

# The remarkable effectiveness of time-dependent damping terms for second order evolution equations

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## Abstract

We consider a second order linear evolution equation with a dissipative term multiplied by a time-dependent coefficient. Our aim is to design the coefficient in such a way that *all* solutions decay in time as fast as possible.

We discover that constant coefficients do not achieve the goal, as well as time-dependent coefficients that are too big. On the contrary, pulsating coefficients which alternate big and small values in a suitable way prove to be more effective.

Our theory applies to ordinary differential equations, systems of ordinary differential equations, and partial differential equations of hyperbolic type.

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**Key words:** damping, linear evolution equations, dissipative hyperbolic equation, decay rates, exponentially decaying solutions.

# 1 Introduction

In this paper we consider abstract evolution equations of the form

$$u''(t) + 2\delta(t)u'(t) + Au(t) = 0, \quad (1.1)$$

with initial data

$$u(0) = u_0 \in D(A^{1/2}), \quad u'(0) = u_1 \in H, \quad (1.2)$$

where  $H$  is a Hilbert space,  $A$  is a self-adjoint linear operator on  $H$  with dense domain  $D(A)$ , and  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  is a measurable function. We always assume that the spectrum of  $A$  is a finite set, or an unbounded increasing sequence of positive eigenvalues.

We are interested in the decay rate of all solutions to problem (1.1)–(1.2). In particular, we are interested in designing the coefficient  $\delta(t)$  so that *all* solutions decay as fast as possible as  $t \rightarrow +\infty$ .

*Motivation and related literature* One of the motivations of the present work originates from control theory. Considering an equation of the form

$$u''(t) + \delta B u'(t) + Au(t) = 0,$$

where  $B$  is a linear operator on  $H$ , it is known (see for example [3]) that uniform exponential decay of the solutions in the energy space is independent of  $\delta > 0$ . At this level it is already natural to seek for an optimal damping term in the class of multiples of  $B$  and even, more generally, of operators of the form  $\Lambda B$  where  $\Lambda$  is some symmetric isomorphism from  $H$  to  $H$  commuting with  $B$ . Nevertheless such optimization suffers from strong limitations, as we shall see below.

In [5] an attempt was done to optimize the decay rate by a somewhat different method. It consists in perturbing the conservative part, namely considering an equation of the form

$$u''(t) + \delta B u'(t) + Au(t) + cu(t) = 0,$$

where  $c > 0$ . When  $c$  and  $\delta$  tend to infinity in a certain way the decay rate can be made arbitrarily large. The problem here is that the equation with  $c$  large can be considered as driven by the operator  $cI$  rather than  $A$ : the nature of the problem is altered. Some of the results from [5] were later improved in [9].

In [2] a different strategy was used in the special case of the string equation

$$u_{tt} - u_{xx} + \delta u_t = 0.$$

The constant dissipation  $\delta u_t$  was replaced by  $\delta(x)u_t$ , and as  $\delta(x)$  approaches the singular potential  $1/x$ , it was shown that the exponential decay rate can be made as large as prescribed.

In the present work we try a different approach consisting in making  $\delta = \delta(t)$  time-dependent. This approach will turn out to be as fruitful as the  $x$ -dependence and basically applicable to any self-adjoint operator  $A$ . Earlier in [4] (see also [8]), time-dependent damping terms of intermittent type (namely with  $\delta(t)$  that vanishes on some intervals up to infinity) were shown to produce exponential decay provided some condition on the length of the intervals of effective damping (together with the maxima and the minima of  $\delta$  on those intervals) is satisfied. This result was extended in [6] to some cases where the damping operator involves a delay term. This was one more motivation to examine the case of time-dependent damping.

*A simple toy model: ODEs with constant dissipation* We begin our investigation by recalling the behavior of solutions in the simplest example where  $H = \mathbb{R}$  and  $\delta(t)$  is constant, so that (1.1) reduces to the ordinary differential equation

$$u''(t) + 2\delta u'(t) + \lambda^2 u(t) = 0, \quad (1.3)$$

where  $\delta$  and  $\lambda$  are positive parameters. This equation can be explicitly integrated. It turns out that the asymptotic behavior of solutions depends on the real part of the roots of the characteristic equation

$$x^2 + 2\delta x + \lambda^2 = 0. \quad (1.4)$$

When  $\delta < \lambda$ , the characteristic equation (1.4) has two complex conjugate roots with real part equal to  $-\delta$ . As a consequence, all nonzero solutions to (1.3) decay as  $e^{-\delta t}$ . When  $\delta > \lambda$ , the characteristic equation (1.4) has two real roots  $r_1$  and  $r_2$ , with

$$r_1 := -\delta - \sqrt{\delta^2 - \lambda^2} \sim -2\delta, \quad r_2 := -\delta + \sqrt{\delta^2 - \lambda^2} \sim -\frac{\lambda^2}{2\delta}.$$

Every solution to (1.3) is a linear combination of  $e^{-r_1 t}$  and  $e^{-r_2 t}$ . Since we are looking for a decay rate valid for *all* solutions, in this case  $e^{-r_2 t}$  is the best possible estimate, and it is also optimal for the generic solution to (1.3). We recall that *for linear equations the slowest behavior is always generic*, and therefore the best estimate valid for all solutions is sharp for the generic solution.

A similar argument shows that for  $\delta = \lambda$  all solutions to (1.3) are linear combinations of  $e^{-\lambda t}$  and  $te^{-\lambda t}$ , so that  $te^{-\lambda t}$  is the estimate valid for all solutions in that case.

The graph in Figure 1 represents the exponent  $r$  in the optimal decay rate  $e^{-rt}$  as a function of  $\delta$  (namely  $r = \delta$  when  $\delta < \lambda$  and  $r = r_2$  when  $\delta > \lambda$ ).

It clarifies that, if we limit ourselves to constant coefficients, we cannot hope that all solutions to (1.3) decay better than  $e^{-\lambda t}$ , and actually neither better than  $te^{-\lambda t}$ . Moreover, beyond the threshold  $\lambda$ , the larger is  $\delta$  the worse is the decay rate. This shows also that in the general setting of equation (1.1), if we restrict ourselves to constant damping coefficients  $\delta(t)$ , the best possible decay rate valid for all solutions is  $te^{-\nu t}$ , where  $\nu^2$  is the smallest eigenvalue of the operator  $A$ . In [2] this remark was summed up effectively by saying that “more is not better”.

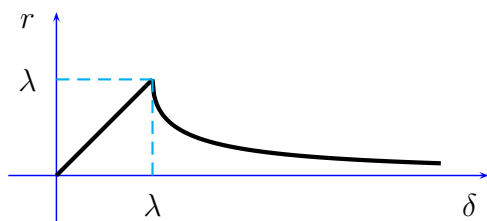


Figure 1: Decay rate  $r$  as a function of the constant dissipation  $\delta$

*ODEs with nonconstant dissipation* The first nontrivial case we consider is the ordinary differential equation

$$u''(t) + 2\delta(t)u'(t) + \lambda^2 u(t) = 0, \quad (1.5)$$

where now the coefficient  $\delta(t)$  is time-dependent. As long as  $\delta \in L^1_{\text{loc}}((0, +\infty))$  the energy of a non-trivial solution can not vanish at any finite time. Moreover, if we limit ourselves to coefficients  $\delta(t) \geq \lambda$ , once again there exists a solution which decays at most as  $te^{-\lambda t}$ , exactly as in the case where  $\delta(t)$  is constant (see Proposition 2.7). In other words, once again “more is not better” and the overdamping prevents a faster stabilization.

Things change when we consider damping coefficients  $\delta(t)$  which alternate intervals where they are big and intervals where they are small (below  $\lambda$ ). We obtain two results.

- In Theorem 2.1 we prove that every exponential decay rate can be achieved through a periodic damping coefficient. More precisely, for every real number  $R$  there exists a periodic function  $\delta(t)$  for which all solutions to (1.5) decay at least as  $e^{-Rt}$ . A possible choice for the period of  $\delta(t)$  is

$$t_0 = \frac{\pi}{2\lambda}, \quad (1.6)$$

hence it does *not* depend on  $R$ . We can also ask further requirements on  $\delta(t)$ , for example being of class  $C^\infty$ , or taking alternatively only two values, 0 and a sufficiently large positive number  $K$ .

- In Theorem 2.2 we obtain even better decay rates. Indeed, we prove that for every nonincreasing function  $\varphi : [0, +\infty) \rightarrow (0, +\infty)$  one can design  $\delta(t)$  in such a way that all solutions to (1.5) decay at least as  $\varphi(t)$ . In this case  $\delta(t)$  is necessarily non-periodic and unbounded, but one can choose it of class  $C^\infty$  or piecewise constant. The proof of this second result relies on the first one, and the key point is that in the first result we can achieve any exponential decay rate  $e^{-Rt}$  with a coefficient whose period does not depend on  $R$ .

The results for the single equation can be easily extended to systems of the form

$$u_k''(t) + 2\delta(t)u_k'(t) + \lambda_k^2 u_k(t) = 0 \quad k = 1, \dots, n. \quad (1.7)$$

The only difference is that in the first result (the one with a fixed exponential decay rate  $e^{-Rt}$ ) the period of  $\delta(t)$  is now

$$t_0 = \frac{\pi}{2} \sum_{k=1}^n \frac{1}{\lambda_k}. \quad (1.8)$$

Once again the period is independent of  $R$ , and this is the key point to achieve any given decay rate through a non-periodic and unbounded coefficient. We refer to Theorem 2.3 and Theorem 2.4 for the details.

The results obtained for (systems of) ordinary differential equations can be extended word for word to the Hilbert setting of (1.1), provided that  $A$  has a finite number of eigenvalues, even with infinite dimensional eigenspaces.

*PDEs with nonconstant dissipation* It remains to consider the case of operators with an infinite number of eigenvalues. Due to our assumption on the spectrum, in this case  $H$  admits an orthonormal system made by eigenvectors of  $A$ , and therefore the evolution equation (1.1) is equivalent to a system of countably many ordinary differential equations. Looking at (1.8) one could naturally guess that our theory extends to the general setting when the series of  $1/\lambda_k$  is convergent. This is actually true, but not so interesting because in most applications the series is divergent (with the notable exception of the beam equation, see section 5). This leads us to follow a partially different path.

In Theorem 2.5 we prove once again that any exponential decay rate  $e^{-Rt}$  can be achieved through a periodic damping coefficient. The main difference is that now the period of  $\delta(t)$  is

$$t_0 = \pi \sum_{\lambda_k^2 \leq 2(R+\lambda_1)^2} \frac{1}{\lambda_k}, \quad (1.9)$$

and hence it does depend on  $R$ . In analogy with the previous results, we can again ask further structure on  $\delta(t)$ , for example being of class  $C^\infty$ , or taking alternatively only three values (instead of two).

The fact that the period depends on  $R$  complicates the search for better decay rates. Our best result is stated in Theorem 2.6, where we prove that there exists  $\delta(t)$  such that all solutions to (1.1) decay at least as a nonincreasing function  $\varphi : [0, +\infty) \rightarrow (0, 1)$  that tends to zero faster than all exponentials. This universal decay rate is independent of the solution, but it does depend on the operator  $A$ , more precisely on its spectrum.

*More requirements on the damping coefficient* The coefficients introduced in the proofs of the results quoted so far alternate intervals where they are close to 0 and intervals where they are very large. Even the coefficients of class  $C^\infty$  are just smooth approximations of the discrete ones. On the other hand, we already know that both large values and values below  $\lambda$  are needed if we want fast decay rates (see Proposition 2.7).

