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ for some $x_0 \in \mathbb{R}^N$;*

(ii) *if $c_{11} = -1$ and $c_{22} < -1$, then $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ for some $x_0 \in \mathbb{R}^N$, and $f_2 = \chi_{B^{m_2}(y_0)}$ for some $y_0 \in \mathbb{R}^N$ with $B^{m_2}(y_0) \subset B^{m_1+m_2}(x_0)$;*

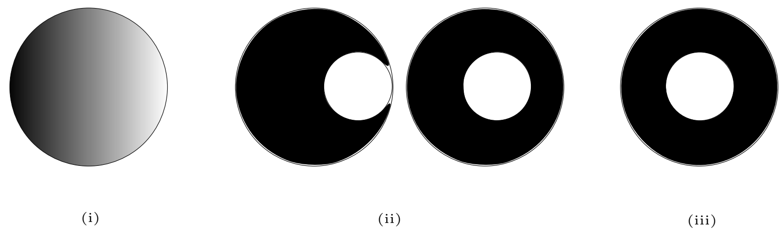


FIG. 3. *The phase f_1 is the black one, whereas the phase f_2 is white. The first image represents the minimizers in* (i). *In this case, all the configurations (f_1, f_2) such that $f_1 + f_2 = \chi_{B^{m_1+m_2}}$ are minimizers of the energy. The second and third images are two examples of minimizers in case* (ii). *The last image is the unique minimizer in case* (iii). *Minimizers in cases* (ii′) *and* (iii′) *can be obtained by the balls above switching the balck parts with the white ones.*

(ii′) *if $c_{22} = -1$ and $c_{11} < -1$, then $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ for some $x_0 \in \mathbb{R}^N$ and $f_1 = \chi_{B^{m_1}(y_0)}$ for some $y_0 \in \mathbb{R}^N$ with $B^{m_1}(y_0) \subset B^{m_1+m_2}(x_0)$;*

(iii) *if $c_{22} < -1$ and $-1 < c_{11} \leq 0$, then $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ and $f_2 = \chi_{B^{m_2}(x_0)}$ for some $x_0 \in \mathbb{R}^N$;*

(iii′) *if $c_{11} < -1$ and $-1 < c_{22} \leq 0$, then $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$ is a minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if $f_1 + f_2 = \chi_{B^{m_1+m_2}(x_0)}$ and $f_1 = \chi_{B^{m_1}(x_0)}$ for some $x_0 \in \mathbb{R}^N$.*

Proof. We prove only (i), (ii), and (iii), the proofs of (ii′) and (iii′) being the same as those of (ii) and (iii), respectively.

It is easy to see that

$$\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) = c_{11}\, J_K(f_1 + f_2, f_1 + f_2) - 2(c_{11} + 1)\, J_K(f_2, f_1 + f_2) + (c_{11} + c_{22} + 2)\, J_K(f_2, f_2).$$

Claim (iii) follows immediately by applying Lemma 1.6 to each of the three addenda above. Moreover,

$$\mathcal{E}_K^{-1,-1}(f_1, f_2) = -J_K(f_1 + f_2, f_1 + f_2),$$
$$\mathcal{E}_K^{-1,c_{22}}(f_1, f_2) = -J_K(f_1 + f_2, f_1 + f_2) + (c_{22} + 1)\, J_K(f_2, f_2)$$

and hence (i) and (ii) easily follow by applying once again Lemma 1.6. ∎

The next proposition gives a characterization for $N = 1$ of the minimizer of $\mathcal{E}_K^{c_{11},c_{22}}$ in the case c_{11}, $c_{22} < -1$ which is left open in Theorem 2.3.

PROPOSITION 2.4. *Let $N = 1$ and c_{11}, $c_{22} < -1$. Then*

$$(f_1, f_2) = (\chi_{[-m_1,0]}, \chi_{[0,m_2]}) \text{ and } (f_1, f_2) = (\chi_{[0,m_1]}, \chi_{[-m_2,0]})$$

are (up to a translation) the unique minimizers of $\mathcal{E}_K^{c_{11},c_{22}}$ in $\mathcal{A}_{m_1,m_2}$.

Proof. It is easy to see that for any (f_1, f_2)

$$\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) = \mathcal{E}_K^{c_{11},-1}(f_1, f_2) + (c_{22} + 1) J_K(f_2, f_2);$$

since the second addendum is minimized when f_2 is the characteristic function of an interval, the claim follows by Theorem 2.3(ii′). ∎

Remark 2.5. In the general multidimensional case, we do not know the explicit form of the minimizers if c_{11}, $c_{22} < -1$. One could guess that f_i are characteristic functions as in the Coulomb case considered in Proposition 3.2. By means of first variation techniques, we can exclude that the solution is given by two tangent balls as well as by a ball and a concentric annulus around it. A natural issue to consider is then the asymptotic behavior of minimizers for c_{11}, c_{22} which tend to the boundary (and at infinity) of the region $\{c_{11}, c_{22} < -1\}$. In fact, there are many interesting limits that one could study:

(1) $c_{11} < -1$, $c_{22\,n} \nearrow -1$. Let $(f_1^n, f_2^n) \in \mathcal{A}_{m_1,m_2}$ be a minimizer of $\mathcal{E}^{c_{11},c_{22\,n}}$ in $\mathcal{A}_{m_1,m_2}$. Notice that the limit problem does not admit a unique solution. Nevertheless, we expect that, up to a unique translation, f_1^n and $f_1^n + f_2^n$ converge strongly in L^1 to characteristic functions of two innerly tangent balls. Indeed, this is the minimizer for $c_{11}, c_{22} < 1$, among the family of pairs of nested balls.

(2) $c_{11}, c_{22} < -1$, $c_{11}, c_{22} \nearrow -1$. In this case the limit problem is the most degenerate one for which it seems difficult to have a clear guess.

(3) $c_{11} < -1$, $c_{22\,n} \to -\infty$. In this case we expect that the second phase tends to a ball, while the first phase tends to the characteristic function of a set which is not a ball.

(4) c_{11}, $c_{22} \to -\infty$. In this case we have that the two phases converge to two tangent balls. This is precisely the content of Proposition 2.6 below.

PROPOSITION 2.6. *Let $\{c_{11\,n}\}, \{c_{22\,n}\} \subset \mathbb{R}$ be such that $c_{11\,n}$, $c_{22\,n} \to -\infty$. For any $n \in \mathbb{N}$, let $(f_1^n, f_2^n) \in \mathcal{A}_{m_1,m_2}$ be a minimizer of $\mathcal{E}_K^{c_{11\,n},c_{22\,n}}$ in $\mathcal{A}_{m_1,m_2}$. Then, up to a unique translation, f_1^n, f_2^n converge strongly in L^1 to characteristic functions of two tangent balls, i.e., there exists a family of translations $\{\tau_n\}$ and a unitary vector $\nu \in \mathbb{R}^N$, such that*

$$f_1^n(\cdot - \tau_n) \to \chi_{B^{m_1}}, \quad f_2^n(\cdot - \tau_n) \to \chi_{B^{m_2}(r\,\nu)} \quad \text{with } r := \left(\tfrac{m_1}{\omega_N}\right)^{\frac{1}{N}}.$$

Proof. First, notice that there exists a constant C such that

$$-2J_K(f_1^n, f_2^n) \ge C, \quad c_{11\,n}\, J_K(f_1^n, f_1^n) \ge c_{11\,n}\, I_{m_1,0}^{-1,0}, \quad c_{22\,n}\, J_K(f_2^n, f_2^n) \ge c_{22\,n}\, I_{0,m_2}^{0,-1},$$

so that

$$I_{m_1,m_2}^{c_{11\,n},c_{22\,n}} = \mathcal{E}_K^{c_{11\,n},c_{22\,n}}(f_1^n, f_2^n) \ge c_{11\,n}\, I_{m_1,0}^{-1,0} + c_{22\,n}\, I_{0,m_2}^{0,-1} + C. \tag{2.2}$$

On the other hand,

$$I_{m_1,m_2}^{c_{11\,n},c_{22\,n}} \le \mathcal{E}_K^{c_{11\,n},c_{22\,n}}(\chi_{B^{m_1}}, \chi_{B^{m_2}(r\,\nu)}) = c_{11\,n}\, I_{m_1,0}^{-1,0} + c_{22\,n}\, I_{0,m_2}^{0,-1} + C, \tag{2.3}$$

which, together with (2.2), yields

$$J_K(f_1^n, f_1^n) \to I_{m_1,0}^{-1,0}, \qquad J_K(f_2^n, f_2^n) \to I_{0,m_2}^{0,-1}.$$

Therefore, by Theorem 1.9 applied to $I_{m_1,0}^{-1,0}$ and $I_{0,m_2}^{0,-1}$, there exist two sequences of translations $\{\tau_i^n\}$ (for $i = 1, 2$) such that

$$f_1^n(\cdot - \tau_1^n) \to \chi_{B^{m_1}}, \qquad f_2^n(\cdot - \tau_2^n) \to \chi_{B^{m_2}} \qquad \text{strongly in } L^1.$$

