On the high regularity of solutions to the p-Laplacian boundary value problem in exterior domains

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On the high regularity of solutions to the p-Laplacian boundary value problem in exterior domains.

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Abstract In this note, we consider the boundary value problem in exterior domains for the p-Laplacian system. For suitable p and L^r-spaces, $r > n$, we furnish existence of a high regular solution, that is a solution whose second derivatives belong to $L^r(\Omega)$. Hence, in particular we get λ-Hölder continuity of the gradient of the solution, with $\lambda = 1 - \frac{n}{r}$. Further, we improve previous results on $W^{2,2}$-regularity in a bounded domain.

Keywords: p-Laplacian system, exterior domain, higher integrability, global regularity.

Mathematics Subject Classification: 35B65, 35J55, 35J92

1 Introduction

We consider the p-Laplacian boundary value problem, $p \in (1, 2)$, in a C^2-smooth exterior domain $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$:

$$\begin{aligned} -\nabla \cdot (|\nabla u|^{p-2}\nabla u) &= f \quad \text{in } \Omega\,, \\ u &= 0 \quad \text{on } \partial\Omega\,, \end{aligned} \qquad p \in (1,2) \tag{1.1}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$, $N \geq 1$, is a vector field.

We prove existence and uniqueness of solutions which are high regular, in the following sense:

Definition 1.1. *Given a distribution f, by high-regular solution of system* (1.1) *we mean a field u such that*

i) $\nabla u \in L^p(\Omega)$, $u = 0$ *on* $\partial\Omega$*;*

ii) *for some* $r \in (n, +\infty)$, $D^2u \in L^r(\Omega)$*;*

iii) $(|\nabla u|^{p-2}\nabla u, \nabla\varphi) = (f, \varphi)$, *for all* $\varphi \in C_0^\infty(\Omega)$.

There are several contributions to the local regularity of weak solutions of (1.1), particularly when $N = 1$. We just recall papers [1], [13], [25] and [26], for integrability of second derivatives in the interior. The global regularity, in the sense of regularity up to the boundary, has been less investigated. In the case of a bounded domain, we particularly refer to [7], [8], [9] for global boundedness or Hölder continuity of the gradient, to [21] for L^r integrability of second derivatives in the scalar case, to [4], [5] and [11], where, for systems, L^r integrability of second derivatives has been studied, under the assumption of p close enough to the limit exponent 2. As far as we know, with the exception of a result in [12], where an even more involved equation is studied in the whole $\mathbb{R}^n$, this paper is the first contribution for global regularity in an exterior domain, and more in general in an unbounded domain.

The paper has different aims.

Firstly, to fill the gap in the theory between bounded and exterior domains.

Secondly, to derive solutions with high regularity, that is solutions whose second derivatives belong to $L^r(\Omega)$, $r > n$. Hence, in particular, $u \in C^{1,\lambda}(\overline{\Omega})$, $\lambda = 1 - \frac{n}{r}$. As in the case of a bounded domain, the result holds for exponents p constrained with the value of r. We remark that this result is a first step for the study of the corresponding parabolic problem in exterior domains, following the approach used in [10].

Last, but not least, to investigate the nature of the estimate of second derivatives. To better explain this last task, it should be recalled the analogous question concerning the linear case. Let us consider the Dirichlet boundary value problem in an exterior domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$:

$$-\Delta u = f, \text{ in } \Omega, \ u = 0 \text{ on } \partial\Omega,$$

with $u \to 0$ as $|x| \to \infty$ if $n > 2$, $u \to u_\infty$ as $|x| \to \infty$ if $n = 2$. Then the estimate

$$\|D^2 u\|_r \leq c\|f\|_r,$$

holds for $r \in (1, \frac{n}{2})$, $n \geq 3$, and fails for $r \in [\frac{n}{2}, \infty)$, $n \geq 2$. A first contribute in this sense is given in [22]. Subsequently, several contributes have been given. For a full enough list of references on the topic, we refer the reader to the monograph [23]. However, in [22], for all u such that $u \in W^{1,r}(\Omega_1)$, $\Omega_1 \subset \Omega$ bounded domain such that $\partial(\Omega - \Omega_1) \cap \partial\Omega = \emptyset$, $u = 0$ on $\partial\Omega$ and $D^2 u \in L^r(\Omega)$, it is proved, as a priori estimate, that the following estimate holds for any exponent $r \in [\frac{n}{2}, \infty)$ ($r \in (1, \infty)$ for $n = 2$):

$$\|D^2 u\|_r \leq c\big(\|f\|_r + \|u\|_{L^r(\Omega_1)}\big), \tag{1.2}$$

where $f := \Delta u$.