In the last part of the paper we ask ourselves whether it is essential that the coefficient approaches 0 or exhibits sudden oscillations between big and small values. The answer to both questions is negative. In the case of the ordinary differential equation (1.5) we show that we can achieve any exponential decay rate  $e^{-Rt}$  through a periodic coefficient  $\delta(t)$  which is always greater than or equal to  $\lambda - \varepsilon$  and has Lipschitz constant equal to  $\varepsilon$  (where  $\varepsilon$  is a fixed parameter). Of course the period of the coefficient now depends on  $\varepsilon$  and  $R$ . We refer to Theorem 2.8 for the details.

*Perspectives and open problems* We consider this paper as a starting point of a research project. Several related questions are not addressed here but could probably deserve future investigations. Just to give some examples, we mention finding the optimal decay rates that can be achieved through damping coefficients with reasonable restrictions, proving or disproving that in the infinite dimensional setting there is a bound on the decay rate one can achieve, extending if possible some parts of the theory to operators with continuum spectrum, proving or disproving that random damping coefficients are ineffective and never better than constant ones.

*Structure of the paper* This paper is organized as follows. In section 2 we state all our results. In section 3 we give a rough explanation of why “pulsating is better”. In section 4 we provide rigorous proofs. In section 5 we present some simple applications to partial differential equations.

## 2 Statements

For the sake of clarity we present our results in increasing order of complexity. We start with ordinary differential equations, we continue with systems of ordinary differential equations, and finally we consider the more general Hilbert setting. In the last subsection we investigate the same problems with additional constraints on the damping coefficients.

In the sequel  $(t - t_0)^+$  stands for  $\max\{t - t_0, 0\}$ .

### 2.1 Ordinary differential equations

To begin with, we consider the ordinary differential equation (1.5). In the first result we achieve any given exponential decay through a periodic damping coefficient.

**Theorem 2.1** (Single ODE, fixed exponential decay rate). *Let  $\lambda$  and  $R$  be positive real numbers, and let  $t_0$  be defined by (1.6).*

*Then there exists a  $t_0$ -periodic damping coefficient  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  (which one can choose either of class  $C^\infty$  or piecewise constant) such that every solution  $u(t)$  to (1.5) satisfies*

$$|u'(t)|^2 + \lambda^2 |u(t)|^2 \leq (|u'(0)|^2 + \lambda^2 |u(0)|^2) \exp(-R(t - t_0)^+) \quad \forall t \geq 0. \quad (2.1)$$

The typical profile of a damping coefficient realizing a fixed exponential decay rate is shown in Figure 2 on the left. In each period there are two impulses of suitable height  $K$  and duration  $\rho$ , one at the beginning and one at the end of the period. For the rest of the time the damping coefficient vanishes, which means that there is no dissipation. Of course, when the coefficient is extended by periodicity, the impulse at the end of each period continues with the impulse at the beginning of next period, thus giving rise to a single impulse with double time-length.

These piecewise constant coefficients with only two values sound good for applications. In any case, since (1.5) is stable under  $L^2$  perturbations of the coefficient (see Lemma 4.1), the same effect can be achieved through a smooth approximation of the piecewise constant coefficient.

In the second result we show that every fixed decay rate can be achieved if we are allowed to exploit non-periodic and unbounded damping coefficients.

**Theorem 2.2** (Single ODE, any given decay rate). *Let  $\lambda$  be a positive real number, let  $\varphi : [0, +\infty) \rightarrow (0, +\infty)$  be a nonincreasing function, and let  $t_0$  be defined by (1.6).*

*Then there exists a damping coefficient  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  (which one can choose either of class  $C^\infty$  or piecewise constant) such that every solution  $u(t)$  to (1.5) satisfies*

$$|u'(t)|^2 + \lambda^2 |u(t)|^2 \leq (|u'(0)|^2 + \lambda^2 |u(0)|^2) \cdot \varphi(t) \quad \forall t \geq t_0. \quad (2.2)$$

The typical profile of a damping coefficient realizing a given decay rate in Theorem 2.2 is shown in Figure 2 on the right. It consists in a sequence of blocks of the type described after Theorem 2.1, the only difference being that now the values of the parameters  $K$  and  $\rho$  are different in each block.

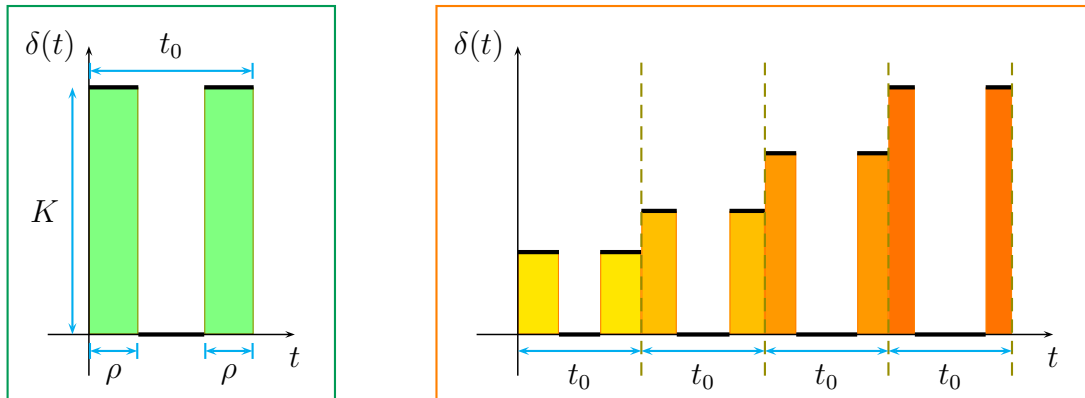


Figure 2: possible profiles of  $\delta(t)$  in Theorem 2.1 (left) and Theorem 2.2 (right)

## 2.2 Systems of ordinary differential equations

The results for a single ordinary differential equation can be extended to systems. The following statement is the generalization of Theorem 2.1.



**Theorem 2.3** (System of ODEs, fixed exponential decay rate). *Let  $n$  be a positive integer, let  $(\lambda_1, \dots, \lambda_n) \in (0, +\infty)^n$ , let  $R$  be a positive real number, and let  $t_0$  be defined by (1.8).*

*Then there exists a  $t_0$ -periodic damping coefficient  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  (which one can choose either of class  $C^\infty$  or piecewise constant) such that every solution  $(u_1(t), \dots, u_n(t))$  to system (1.7) satisfies*

$$\sum_{k=1}^n (|u'_k(t)|^2 + \lambda_k^2 |u_k(t)|^2) \leq \left[ \sum_{k=1}^n (|u'_k(0)|^2 + \lambda_k^2 |u_k(0)|^2) \right] \exp(-R(t - t_0)^+)$$

for every  $t \geq 0$ .

The typical profile of a damping coefficient realizing a fixed exponential decay rate for a system is shown in Figure 3 on the left. Now the period  $[0, t_0]$  is the union of  $k$  subintervals ( $k = 3$  in the figure), where the  $i$ -th subinterval has length  $\pi/(2\lambda_i)$ . In each subinterval we exploit once again the profile with two impulses described after Theorem 2.1. We can assume that in all subintervals the values of the parameters  $K$  and  $\rho$  are the same (and hence the time-length of the vanishing phase is different), so we end up once again with a damping coefficient with just two values.

The following statement is the generalization of Theorem 2.2 to systems.

**Theorem 2.4** (System of ODEs, any given decay rate). *Let  $n$  be a positive integer, let  $(\lambda_1, \dots, \lambda_n) \in (0, +\infty)^n$ , let  $\varphi : [0, +\infty) \rightarrow (0, +\infty)$  be a nonincreasing function, and let  $t_0$  be defined by (1.8).*

*Then there exists a damping coefficient  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  (which one can choose either of class  $C^\infty$  or piecewise constant) such that every solution  $(u_1(t), \dots, u_n(t))$  to system (1.7) satisfies*

$$\sum_{k=1}^n (|u'_k(t)|^2 + \lambda_k^2 |u_k(t)|^2) \leq \left[ \sum_{k=1}^n (|u'_k(0)|^2 + \lambda_k^2 |u_k(0)|^2) \right] \cdot \varphi(t) \quad \forall t \geq t_0.$$

As in the case of a single equation, the typical profile of a damping coefficient realizing a given decay rate for a system is a sequence of blocks of the same type used in order to realize exponential decay rates for the same system, just with different values of the parameters  $K$  and  $\rho$  in different blocks (see Figure 3 on the right).

## 2.3 Partial differential equations

We examine now equation (1.1) in the general Hilbert setting. As usual, we consider weak solutions with regularity

$$u \in C^0([0, +\infty), D(A^{1/2})) \cap C^1([0, +\infty), H).$$

In the first result we achieve once again a given exponential decay rate through a periodic damping coefficient. In contrast with the case of ordinary differential equations or systems, the period of the coefficient now does depend on the decay rate.

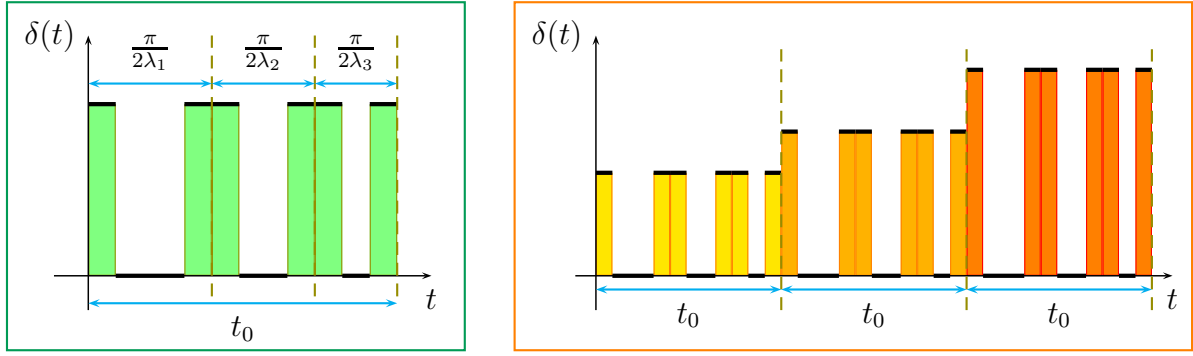


Figure 3: possible profiles of  $\delta(t)$  in Theorem 2.3 (left) and Theorem 2.4 (right)

**Theorem 2.5** (PDE, fixed exponential decay rate). *Let  $H$  be a Hilbert space, and let  $A$  be a self-adjoint nonnegative operator on  $H$  with dense domain  $D(A)$ . Let us assume that the spectrum of  $A$  is an increasing unbounded sequence of positive real numbers  $\{\lambda_k^2\}_{k \geq 1}$  (with the agreement that  $\lambda_k > 0$  for every  $k \geq 1$ ).*

*Let  $R$  be a positive real number, and let  $t_0$  be defined by (1.9).*

*Then there exists a  $t_0$ -periodic damping coefficient  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  (which one can choose either of class  $C^\infty$  or piecewise constant) such that every weak solution to equation (1.1) satisfies*

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 \leq (|u'(0)|^2 + |A^{1/2}u(0)|^2) \exp(-R(t - t_0)^+) \quad (2.3)$$

for every  $t \geq 0$ .

The typical profile of a damping coefficient realizing a fixed exponential decay rate for the full equation (1.1) is shown in Figure 4. Now the period  $[0, t_0]$  is divided into two subintervals of the same length  $t_0/2$ . In the second subinterval the damping coefficient is constant. The first half of the period is in turn divided into subintervals of length  $\pi/(2\lambda_i)$ , where the  $\lambda_i$ 's are those which contribute to the sum in the right-hand side of (1.9). In each of these subintervals we have again the same profile with two impulses described after Theorem 2.1. We can assume that the parameters  $K$  and  $\rho$  are the same in all subintervals of the first half of the period, but the constant in the second half of the period might differ from  $K$ . Therefore, now the damping coefficient takes in general three values instead of two.