It remains to prove that $|\tau_1^n - \tau_2^n| \to r$ as $n \to \infty$. Set

$$\lambda_n := \frac{|\tau_1^n - \tau_2^n|}{r}.$$

Notice that $\liminf_{n\to\infty} \lambda_n \geq 1$ (otherwise, for n large, f_1^n and f_2^n would be close in L^1 to characteristic functions of two intersecting balls, so that (f_1^n, f_2^n) would not be admissible). Up to a subsequence, we can assume that $\limsup_{n\to\infty} \lambda_n = \lim_{n\to\infty} \lambda_n =: \lambda$ with $\lambda \geq 1$. Then, set

$$\tilde{f}_1^n := \chi_{B^{m_1}(\tau_1^n)}, \quad \tilde{f}_2^n := \chi_{B^{m_2}(\tau_2^n)};$$

notice that $\|\tilde{f}_i^n - f_i^n\|_{L^1} \to 0$ as $n \to \infty$ for $i = 1, 2$. Then, by the lower semicontinuity property of J_K with respect to the strong L^1 convergence, we get

$$\liminf_n J_K(f_1^n, f_2^n) - J_K(\tilde{f}_1^n, \tilde{f}_2^n) \geq 0.$$

We conclude

$$I_{m_1,m_2}^{c_{11n},c_{22n}} \geq \mathcal{E}_K^{c_{11n},c_{22n}}(\tilde{f}_1^n, \tilde{f}_2^n) + \rho(n) \geq \mathcal{E}_K^{c_{11n},c_{22n}}(\chi_{B^{m_1}}, \chi_{B^{m_2}(r\nu)}) + \rho(n) + \omega(\lambda_n),$$

where $\rho(n) \to 0$ as $n \to \infty$ and $\omega : [1, +\infty) \to [0, +\infty)$ is an increasing function vanishing at 1. By minimality it easily follows that $\lambda = 1$ and hence the claim. ∎

2.3. The weakly attractive case $c_{11} + c_{22} > -2$. Here we will consider the case $c_{11} + c_{22} > -2$, and we will characterize the solution only for the purely weakly attractive case $0 \geq c_{11}, c_{22} > -1$ with $(c_{11} + 1)m_1 = (c_{22} + 1)m_2$. Moreover, we will assume that K is positive definite, according to definition (1.1). Notice that this implies that the functional $J_K(\varphi, \varphi)$ is strictly convex.

LEMMA 2.7. *Let K be positive definite. For any $-1 < c < 1$ and for any $m > 0$, the (unique up to a translation) minimizer of $\mathcal{E}_K^{c,c}$ in $\mathcal{A}_{m,m}$ is given by the pair $(f_1^0, f_2^0) = \left(\frac{1}{2}\chi_{B^{2m}}, \frac{1}{2}\chi_{B^{2m}}\right)$.*

Proof. Let $(f_1, f_2) \in \mathcal{A}_{m,m}$. We first notice that the convexity of the functional $J_K(f, f)$ immediately implies that

$$J_K(f_1, f_2) = 2J_K\left(\tfrac{f_1+f_2}{2}, \tfrac{f_1+f_2}{2}\right) - \tfrac{J_K(f_1,f_1)}{2} - \tfrac{J_K(f_2,f_2)}{2} \leq J_K\left(\tfrac{f_1+f_2}{2}, \tfrac{f_1+f_2}{2}\right). \tag{2.4}$$

Moreover

$$\mathcal{E}_K^{c,c}(f_1, f_2) = c\, J_K(f_1 + f_2, f_1 + f_2) - 2\,(1 + c)\, J_K(f_1, f_2),$$

which, together with (2.4), yields

$$\begin{aligned}\mathcal{E}_K^{c,c}(f_1, f_2) \geq c\, J_K &\left(\frac{f_1 + f_2}{2} + \frac{f_1 + f_2}{2}, \frac{f_1 + f_2}{2} + \frac{f_1 + f_2}{2}\right) \\ &-2(1 + c)\, J_K\left(\frac{f_1 + f_2}{2}, \frac{f_1 + f_2}{2}\right) = \mathcal{E}_K^{c,c}\left(\frac{f_1 + f_2}{2}, \frac{f_1 + f_2}{2}\right),\end{aligned}$$

where in the inequality we have also used that $c + 1 > 0$. By the strict convexity of $J_K(f, f)$, the inequality is strict whenever $f_1 \neq f_2$. It follows that, if (f_1, f_2) is a minimizer, then $f_1 = f_2 = \frac{f_1+f_2}{2} =: f$. Finally, since $\mathcal{E}_K^{c,c}(f, f) = 2(c - 1)\, J_K(f, f)$, by Lemma 1.6, we conclude that $\mathcal{E}_K^{c,c}(f_1, f_2)$ attains its unique minimum when $f_1 = f_2 = \frac{1}{2}\chi_{B^{2m}}$. ∎

Let us introduce the coefficients a_i (depending on c_{11} and c_{22}) which represent the volume fractions of the two phases where they mix:

$$a_1 := \frac{c_{22} + 1}{c_{11} + c_{22} + 2}, \qquad a_2 := \frac{c_{11} + 1}{c_{11} + c_{22} + 2}. \tag{2.5}$$

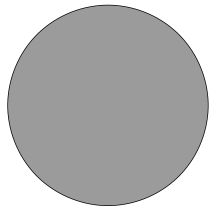


FIG. 4. *Under the assumptions of Proposition* 2.8*, the minimizer is given by a ball (represented in gray color) where the two phases mix each other.*

Notice that if $(c_{11}+1)m_1 = (c_{22}+1)m_2$, then

$$a_1 = \frac{m_1}{m_1+m_2}, \qquad a_2 = \frac{m_2}{m_1+m_2}.$$

PROPOSITION 2.8. *Let* $-1 < c_{11},\, c_{22} \le 0$*. If* $(c_{11}+1)m_1 = (c_{22}+1)m_2$*, then the (unique up to a translation) minimizer of* $\mathcal{E}_K^{c_{11},c_{22}}$ *in* $\mathcal{A}_{m_1,m_2}$ *is given by the pair*

$$(f_1, f_2) = (a_1\chi_{B^{m_1+m_2}}, a_2\chi_{B^{m_1+m_2}}).$$

Proof. By Lemma 2.7 we get directly the claim in the case $c_{11} = c_{22}$, since by assumption this implies $m_1 = m_2$.

We now prove the result in the general case. For any $(f_1, f_2) \in \mathcal{A}_{m_1,m_2}$, we set

$$h_1 := (1 + \tfrac{c_{11}}{2})f_1 - \tfrac{c_{22}}{2}f_2, \quad h_2 := -\tfrac{c_{11}}{2}f_1 + (1 + \tfrac{c_{22}}{2})f_2. \tag{2.6}$$

It is easy to see that $h_1, h_2 \ge 0$, $h_1 + h_2 = f_1 + f_2 \le 1$, and, by assumption,

$$\int_{\mathbb{R}^N} h_1(x)\,\mathrm{d}x = \int_{\mathbb{R}^N} h_2(x)\,\mathrm{d}x = \frac{m_1+m_2}{2} =: m.$$

By straightforward computations it follows that, setting $c := \frac{c_{11}c_{22}}{2-c_{11}c_{22}}$,

$$\mathcal{E}_K^{c_{11},c_{22}}(f_1, f_2) = \frac{2 - c_{11}c_{22}}{2 + c_{11} + c_{22}}\,\mathcal{E}_K^{c,c}(h_1, h_2).$$

Notice that since $-1 < c_{11}, c_{22} < 0$, we have that $0 < c < 1$ and $\frac{2-c_{11}c_{22}}{2+c_{11}+c_{22}} > 0$; therefore, (f_1, f_2) minimizes $\mathcal{E}_K^{c_{11},c_{22}}$ (in $\mathcal{A}_{m_1,m_2}$) if and only if (h_1, h_2) minimizes $\mathcal{E}_K^{c,c}$ in $\mathcal{A}_{m,m}$. By Lemma 2.7, the unique minimizer of $\mathcal{E}_K^{c,c}$ in $\mathcal{A}_{m,m}$ is given by $(h_1, h_2) = (\frac{1}{2}\chi_{B^{2m}}, \frac{1}{2}\chi_{B^{2m}})$. Hence the claim for $c_{11} \ne c_{22}$ follows by (2.6). ∎

Remark 2.9. Proposition 2.8 establishes that, for very special coefficients c_{11} and c_{22} depending on the masses m_1, m_2, the minimizer is given by a homogeneous density that mixes the two phases with specific volume fractions (see Figure 4). The proof is based on the convexity of J_K. One may wonder whether, under this assumption, the result still holds for generic c_{11} and c_{22}. We will see that this is not the case, not even for the Coulomb kernel (see Corollary 3.5 and Theorem 3.9).

3. The Coulomb kernel.

In this section we will assume that $K = K_{C_N}$ is the Coulomb kernel defined in (1.7). We will provide the explicit form of the solutions for all the choices of the (nonpositive) parameters c_{11}, c_{22}, except when they are both strictly less than -1, in which case we will only be able to say that f_i are characteristic functions of sets.

3.1. Consequences of the first variation. We specialize the results of section 1.2 to the case of Coulomb kernels. We recall that the sets G_i, F_i, and S are defined in (1.8), (1.9).