Motivated by the above considerations, it appears interesting to understand if an estimate similar to (1.2) holds for solutions to the p-Laplacian problem, or if the nonlinear character of the operator entails new difficulties to the question. Our main theorem can be stated as follows.

Theorem 1.1. *Let Ω be an exterior domain of class C^2. Assume that $f \in L^r(\Omega) \cap (\widehat{W}^{1,p}(\Omega))'$, with $r \in (n, \infty)$. Then, there exists $\overline{p} := \overline{p}(r) \in (1, 2)$ such that if $p \in (\overline{p}, 2)$ there exists a unique high-regular solution u of system* (1.1), *with*

$$\|\nabla u\|_p \leq c\,\|f\|_{-1,p'}^{\frac{1}{p-1}}, \tag{1.3}$$

$$\|D^2 u\|_r \leq c\,(\|f\|_{-1,p'}^{\frac{1}{p-1}} + \|f\|_r^{\frac{1}{p-1}}), \tag{1.4}$$

where c is a constant independent of u.

Moreover, the solution is unique in the class of weak solutions.

The restriction on the exponent p arises from some L^r-estimates for second order derivatives of solutions to the Dirichlet problem for the Poisson equation in bounded domains. However, in the case of an exterior domain one cannot directly apply the corresponding estimates for the Poisson equation, since they rely on the finite measure of Ω. Hence, we need to introduce some new ideas.

The result is deduced by means of the technique of invading domains. We establish regularity properties of solutions to approximating problems in $\Omega \cap B_R$. The task is to deduce these properties uniformly in R.

Even though our main interest is to deal with high regularity in exterior domains in the sense specified before, our approach enables us to obtain a L^2-regularity result for second derivatives either in exterior and in bounded domains of class C^2, under the unique assumption $p \in (1,2)$. More precisely, we have the following result.

Set

$$\widehat{r} \begin{cases} = \dfrac{2n}{n(p-1)+2(2-p)}, & \text{if } \ n \geq 3, \\ \in \left(2, \dfrac{2}{p-1}\right), & \text{if } n = 2\,. \end{cases} \tag{1.5}$$

Theorem 1.2. *Let Ω be an exterior domain of class C^2 and $p \in (1,2)$. Assume that $f \in L^{\widehat{r}}(\Omega) \cap (\widehat{W}^{1,p}(\Omega))'$. Then, denoting by u the unique weak solution of* (1.1) *the following regularity estimates hold*

$$\|\nabla u\|_p \leq c\,\|f\|_{-1,p'}^{\frac{1}{p-1}},$$

$$\|D^2 u\|_2 \leq c\,(\|f\|_{-1,p'}^{\frac{1}{p-1}} + \|f\|_{\widehat{r}}^{\frac{1}{p-1}})\,.$$

The same result holds for a C^2 bounded domain E.

The interest of Theorem 1.2 is twofold. On one hand, it improves the known $W^{2,2}$-regularity for bounded domains, for which we refer to papers [4, Corollary 2.2] and [11, Theorem 1.1, $\widehat{q} = 2$]). Actually, in these papers it is shown that the unique weak solution of (1.1) belongs to $W^{2,2}$, for any $p \in (1,2)$ if the (sufficiently regular) domain E is bounded and convex, only for p close enough to 2 if E is bounded but non-convex. We are able to overcome the latter restriction, achieving the result for any $p \in (1,2)$. The second reason of interest for such a L^2-integrability result is connected with the behavior of the solution and its gradient as $|x| \to \infty$ when the space dimension is $n = 2$. It is well known that, in the case of the Laplacian, a L^2-theory for ∇u is not sufficient to ensure that $u \to 0$ as $|x| \to \infty$, as well as a L^2-theory fo $D^2 u$ is not sufficient for $\nabla u \to 0$ as $|x| \to \infty$. So, it is remarkable (in this regard see [14] and [15]) that for the p-Laplacian ($p \in (1,2)$), the existence class enables to deduce an asymptotic behavior.