In the second result as usual we allow ourselves to exploit non-periodic and unbounded damping coefficients. What we obtain is a decay rate which is faster than all exponentials. This decay rate does depend on the operator.

**Theorem 2.6** (PDE, decay rate faster than all exponentials). *Let the Hilbert space  $H$  and the operator  $A$  be as in Theorem 2.5.*

*Then there exist a damping coefficient  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  (which one can choose either of class  $C^\infty$  or piecewise constant) and a nonincreasing function  $\varphi : [0, +\infty) \rightarrow (0, +\infty)$  such that*

$$\lim_{t \rightarrow +\infty} \varphi(t)e^{Rt} = 0 \quad \forall R > 0, \quad (2.4)$$

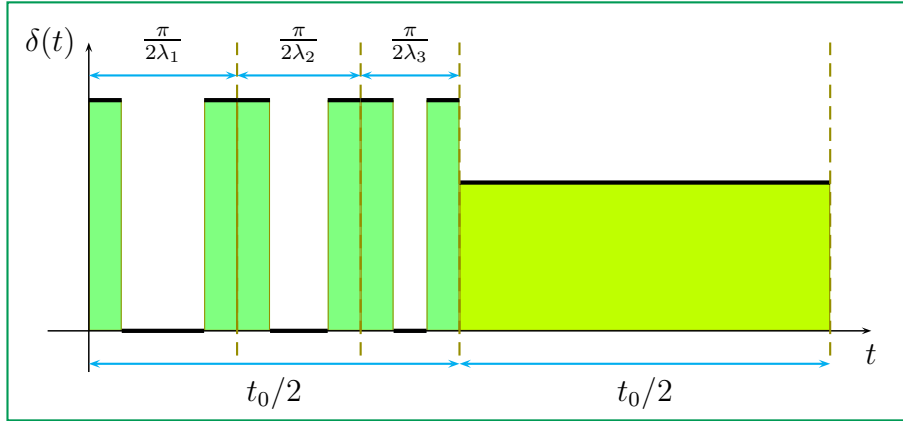


Figure 4: possible profiles of  $\delta(t)$  in Theorem 2.5

and such that every weak solution to equation (1.1) satisfies

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 \leq (|u'(0)|^2 + |A^{1/2}u(0)|^2) \cdot \varphi(t) \quad \forall t \geq 0. \quad (2.5)$$

As in the case of ordinary differential equations or systems, the standard profile of a damping coefficient provided by Theorem 2.6 is a sequence of blocks of the same type as those introduced for Theorem 2.5. The main difference is that now also the time-length  $t_0$  is different in different blocks, and increases with time. The reason is that now  $t_0$  depends on  $R$ , and when  $R$  increases a larger number of eigenvalues contributes to (1.9), and hence  $t_0$  increases as well.

## 2.4 Further requirements on the damping coefficient

In the last part of the paper we investigate which features of the damping coefficient are essential when one wants to achieve fast decay rates. To begin with, in the following statement we list three simple situations where the decay rate of solutions can be bounded *from below*. We point out that, when looking for estimates from below, there is almost no loss of generality in considering just the ordinary differential equation (1.5).

**Proposition 2.7** (Estimates of the decay rate from below). *Let us consider equation (1.5) for some positive real number  $\lambda$  and some nonnegative damping coefficient  $\delta \in L^1_{loc}((0, +\infty))$ .*

- (1) *If  $\delta \in L^1((0, +\infty))$ , then all nonzero solutions do not decay to zero.*
- (2) *If there exists a constant  $M > 0$  such that  $\delta(t) \leq M$  for every  $t \geq 0$ , then all solutions satisfy*

$$|u'(t)|^2 + \lambda^2|u(t)|^2 \geq (|u'(0)|^2 + \lambda^2|u(0)|^2) e^{-4Mt} \quad \forall t \geq 0.$$

(3) If there exists  $T \geq 0$  such that  $\delta(t) \geq \lambda$  for every  $t \geq T$ , then there exist  $T_* \geq T$  and a solution  $u(t)$  to (1.5) such that

$$|u(t)| \geq te^{-\lambda t} \quad \forall t \geq T_*. \quad (2.6)$$

As a consequence, if we want *all* solutions to (1.5) to decay faster than a given exponential, we are forced to choose a damping coefficient  $\delta(t)$  which alternates intervals where it is large enough, and intervals where it is smaller than  $\lambda$ . If we want solutions to decay faster than all exponentials, we also need  $\delta(t)$  to be unbounded.

In some sense these are the unique essential features. In the following result we show that any fixed exponential decay rate can be achieved through a periodic damping coefficient which is greater than or equal to  $\lambda - \varepsilon$ , and has Lipschitz constant equal to  $\varepsilon$ , and hence it exhibits very slow transitions from small to large values.

**Theorem 2.8** (Single ODE, with borderline constraints on  $\delta(t)$ ). *Let  $\lambda$  and  $R$  be positive real numbers, and let  $\varepsilon \in (0, \lambda)$ . Then there exist a positive real number  $t_0$ , with*

$$t_0 \leq 16 \left( \frac{\pi}{\varepsilon} + 1 \right) R + \frac{2(\pi + 1)}{\varepsilon} + \frac{8}{\lambda} + 2 + 8 \log 2, \quad (2.7)$$

and a  $t_0$ -periodic function  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  such that

$$|\delta(t) - \delta(s)| \leq \varepsilon |t - s| \quad \forall t \geq 0 \quad \forall s \geq 0,$$

$$\delta(t) \geq \lambda - \varepsilon \quad \forall t \geq 0,$$

and such that every solution to (1.5) satisfies

$$|u'(t)|^2 + \lambda^2 |u(t)|^2 \leq (|u'(0)|^2 + \lambda^2 |u(0)|^2) \exp(-R(t - t_0)^+) \quad \forall t \geq 0. \quad (2.8)$$

Theorem 2.8 deals with the case of a single ordinary differential equation, but an analogous result holds true also for systems or the abstract equation (1.1). In order to contain this paper in a reasonable length we spare the reader from the details.

More delicate is achieving decay rates faster than all exponentials through damping coefficients with small Lipschitz constant. This could be done at most in the same spirit of Theorem 2.6, the reason being as usual that now  $t_0$  depends on  $R$ . We do not address this issue in this paper.

### 3 Heuristics

In this section we provide an informal description of the strategy of our proofs. Our aim is clarifying why pulsating damping coefficients are more effective when we are interested in damping all solutions to an equation or system.

Let us start with the single ordinary differential equation (1.5). We want to design  $\delta(t)$  so that all solutions decay as fast as possible. The first naive idea is to choose  $\delta(t)$

very large. Bending the rules a little bit, we can even imagine to choose a damping coefficient which is not a function, but a Dirac delta function (which is actually a measure) concentrated at time  $t = 0$ , or even better a delta function multiplied by a large enough constant  $k$ .

An easy calculation shows that such an extreme damping has a great effect on the solution with initial data  $u(0) = 0$  and  $u'(0) = 1$ , whose energy is instantly reduced by a factor  $e^{-k}$ . On the contrary, it has no effect on the orthogonal solution with initial data  $u(0) = 1$  and  $u'(0) = 0$ . This can be explained by observing that the damping coefficient multiplies  $u'(t)$ , and in the case of the second solution this time-derivative vanishes when the delta function acts. The effect on any other solution is a linear combination of the two, namely highly reducing the time-derivative but leaving the function untouched.

This apparently inconclusive approach suggests a first strategy: if we want to dampen a single solution, we can use a delta function acting when the energy of the solution is concentrated on the time-derivative. So we take again the solution with initial data  $u(0) = 1$  and  $u'(0) = 0$ , and we apply no dissipation until  $u(t)$  vanish. This happens for the first time when  $t = \pi/(2\lambda)$ . At that point we apply a second delta function.

Summing up, a first delta function at time  $t = 0$  cuts the first solution, then the damping coefficient vanishes until the second delta function at time  $t = \pi/(2\lambda)$  cuts the second solution. What happens to all other solutions? Since the equation is linear, cutting two linearly independent solutions is equivalent to cutting all solutions. If we repeat this procedure by periodicity, we can achieve any exponential decay rate. If at each reiteration we increase the multiplicative constant in front of the delta functions, we can achieve any given decay rate. This is the idea behind the proofs of the results stated in section 2.1, and the point where the special time (1.6) and the profiles of Figure 2 come into play.

The idea for systems is a simple generalization. We use a first block of two delta functions with time-gap of  $\pi/(2\lambda_1)$  in order to dampen solutions of the first equation, then we use two more delta functions with time-gap of  $\pi/(2\lambda_2)$  in order to dampen solutions of the second equation, and so on. In other words, we take care of the equations of the system one by one. We end up with the special time (1.8) and the profiles of Figure 3. We are quite skeptic about the possibility of reducing the time (1.8), unless  $\lambda_i$ 's satisfy special rationality conditions.

When we deal with a partial differential equation, which we regard as a system of countably many ordinary differential equations, we exploit a mixed strategy. If we want to achieve a given exponential decay rate, a suitable constant damping coefficient does the job for all components corresponding to large enough eigenvalues. Thus we are left with cutting a finite number of components, and this can be done as in the case of finite systems. As a consequence, now a good damping coefficient consists in a constant damping half the time, alternated with a train of delta functions in the remaining half of the time. This is the idea behind the proof of Theorem 2.5 and the profile of Figure 4.

In the case of partial differential equations, things are more complex if we want to achieve a decay rate faster than all exponentials. Indeed, when we reiterate the procedure, a better decay requires more components to be treated separately, and in

turn this implies a longer wait. The compromise between faster decay rates and longer waiting times gives rise to the operator dependent rate  $\varphi(t)$  of Theorem 2.6.

This discussion motivates also the last part of the paper. Indeed, a train of delta functions (or a suitable approximation) emerged as a common pattern of the damping coefficients which realize fast decay rates. In a first stage this led us to suspect that a bound on the Lipschitz constant of the coefficient, or the impossibility to attain values close to zero, could yield a bound from below on the decay rate of solutions.

In Theorem 2.8 we show that this is not the case, because any exponential decay rate can be realized through a damping coefficient  $\delta(t)$  with arbitrarily small Lipschitz constant. The construction of  $\delta(t)$  is more involved, but once again it acts in two steps. In a first phase the coefficient grows and kills a first solution. Then the coefficients goes below  $\lambda$  and stays there until the orthogonal solution has rotated enough so that it is ready to be damped by a second growth of the coefficient.

This two-phase action (destroy the first solution, wait for rotation, destroy the second solution) seems to be the quintessence of all the story.

## 4 Proofs

In this section we prove our main results, following the same scheme of the statement section. We begin by investigating how solutions to (1.1) depend on the damping coefficient.