PROPOSITION 3.1. *Let (f_1, f_2) be a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$. The following facts hold true:*

(i) $(c_{11}+1)f_1 - (c_{22}+1)f_2 = 0$ *a.e. in* $G_1 \cap G_2$. *In particular, if either* $(c_{11}+1)(c_{22}+1) < 0$ *or* $c_{11} = -1 \neq c_{22}$ *or* $c_{22} = -1 \neq c_{11}$, *then* $|G_1 \cap G_2| = 0$.
(ii) *If* $c_{11} \neq 0$, *then* $|G_1 \setminus G_2| = 0$, *while if* $c_{22} \neq 0$, *then* $|G_2 \setminus G_1| = 0$.
(iii) $|(G_1 \cap G_2) \setminus S| = 0$.
(iv) *If* $c_{11} \neq -1$ *or* $c_{22} \neq -1$, *then*

$$f_1 = a_1, \quad f_2 = a_2 \quad \text{a.e. in } G_1 \cap G_2, \tag{3.1}$$

where a_i are defined in (2.5).

Proof. Fact (i) is a consequence of (1.14) differentiated twice. To prove (ii) notice that $G_1 \setminus G_2 \subset G_1 \setminus S$, which implies by (1.11) that $c_{11} f_1 = f_2$ in $G_1 \setminus G_2$. Furthermore, in this region $f_2 = 0$, so that (since $c_{11} \neq 0$) also $f_1 = 0$. The case $c_{22} \neq 0$ is proved in the same way.

The proof of (iii) follows recalling that by (1.11) we have $0 > c_{11} f_1 - f_2 = 0$ in $(G_1 \cap G_2) \setminus S$ and hence $|(G_1 \cap G_2) \setminus S| = 0$. The claim in (iv) follows by (1.14) recalling that, in view of (iii), $f_1 + f_2 = 1$. ∎

3.2. The strongly attractive case $c_{11} + c_{22} \leq -2$. In Theorem 2.3 we have characterized the minimizers for every c_{11}, c_{22} such that $c_{11} + c_{22} \leq -2$ and $\max\{c_{11}, c_{22}\} \geq -1$. Clearly such result applies also to Coulomb kernels. The (general N dimensional) case $c_{11}, c_{22} < -1$ was left open. In the following proposition, we show that for Coulomb kernels the minimizers f_i are characteristic functions of sets E_i whose shape is unknown (see Remark 2.5 for some further comments in this direction).

PROPOSITION 3.2. *Let $c_{11} + c_{22} \leq -2$ with $(c_{11}, c_{22}) \neq (-1,-1)$. If (f_1, f_2) is a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$, then $f_1 = \chi_{F_1}$ and $f_2 = \chi_{F_2}$ for some F_1, $F_2 \subset \mathbb{R}^N$.*

Proof. By Theorem 2.3 and Proposition 2.4 the claim holds true in the one-dimensional case and in the general N dimensional case for $\max\{c_{11}, c_{22}\} \geq -1$, so that it is enough to prove the claim in the case $N \geq 2$ and $c_{11}, c_{22} < -1$. Since $c_{11} + c_{22} + 2 < 0$, by applying Lemma 1.5 with $\varphi \in L^1(\mathbb{R}^N; \mathbb{R})$, $\varphi = 0$ a.e. in $\mathbb{R}^N \setminus (G_1 \cap G_2)$ and $\int_{\mathbb{R}^N} \varphi \, \mathrm{d}x = 0$, we get

$$\int_{G_1 \cap G_2} \int_{G_1 \cap G_2} K_{C_N}(x-y)\, \varphi(x)\, \varphi(y) \, \mathrm{d}x \, \mathrm{d}y \leq 0. \tag{3.2}$$

By Remark 1.1 we deduce that the above inequality is actually an equality and that $\varphi = 0$ in $G_1 \cap G_2$. By the arbitrariness of φ, it follows that $|G_1 \cap G_2| = 0$. Finally, by Proposition 3.1(ii), we have that $|G_1 \setminus G_2| = |G_2 \setminus G_1| = 0$, so we conclude that $|G_1| = |G_2| = 0$. ∎

3.3. The weakly attractive case $c_{11} + c_{22} > -2$ (preliminary results). For any measurable set $E \subset \mathbb{R}^N$, we set $V_E := \chi_E * K$.

LEMMA 3.3. *Let $-1 \leq c_{11}, c_{22} \leq 0$ with $c_{11} \neq -1$ or $c_{22} \neq -1$. Then, there exists a minimizer (f_1, f_2) of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$ such that $|G_1 \setminus G_2| = |G_2 \setminus G_1| = 0$ and either $|F_1| = 0$ or $|F_2| = 0$.*

Moreover, any minimizer (f_1, f_2) *of* $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ *in* $\mathcal{A}_{m_1,m_2}$ *is such that either* $|G_1 \setminus G_2| + |F_1| = 0$ *or* $|G_2 \setminus G_1| + |F_2| = 0$.

Proof. Let (f_1, f_2) be a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$. By Proposition 3.1(ii) and Corollary 2.2 we can always assume

$$\{f_1 \neq 0\} = (G_1 \cap G_2) \cup F_1, \qquad \{f_2 \neq 0\} = (G_1 \cap G_2) \cup F_2 \quad \text{a.e.} \tag{3.3}$$

so that $|G_1 \setminus G_2| = |G_2 \setminus G_1| = 0$.

Now, let us prove that either $|F_1| = 0$ or $|F_2| = 0$. We first focus on the case $N \geq 3$. By (3.3) and (3.1) we have

$$f_1 = a_1 \chi_{G_1 \cap G_2} + \chi_{F_1}, \quad f_2 = a_2 \chi_{G_1 \cap G_2} + \chi_{F_2}. \tag{3.4}$$

It follows that

$$V_1 = a_1 V_{G_1 \cap G_2} + V_{F_1}, \quad V_2 = a_2 V_{G_1 \cap G_2} + V_{F_2},$$

which together with (1.13) easily yields

$$(c_{11}+1)V_{F_1}(x_2) - (c_{22}+1)V_{F_2}(x_2) \geq (c_{11}+1)V_{F_1}(x_1) - (c_{22}+1)V_{F_2}(x_1) \tag{3.5}$$

for any $x_1 \in \bar{F}_1$ and any $x_2 \in \bar{F}_2$. Set $U(x) := (c_{11}+1)V_{F_1}(x) - (c_{22}+1)V_{F_2}(x)$. Then U solves

$$\begin{cases} -\Delta U = (c_{11}+1)\chi_{F_1} - (c_{22}+1)\chi_{F_2} & \text{in } \mathbb{R}^N, \\ U(x) \to 0 & \text{if } |x| \to \infty. \end{cases} \tag{3.6}$$

So, U is subharmonic in $\mathbb{R}^N \setminus \overline{F}_1$ and hence either $U \leq 0$ or U reaches its maximum on $\overline{F}_1$. Analogously, since U is superharmonic in $\mathbb{R}^N \setminus \overline{F}_2$, either $U \geq 0$ or U reaches its minimum on $\overline{F}_2$. Now, if $U \equiv 0$, then $|F_1| = |F_2| = 0$; otherwise, assume, for instance, that U reaches its maximum on $\overline{F}_1$. By (3.5) and by (3.6), it follows that U is constant in F_2, and hence $|F_2| = 0$. Analogously, if U reaches its minimum on $\overline{F}_2$, we get that $|F_1| = 0$.

The proofs for the cases $N = 1, 2$ are analogous, the only care being that, for $N = 2$, the boundary condition in (3.6) should be replaced by either $U(x) \to 0$ or $U(x) \to \pm\infty$, according to the sign of $(c_{11}+1)|F_1| - (c_{22}+1)|F_2|$. For $N = 1$ a direct proof shows that U reaches its maximum on $\overline{F}_1$ and its minimum on $\overline{F_2}$.

We pass to the proof of the last claim of the lemma. Assume by contradiction that $|G_1 \setminus G_2| + |F_1| > 0$ and $|G_1 \setminus G_2| + |F_2| > 0$. By Proposition 3.1(ii) and Corollary 2.2 we deduce that there exists a minimizer satisfying (3.3) with both F_1 and F_2 with positive measure. Following the lines of the proof of the first claim of the lemma, this provides a contradiction. ∎

The remaining part of this section is devoted to the uniqueness and characterization of the minimizer. In particular, we will see that the unique minimizer in the purely weakly attractive case, corresponding to $-1 < c_{11}, c_{22} \leq 0$, is given by a ball where the two phases are mixed proportionally to their self attraction coefficents and by an annulus around this ball (see Corollary 3.5 for the case $N = 1$ and Theorem 3.9

and Corollary 3.12 for the case $N \geq 2$). Moreover, we will see that also in the remaining cases, i.e., $c_{11} \leq -1 \leq c_{22} \leq 0$ and $c_{22} \leq -1 \leq c_{11} \leq 0$, with $c_{11} + c_{22} > -2$, the unique minimizer is given once again by a ball and an annulus around it, where the internal ball corresponds to the phase having the stronger self attraction coefficient (see Proposition 3.4 for the case N =1 and Corollary 3.10 for the case $N \geq 2$).

3.4. The weakly attractive case $c_{11}+c_{22} > -2$ (in dimension $N = 1$). In the following proposition we study the minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_1}}$ when $c_{11} \leq -1 \leq c_{22} \leq 0$ and $c_{11} + c_{22} > -2$. In the subsequent corollary we take advantage of this result via a reparameterization of the energies to study the case $-1 < c_{11}, c_{22} \leq 0$.