The plan of the paper is the following. In sec. 2 we give the notation used throughout the paper and introduce or recall some auxiliary results. In sec. 3 we introduce an approximating problem, whose solution has second derivatives in $L^2(\Omega \cap B_R)$, and satisfies R-uniform estimates. This enables us to prove Theorem 1.2. In sec. 4 we improve the regularity of the solution of the approximating system up to $L^r(\Omega \cap B_R)$, $r > n$, and in the final section we prove our main result passing to the limit as R tends to infinity.

2 Notation and some preliminary results

Throughout the paper we will assume $p \in (1,2)$. Moreover $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, will denote an exterior domain, that is an unbounded domain with C^2 compact boundary. $E \subset \mathbb{R}^n$ will denote a C^2 bounded domain and $|E|$ denotes its Lebesgue measure.

We choose the origin of coordinates lying in the interior of $\mathbb{R}^n \setminus \Omega$. For any $\sigma > 0$ let us denote by $B_\sigma := B(O,\sigma)$ the n-dimensional open ball of radius σ centered at the origin. We define $\Omega_\sigma := \Omega \cap B_\sigma$. We fix $R_0 > 0$ such that $\mathbb{R}^n \setminus \Omega \subset B_{\frac{R_0}{2}}$. Throughout the paper R will be a real number greater than $2R_0$. Without loss of generality we require that any Ω_R satisfies the same cone property as Ω does.

We decompose ∂E in two disjoint parts, $\partial E := \Gamma_1 \cup \Gamma_2$, where Γ_2 is a surface such that, at any point of Γ_2, E is locally the subgraph of a concave function (see [18, §1.1.5]). In particular if $E = \Omega_R$, we set $\Gamma_2 := \partial B_R$.

Let $d > 1$ be an integer. We define an infinitely differentiable function $\chi_d : [0,+\infty) \to [0,1]$ satisfying the conditions $\chi_d(x) = 0$ for $x \leq (d-1)$, $\chi_d(x) = 1$ for $x \geq d$. If θ is a positive constant and x is a point of $\mathbb{R}^n$, we let $\chi_d^\theta(x) = \chi_d(\frac{|x|}{\theta})$.

We use the summation convention on repeated indices. For a function $v(x)$, by $\partial_i v$ we mean $\frac{\partial v}{\partial x_i}$, and, if $v(x)$ is a vector valued function, by $\partial_j v_i$ we mean $\frac{\partial v_i}{\partial x_j}$. If v and w are two vector fields, by $w \cdot \nabla v$ we mean $w_j \partial_j v_i$. Further, by the symbol $(\nabla v \otimes \nabla v)\, D^2 w$ we mean $\partial_j v_i\, \partial_k v_h \partial^2_{jk} w_h$.

By $L^r(\Omega)$ and $W^{m,r}(\Omega)$, m nonnegative integer and $r \in [1,\infty]$, we denote the usual Lebesgue and Sobolev spaces, with norms $\|\cdot\|_{r,\Omega}$ and $\|\cdot\|_{m,r,\Omega}$, respectively. The L^2-norm, L^r-norm and $W^{m,r}$-norm on Ω will be simply denoted, respectively, by $\|\cdot\|$, $\|\cdot\|_r$ and $\|\cdot\|_{m,r}$, when no danger of confusion is possible.