**Lemma 4.1** (Continuous dependence on the damping coefficient). *Let  $H$  be a Hilbert space, let  $A$  be a self-adjoint nonnegative linear operator on  $H$  with dense domain  $D(A)$ , and let  $T$  be a positive real number. Let  $\delta_1 : [0, T] \rightarrow [0, +\infty)$  and  $\delta_2 : [0, T] \rightarrow [0, +\infty)$  be two bounded measurable functions. Let  $u_1(t)$  and  $u_2(t)$  be the solutions to (1.1) with  $\delta(t)$  replaced by  $\delta_1(t)$  and  $\delta_2(t)$ , respectively, and with initial data*

$$u_1(0) = u_2(0) = u_0 \in D(A^{1/2}), \quad u_1'(0) = u_2'(0) = u_1 \in H.$$

*Then for every  $t \in [0, T]$  the following estimate holds true*

$$\begin{aligned} |u_2'(t) - u_1'(t)|^2 + |A^{1/2}(u_2(t) - u_1(t))|^2 &\leq 2(|u_1|^2 + |A^{1/2}u_0|^2) e^{2t} \cdot \\ &\cdot \int_0^t |\delta_2(s) - \delta_1(s)|^2 ds. \end{aligned} \quad (4.1)$$

*Proof* To begin with, we observe that

$$|u_1'(t)|^2 + |A^{1/2}u_1(t)|^2 \leq |u_1|^2 + |A^{1/2}u_0|^2 \quad \forall t \in [0, T]. \quad (4.2)$$

This inequality holds true because the left-hand side is a nonincreasing function of time. Now let us set

$$E(t) := |u_2'(t) - u_1'(t)|^2 + |A^{1/2}(u_2(t) - u_1(t))|^2.$$

An easy computation shows that

$$E'(t) = -4\delta_2(t)|u_2'(t) - u_1'(t)|^2 + 4(\delta_1(t) - \delta_2(t)) \cdot \langle u_1'(t), u_2'(t) - u_1'(t) \rangle.$$

The first term in the right-hand side is less than or equal to zero. Keeping (4.2) into account, we can estimate the second term and obtain that

$$\begin{aligned} E'(t) &\leq 2|u_2'(t) - u_1'(t)|^2 + 2|u_1'(t)|^2 \cdot |\delta_2(t) - \delta_1(t)|^2 \\ &\leq 2E(t) + 2(|u_1|^2 + |A^{1/2}u_0|^2) |\delta_2(t) - \delta_1(t)|^2. \end{aligned}$$

Integrating this differential inequality, and recalling that  $E(0) = 0$  because the initial conditions of  $u_2(t)$  and  $u_1(t)$  are the same, we conclude that

$$E(t) \leq 2(|u_1|^2 + |A^{1/2}u_0|^2) e^{2t} \int_0^t |\delta_2(s) - \delta_1(s)|^2 ds \quad \forall t \in [0, T],$$

which proves (4.1).  $\square$

## 4.1 Ordinary differential equations

The following result is the fundamental tool in our theory.

**Lemma 4.2** (Decay for two orthogonal solutions to a single ODE). *Let  $\lambda$  and  $M$  be positive real numbers, and let  $t_0$  be defined by (1.6). For every positive integer  $n$ , let us consider the function  $\delta_n : [0, t_0] \rightarrow [0, +\infty)$  defined by*

$$\delta_n(t) := \begin{cases} Mn & \text{if } t \in [0, 1/n] \cup [t_0 - 1/n, t_0], \\ 0 & \text{otherwise,} \end{cases} \quad (4.3)$$

and the differential equation

$$u''(t) + 2\delta_n(t)u'(t) + \lambda^2u(t) = 0. \quad (4.4)$$

Let  $v_n(t)$  be the solution with initial data  $v_n(0) = 0$  and  $v_n'(0) = 1$ . Let  $w_n(t)$  be the solution with initial data  $w_n(0) = 1/\lambda$  and  $w_n'(0) = 0$ .

Then

$$\limsup_{n \rightarrow +\infty} (|v_n'(t_0)|^2 + \lambda^2|v_n(t_0)|^2) \leq e^{-4M}, \quad (4.5)$$

$$\limsup_{n \rightarrow +\infty} (|w_n'(t_0)|^2 + \lambda^2|w_n(t_0)|^2) \leq e^{-4M}. \quad (4.6)$$

*Proof* For every solution  $u(t)$  to (4.4), let us consider its energy

$$E_u(t) := |u'(t)|^2 + \lambda^2|u(t)|^2.$$

A simple computation of the time-derivative shows that  $E_u(t)$  is nonincreasing. Let us set for simplicity

$$t_n := 1/n, \quad s_n := t_0 - 1/n,$$

and let us assume that  $n$  is large enough to that  $t_n < s_n$ , and hence the two intervals where  $\delta_n(t) = Mn$  are disjoint.

*Estimate on  $v_n(t)$*  Since  $E_{v_n}(t) \leq E_{v_n}(0) = 1$ , it follows that  $|v'_n(t)| \leq 1$  for every  $t \in [0, t_0]$ . Thus from the mean value theorem we obtain that  $|v_n(0) - v_n(t_n)| \leq t_n$ , and hence

$$\lim_{n \rightarrow +\infty} v_n(t_n) = 0. \quad (4.7)$$

Let us consider now the time-derivative. To this end, we interpret (4.4) as a first order linear equation in  $u'(t)$ , with forcing term  $-\lambda^2 u(t)$ . Integrating this differential equation we obtain that

$$v'_n(t) = v'_n(0)e^{-2Mnt} + \int_0^t e^{2Mn(s-t)} \lambda^2 v_n(s) ds \quad \forall t \in [0, t_n].$$

Now we set  $t = t_n$ , we recall that  $v'_n(0) = 1$ , and we pass to the limit as  $n \rightarrow +\infty$ . The integrand is bounded because  $s \leq t$  and  $|v_n(s)|$  is bounded owing to the energy estimate. Since  $t_n \rightarrow 0$  the integral tends to 0, and hence

$$\lim_{n \rightarrow +\infty} v'_n(t_n) = e^{-2M}. \quad (4.8)$$

From (4.7) and (4.8) we conclude that

$$\lim_{n \rightarrow +\infty} (|v'_n(t_n)|^2 + \lambda^2 |v_n(t_n)|^2) = e^{-4M},$$

which implies (4.5) because the energy is nonincreasing with time.

*Estimate on  $w_n(t)$*  Let us begin with the interval  $[0, t_n]$ . The same argument exploited in the case of  $v_n(t)$  now leads to

$$\lim_{n \rightarrow +\infty} w_n(t_n) = 1/\lambda, \quad \lim_{n \rightarrow +\infty} w'_n(t_n) = 0. \quad (4.9)$$

Roughly speaking, this means that nothing changes for  $w_n(t)$  in the interval  $[0, t_n]$ .

Let us consider now the interval  $(t_n, s_n)$ , where  $\delta_n(t)$  is identically 0. An easy computation shows that in this interval  $w_n(t)$  is given by the explicit formula

$$w_n(t) = w_n(t_n) \cos(\lambda(t - t_n)) + \frac{w'_n(t_n)}{\lambda} \sin(\lambda(t - t_n)).$$

Setting  $t = s_n$  we find that

$$\begin{aligned} w_n(s_n) &= w_n(t_n) \cos\left(\frac{\pi}{2} - \frac{2\lambda}{n}\right) + \frac{w'_n(t_n)}{\lambda} \sin\left(\frac{\pi}{2} - \frac{2\lambda}{n}\right), \\ w'_n(s_n) &= -\lambda w_n(t_n) \sin\left(\frac{\pi}{2} - \frac{2\lambda}{n}\right) + w'_n(t_n) \cos\left(\frac{\pi}{2} - \frac{2\lambda}{n}\right). \end{aligned}$$

Passing to the limit as  $n \rightarrow +\infty$ , and keeping (4.9) into account, we deduce that

$$\lim_{n \rightarrow +\infty} w_n(s_n) = 0, \quad \lim_{n \rightarrow +\infty} w'_n(s_n) = -1.$$



Roughly speaking, this means that the interval  $[t_n, s_n]$  has produced a rotation of  $w_n(t)$  in the phase space, with the effect of moving all the energy on the derivative.

Let us finally consider the interval  $[s_n, t_0]$ , where we argue as we did in  $[0, t_n]$  with the function  $v_n(t)$ . Due to the uniform bound on  $w'_n(t)$  coming from the energy estimate, from the mean value theorem we deduce that

$$\lim_{n \rightarrow +\infty} w_n(t_0) = \lim_{n \rightarrow +\infty} w_n(s_n) = 0. \quad (4.10)$$

As for the derivative, once again we interpret (4.4) as a first order linear equation in  $w'_n(t)$ , and we find that

$$w'_n(t) = w'_n(s_n)e^{-2Mn(t-s_n)} + \int_{s_n}^t e^{2Mn(s-t)} \lambda^2 w_n(s) ds \quad \forall t \in [s_n, t_0].$$

Once again the integrand is bounded because  $s \leq t$  and  $w_n(s)$  is uniformly bounded owing to the energy estimate. Setting  $t = t_0$ , and passing to the limit as  $n \rightarrow +\infty$ , we conclude that

$$\lim_{n \rightarrow +\infty} w'_n(t_0) = -e^{-2M}. \quad (4.11)$$

From (4.10) and (4.11) we deduce (4.6).  $\square$

A careful inspection of the proof reveals that the inequality in (4.6) is actually an equality, and the limsup is actually a limit. With similar arguments one could show that the same is true in (4.5) (it is enough to follow the solution  $v_n(t)$  until the end of the interval). In any case, (4.5) and (4.6) are what we need in the sequel.

In the next result we apply Lemma 4.2 in order to dampen all solutions to (1.5).

**Lemma 4.3** (Decay for all solutions to a single ODE). *Let  $\lambda$  and  $M$  be positive real numbers, and let  $t_0$  be defined by (1.6).*

*Then there exists a bounded measurable damping coefficient  $\delta : [0, t_0] \rightarrow [0, +\infty)$  such that every solution to (1.5) satisfies*

$$|u'(t_0)|^2 + \lambda^2 |u(t_0)|^2 \leq (|u'(0)|^2 + \lambda^2 |u(0)|^2) \cdot 2e^{-M}. \quad (4.12)$$

*Furthermore, one can choose  $\delta(t)$  such that*

- *either it is of the form (4.3) for a large enough  $n$ ,*
- *or it is of class  $C^\infty$  and all its time-derivatives vanish both in  $t = 0$  and in  $t = t_0$ .*

*Proof* We claim that (4.12) holds true for all solutions to (1.5) if the damping coefficient is of the form (4.3) with  $n$  large enough. To this end, let  $\delta_n(t)$ ,  $v_n(t)$  and  $w_n(t)$  be defined as in Lemma 4.2. Due to (4.5) and (4.6), the two inequalities

$$|v'_n(t_0)|^2 + \lambda^2 |v_n(t_0)|^2 \leq e^{-M}, \quad |w'_n(t_0)|^2 + \lambda^2 |w_n(t_0)|^2 \leq e^{-M} \quad (4.13)$$

hold true provided that  $n$  is large enough. Every solution  $u(t)$  to equation (1.5) with  $\delta(t) := \delta_n(t)$  is a linear combination of  $v_n(t)$  and  $w_n(t)$ , and more precisely

$$u(t) = u'(0)v_n(t) + \lambda u(0)w_n(t).$$

It follows that

$$\begin{aligned} |u'(t_0)|^2 + \lambda^2 |u(t_0)|^2 &\leq 2|u'(0)|^2 \cdot |v'_n(t_0)|^2 + 2\lambda^2 |u(0)|^2 \cdot |w'_n(t_0)|^2 \\ &\quad + 2\lambda^2 |u'(0)|^2 \cdot |v_n(t_0)|^2 + 2\lambda^4 |u(0)|^2 \cdot |w_n(t_0)|^2 \\ &= 2|u'(0)|^2 \cdot (|v'_n(t_0)|^2 + \lambda^2 |v_n(t_0)|^2) \\ &\quad + 2\lambda^2 |u(0)|^2 \cdot (|w'_n(t_0)|^2 + \lambda^2 |w_n(t_0)|^2), \end{aligned}$$

so that (4.12) follows from (4.13). This proves our original claim.