PROPOSITION 3.4. *Let $c_{11} \leq -1$ and $-1 \leq c_{22} \leq 0$ (resp., $c_{22} \leq -1$ and $-1 \leq c_{11} \leq 0$) with $c_{11} + c_{22} > -2$. Then the (unique up to a translation) minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_1}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair*

$$(f_1, f_2) = (\chi_{B^{m_1}}, \chi_{B^{m_1+m_2}\setminus B^{m_1}}) \quad \textit{(resp., } (f_1, f_2) = (\chi_{B^{m_1+m_2}\setminus B^{m_2}}, \chi_{B^{m_2}})\textit{)}.$$

Proof. We prove the claim only for $c_{11} \leq -1$ and $-1 \leq c_{22} \leq 0$ with $c_{11} + c_{22} > -2$, the proof of the other case being analogous. Let (f_1, f_2) be a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_1}}$ in $\mathcal{A}_{m_1,m_2}$. By (i) and (ii) of Proposition 3.1, we have that $f_1 = \chi_{F_1}$ and $f_2 = \chi_{F_2} + f_2 \llcorner G_2$. We can assume without loss of generality that $F_1 \cup F_2 \cup G_2$ is an interval, since reducing the distances decreases the energy. For the same reason, it is easy to see that $|G_2| = 0$. Notice that

$$\mathcal{E}^{c_{11},c_{22}}_{K_{C_1}}(f_1, f_2) = \mathcal{E}^{-1,c_{22}}_{K_{C_1}}(f_1, f_2) + (c_{11} + 1)J_{K_{C_1}}(f_1, f_1),$$

so it is enough to prove the claim for $c_{11} = -1$. We now prove that $V_2' = 0$ in F_1. By (1.13), we have

$$V_2(x_1) \geq V_2(x_2) \quad \text{for any } x_1 \in \overline{F_1} \text{ and } x_2 \in \overline{F_2},$$

and, by the maximum principle, V_2 attains its maximum in $\overline{F_2}$ (notice that $V_2 \to -\infty$ as $|x| \to +\infty$). It follows that for any $x \in F_1$, $V_2(x) = \max V_2$. We have

$$0 = V_2'(x) = \frac{1}{2}\left(|F_2 \cap (-\infty, x]| - |F_2 \cap [x, \infty)|\right) \qquad \text{for any } x \in F_1,$$

and hence F_1 is connected and centered in $F_1 \cup F_2$. ∎

COROLLARY 3.5. *Let $-1 < c_{11}, c_{22} \leq 0$. Then, the following results hold true (recall that a_i are defined in (2.5)):*

(i) *If $(c_{22} + 1)m_2 \geq (c_{11} + 1)m_1$, then (up to a translation)*

$$(f_1, f_2) = \left(a_1\, \chi_{B^{\frac{m_1}{a_1}}}, \chi_{B^{m_2+m_1}} - a_1\, \chi_{B^{\frac{m_1}{a_1}}}\right) \tag{3.7}$$

is the (unique) minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{a_2}}$ in $\mathcal{A}_{m_1,m_2}$.

(ii) *If $(c_{11} + 1)m_1 > (c_{22} + 1)m_2$, then (up to a translation)*

$$(f_1, f_2) = \left(\chi_{B^{m_2+m_1}} - a_2\, \chi_{B^{\frac{m_2}{a_2}}}, a_2\, \chi_{B^{\frac{m_2}{a_2}}}\right)$$

is the (unique) minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{a_2}}$ in $\mathcal{A}_{m_1,m_2}$.

Proof. We only prove (i) since the proof of (ii) is analogous.

Let (f_1, f_2) be a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_1}}$ in $\mathcal{A}_{m_1,m_2}$. Arguing as in the proof of Proposition 3.4, one can show that $|G_2 \setminus G_1| + |G_1 \setminus G_2| = 0$, and hence

$$f_1 = a_1\chi_{G_1\cap G_2} \text{ and } f_2 = a_2\chi_{G_1\cap G_2} + \chi_{F_2}. \tag{3.8}$$

Set $A := G_1 = G_2$, $B := F_2$, $\tilde{m}_1 := \frac{m_1}{a_1}$, and $\tilde{m}_2 := m_2 - \frac{c_{11}+1}{c_{22}+1}m_1 > \tilde{m}_1$; then, by easy computations, it follows that

$$\begin{aligned}\mathcal{E}^{c_{11},c_{22}}_{K_{C_1}}(f_1, f_2) &= \frac{1 - c_{11}c_{22}}{c_{11} + c_{22} + 2}[-J_{K_{C_1}}(A, A) + c_{22}\frac{c_{11} + c_{22} + 2}{1 - c_{11}c_{22}}J_{K_{C_1}}(B, B) \\ &\quad - 2J_{K_{C_1}}(A, B)] \\ &= \frac{1 - c_{11}c_{22}}{c_{11} + c_{22} + 2}\mathcal{E}^{-1,\tilde{c}_{22}}_{K_{C_1}}(\chi_A, \chi_B)\end{aligned}$$

with $\tilde{c}_{22} := c_{22}\,\frac{c_{11}+c_{22}+2}{1-c_{11}c_{22}} \in (-1, 0)$. Since $\frac{1-c_{11}c_{22}}{c_{11}+c_{22}+2} > 0$, it follows that (f_1, f_2) is a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_1}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if (χ_A, χ_B) minimizes $\mathcal{E}^{-1,\tilde{c}_{22}}_{K_{C_1}}$ in $\mathcal{A}_{\tilde{m}_1,\tilde{m}_2}$. By Proposition 3.4, the unique minimizer of $\mathcal{E}^{-1,\tilde{c}_{22}}_{K_{C_1}}(\chi_A, \chi_B)$ among the pairs (A, B) with $|A| = \tilde{m}_1$ and $|B| = \tilde{m}_2$ is given by $(B^{\tilde{m}_1}, B^{\tilde{m}_1+\tilde{m}_2} \setminus B^{\tilde{m}_1})$. The claim follows thanks to formula (3.8). ∎

One might wonder whether the assumption that $K = K_{C_1}$ is crucial in order to prove Proposition 3.4 and Corollary 3.5. In the following remark, we exhibit an example of a kernel for which the pair (f_1, f_2) in (3.7) is not the minimizer of $\mathcal{E}^{0,0}_K$ in $\mathcal{A}_{m_1,m_2}$ for suitably chosen $m_1, m_2 > 0$.

Remark 3.6. Let $\rho > 0$ and let $m_1, m_2 > 0$ be such that $m_1 > 2\rho$, $m_2 > m_1 + 4\rho$. Consider the kernel $K := \chi_{[-\rho,\rho]}$ and set $A := (-m_1, m_1)$, $B := (-\frac{m_1+m_2}{2}, \frac{m_1+m_2}{2})$, $(f_1, f_2) = (\frac{1}{2}\chi_A, \chi_B - \frac{1}{2}\chi_A)$. Then,

$$\mathcal{E}^{0,0}(f_1, f_2) = -[-\tfrac{1}{2}J_K(A, A) + J_K(A, B)].$$

One can easily check that $J_K(A, A) = 4\rho m_1 - \rho^2$ and $J_K(A, B) = 4\rho m_1$; it follows that

$$\mathcal{E}^{0,0}_K(f_1, f_2) = -\rho(2m_1 + \tfrac{\rho}{2}).$$

Now split A into two intervals $A_1 := (-c\rho - m_1, -c\rho)$ and $A_2 := (c\rho, c\rho + m_1)$, with $\frac{1}{2} < c < 1$, and consider the energy of the admissible pair

$$(g_1, g_2) := (\tfrac{1}{2}\chi_{A_1} + \tfrac{1}{2}\chi_{A_2}, \chi_B - \tfrac{1}{2}\chi_{A_1} - \tfrac{1}{2}\chi_{A_2}).$$

By symmetry $J_K(A_2, A_2) = J_K(A_1, A_1)$ and $J_K(A_2, B) = J_K(A_1, B)$. Hence

$$\mathcal{E}^{0,0}_K(g_1, g_2) = -[-J_K(A_1, A_1) - J_K(A_1, A_2) + 2J_K(A_1, B)],$$

where $J_K(A_1, A_1) = 2\rho m_1 - \rho^2$, $J_K(A_1, A_2) = 0$ (since $c > \frac{1}{2}$) and $J_K(A_1, B) = 2\rho m_1$. It follows that $\mathcal{E}^{0,0}_K(g_1, g_2) = -\rho(2m_1 + \rho) < \mathcal{E}^{0,0}_K(f_1, f_2)$ and therefore (f_1, f_2) is not the minimizer of $\mathcal{E}^{0,0}_K$ in $\mathcal{A}_{m_1,m_2}$. One can easily check that the above result holds true also taking $K(x) := \chi_{[-\rho,\rho]}(x)\,(\rho - |x|)$ and m_1, m_2 as above.