For $r \in (1,\infty)$ we set $\widehat{W}^{1,r}(\Omega) :=$ completion of $C_0^\infty(\Omega)$ in $\|\nabla \cdot\|_r$-norm. By $(\widehat{W}^{1,r}(\Omega))'$ we denote the normed dual of $\widehat{W}^{1,r}(\Omega)$, by $\|\cdot\|_{-1,r'}$ its norm and by $\langle\cdot,\cdot\rangle$ the duality pairing. Note that in the case of a bounded domain E, the space $(\widehat{W}^{1,r}(E))'$ and $W^{-1,r'}(E)$ are isomorphic.

We use the symbols $\rightharpoonup$ and $\to$ to denote weak and strong convergences, respectively.

We shall use the lower case letter c to denote a positive constant whose numerical value (and dependence on some parameters) is unessential for our aims. As well as, we can find in the same line $k > 1$ and $k\,c \leq c$.

We begin by recalling some known results, related to the regularity theory for linear elliptic equations and systems.

We will make use of the following L^q-estimate in bounded domains

$$\|D^2\, u\|_{q,E} \leq\, K_1(q,E)\|\Delta u\|_{q,E}\,, \tag{2.1}$$

for $u \in W^{2,q}(E) \cap W_0^{1,q}(E)$, $q > 1$, where the constant K_1 depends on q and E. It relies on standard estimates for solutions to the Dirichlet problem for the Poisson equation in bounded domains. For details we refer to [2], [17] and, for the particular case $q = 2$, also to [18, §1.1.5] and [19, §3.8]. Estimate (2.1) is improved in the following three lemmas.

Lemma 2.1. *Assume that $v \in W_0^{1,2}(E) \cap W^{2,2}(E)$. Then, we have*

$$\|D^2 v\|_{2,E} \,\leq \|\Delta v\|_{2,E} + C\|\nabla v\|_{2,\Gamma_1}. \tag{2.2}$$

Proof. The result easily follows from the proof of estimate (2.1), with $q = 2$, for which we refer to [18, §1.1.5]. From this proof we can infer that $\|D^2 v\|_{2,E}$ can be controlled by $\|\Delta v\|_{2,E}$ and by the integral of the normal derivative on the boundary multiplied by a term $\kappa(\partial E)$ involving the curvature:

$$\|D^2 v\|^2_{2,E} = \|\Delta v\|^2_{2,E} + \int_{\Gamma_1} (n \cdot \nabla v)^2 \kappa(\Gamma_1) ds + \int_{\Gamma_2} (n \cdot \nabla v)^2 \kappa(\Gamma_2)\, ds, \tag{2.3}$$

where we have used the decomposition $\partial E = \Gamma_1 \cup \Gamma_2$ introduced in the notation. Since $\kappa(\Gamma_2) \le 0$, we immediately get the result. $\square$

We point out that, for $E = \Omega_R$, ∂B_R does not contribute to the estimate (2.2) since $\partial B_R = \Gamma_2$.

Lemma 2.2. *For any $q \in (1, +\infty)$ there exists a constant $K_2(q)$, not depending on R, such that*

$$\|D^2 u\|_{q,B_R} \le K_2(q) \|\Delta u\|_{q,B_R}$$

for any $u \in W^{1,q}_0(B_R) \cap W^{2,q}(B_R)$.

Proof. The proof follows by (2.1) applied to the unit ball B_1, and then using an homothetic transformation on the ball B_R. $\square$

¿From estimate (2.1) and Lemma 2.2, we show the following

Lemma 2.3. *For any $q \in [p, +\infty)$, setting*

$$K_3(q) := 4(K_1(q, \Omega_{2R_0}) + K_2(q)),$$

for any $u \in W^{1,p}_0(\Omega_R) \cap W^{2,q}(\Omega_R)$ the following estimate holds

$$\|D^2 u\|_{q,\Omega_R} \le K_3(q) \|\Delta u\|_{q,\Omega_R} + C(q, R_0) \|\nabla u\|_{p,\Omega_{2R_0}}.$$