If we want a smooth damping coefficient, we need a two steps approximation. First of all we choose  $n_0 \in \mathbb{N}$  such that

$$|v'_{n_0}(t_0)|^2 + \lambda^2 |v_{n_0}(t_0)|^2 \leq e^{-2M}, \quad |w'_{n_0}(t_0)|^2 + \lambda^2 |w_{n_0}(t_0)|^2 \leq e^{-2M}.$$

Then we approximate  $\delta_{n_0}(t)$  with a nonnegative function  $\delta(t)$  of class  $C^\infty$  with the property that all its time-derivatives vanish both in  $t = 0$  and in  $t = t_0$ . Let  $v(t)$  and  $w(t)$  denote the corresponding solutions to (1.5) with the same initial data of  $v_{n_0}(t)$  and  $w_{n_0}(t)$ , respectively. From Lemma 4.1 we deduce that, if  $\delta(t)$  is close enough to  $\delta_{n_0}(t)$  in  $L^2((0, t_0))$ , then

$$|v'(t_0)|^2 + \lambda^2 |v(t_0)|^2 \leq e^{-M}, \quad |w'(t_0)|^2 + \lambda^2 |w(t_0)|^2 \leq e^{-M}.$$

Since every solution to (1.5) is a linear combination of  $v(t)$  and  $w(t)$ , we can conclude exactly as before.  $\square$

### Proof of Theorem 2.1

Let  $M$  be such that  $2e^{-M} = e^{-Rt_0}$ . Let us consider any function  $\delta(t)$  provided by Lemma 4.3 with this choice of  $M$ , and let us extend it by periodicity to the half line  $t \geq 0$ . Let  $u(t)$  be any corresponding solution to (1.5), and let us consider the usual energy  $E(t) := |u'(t)|^2 + \lambda^2 |u(t)|^2$ .

The estimate of Lemma 4.3 can be applied in all intervals of the form  $[kt_0, (k+1)t_0]$ , yielding that

$$E((k+1)t_0) \leq E(kt_0) \cdot 2e^{-M} = E(kt_0) \cdot e^{-Rt_0} \quad \forall k \in \mathbb{N}.$$

Therefore, an easy induction proves that

$$E(kt_0) \leq E(0) \cdot e^{-kRt_0} \quad \forall k \in \mathbb{N}.$$

Since  $E(t)$  is nonincreasing, this implies that

$$E(t) \leq E(0) \exp(-R(t - t_0)^+) \quad \forall t \geq 0,$$

which is equivalent to (2.1).

If we want  $\delta(t)$  to be piecewise constant, it is enough to take a function with this property from Lemma 4.3. If we want  $\delta(t)$  to be of class  $C^\infty$ , it is enough to take from Lemma 4.3 a function of class  $C^\infty$  whose time-derivatives of any order vanish at the endpoints of the interval. This condition guarantees that the periodic extension remains of class  $C^\infty$ .  $\square$

### Proof of Theorem 2.2

Let  $M_k$  be a sequence of positive real numbers such that

$$2e^{-M_0} \leq \varphi(2t_0), \quad (4.14)$$

and

$$\varphi((k+1)t_0) \cdot 2e^{-M_k} \leq \varphi((k+2)t_0) \quad \forall k \geq 1. \quad (4.15)$$

For every  $k \in \mathbb{N}$ , let  $\delta_k : [0, t_0] \rightarrow [0, +\infty)$  be one of the functions provided by Lemma 4.3, applied with  $M := M_k$ . Let us define  $\delta : [0, +\infty) \rightarrow [0, +\infty)$  by glueing together all these functions, namely by setting

$$\delta(t) := \delta_k(t - kt_0) \quad \forall t \in [kt_0, (k+1)t_0).$$

Let  $u(t)$  be any corresponding solution to (1.5), and let us consider the usual energy

$$E(t) := |u'(t)|^2 + \lambda^2 |u(t)|^2.$$

We claim that  $E(t) \leq E(0)\varphi(t)$  for all  $t \geq t_0$ , which is equivalent to (2.2). In order to prove this result, it is enough to show that

$$E(kt_0) \leq E(0) \cdot \varphi((k+1)t_0) \quad \forall k \geq 1. \quad (4.16)$$

Indeed, since both  $E(t)$  and  $\varphi(t)$  are nonincreasing, when  $t \in [kt_0, (k+1)t_0]$  for some  $k \geq 1$  it turns out that

$$E(t) \leq E(kt_0) \leq E(0) \cdot \varphi((k+1)t_0) \leq E(0) \cdot \varphi(t).$$

In order to prove (4.16), we repeatedly apply Lemma 4.3. Since in the interval  $[0, t_0]$  the function  $\delta(t)$  coincides with  $\delta_0(t)$ , from Lemma 4.3 and (4.14) we deduce that

$$E(t_0) \leq E(0) \cdot 2e^{-M_0} \leq E(0) \cdot \varphi(2t_0),$$

which proves (4.16) in the case  $k = 1$ .

Now we proceed by induction. Let us assume that (4.16) holds true for some positive integer  $k$ . In the interval  $[kt_0, (k+1)t_0]$  the function  $\delta(t)$  is a time translation of  $\delta_k(t)$ , and thus we can apply Lemma 4.3 up to this time translation. Keeping (4.15) into account, we deduce that

$$E((k+1)t_0) \leq E(kt_0) \cdot 2e^{-M_k} \leq E(0) \cdot \varphi((k+1)t_0) \cdot 2e^{-M_k} \leq E(0) \cdot \varphi((k+2)t_0),$$

which completes the induction.

The possibility of choosing a damping coefficient satisfying further requirements depends on the analogous possibility in Lemma 4.3  $\square$

## 4.2 Systems of ordinary differential equations

The following result is the natural generalization of Lemma 4.3 to systems.

**Lemma 4.4** (Decay for all solutions to a system). *Let  $n$  be a positive integer, let  $(\lambda_1, \dots, \lambda_n) \in (0, +\infty)^n$ , let  $M$  be a positive real number, and let  $t_0$  be defined by (1.8).*

*Then there exists a bounded measurable damping coefficient  $\delta : [0, t_0] \rightarrow [0, +\infty)$  such that every solution to (1.7) satisfies*

$$\sum_{k=1}^n (|u'_k(t_0)|^2 + \lambda_k^2 |u_k(t_0)|^2) \leq \left[ \sum_{k=1}^n (|u'_k(0)|^2 + \lambda_k^2 |u_k(0)|^2) \right] \cdot 2e^{-M}. \quad (4.17)$$

Furthermore, one can choose  $\delta(t)$  such that

- either it has the profile shown in Figure 3 on the left,
- or it is of class  $C^\infty$  and all its time-derivatives vanish both in  $t = 0$  and in  $t = t_0$ .

*Proof* Let us apply Lemma 4.3 to the  $k$ -th equation of the system. Setting  $t_{0k} := \pi/(2\lambda_k)$ , we obtain a function  $\delta_k : [0, t_{0k}] \rightarrow [0, +\infty)$  such that every solution to the  $k$ -th equation of the system satisfies

$$|u'_k(t_{0k})|^2 + \lambda_k^2 |u_k(t_{0k})|^2 \leq (|u'_k(0)|^2 + \lambda_k^2 |u_k(0)|^2) \cdot 2e^{-M}.$$

Now we observe that  $t_0 = t_{01} + \dots + t_{0n}$ , and we define  $\delta : [0, t_0] \rightarrow [0, +\infty)$  by glueing together the functions  $\delta_k(t)$  defined above. More precisely, we partition  $[0, t_0]$  into  $n$  subintervals  $[s_{k-1}, s_k]$  with  $s_0 = 0$ ,  $s_n = t_0$  and  $s_k - s_{k-1} = t_{0k}$ , and then we set

$$\delta(t) := \delta_k(t - s_{k-1}) \quad \forall t \in [s_{k-1}, s_k].$$

Let  $(u_1(t), \dots, u_n(t))$  be a corresponding solution to the system (1.7), and let us set

$$E_k(t) := |u'_k(t)|^2 + \lambda_k^2 |u_k(t)|^2.$$

We claim that the energy  $E_k(t)$  of the  $k$ -th component is reduced by a factor  $2e^{-M}$  in the interval  $[s_{k-1}, s_k]$ . Indeed in this interval  $\delta(t)$  coincides with  $\delta_k(t)$  up to a time translation, hence from Lemma 4.3 we deduce that

$$E_k(t_0) \leq E_k(s_k) \leq E_k(s_{k-1}) \cdot 2e^{-M} \leq E_k(0) \cdot 2e^{-M}.$$

Summing over all indices  $k$  from 1 to  $n$  we obtain (4.17).

If we want a damping coefficient with the profile shown in Figure 3 on the left, it is enough that all functions  $\delta_k(t)$  are of the form (4.3), with the same value of  $n$  for all  $k$ 's (this is possible provided that  $n$  is large enough).

If we want a damping coefficient of class  $C^\infty$ , it is enough to choose all functions  $\delta_k(t)$  of class  $C^\infty$  with all time-derivatives vanishing at the endpoints of the interval. This condition guarantees that the glueing procedure yields a function which is still of class  $C^\infty$ .  $\square$

### Proof of Theorem 2.3 and Theorem 2.4

The arguments are analogous to the proofs of Theorem 2.1 and Theorem 2.2, just starting from Lemma 4.4 instead of Lemma 4.3.  $\square$

**Remark 4.5.** As already mentioned in the introduction, now it should be clear from the proofs that the conclusions of Theorem 2.3 and Theorem 2.4 hold true for the general equation (1.1), provided that the spectrum of  $A$  is finite, even if the dimension of eigenspaces is infinite.

## 4.3 Partial differential equations

When the operator  $A$  has infinitely many eigenvalues, a constant dissipation is enough to dampen all components corresponding to large enough eigenvalues. This is the content of next result.

**Lemma 4.6** (PDE with constant dissipation). *Let  $H$  be a Hilbert space, and let  $A$  be a self-adjoint nonnegative operator on  $H$  with dense domain  $D(A)$ . Let  $M$  be a positive real number, and let us assume that*

$$|A^{1/2}x|^2 \geq 2M^2|x|^2 \quad \forall x \in D(A^{1/2}). \quad (4.18)$$

*Then every weak solution  $u(t)$  to*

$$u''(t) + 2Mu'(t) + Au(t) = 0$$

*satisfies*

$$|u'(t)|^2 + |A^{1/2}u(t)|^2 \leq (|u'(0)|^2 + |A^{1/2}u(0)|^2) \cdot 8e^{-2Mt} \quad \forall t \geq 0. \quad (4.19)$$

*Proof* Let us consider the usual energy

$$E(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2,$$

and the modified energy

$$\widehat{E}(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2 + 2M\langle u'(t), u(t) \rangle.$$

An easy calculation shows that

$$\widehat{E}'(t) = -2M\widehat{E}(t) \quad \forall t \geq 0. \quad (4.20)$$

We claim that

$$\frac{1}{4}E(t) \leq \widehat{E}(t) \leq 2E(t) \quad \forall t \geq 0. \quad (4.21)$$

Indeed, from the inequality

$$2M\langle u'(t), u(t) \rangle \leq \frac{1}{2}|u'(t)|^2 + 2M^2|u(t)|^2$$

and assumption (4.18) it follows that

$$\widehat{E}(t) \leq E(t) + \frac{1}{2}|u'(t)|^2 + |A^{1/2}u(t)|^2 \leq 2E(t).$$

On the other hand, from the inequality

$$2M\langle u'(t), u(t) \rangle \geq -\frac{3}{4}|u'(t)|^2 - \frac{4}{3}M^2|u(t)|^2$$

and assumption (4.18) it follows that

$$\widehat{E}(t) \geq E(t) - \frac{3}{4}|u'(t)|^2 - \frac{2}{3}|A^{1/2}u(t)|^2 \geq \frac{1}{4}E(t).$$

From (4.20) and (4.21) we conclude that

$$E(t) \leq 4\widehat{E}(t) = 4\widehat{E}(0)e^{-2Mt} \leq 8E(0)e^{-2Mt},$$

which is equivalent to (4.19).  $\square$

The following result is the generalization of Lemma 4.3 and Lemma 4.4 to the infinite dimensional setting.