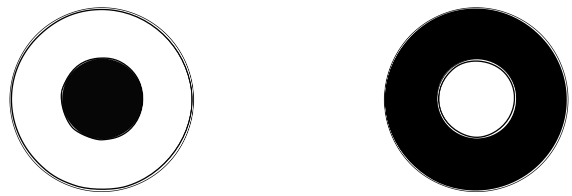
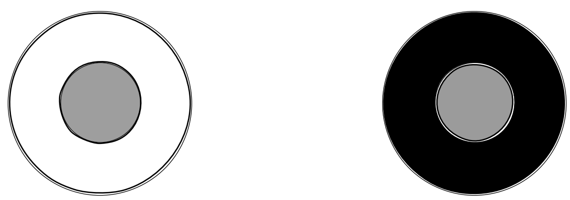


FIG. 5. *The phase f_1 is black and the phase f_2 is white. The minimizer in the case $c_{11} \leq -1 \leq c_{22} \leq 0$ is represented on the left, whereas on the right there is the minimizer in the case $c_{22} \leq -1 \leq c_{11} \leq 0$.*



(i) (ii)

FIG. 6. *The phase f_1 is black and the phase f_2 is white. The mixing of the two phases is represented by gray. The image on the left represents the unique minimizer in* (i)*. In this case, the two phases mix each other in the inner ball, and the remainig mass of f_2 is arranged in an annulus around such ball. In case* (ii)*, the minimizer has the same form, but replacing f_2 (white) with f_1 (black).*

3.5. The weakly attractive case $c_{11} + c_{22} > -2$ (the case $N \geq 2$). Now we focus on the case $N \geq 2$, considering first the case $c_{11} = c_{22} = 0$ (Theorem 3.9) and then the remaining cases (see Corollaries 3.10 and 3.12, and Figures 5 and 6).

We first introduce some preliminary notation and recall some well known results we will use in this section. For any $g \in L^2(\mathbb{R}^N;\mathbb{R}^+)$, we set $V := K_{C_N} * g$. Moreover, we recall that for every function $u \in L^1(\mathbb{R}^N;\mathbb{R}^+)$, u^* is the spherical symmetric nonincreasing rearrangement of u defined in (1.21). Clearly, the notion of spherical symmetric nonincreasing rearrangement can be extended in the obvious way to functions $u \in L^1_{loc}(\mathbb{R}^N;\mathbb{R})$ tending to $-\infty$ for $x \to +\infty$.

LEMMA 3.7. *Let $g \in L^2(\mathbb{R}^N;\mathbb{R}^+)$, let $m := \int_{\mathbb{R}^N} g(x)\,\mathrm{d}x$, and let $V := K_{C_N} * g$. Moreover, for $N = 2$ assume that g has compact support. Then,*

$$V(x) \to 0 \ \text{as}\ |x| \to +\infty \qquad \text{for } N \geq 3; \tag{3.9}$$

$$V(x) = -\frac{m}{2\pi}\log|x| + r(x) \qquad \text{for } N = 2; \tag{3.10}$$

where $r(x) \to 0$ as $|x| \to +\infty$. As a consequence, $V - V^* \to 0$ as $|x| \to +\infty$.

Let now $f \in L^1(\mathbb{R}^N;\mathbb{R}^+)$. For any $r > 0$ we denote by $t(r)$ the unique $t \in \mathbb{R}$ such that $|\{V > t\}| \leq \omega_N r^N \leq |\{V \geq t\}|$. Let $\tilde{f} : \mathbb{R}^N \to \mathbb{R}$ be defined by

$$\tilde{f}(x) := \tfrac{1}{N\omega_N|x|^{N-1}} \tfrac{\mathrm{d}t}{\mathrm{d}r}\Big|_{t=t(|x|)} \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\{V>t\}} f(y)\,\mathrm{d}y\right)\Big|_{t=t(|x|)}. \tag{3.11}$$

We notice that $B_r = \{V^* > t(r)\}$ and that

$$\int_{\{V^*>t\}} \tilde{f}(x)\,\mathrm{d}x = \int_{\{V>t\}} f(x)\,\mathrm{d}x \qquad \text{for every } t \in \mathbb{R}. \tag{3.12}$$

Moreover, one can easily check that also $\tilde{f}$ takes values in $\mathbb{R}^+$, and

$$\|\tilde{f}\|_1 = \|f\|_1, \qquad \|\tilde{f}\|_p \le \|f\|_p \quad \text{for all } 1 < p \le +\infty. \tag{3.13}$$

LEMMA 3.8. *Let $f \in L^2(\mathbb{R}^N;\mathbb{R}^+)$, with $N \ge 2$, and let $V := K_{C_N} * f$. Moreover, let $\tilde{f} \in L^2(\mathbb{R}^N;\mathbb{R}^+)$ be defined as in* (3.11), *and let $\tilde{V} := K_{C_N} * \tilde{f}$. Then, $\tilde{V} \ge V^*$, and*

$$\tilde{V}(x) > V^*(x) \qquad \textit{for a.e. } x \in B_{r(t_{max})},$$

where t_{max} is the maximal level such that $\{V > t\}$ is a ball for every $t \le t_{max}$.

Proof. By the coarea formula and the isoperimetric inequality, for almost every $t \in \mathbb{R}$ we have

$$\int_{\partial\{V>t\}} |\nabla V(x)|\ \mathrm{d}\mathcal{H}^{N-1} \ge \int_{\partial\{V^*>t\}} |\nabla V^*(x)|\ \mathrm{d}\mathcal{H}^{N-1},$$

with strict inequality whenever $\{V > t\}$ is not a ball. Therefore, by (3.12)

$$\begin{aligned}\int_{\partial\{V^*>t\}} |\nabla\tilde{V}(x)|\ \mathrm{d}\mathcal{H}^{N-1} &\ge -\int_{\{V^*>t\}} \Delta\tilde{V}(x)\,\mathrm{d}x = -\int_{\{V>t\}} \Delta V(x)\,\mathrm{d}x \\ &= \int_{\partial\{V>t\}} |\nabla V(x)|\ \mathrm{d}\mathcal{H}^{N-1} \ge \int_{\partial\{V^*>t\}} |\nabla V^*(x)|\ \mathrm{d}\mathcal{H}^{N-1}\end{aligned} \tag{3.14}$$

with strict inequalities whenever $\{V > t\}$ is not a ball. Since $\tilde{V} - V^*$ is radial and in view of Lemma 3.7 it vanishes at infinity, we have

$$\begin{aligned}\tilde{V}(r) - V^*(r) &= \int_r^{+\infty} \frac{\mathrm{d}}{\mathrm{d}s}\Big(V^*(s) - \tilde{V}(s)\Big)\ \mathrm{d}s \\ &= \int_r^{+\infty} \frac{1}{N\omega_N s^{N-1}}\ \mathrm{d}s \int_{\partial B_s} -|\nabla V^*(x)| + |\nabla\tilde{V}(x)|\,\mathrm{d}x.\end{aligned} \tag{3.15}$$

The claim follows since the integrand is nonnegative, and it is strictly positive in a subset of positive measure of $(r, +\infty)$ for all $r < r_{t_{max}}$. ∎

Lemma 3.8 establishes that we can rearrange the mass of f_1 in order to obtain a new radial charge configuration $\tilde{f}_1$, increasing the corresponding potential. Exploiting such a result, we deduce that the minimizer of $\mathcal{E}^{0,0}_K$ has radial symmetry. This is done in the next theorem.

THEOREM 3.9. *For $m_2 \ge m_1$, the (unique up to a translation) minimizer of $\mathcal{E}^{0,0}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair (f_1, f_2), where*

$$f_1 := \tfrac{1}{2}\chi_{B^{2m_1}}, \qquad f_2 := \chi_{B^{m_1+m_2}} - \tfrac{1}{2}\chi_{B^{2m_1}}.$$

Proof. Let (f_1, f_2) be a minimizer of $\mathcal{E}^{0,0}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$. Let V_1 be the potential generated by f_1 and let $\tilde{f}_1$ and $\tilde{V}_1$ be defined according to Lemma 3.8. Notice that $0 \le \tilde{f}_1 \le 1$ and that $\|\tilde{f}_1\|_{L^1(\mathbb{R}^N)} = m_1$. Let us observe that by standard regularity theory, $\tilde{V}_1$ attains a maximum. We denote it by $\tilde{M}_1$. We first show that there exists $\tilde{t} < \tilde{M}_1$ such that

$$\int_{\{\tilde{V}_1 > \tilde{t}\}} (1 - \tilde{f}_1(x))\, \mathrm{d}x = m_2. \tag{3.16}$$