Proof. We introduce the infinitely differentiable function $\chi^{R_0}_2 : \mathbb{R}^n \to [0, 1]$, and we decompose u as

$$u := u\chi^{R_0}_2 + u(1 - \chi^{R_0}_2).$$

Extending u to 0 in $\mathbb{R}^n \setminus \Omega$, we get that $(u\chi^{R_0}_2) \in W^{2,q}(B_R) \cap W^{1,p}_0(B_R)$ and

$$\Delta(u\chi^{R_0}_2) = \chi^{R_0}_2 \Delta u + 2\nabla\chi^{R_0}_2 \cdot \nabla u + u\Delta\chi^{R_0}_2 .$$

By applying Lemma 2.2

$$\|D^2(u\chi^{R_0}_2)\|_{q,B_R} \le K_2(q)(\|\Delta u\|_{q,B_R\setminus B_{R_0}} + \|\nabla u\|_{q,B_{2R_0}\setminus B_{R_0}} + \|u\|_{q,B_{2R_0}\setminus B_{R_0}}) . \tag{2.4}$$

By applying Gagliardo-Nirenberg's inequality and then Young's inequality

$$\begin{aligned} \|\nabla u\|_{q,B_R\setminus B_{2R_0}} &\le c(R_0) \|D^2 u\|^a_{q,B_{2R_0}\setminus B_{R_0}} \|\nabla u\|^{1-a}_{p,B_{2R_0}\setminus B_{R_0}} + c\|\nabla u\|_{p,B_{2R_0}\setminus B_{R_0}} \\ &\le \varepsilon \|D^2 u\|_{q,B_{2R_0}\setminus B_{R_0}} + c(\varepsilon, R_0) \|\nabla u\|_{p,B_{2R_0}\setminus B_{R_0}}, \end{aligned}$$

with $a = \frac{n(q-p)}{nq+pq-np}$. Further, by applying Poincaré's inequality and then reasoning as above we get

$$\begin{aligned} \|u\|_{q,B_{2R_0}\setminus B_{R_0}} &\le \|u\|_{q,\Omega_{2R_0}} \le c(R_0) \|\nabla u\|_{q,\Omega_{2R_0}} \\ &\le \varepsilon \|D^2 u\|_{q,\Omega_{2R_0}} + c(\varepsilon, R_0) \|\nabla u\|_{p,\Omega_{2R_0}}. \end{aligned}$$

Collecting the previous estimates, (2.4) gives

$$\|D^2(u\chi_2^{R_0})\|_{q,B_R} \le K_2(q)\|\Delta u\|_{q,B_R\setminus B_{R_0}} + \varepsilon\|D^2u\|_{q,\Omega_{2R_0}} + c(\varepsilon,R_0)\|\nabla u\|_{p,\Omega_{2R_0}}. \tag{2.5}$$

Let us estimate $D^2(u(1-\chi_2^{R_0}))$. By using estimate (2.1), we readily get

$$\begin{aligned}\|D^2(u(1-\chi_2^{R_0}))\|_{q,\Omega_{2R_0}} \le \quad & K_1(q,\Omega_{2R_0})(\|\Delta u\|_{q,\Omega_{2R_0}} \\ & + \|\nabla u\|_{q,B_{2R_0}\setminus B_{R_0}} + \|u\|_{q,B_{2R_0}\setminus B_{R_0}}).\end{aligned}$$

Treating the last two terms as before, we find

$$\begin{aligned}\|D^2(u(1-\chi_2^{R_0}))\|_{q,\Omega_{2R_0}} \le \quad & K_1(q,\Omega_{2R_0})(\|\Delta u\|_{q,\Omega_{2R_0}} \\ & + \varepsilon\|D^2u\|_{q,\Omega_{2R_0}} + c(\varepsilon,R_0)\|\nabla u\|_{q,\Omega_{2R_0}}).\end{aligned} \tag{2.6}$$