**Lemma 4.7** (Decay for all solutions to a PDE). *Let  $H$ ,  $A$ ,  $\lambda_k$  be as in Theorem 2.5. Let  $R$  be a positive real number, and let*

$$T_R := \frac{\pi}{2} \sum_{\lambda_k^2 \leq 2(R+\lambda_1)^2} \frac{1}{\lambda_k}. \quad (4.22)$$

*Then there exists a bounded measurable damping coefficient  $\delta : [0, 2T_R] \rightarrow [0, +\infty)$  such that every weak solution to (1.1) satisfies*

$$|u'(2T_R)|^2 + |A^{1/2}u(2T_R)|^2 \leq (|u'(0)|^2 + |A^{1/2}u(0)|^2) \cdot e^{-R \cdot 2T_R}. \quad (4.23)$$

*Furthermore, one can choose  $\delta(t)$  such that*

- *either it has the profile shown in Figure 4,*
- *or it is of class  $C^\infty$  and all its time-derivatives vanish both in  $t = 0$  and in  $t = 2T_R$ .*

*Proof* Let us write  $H$  as a direct sum

$$H = H_{R,-} \oplus H_{R,+},$$

where  $H_{R,-}$  is the subspace generated by all eigenvectors of  $A$  corresponding to eigenvalues  $\lambda_k^2 \leq 8(R + \lambda_1)^2$ , and  $H_{R,+}$  is the closure of the subspace generated by the remaining eigenvectors. We point out that  $H_{R,-}$  and  $H_{R,+}$  are  $A$ -invariant subspaces of  $H$ . Moreover, the restriction of  $A$  to  $H_{R,-}$  has only a finite number of eigenvalues, while the restriction of  $A$  to  $H_{R,+}$  satisfies the coercivity condition

$$|A^{1/2}x|^2 \geq 2(R + \lambda_1)^2|x|^2 \quad \forall x \in D(A^{1/2}) \cap H_{R,+},$$

and even a stronger condition

$$|A^{1/2}x|^2 \geq 2(R' + \lambda_1)^2|x|^2 \quad \forall x \in D(A^{1/2}) \cap H_{R,+}, \quad (4.24)$$

for some  $R' > R$  (actually we can choose  $8(R' + \lambda_1)^2$  to be the smallest eigenvalue of  $A$  greater than  $2(R + \lambda_1)^2$ ).

Let  $u_{R,-}(t)$  and  $u_{R,+}(t)$  denote the components of  $u(t)$  with respect to the decomposition, let  $E(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2$  be the usual energy of  $u(t)$ , and let  $E_{R,\pm}(t)$  denote the energy of the two components.

Since the restriction of  $A$  to  $H_{R,-}$  has only a finite number of eigenvalues, the component  $u_{R,-}(t)$  can be regarded as a solution to a system of finitely many ordinary differential equations. For this system, the time  $T_R$  defined by (4.22) coincides with the time  $t_0$  defined by (1.8). Therefore, from Lemma 4.4 (see also Remark 4.5) we deduce the existence of a function  $\delta : [0, T_R] \rightarrow [0, +\infty)$  which reduces the energy of  $u_{R,-}(t)$  at time  $t = T_R$  by any given factor. In particular, we can choose this factor equal to  $8e^{-2(R+\lambda_1)T_R}$  and obtain that

$$E_{R,-}(T_R) \leq E_{R,-}(0) \cdot 8e^{-2(R+\lambda_1)T_R}. \quad (4.25)$$

Now we extend  $\delta(t)$  to the interval  $[0, 2T_R]$  by setting  $\delta(t) := R + \lambda_1$  in the second half of the interval, namely for every  $t \in (T_R, 2T_R]$ . Since the restriction of  $A$  to  $H_{R,+}$  satisfies (4.18), we can apply Lemma 4.6 with  $M := R + \lambda_1$ . We obtain that

$$E_{R,+}(2T_R) \leq E_{R,+}(T_R) \cdot 8e^{-2(R+\lambda_1)T_R}. \quad (4.26)$$

Keeping into account that in both cases the energy is nonincreasing in the whole interval, from (4.25) and (4.26) we deduce that

$$E(2T_R) \leq E(0) \cdot 8e^{-2(R+\lambda_1)T_R} = E(0) \cdot 8e^{-2\lambda_1 T_R} \cdot e^{R \cdot 2T_R}. \quad (4.27)$$

On the other hand, from (4.22) it is clear that  $2\lambda_1 T_R \geq \pi$ , and hence  $8e^{-2\lambda_1 T_R} \leq 1$ . Therefore, now (4.27) reads as

$$E(2T_R) \leq E(0) \cdot e^{-R \cdot 2T_R},$$

which is exactly (4.23).

If we want  $\delta(t)$  with the profile of Figure 4, it is enough to reduce the energy of  $u_{R,-}(t)$  through a damping coefficient in  $[0, T_R]$  with the profile shown in Figure 3 on the left.

If we want a damping coefficient of class  $C^\infty$  with all time-derivatives vanishing at the endpoints, we need an approximation procedure. To this end, we first choose a bounded measurable function  $\delta(t)$  in  $[0, 2T_R]$  for which (4.23) holds true for all solutions with  $2RT_R$  replaced by a larger constant. This is possible because in the first half of the interval we can reduce the energy of  $u_{R,-}(t)$  by any given factor, and in the second half of the interval we can reduce the energy of  $u_{R,+}(t)$  by a factor  $e^{-2(R'+\lambda_1)T_R}$ , where  $R' > R$  is the constant that appears in the reinforced coercivity inequality (4.24) (of course we need to set  $\delta(t) := R' + \lambda_1$  in the second half of the period, as required by Lemma 4.6). At this point we can approximate  $\delta(t)$  in  $L^2$ -norm through a damping coefficient with the required smoothness, and conclude with the aid of Lemma 4.1.  $\square$

### Proof of Theorem 2.5

Let us observe that the time  $2T_R$ , with  $T_R$  given by (4.22), coincides with  $t_0$  as defined in (1.9). Therefore, from Lemma 4.7 we obtain a function  $\delta : [0, t_0] \rightarrow [0, +\infty)$  such that every weak solution to (1.1) satisfies

$$E(t_0) \leq E(0) \cdot e^{-Rt_0},$$

where  $E(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2$  denotes the usual energy of the solution.

Now we extend  $\delta(t)$  by periodicity to the whole half line  $t \geq 0$ , and we obtain by an easy induction that

$$E(kt_0) \leq E(0) \cdot e^{-kRt_0} \quad \forall k \in \mathbb{N}.$$

Since  $E(t)$  is nonincreasing, this implies that

$$E(t) \leq E(0) \cdot \exp(-R(t - t_0)^+) \quad \forall t \geq 0,$$

which is equivalent to (2.3).

In analogy with the proof of Theorem 2.1, if we want a piecewise constant damping coefficient, we can take a function with this property from Lemma 4.7. If we want a damping coefficient of class  $C^\infty$ , it is enough to take from Lemma 4.7 a function of class  $C^\infty$  whose time-derivatives of any order vanish at the endpoints of the interval. This condition guarantees that the periodic extension remains of class  $C^\infty$ .  $\square$

### Proof of Theorem 2.6

Let us start by defining  $\varphi(t)$ . To this end, let  $n_0$  be the smallest integer such that  $\lambda_{n_0}^2 > 2\lambda_1^2$ . For every  $n \geq n_0$  we consider the positive real number

$$R_n := \frac{\lambda_n}{\sqrt{2}} - \lambda_1,$$



and we define  $T_n$  as in (4.22) with  $R := R_n$ . Since  $2(R_n + \lambda_1)^2 = \lambda_n^2$ , this means that

$$T_n = \frac{\pi}{2} \sum_{k=1}^n \frac{1}{\lambda_k} \quad \forall n \geq n_0.$$

Then we set  $S_{n_0-1} := 0$ ,  $U_{n_0-1} := 0$ , and

$$S_n := 2 \sum_{k=n_0}^n T_k, \quad U_n := 2 \sum_{k=n_0}^n R_k T_k$$

for  $n \geq n_0$ . It is clear that  $S_n$  and  $U_n$  are unbounded increasing sequences, with

$$2T_{n+1} = 2T_n + \frac{\pi}{\lambda_{n+1}} \leq 2T_n + 2T_{n_0} \leq S_n$$

for every  $n \geq n_0 + 1$ , and therefore

$$S_{n+1} = S_n + 2T_{n+1} \leq 2S_n \quad \forall n \geq n_0 + 1.$$

Finally, we define  $\varphi : [0, +\infty) \rightarrow (0, +\infty)$  as the piecewise constant function such that

$$\varphi(t) := e^{-U_n} \quad \text{if } t \in [S_n, S_{n+1}) \text{ for some } n \geq n_0 - 1.$$

It is clear that  $\varphi(t)$  is nonincreasing. We claim that it satisfies (2.4). Indeed for every  $n \geq n_0 - 1$  it turns out that

$$\varphi(t)e^{Rt} = e^{-U_n + Rt} \leq e^{-U_n + RS_{n+1}} \quad \forall t \in [S_n, S_{n+1}), \quad (4.28)$$

and for  $n \geq n_0 + 1$  it turns out that

$$-U_n + RS_{n+1} \leq -U_n + 2RS_n = \sum_{k=n_0}^n (4R - 2R_k)T_k.$$

Since  $R_k \rightarrow +\infty$  and  $T_k \geq T_{n_0} > 0$ , we deduce that  $-U_n + RS_{n+1} \rightarrow -\infty$ . Therefore, the right-hand side of (4.28) tends to 0 as  $n \rightarrow +\infty$ , which proves (2.4).

It remains to define  $\delta(t)$ . To begin with, for every  $n \geq n_0$  we apply Lemma 4.7. Since the time  $2T_{R_n}$  of Lemma 4.7 coincides with  $2T_n$  as defined above, from the lemma we obtain a damping coefficient  $\delta_n : [0, 2T_n] \rightarrow [0, +\infty)$  which reduces the energy of all solutions to (1.1) by a factor  $e^{-2R_n T_n}$  in the interval  $[0, 2T_n]$ .

Now we glue all these functions by setting

$$\delta(t) := \delta_{n+1}(t - S_n) \quad \text{if } t \in [S_n, S_{n+1}) \text{ for some } n \geq n_0 - 1.$$

Let us consider equation (1.1) with this choice of  $\delta(t)$ , let  $u(t)$  be any solution, and let  $E(t) := |u'(t)|^2 + |A^{1/2}u(t)|^2$  be its usual energy. The effect of  $\delta(t)$  in the interval  $[S_n, S_{n+1}]$  is the same as the effect of  $\delta_{n+1}(t)$  in the interval  $[0, 2T_{n+1}]$ , and therefore

$$E(S_{n+1}) \leq E(S_n) \cdot e^{-2R_{n+1}T_{n+1}} \quad \forall n \geq n_0 - 1.$$

At this point an easy induction shows that

$$E(S_n) \leq E(0) \cdot \exp(-2(R_{n_0}T_{n_0} + \dots + R_n T_n)) = E(0) \cdot e^{-U_n} \quad \forall n \geq n_0.$$

On the other hand, also for  $n = n_0 - 1$  it is true that  $E(S_n) = E(0) \cdot e^{-U_n}$ , for the trivial reason that  $U_{n_0-1} = 0$ . Since  $E(t)$  is nonincreasing for  $t \geq 0$ , we can finally conclude that

$$E(t) \leq E(S_n) \leq E(0)e^{-U_n} = E(0) \cdot \varphi(t)$$

for every  $n \geq n_0 - 1$  and every  $t \in [S_n, S_{n+1}]$ , which proves (2.5).  $\square$

## 4.4 Further requirements on the damping coefficient

### Proof of Proposition 2.7

Let us consider the usual energy  $E(t) := |u'(t)|^2 + \lambda^2|u(t)|^2$ . Its time-derivative is  $E'(t) = -4\delta(t)|u'(t)|^2$ , so that

$$E'(t) \geq -4\delta(t)E(t).$$

Integrating this differential inequality we deduce that

$$E(t) \geq E(0) \cdot \exp\left(-4 \int_0^t \delta(s) ds\right) \quad \forall t \geq 0.$$

This is enough to deal with the first two cases.