Suppose by contradiction that there does not exist $\tilde{t}$ such that (3.16) holds true. Notice that $-\Delta \tilde{V} = \tilde{f}$ and that $\tilde{f}$ and $\tilde{V}$ are radially symmetric. Therefore, $\tilde{V}$ may have a flat region only in a ball centered at the origin, whereas it is strictly decreasing with respect to $|x|$ elsewhere. We deduce that $\int_{\{\tilde{V}_1 > t\}} (1 - \tilde{f}_1(x))\, \mathrm{d}x > m_2$ for any $t < \tilde{M}_1$ and in particular that $\int_{\{\tilde{V}_1 = \tilde{M}_1\}} (1 - \tilde{f}_1(x))\, \mathrm{d}x \geq m_2$. It follows that $|\{\tilde{V}_1 = \tilde{M}_1\}| \geq m_2$ and, since $\tilde{V}_1$ is radially symmetric, $\{\tilde{V}_1 = \tilde{M}_1\}$ is a ball centered at the origin containing B^{m_2}. Set $\hat{f}_2 := \chi_{B^{m_2}}$, then $(\tilde{f}_1, \hat{f}_2) \in \mathcal{A}_{m_1,m_2}$. Set $M_1 := \max V_1$, by Lemma 3.8, $\tilde{M}_1 \geq \max V_1^* = M_1$; it follows that

$$\begin{aligned}
\mathcal{E}^{0,0}_{K_{C_N}}(\tilde{f}_1, \hat{f}_2) &= -2 \int_{\mathbb{R}^N} \hat{f}_2(x)\, \tilde{V}_1(x)\, \mathrm{d}x = -2 \int_{B^{m_2}} \tilde{V}_1(x)\, \mathrm{d}x \leq -2\, \tilde{M}_1\, m_2 \\
&\leq -2\, M_1 \int_{\mathbb{R}^N} f_2(x)\, \mathrm{d}x \leq -2 \int_{\mathbb{R}^N} f_2(x)\, V_1(x)\, \mathrm{d}x = \mathcal{E}^{0,0}_{K_{C_N}}(f_1, f_2),
\end{aligned}$$

and hence $(\tilde{f}_1, \hat{f}_2)$ is a minimizer. Since the supports of $\tilde{f}_1$ and $\hat{f}_2$ are disjoint, we come to a contradiction using Proposition 3.3. We conclude that there exists $\tilde{t}$ satisfying (3.16). Set

$$\tilde{f}_2(x) := \begin{cases} 1 - \tilde{f}_1(x) & \text{for } x \in \{\tilde{V}_1 > \tilde{t}\}, \\ 0 & \text{otherwise} \end{cases}$$

by construction $(\tilde{f}_1, \tilde{f}_2) \in \mathcal{A}_{m_1,m_2}$ ($\int_{\mathbb{R}^N} \tilde{f}_2(x)\, \mathrm{d}x = m_2$ by (3.16)).

Let now $\hat{t} \leq \tilde{t}$ be such that

$$\{V_1^* > \hat{t}\} \subseteq \{\tilde{V}_1 > \tilde{t}\} \subseteq \{V_1^* \geq \hat{t}\}\,.$$

This is possible since the superlevel set $\{\tilde{V}_1 > \tilde{t}\}$ is a ball centered at the origin. Let $A := \{\tilde{V}_1 > \tilde{t}\} \setminus \{V_1^* > \hat{t}\}$. Since $A \subseteq \{V_1^* = \hat{t}\}$, we have $\tilde{f}_1 = 0$ a.e. on A, and hence $\tilde{f}_2 = 1$ a.e. on A. Moreover, by Corollary 2.2 we can always assume that

$$\operatorname{supp} f_2 = \{V_1 > \hat{t}\} \cup A', \quad f_2 = 1 - f_1 \text{ on } \{V_1 > \hat{t}\}, \quad f_2 \equiv 1 \text{ on } A', \tag{3.17}$$

for some set $A' \subseteq \{V_1 = \hat{t}\}$ with $|A'| = |A|$. By the coarea formula and Lemma 3.8 we have

$$\begin{aligned}
\mathcal{E}^{0,0}_{K_{C_N}}(\tilde{f}_1, \tilde{f}_2) &= -2 \int_{\mathbb{R}^N} \tilde{f}_2(x) \tilde{V}_1(x)\, \mathrm{d}x \\
&\leq -2 \int_{\mathbb{R}^N} \tilde{f}_2(x) V_1^*(x)\, \mathrm{d}x \\
&= -2\, \hat{t}\, |A| - 2 \int_{\hat{t}}^{+\infty} t \int_{\{V_1^* > t\}} (1 - \tilde{f}_1(x))\, \mathrm{d}x\, \mathrm{d}t
\end{aligned} \tag{3.18}$$

$$= -2\,\hat{t}\,|A| - 2\int_{\hat{t}}^{+\infty} t \int_{\{V_1>t\}} (1 - f_1(x))\,\mathrm{d}x\,\mathrm{d}t$$

$$= -2\,\hat{t}\,|A| - 2\int_{\{V_1>\hat{t}\}} (1 - f_1(x))V_1(x)\,\mathrm{d}x$$

$$= -2\int_{A'} f_2(x)\,\hat{t}\,\mathrm{d}x - 2\int_{\{V_1>\hat{t}\}} (1 - f_1(x))V_1(x)\,\mathrm{d}x \tag{3.19}$$

$$= -2\int_{\mathbb{R}^N} f_2(x)V_1(x)\,\mathrm{d}x = \mathcal{E}^{0,0}_{K_{C_N}}(f_1, f_2), \tag{3.20}$$

where the equality in (**??**) follows from (3.17). By minimality, the inequality in (3.18) is actually an equality, and hence $V_1^* \equiv \tilde{V}_1$. It follows that all the superlevels of V_1 are balls. By Proposition 3.1, $f_1 = \frac{1}{2}$ in $G_1 \cap G_2$, whereas by Lemma 3.3 $G_1 \cup F_1 = G_1 \cap G_2$, so that $f_1 := \frac{1}{2}\chi_E$ for some set E. Since all the superlevel sets of V_1 are balls, we conclude that, up to a translation, $f_1 := \frac{1}{2}\chi_{B^{2m_1}}$. By Lemma 2.1 we also deduce that $f_2 := \chi_{B^{m_1+m_2}} - \frac{1}{2}\chi_{B^{2m_1}}$ and this concludes the proof. ∎

COROLLARY 3.10. *Let $c_{11} \le -1$ and $-1 \le c_{22} \le 0$ (resp., $c_{22} \le -1$ and $-1 \le c_{11} \le 0$) with $c_{11} + c_{22} > -2$. Then, the (unique up to a translation) minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair*

$$(f_1, f_2) = (\chi_{B^{m_1}}, \chi_{B^{m_1+m_2}\setminus B^{m_1}}) \quad \textit{(resp., } (f_1, f_2) = (\chi_{B^{m_1+m_2}\setminus B^{m_2}}, \chi_{B^{m_2}})\textit{)}.$$

Proof. We prove the claim only for $c_{11} \le -1$ and $-1 \le c_{22} \le 0$ with $c_{11} + c_{22} > -2$, the proof of the other case being fully analogous. Let (f_1, f_2) be a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$.

Set $\tilde{m}_1 := \frac{m_1}{2}$, $\tilde{m}_2 := \frac{m_1}{2} + m_2 > \tilde{m}_1$,

$$g_1 := \frac{f_1}{2}, \qquad g_2 := \frac{f_1}{2} + f_2. \tag{3.21}$$

It is easy to see that $g_i \ge 0$, $\int_{\mathbb{R}^N} g_i(x)\,\mathrm{d}x = \tilde{m}_i$ (for $i = 1, 2$) and $g_1 + g_2 = f_1 + f_2 \le 1$, so that $(g_1, g_2) \in \mathcal{A}_{\tilde{m}_1,\tilde{m}_2}$. A straightforward computation yields

$$\begin{aligned}
\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}(f_1, f_2) &= (c_{11} + 1)J_{K_{C_N}}(f_1, f_1) + \mathcal{E}^{-1,c_{22}}_{K_{C_N}}(f_1, f_2)\\
&= (c_{11} + 1)J_{K_{C_N}}(f_1, f_1) + c_{22}J_{K_{C_N}}(f_1 + f_2, f_1 + f_2) \qquad (3.22)\\
&\quad + (1 + c_{22})(-J_{K_{C_N}}(f_1, f_1) - 2J_{K_{C_N}}(f_1, f_2))\\
&= 4\,(c_{11} + 1)J_{K_{C_N}}(g_1, g_1) + c_{22}J_{K_{C_N}}(g_1 + g_2, g_1 + g_2) \qquad (3.23)\\
&\quad + 2(1 + c_{22})\,\mathcal{E}^{0,0}_{K_{C_N}}(g_1, g_2),
\end{aligned}$$

and hence (f_1, f_2) is a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if (g_1, g_2) minimizes the energy

$$4\,(c_{11} + 1)J_{K_{C_N}}(g_1, g_1) + c_{22}J_{K_{C_N}}(g_1 + g_2, g_1 + g_2) + 2(1 + c_{22})\,\mathcal{E}^{0,0}_{K_{C_N}}(g_1, g_2) \tag{3.24}$$

in $\mathcal{A}_{\tilde{m}_1,\tilde{m}_2}$. By Theorem 3.9, the third addendum in (3.24) is minimized (in $\mathcal{A}_{\tilde{m}_1,\tilde{m}_2}$) if and only if

$$(g_1, g_2) = (\tfrac{1}{2}\chi_{B^{2\tilde{m}_1}}, \tfrac{1}{2}\chi_{B^{2\tilde{m}_1}} + \chi_{B^{\tilde{m}_1+\tilde{m}_2}\setminus B^{2\tilde{m}_1}}).$$

We notice that such configuration minimizes also the first and the second addendum. The claim follows directly by (3.21). ∎

Quantitative Riesz inequalities have been recently studied in [6, Theorem 1.5]. For any measurable set $E \subset \mathbb{R}^N$ with finite measure, let $E^* := B^{|E|}$ be the ball centered at the origin such that $|E^*| = |E|$. From Corollary 3.10 with $c_{11} = -1$ and $c_{22} = 0$ we immediately get the following improved Riesz inequality.

COROLLARY 3.11. *For any measurable sets $E_1 \subseteq E_2 \subset \mathbb{R}^N$ with finite measure, there holds*

$$J_{K_{C_N}}(E_1^*, E_2^*) - J_{K_{C_N}}(E_1, E_2) \geq \frac{1}{2}\Big(J_{K_{C_N}}(E_1^*, E_1^*) - J_{K_{C_N}}(E_1, E_1)\Big). \tag{3.25}$$

Moreover, for any measurable sets $A_1 \subseteq A_2 \subset \mathbb{R}^N$ with finite measure, there holds

$$\begin{aligned} J_{K_{C_N}}(A_2, A_2) - J_{K_{C_N}}(A_1, A_1) \leq{}& J_{K_{C_N}}(B^{|A_2|}, B^{|A_2|}) \\ &- J_{K_{C_N}}(B^{|A_2|} \setminus B^{|A_2|-|A_1|}, B^{|A_2|} \setminus B^{|A_2|-|A_1|}). \end{aligned} \tag{3.26}$$

Proof. We prove only (3.25), since (3.26) is indeed equivalent to (3.25) replacing E_1 with $A_2 \setminus A_1$ and E_2 with A_2.

Let $f_1 := \chi_{E_1}$, $f_2 := \chi_{E_2 \setminus E_1}$. By Corollary 3.10 we have

$$\begin{aligned} J_{K_{C_N}}&(E_1, E_1) - 2J_{K_{C_N}}(E_1, E_2) = J_{K_{C_N}}(f_1, f_1) - 2J_{K_{C_N}}(f_1, f_1 + f_2) \\ &= \mathcal{E}^{-1,0}_{K_{C_N}}(f_1, f_2) \geq \mathcal{E}^{-1,0}_{K_{C_N}}(\chi_{E_1^*}, \chi_{E_2^* \setminus E_1^*}) = J_{K_{C_N}}(E_1^*, E_1^*) - 2J_{K_{C_N}}(E_1^*, E_2^*). \quad \square \end{aligned}$$

In the next corollary we will consider the case $-1 < c_{11} \leq 0$, $-1 < c_{22} \leq 0$, completing the analysis of the weakly attractive case for the Coulomb interaction kernel. Recall the coefficients a_i defined in (2.5).

COROLLARY 3.12. *Let $-1 < c_{11} \leq 0$, $-1 < c_{22} \leq 0$. The following results hold true:*

(i) *If $(c_{22}+1)m_2 > (c_{11}+1)m_1$, then the (unique up to a translation) minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair*

$$(f_1, f_2) = \Big(a_1\,\chi_{B^{\frac{m_1}{a_1}}}, \chi_{B^{m_2+m_1}} - a_1\,\chi_{B^{\frac{m_1}{a_1}}}\Big).$$

(ii) *If $(c_{11}+1)m_1 > (c_{22}+1)m_2$, then the (unique up to a translation) minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$ is given by the pair*

$$(f_1, f_2) = \Big(\chi_{B^{m_2+m_1}} - a_2\,\chi_{B^{\frac{m_2}{a_2}}}, a_2\,\chi_{B^{\frac{m_2}{a_2}}}\Big).$$

Proof. We prove only (i) since the proof of (ii) is analogous. Let (f_1, f_2) be a minimizer of $\mathcal{E}^{c_{11},c_{22}}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$. We first notice that, in the case $c_{22} < 0$, by (i) and (ii) of Proposition 3.1 we have $f_1 = a_1\chi_A$, $f_2 = a_2\chi_A + \chi_B$ for some measurable sets A, $B \subset \mathbb{R}^N$. Then, one can argue as in the proof of Corollary 3.5 (applying Corollary 3.10 instead of Proposition 3.4). The details are left to the reader.

It remains to prove the claim for $c_{22} = 0$. In this case set $\tilde{m}_1 := \frac{c_{11}+2}{2}m_1$ and $\tilde{m}_2 := -\frac{c_{11}}{2}m_1 + m_2$. By assumption $\tilde{m}_2 > \tilde{m}_1$. Set moreover

$$g_1 := \frac{c_{11}+2}{2}f_1, \qquad g_2 := -\frac{c_{11}}{2}f_1 + f_2. \tag{3.27}$$

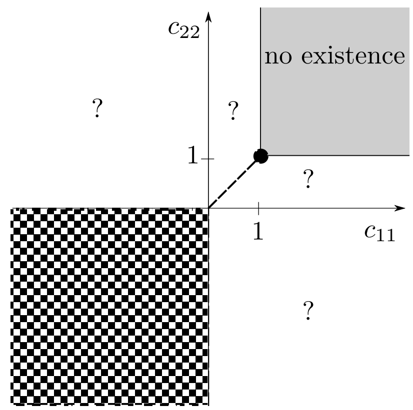


FIG. 7. *Existence/nonexistence regions of parameters* c_{11}, c_{22}.

It is easy to see that $g_i \geq 0$, $\int_{\mathbb{R}^N} g_i(x)\,\mathrm{d}x = \tilde{m}_i$ (for $i = 1, 2$) and $g_1 + g_2 = f_1 + f_2 \leq 1$, so that $(g_1, g_2) \in \mathcal{A}_{\tilde{m}_1,\tilde{m}_2}$. Moreover, a straightforward computation yields

$$\mathcal{E}^{c_{11},0}_{K_{C_N}}(f_1, f_2) = \frac{2}{c_{11} + 2}\,\mathcal{E}^{0,0}_{K_{C_N}}(g_1, g_2),$$

and hence (f_1, f_2) is a minimizer of $\mathcal{E}^{c_{11},0}_{K_{C_N}}$ in $\mathcal{A}_{m_1,m_2}$ if and only if (g_1, g_2) is a minimizer of $\mathcal{E}^{0,0}_{K_{C_N}}$ in $\mathcal{A}_{\tilde{m}_1,\tilde{m}_2}$. By Theorem 3.9, the unique (up to a translation) minimizer of $\mathcal{E}^{0,0}_{K_{C_N}}$ in $\mathcal{A}_{\tilde{m}_1,\tilde{m}_2}$ is given by $(g_1, g_2) = (\frac{1}{2}\chi_{B^{2\tilde{m}_1}}, \frac{1}{2}\chi_{B^{2\tilde{m}_1}} + \chi_{B^{\tilde{m}_1+\tilde{m}_2}\setminus B^{\tilde{m}_1}})$. This, together with (3.27), concludes the proof. $\square$

Conclusions and perspectives. We have studied existence and qualitative properties of minimizers of the energy

$$\mathcal{E}^{c_{11},c_{22}}_K(f_1, f_2) = c_{11}\, J_K(f_1, f_1) + c_{22}\, J_K(f_2, f_2) - 2J_K(f_1, f_2)$$

in the class of densities $(f_1, f_2) \in L^1(\mathbb{R}^N; [0, 1]) \times L^1(\mathbb{R}^N; [0, 1])$ with fixed masses m_1, m_2 and satisfying the constraint $f_1 + f_2 \leq 1$. We have focused on the attractive case $c_{11}, c_{22} \leq 0$ (the checkerboard region in Figure 7) and proved the existence of a minimizer in this case for all the values of masses m_1, m_2 (see Theorem 1.9). Moreover, for $0 < c_{11} = c_{22} \leq 1$, $m_1 = m_2$, and K positive definite (the dashed segment in Figure 7), we have proved that there exists a minimizer (see Lemma 2.7 and Remark 1.12). Eventually, for $c_{11}, c_{22} \geq 1$ with $\max\{c_{11}, c_{22}\} > 1$ (gray region in Figure 7), the energy $\mathcal{E}^{c_{11},c_{22}}_K$ does not admit a minimizer for any pair of values m_1 and m_2 (see Remark 1.12).

A natural question arising from these (partial) results is whether existence of minimizers can be proven in the remaining cases. A general existence result, i.e., independent of the masses, seems to be false if at least one of the coefficients is strictly positive. Indeed, the corresponding phase would lose some of its (if too large) mass. In this case, existence results depending on the masses seem to be an interesting issue.

A relevant aspect of our analysis is that for the Coulomb interaction kernel, we have found the explicit shape of minimizers for all choices of negative coefficients, except when they are both strictly less than -1 (see Figure 8). In this case, we can

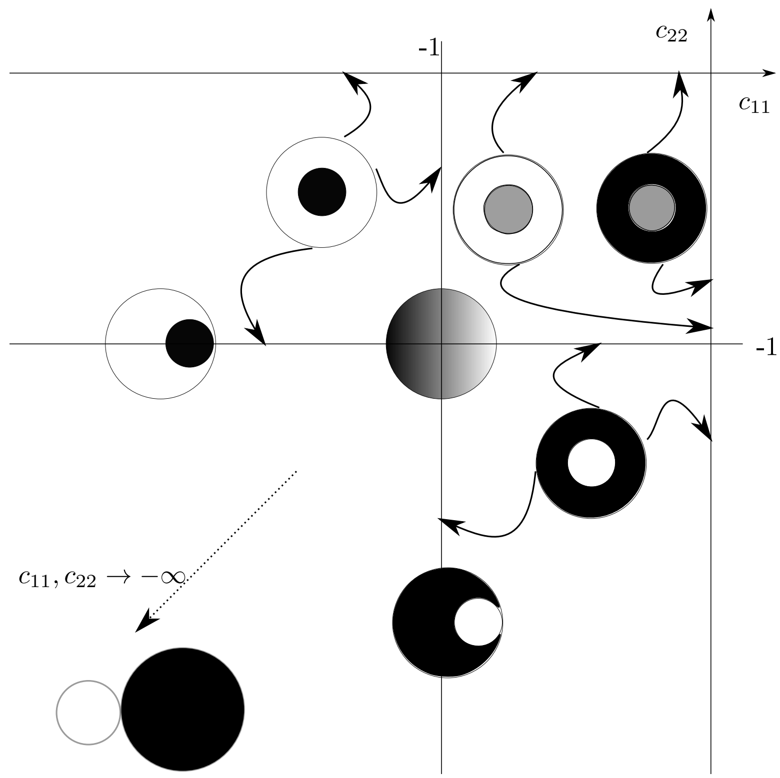


FIG. 8. *Minimizers for Coulomb interactions.*

still say that f_i are characteristic functions of two pairwise disjoint sets. But their specific shape is unknown and could be analyzed using numerical methods.