For the third statement, let us assume now that  $\delta(t) \geq \lambda$  for every  $t \geq T$ . In this case we consider the Riccati equation

$$\varphi'(t) = \lambda^2 - 2\delta(t)\varphi(t) + \varphi^2(t). \quad (4.29)$$

Let us set  $T_* := \max\{T, 1/\lambda\}$ . It is not difficult to check that  $\varphi(t) \equiv 0$  is a subsolution of (4.29) for  $t \geq 0$ , while  $\varphi(t) := \lambda - 1/t$  is a supersolution for  $t \geq T_*$  owing to our assumptions on  $\delta(t)$ . As a consequence, the solution to (4.29) with  $\varphi(T_*) = 0$  is defined for every  $t \geq T_*$  and satisfies

$$0 \leq \varphi(t) \leq \lambda - \frac{1}{t} \quad \forall t \geq T_*. \quad (4.30)$$

Finally, one can check that

$$u(t) := \exp\left(-\int_{T_*}^t \varphi(s) ds\right) \quad \forall t \geq T_*$$

defines a solution to (1.5), which can be easily extended to the whole half line  $t \geq 0$ .

Due to the estimate from above in (4.30), this solution satisfies

$$|u(t)| \geq cte^{-\lambda t} \quad \forall t \geq T_*$$

for a suitable positive constant  $c$ . Since equation (1.5) is linear,  $c^{-1}u(t)$  is again a solution, and satisfies (2.6).  $\square$

The rest of this section is devoted to the proof of Theorem 2.8. This requires two preliminary results. In the first one we show how to dampen one solution to (1.5) through a slowly increasing damping coefficient.

**Lemma 4.8** (Supercritical energy reduction of a solution). *Let  $\lambda$  and  $M$  be positive real numbers, and let  $\varepsilon \in (0, \lambda)$ . Let us define*

$$t_1 := \frac{M}{2\varepsilon} + \frac{2}{\lambda}, \quad (4.31)$$

$$\delta(t) := \lambda + \varepsilon t \quad \forall t \in [0, t_1]. \quad (4.32)$$

Then there exists a nonzero solution  $v(t)$  to (1.5) such that

$$|v'(t_1)|^2 + \lambda^2 |v(t_1)|^2 \leq (|v'(0)|^2 + \lambda^2 |v(0)|^2) e^{-Mt_1}. \quad (4.33)$$

*Proof* Let us start by proving that (4.31) implies that

$$(3\lambda^2 + 8\varepsilon^2 t_1^2) \exp(-2\lambda t_1 - 2\varepsilon t_1^2) \leq 2\lambda^2 \exp(-Mt_1). \quad (4.34)$$

Indeed from (4.31) it turns out that

$$2\lambda t_1 \geq 4, \quad 2\varepsilon t_1^2 \geq Mt_1, \quad 2\varepsilon t_1^2 \geq Mt_1 + \frac{4\varepsilon}{\lambda} t_1.$$

From the first two inequalities it follows that

$$3\lambda^2 \exp(-2\lambda t_1 - 2\varepsilon t_1^2) \leq \lambda^2 \cdot 3e^{-4} \cdot e^{-Mt_1} \leq \lambda^2 e^{-Mt_1}. \quad (4.35)$$

From the third inequality it follows that

$$\exp(-2\lambda t_1 - 2\varepsilon t_1^2) \leq e^{-2\varepsilon t_1^2} \leq e^{-4\varepsilon t_1/\lambda} \cdot e^{-Mt_1}.$$

Therefore, from the inequality

$$t^2 e^{-ct} \leq \frac{4}{e^2 c^2} \quad \forall t \geq 0, \forall c > 0,$$

applied with  $c := 4\varepsilon/\lambda$  and  $t := t_1$ , we obtain that

$$\begin{aligned} 8\varepsilon^2 t_1^2 \exp(-2\lambda t_1 - 2\varepsilon t_1^2) &\leq 8\varepsilon^2 t_1^2 \exp(-4\varepsilon t_1/\lambda) \cdot \exp(-Mt_1) \\ &\leq 8\varepsilon^2 \frac{4\lambda^2}{16e^2 \varepsilon^2} \exp(-Mt_1) \\ &\leq \lambda^2 \exp(-Mt_1). \end{aligned} \quad (4.36)$$

Adding (4.35) and (4.36) we obtain exactly (4.34).

We are now ready to prove the existence of  $v(t)$ . Let us consider again the Riccati equation (4.29). When  $\delta(t)$  is given by (4.32), a simple computation shows that  $\varphi(t) := \lambda + 2\varepsilon t$  is a supersolution for  $t \geq 0$ , and the constant function  $\varphi(t) := 2(\lambda + 2\varepsilon t_1)$  is a subsolution for  $t \in [0, t_1]$ . It follows that the solution to (4.29) with “final” condition

$$\varphi(t_1) = \lambda + 2\varepsilon t_1 \quad (4.37)$$

is defined for every  $t \in [0, t_1]$  and satisfies

$$\varphi(t) \geq \lambda + 2\varepsilon t \quad \forall t \in [0, t_1]. \quad (4.38)$$

At this point a simple computation shows that

$$v(t) := \exp\left(-\int_0^t \varphi(s) ds\right) \quad \forall t \in [0, t_1]$$

is a solution to (1.5). We claim that this solution satisfies (4.33).

To begin with, we observe that  $v(0) = 1$ , and from (4.38) it follows that

$$v(t_1) = \exp\left(-\int_0^{t_1} \varphi(s) ds\right) \leq \exp(-\lambda t_1 - \varepsilon t_1^2).$$

As for the time-derivative, it is given by

$$v'(t) = -\varphi(t) \exp\left(-\int_0^t \varphi(s) ds\right),$$

hence by (4.38)

$$|v'(0)| = |\varphi(0)| \geq \lambda,$$

and by (4.37) and (4.38)

$$|v'(t_1)| = |\varphi(t_1)| \exp\left(-\int_0^{t_1} \varphi(s) ds\right) \leq (\lambda + 2\varepsilon t_1) \exp(-\lambda t_1 - \varepsilon t_1^2).$$

Recalling (4.34), from all these estimates it follows that

$$\begin{aligned} |v'(t_1)|^2 + \lambda^2 |v(t_1)|^2 &\leq [(\lambda + 2\varepsilon t_1)^2 + \lambda^2] \exp(-2\lambda t_1 - 2\varepsilon t_1^2) \\ &\leq [3\lambda^2 + 8\varepsilon^2 t_1^2] \exp(-2\lambda t_1 - 2\varepsilon t_1^2) \\ &\leq 2\lambda^2 \exp(-Mt_1) \\ &\leq (|v'(0)|^2 + \lambda^2 |v(0)|^2) \exp(-Mt_1), \end{aligned}$$

which proves (4.33).  $\square$

The second preliminary result clarifies that any constant damping coefficient below the threshold  $\lambda$  allows solutions to rotate in the phase space.

**Lemma 4.9** (Subcritical rotation in the phase space). *Let  $\lambda$  be a positive real number, and let  $\varepsilon \in (0, \lambda)$ . Let  $(\alpha, \beta)$  and  $(\gamma, \delta)$  be two pairs of real numbers with  $\alpha^2 + \beta^2 \neq 0$  and  $\gamma^2 + \delta^2 \neq 0$ .*

*Then there exist positive real numbers  $r$  and  $t_2$ , with*

$$t_2 \leq \frac{2\pi}{\sqrt{\varepsilon(2\lambda - \varepsilon)}}, \quad (4.39)$$

*with the following property. The solution  $u(t)$  to equation (1.5) with constant dissipation  $\delta(t) := \lambda - \varepsilon$  and initial data  $u(0) = \alpha$  and  $u'(0) = \beta$  satisfies*

$$u(t_2) = r\gamma, \quad u'(t_2) = r\delta.$$

*Proof* The solution  $u(t)$  can be written in the form  $u(t) := e^{-(\lambda-\varepsilon)t}v(t)$ , where  $v(t)$  is a solution to

$$v''(t) + \varepsilon(2\lambda - \varepsilon)v(t) = 0.$$

Integrating this differential equation we see that the pair  $(v(t), v'(t))$  rotates in the phase space with period equal to the right-hand side of (4.39). During the complete rotation the pair  $(v(t), v'(t))$  turns out to be a positive multiple of any given vector. Since the same is true for  $u(t)$ , this proves the result.  $\square$

### Proof of Theorem 2.8

*Strategy* Let us describe the strategy of the proof before entering into details. Let us consider the function  $\delta(t)$  with the graph as in Figure 5, and then extended by periodicity.

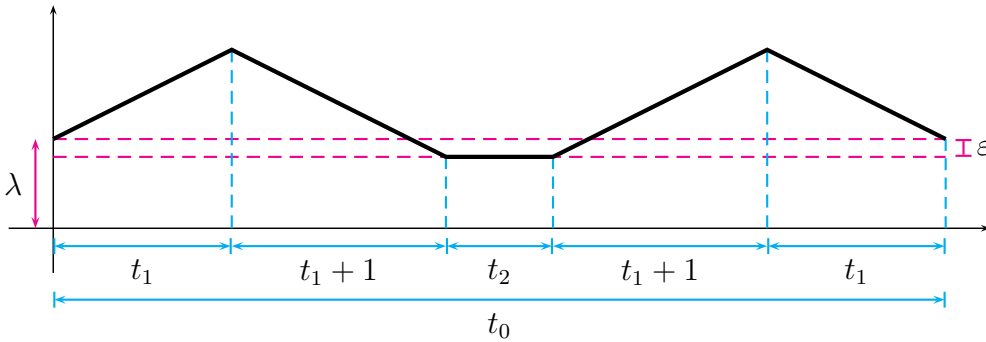


Figure 5: profile of  $\delta(t)$  for the proof of Theorem 2.8

The value at the endpoints is  $\lambda$ , the value in the horizontal part is  $\lambda - \varepsilon$ , the slope in the oblique sections is  $\pm\varepsilon$ . This graph depends on two parameters to be fixed, namely the length  $t_1$  of the first and last oblique sections, and the length  $t_2$  of the horizontal

plateau. The length of each of the two oblique sections attached to the horizontal part is necessarily  $t_1 + 1$ . The total length is therefore

$$t_0 := 4t_1 + t_2 + 2. \quad (4.40)$$

Now the idea is the following. In the first ascending section,  $\delta(t)$  causes a great reduction of the energy of a special solution  $v(t)$  to (1.5). Let  $w(t)$  be the solution with initial data orthogonal to the initial data of  $v(t)$ . In the first two oblique sections we know that the energy of  $w(t)$  is just nonincreasing. On the other hand, we can choose  $t_2$  in such a way that, at the end of the horizontal plateau,  $w(t)$  has the right “initial” data which guarantee a reduction of its energy in the third oblique section.