For general kernels our analysis is far from being complete. Nevertheless, there are many possible generalizations we would like to comment on.

First of all, one may study the minimum problem above for some specific kernels that are used frequently in the context of population dynamics (see, for instance, [9, 13] and the references therein) such as Gaussian, Morse, or power law kernels, or suitable combinations of these. Moreover, one might remove the assumption that the cross and self interaction kernels K_{ij} are all multiples of a given K. Actually, it would be interesting also to understand whether the improved Riesz inequality established in corollary 3.11 holds true for more general kernels. We notice that this corollary is equivalent to Theorem 3.9 once one knows that there is no coexistence of two homogeneous phases, i.e., when f_i are as in Lemma 3.3.

Another interesting direction is the extension of the model to the case of n species, i.e., considering minimizers of functionals of the type

$$\mathcal{E}_K(f_1, \ldots, f_n) := \sum_{i,j=1}^{n} J_{K_{ij}}(f_i, f_j)$$

under the constraint $\sum_{i=1}^n f_i \leq 1$ and $\int_{\mathbb{R}^N} f_i(x)\ \mathrm{d}x = m_i$ for $i = 1, 2, \ldots, n$. We believe that some of the techniques developed here could be slightly modified in order to prove existence and some qualitative properties of the minimizers. As already mentioned, the explicit shape of minimizers might require a specific analysis and could be the subject of numerical investigation.

Finally, we point out that our analysis focuses only on the global minimizers of the functional $\mathcal{E}_K^{c_{11},c_{22}}$. Notice that ground states play a crucial role in the long time asymptotics of nonlinear aggregation-diffusion models. Nevertheless, the analysis of stationary states (rather than minimizers) would provide a better understanding of such problems. In this respect, an interesting analysis would concern the dynamics of two phases governed by the energy proposed in this paper. A suitable notion of Wasserstein gradient flow could be considered, in the spirit of [9, 29].

REFERENCES

[1] L. Ambrosio and A. Braides, *Functionals defined on partitions in sets of finite perimeter.* II, J. Math. Pures Appl., 69 (1990), pp. 307–333.
[2] M. Bonacini and R. Cristoferi, *Local and global minimality results for a nonlocal isoperimetric problem on* $\mathbb{R}^N$, SIAM J. Math. Anal., 46 (2014), pp. 2310–2349.
[3] M. Bonacini, H. Knüpfer, and M. Röger, *Optimal distribution of oppositely charged phases: Perfect screening and other properties*, SIAM J. Math. Anal., 48 (2016), pp. 1128–1154.
[4] A. Burchard, R. Choksi, I. Topaloglu, *Nonlocal shape optimization via interactions of attractive and repulsive potentials*, preprint 2016.
[5] J. A. Cañizo, J. A. Carrillo, and F. S. Patacchini, *Existence of compactly supported global minimisers for the interaction energy*, Arch. Ration. Mech. Anal., 217 (2015), pp. 1197–1217.
[6] E. A. Carlen and F. Maggi, *Stability for the Brunn-Minkowski and Riesz Rearrangement Inequalities, with Applications to Gaussian Concentration and Finite Range Non-local Isoperimetry*, preprint 2015.
[7] E. A. Carlen, M. Carvalho, R. Esposito, J. L. Lebowitz, and R. Marra, *Free energy minimizers for a two-species model with segregation and liquid-vapor transition*, Nonlinearity, 16 (2003), pp. 1075–1105.
[8] J. A. Carrillo, M. Chipot, and Y. Huang, *On global minimizers of repulsive-attractive power-law interaction energies*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 372, (2014).
[9] J. A. Carrillo, M. Di Francesco, A. Figalli, T. Laurent, and D. Slepčev, *Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations*, Duke Math. J., 156 (2011), pp. 229–271.
[10] R. Choksi, R.C. Fetecau, and I. Topaloglu, *On minimizers of interaction functionals with competing attractive and repulsive potentials*, Ann. Inst. H. Poincaré Anal. Non Linéalre, 32 (2015), pp. 1283–1305.
[11] E. Cristiani, B. Piccoli, and A. Tosin, *Multiscale Modeling of Pedestrian Dynamics: Modeling, Simulation and Applications*, Springer, Berlin, 2014.
[12] A. Di Castro, M. Novaga, B. Ruffini, and E. Valdinoci, *Nonlocal quantitative isoperimetric inequalities*, Calc. Var. Partial Differential Equations, 54 (2015), pp. 2421–2464.
[13] M. Di Francesco and S. Fagioli, *Measure solutions for non-local interaction PDEs with two species*, Nonlinearity, 26 (2013), pp. 2777–2808.
[14] C. Escudero, F. Maciá, and J.J.L. Velázquez, *Two-species-coagulation approach to consensus by group level interactions*, Phys. Rev. E, 82 (2010), 016113.
[15] A. Figalli, N. Fusco, F. Maggi, V. Millot, and M. Morini, *Isoperimetry and stability properties of balls with respect to nonlocal energies*, Comm. Math. Phys., 336 (2015), pp. 441–507.
[16] E. H. Lieb and M. Loss, *Analysis*, Grad. Stud. Math. 14, AMS, Providence, RI, 2001.
[17] V. Julin, *Isoperimetric problem with a Coulomb repulsive term*, Indiana Univ. Math. J., 63 (2014), pp. 77–89.
[18] P. Kevrevidis, A. G. Stefanov, and H. Xu, *Traveling waves for the mass in mass model of granular chains*, Lett. Math. Phys., 106 (2016), pp. 1067–1088.
[19] H. Knüpfer and C. Muratov, *On an isoperimetric problem with a competing non-local term.* I. *The planar case*, Commun. Pure Appl. Math., 66 (2013), pp. 1129–1162.
[20] H. Knüpfer and C. Muratov, *On an isoperimetric problem with a competing non-local term.* II. *The general case*, Commun. Pure Appl. Math., 67 (2014), pp. 1974–1994.

[21] T. Kolokolnikov, Y. Huang, and M. Pavlovski, *Singular patterns for an aggregation model with a confining potential*, Phys. D, 260 (2013), pp. 65–76.

[22] T. Kolokolnikov, H. Sun, D. Uminsky, and A. L. Bertozzi, *A theory of complex patterns arising from 2D particle interactions*, Phys. Rev. E, 84 (2011), 015203

[23] N. S. Landkof, *Foundations of Modern Potential Theory*, Grundlehren Math. Wisse. 180, Springer-Verlag, Heidelberg, 1972.

[24] H. Levine, E. Ben-Jacob, I. Cohen, and W.-J. Rappel, *Swarming patterns in Microorganisms: Some new modeling results*, Proceedings of IEEE CDC, 2006, pp. 5073–5077.

[25] T. Liu, M. L. K. Langston, D. Li, J. M. Pigga, C. Pichon, A M. Todea, and A. Müller, *Self-recognition among different polyprotic macroions during assembly processes in dilute solution*, Science, 331 (2011), pp. 1590–1592.

[26] J. Lu and F. Otto, *Nonexistence of minimizers for Thomas-Fermi-Dirac-von Weizsäcker model*, Commun. Pure Appl. Math., 67 (2014), pp. 1605–1617.

[27] A. Mackey, T. Kolokolnikov, and A. L. Bertozzi, *Two-species particle aggregation and stability of co-dimension one solutions*, Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), pp. 1411–1436.

[28] A. Magni and M. Novaga, *A note on non lower semicontinuous perimeter functionals on partitions*, Netw. Heterog. Media, 11 (2016), pp. 501–508.

[29] B. Maury, A. Roudneff-Chupin, F. Santambrogio, and J. Venel, *Handling congestion in crowd motion models*, Netw. Heterog. Media, 6 (2011), pp. 485–519.

[30] X. Ren and J. Wei, *A double bubble assembly as a new phase of a ternary inhibitory system*, Arch. Ration. Mech. Anal., 215 (2015), pp. 967–1034.

[31] F. Riesz, *Sur une inégalité intégrale*, J. Lond. Math. Soc., 5 (1930), pp. 162–168.

[32] R. Simione, D. Slepcev, and I. Topaloglu, *Existence of ground states of nonlocal-interaction energies*, J. Stat. Phys., 159 (2015), pp. 972–986.

[33] J. D. van der Waals and I. Verhandelingen, *Kon. Akad. Wet. Amsterdam* 20, 1880; II *Théorie moléculaire d'une substance composée de deux matiéres différentes*, Arch. Néerl., 24 (1891).