In other words, the first ascending section cuts the energy of  $v(t)$ , while the horizontal plateau rotates  $w(t)$  preparing it to undergo a cut of its energy during the second ascending section. The two descending sections are merely junctions between the “active” parts of the graph.

*Choice of parameters* Let us set

$$M := 8(\pi + \varepsilon)R + 4\varepsilon \log 2 + 1, \quad (4.41)$$

and let  $t_1$  be defined by (4.31). The function  $\delta(t)$  we are considering is equal to  $\lambda + \varepsilon t$  for every  $t \in [0, t_1]$ . Therefore, from Lemma 4.8 we know that (1.5) admits a solution  $v(t)$  satisfying (4.33). Since the equation is linear, we can always assume that

$$|v'(0)|^2 + \lambda^2 |v(0)|^2 = 1. \quad (4.42)$$

In the sequel we also need to consider  $\delta(t)$  as extended to  $t \in [-1, 0]$  with the same expression  $\lambda + \varepsilon t$ . Accordingly, we can extend  $v(t)$  to the interval  $[-1, 0]$ , and consider the pair  $(v(-1), v'(-1))$  of its data for  $t = -1$ .

Let us consider now the solution  $w(t)$  to (1.5) with initial data

$$w(0) = \frac{v'(0)}{\lambda}, \quad w'(0) = -\lambda v(0).$$

We point out that the initial data of  $w(t)$  satisfy

$$|w'(0)|^2 + \lambda^2 |w(0)|^2 = 1 \quad (4.43)$$

and they are orthogonal to the initial data of  $v(t)$  in the sense that

$$w'(0)v'(0) + \lambda^2 w(0)v(0) = 0. \quad (4.44)$$

We are now ready to choose  $t_2$ . Let us consider the end of the second oblique section, corresponding to  $t = 2t_1 + 1$ , and let us set

$$(\alpha, \beta) := (w(2t_1 + 1), w'(2t_1 + 1)), \quad (\gamma, \delta) := (v(-1), v'(-1)).$$

From Lemma 4.9 we deduce the existence of a positive time  $t_2$  satisfying (4.39) such that the effect of the constant dissipation equal to  $\lambda - \varepsilon$  on the solution with “initial” data  $(\alpha, \beta)$  is to transform it in a multiple of  $(\gamma, \delta)$  in a time  $t_2$ . It follows that, at the end of the horizontal plateau at time  $t = 2t_1 + t_2 + 1$ , the solution  $w(t)$  is in the same conditions as (the extension of) the solution  $v(t)$  at time  $t = -1$ . As a consequence, the energy of  $w(t)$  is reduced by a factor  $e^{-Mt_1}$  during the third oblique section, from  $t = 2t_1 + t_2 + 1$  to  $t = 3t_1 + t_2 + 2$ .

*Estimates* From (4.40), (4.31) and (4.39) it follows that

$$t_0 = \frac{2M}{\varepsilon} + \frac{8}{\lambda} + 2 + t_2 \leq \frac{2M}{\varepsilon} + \frac{8}{\lambda} + 2 + \frac{2\pi}{\varepsilon},$$

so that (2.7) follows from (4.41).

As for energy estimates, let  $E_u(t) := |u'(t)|^2 + \lambda^2|u(t)|^2$  denote as usual the energy of a solution to (1.5). The energy of  $v(t)$  has been reduced in the first ascending section. Since it is always nonincreasing, it follows that

$$E_v(t_0) \leq E_v(t_1) \leq E_v(0) \cdot e^{-Mt_1} = e^{-Mt_1}. \quad (4.45)$$

Similarly, the energy of  $w(t)$  has been reduced in the second ascending section, hence

$$E_w(t_0) \leq E_w(3t_1 + t_2 + 2) \leq E_w(2t_1 + t_2 + 1) \cdot e^{-Mt_1} \leq E_w(0) \cdot e^{-Mt_1} = e^{-Mt_1}. \quad (4.46)$$

Let us consider now any solution  $u(t)$  to (1.5). Since  $v(t)$  and  $w(t)$  are linearly independent, we can write

$$u(t) = av(t) + bw(t)$$

for suitable real constants  $a$  and  $b$ . Thanks to the orthonormality relations (4.42), (4.43), and (4.44), it turns out that

$$a^2 + b^2 = E_u(0).$$

Moreover, from (4.45) and (4.46) it follows that

$$\begin{aligned} E_u(t_0) &= |av'(t_0) + bw'(t_0)|^2 + \lambda^2|av(t_0) + bw(t_0)|^2 \\ &\leq 2a^2E_v(t_0) + 2b^2E_w(t_0) \\ &\leq 2(a^2 + b^2)e^{-Mt_1} \\ &= E_u(0) \cdot 2e^{-Mt_1}. \end{aligned}$$

Now we claim that

$$2e^{-Mt_1} \leq e^{-Rt_0}, \quad (4.47)$$

and hence

$$E_u(t_0) \leq E_u(0) \cdot e^{-Rt_0}.$$

Indeed, from (4.41) it follows that  $M \geq 8R$ , and hence

$$\frac{1}{2}Mt_1 \geq 4Rt_1, \quad (4.48)$$

while from (4.41) and (4.31) it follows that

$$\frac{1}{2}Mt_1 \geq \frac{M^2}{4\varepsilon} \geq \frac{M}{4\varepsilon} \geq \left(2 + \frac{2\pi}{\varepsilon}\right)R + \log 2. \quad (4.49)$$

Summing (4.48) and (4.49) we obtain that

$$Mt_1 \geq \left(4t_1 + 2 + \frac{2\pi}{\varepsilon}\right)R + \log 2 \geq (4t_1 + 2 + t_2)R + \log 2 = Rt_0 + \log 2,$$

which is equivalent to (4.47).

Finally, when we extend  $\delta(t)$  by periodicity to the whole half line  $t \geq 0$ , an easy induction gives that

$$E_u(nt_0) \leq E_u(0) \cdot e^{-nRt_0} \quad \forall n \in \mathbb{N}.$$

Since  $E_u(t)$  is nonincreasing, this implies (2.8).  $\square$

## 5 Applications to PDEs

In this section we present some simple examples of application of the abstract results stated in section 2.3.

Let  $\Omega \subseteq \mathbb{R}^d$  be a connected bounded open set with smooth boundary. We consider two model examples: a dissipative wave equation

$$u_{tt} - \Delta u + p(x)u + \delta(t)u_t = 0 \quad t \geq 0, \quad x \in \Omega, \quad (5.1)$$

and a dissipative beam/plate equation

$$u_{tt} + \Delta^2 u + q(x)u + \delta(t)u_t = 0 \quad t \geq 0, \quad x \in \Omega, \quad (5.2)$$

where  $p(x)$  and  $q(x)$  are nonnegative bounded measurable functions. Just to fix ideas, we consider equation (5.1) with homogeneous Dirichlet boundary conditions  $u = 0$ , and equation (5.2) with one of the following boundary conditions: either  $u = \Delta u = 0$  (simply supported beam or plate) or  $u = |\nabla u| = 0$  (clamped beam or plate).

In both cases Theorem 2.5 and Theorem 2.6 imply that

- any exponential decay rate  $e^{-Rt}$  can be realized through a suitable periodic damping coefficient with period  $T_R$  that depends on  $R$ ,
- with a suitable non-periodic coefficient one can achieve a decay rate  $\varphi(t)$  faster than all exponentials.

The period  $T_R$  and the ultra-exponential decay rate  $\varphi(t)$  depend on the growth of eigenvalues, which in turn is known to depend on the space dimension  $d$ . More precisely, when eigenvalues are arranged in increasing order, the  $n$ -th eigenvalue is comparable



with the power of  $n$  indicated in the fifth column of the table below (for a proof we refer to the seminal papers [1, 7, 10]).

In the third and fourth column we state our estimates for  $T_R$  and  $\varphi(t)$ . They have to be interpreted as asymptotic behaviors, namely when we write  $\log R$  in the third column we actually mean that  $T_R = O(\log R)$  in that case. In the same way, we write shortly  $\exp(-t^\alpha)$  instead of  $\exp(-O(t^\alpha))$ .

Equation	$d$	$T_R$	$\varphi(t)$	$\lambda_n$	$T_n$	$S_n$	$U_n$
(5.1)	1	$\log R$	$\exp(-t^2/\log t)$	$n$	$\log n$	$n \log n$	$n^2 \log n$
(5.1)	$\geq 2$	$R^{d-1}$	$\exp(-t^{2d/(2d-1)})$	$n^{1/d}$	$n^{1-1/d}$	$n^{2-1/d}$	$n^2$
(5.2)	1	1	any	$n^2$	—	—	—
(5.2)	2	$\log R$	$\exp(-t^2/\log t)$	$n$	$\log n$	$n \log n$	$n^2 \log n$
(5.2)	$\geq 3$	$R^{d/2-1}$	$\exp(-t^{d/(d-1)})$	$n^{2/d}$	$n^{1-2/d}$	$n^{2-2/d}$	$n^2$

In particular, we observe that for equation (5.2) with  $d = 1$  (beam equation), the series of reciprocals of eigenvalues is convergent. As a consequence, the period  $T_R$  given by (1.9) is bounded independently of  $R$ . At this point, the same argument of the proof of Theorems 2.2 and 2.4 gives that any decay rate  $\varphi(t)$  can be achieved if we are allowed to exploit non-periodic damping coefficients.

In all other cases, the series of reciprocals of eigenvalues is divergent, and  $T_R$  grows with  $R$  according to (1.9). In order to compute  $\varphi(t)$ , we need to estimate the growth of the sequences  $S_n, T_n, U_n$  defined in the proof of Theorem 2.6, and recall that  $\varphi(t)$  decays in such a way that  $\varphi(S_n) = \exp(-U_n)$ . The computations are reported in the last four columns of the table, with the usual agreement that entries have to be intended in the sense of the “big O” notation.

The optimality of the third and fourth column of the table might probably deserve further future investigation.

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## References

- [1] S. Agmon; On kernels, eigenvalues and eigenfunctions of operators related to elliptic problems. *Commun. Pure Appl. Math.* **18** (1965), 627–663.

- [2] C. Castro, S. J. Cox; Achieving arbitrarily large decay in the damped wave equation. *SIAM J. Control Optim.* **39** (2001), no. 6, 1748–1755 (electronic).
- [3] A. Haraux; Une remarque sur la stabilisation de certains systèmes du deuxième ordre en temps. *Portugal. Math.* **46** (1989), no. 3, 245–258.
- [4] A. Haraux, P. Martinez, J. Vancostenoble; Asymptotic stability for intermittently controlled second-order evolution equations. *SIAM J. Control Optim.* **43** (2005), no. 6, 2089–2108.
- [5] V. Komornik; Stabilisation rapide de problèmes d'évolution linéaires. *C. R. Acad. Sci. Paris Sér. I Math.* **321** (1995), no. 5, 581–586.
- [6] S. Nicaise, C. Pignotti; Stability results for second-order evolution equations with switching time-delay. *J. Dynam. Differential Equations* **26** (2014), no. 3, 781–803.
- [7] A. Pleijel; On the eigenvalues and eigenfunctions of elastic plates. *Commun. Pure Appl. Math.* **3** (1950), 1–10.
- [8] R. A. Smith; Asymptotic stability of  $x'' + a(t)x' + x = 0$ . *Quart. J. Math. Oxford (2)* **12** (1961), 123–126.
- [9] A. Vest; Rapid stabilization in a semigroup framework. *SIAM J. Control Optim.* **51** (2013), no. 5, 4169–4188.
- [10] H. Weyl; Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen. *Math. Ann.* **71** (1912), no. 4, 441–479.