Therefore, using (2.5) and (2.6), we end up with

$$\begin{aligned}\|D^2u\|_{q,\Omega_R} \le \quad & \|D^2(u\chi_2^{R_0})\|_{q,\Omega_R} + \|D^2(u(1-\chi_2^{R_0}))\|_{q,\Omega_R} \\ \le \quad & K_2(q)\|\Delta u\|_{q,B_R\setminus B_{R_0}} + K_1(q,\Omega_{2R_0})\|\Delta u\|_{q,\Omega_{2R_0}} \\ & + \varepsilon\|D^2u\|_{q,\Omega_{2R_0}} + c(\varepsilon,R_0)\|\nabla u\|_{p,\Omega_{2R_0}},\end{aligned}$$

which, choosing $\varepsilon = \frac{1}{2}$ gives the result. □

3 $W^{2,2}$-regularity

Let $\mu \in (0,1]$. We introduce the following auxiliary problem

$$\begin{aligned} -\frac{\Delta u_E}{(\mu+|\nabla u_E|^2)^{\frac{2-p}{2}}} - (p-2)\frac{(\nabla u_E \otimes \nabla u_E)D^2u_E}{(\mu+|\nabla u_E|^2)^{\frac{4-p}{2}}} &= f \quad \text{in } E \\ u_E &= 0 \quad \text{on } \partial E. \end{aligned} \tag{3.1}$$

Proposition 3.1. *Let $p \in (1,2)$ and let $f \in L^{\widehat{r}}(E) \cap W^{-1,p'}(E)$, with $\widehat{r}$ defined in (1.5). Then, there exists a solution $u_E \in W_0^{1,2}(E) \cap W^{2,2}(E)$ of system (3.1). Moreover, the following estimates hold*

$$\|\nabla u_E\|_{p,E} \le c\left(\|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}}|E|^{\frac{n}{p}}\right), \tag{3.2}$$

$$\|D^2u_E\|_{2,E} \le c\,(\|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}}|E|^{\frac{n}{p}} + \|f\|_{\widehat{r}}^{\frac{1}{p-1}} + \mu^{\frac{2-p}{2}}\|f\|), \tag{3.3}$$

with c independent of $|E|$.

Proof. Let us consider the following auxiliary problem for fixed $\varepsilon > 0$

$$\begin{aligned} -\varepsilon\Delta u - \nabla\cdot\left((\mu+|\nabla u|^2)^{\frac{p-2}{2}}\nabla u\right) &= f \quad \text{in } E, \\ u &= 0 \quad \text{on } \partial E. \end{aligned} \tag{3.4}$$

By [6, Theorem 8.2], as $f \in L^2(E)$ we determine the solution $u_\varepsilon \in W^{2,2}(E)$ of the above problem. Multiplying (3.4) by u_ε and integrating over E we get

$$\varepsilon\|\nabla u_\varepsilon\|^2 + \int_E \frac{|\nabla u_\varepsilon|^2}{(\mu+|\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}}\,dx = \langle f, u_\varepsilon\rangle.$$

It follows that

$$\int_E \frac{|\nabla u_\varepsilon|^2}{(\mu + |\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}}\, dx \le \|f\|_{-1,p'} \|\nabla u_\varepsilon\|_p.$$

Using the above estimate together with Young's inequality, since $\mu \le 1$, we have

$$\begin{aligned}
\|\nabla u_\varepsilon\|_p^p &= \int_{\{|\nabla u_\varepsilon|^2 \ge \mu\}} |\nabla u_\varepsilon|^p\, dx + \int_{\{|\nabla u_\varepsilon|^2 < \mu\}} |\nabla u_\varepsilon|^p\, dx \\
&\le 2^{\frac{2-p}{2}} \int_E \frac{|\nabla u_\varepsilon|^2}{(\mu + |\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}}\, dx + \mu^{\frac{p}{2}} |E| \le 2^{\frac{2-p}{2}} \|f\|_{-1,p'} \|\nabla u_\varepsilon\|_p + \mu^{\frac{p}{2}} |E| \\
&\le \frac{1}{p} \|\nabla u_\varepsilon\|_p^p + c \|f\|_{-1,p'}^{p'} + \mu^{\frac{p}{2}} |E|
\end{aligned}$$

hence

$$\|\nabla u_\varepsilon\|_p \le c \left(\|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}} |E|^{\frac{1}{p}} \right), \tag{3.5}$$

with the constant c independent of ε, μ and $|E|$. Considering that $u_\varepsilon \in W_0^{1,2}(E) \cap W^{2,2}(E)$ we can state that

$$-\varepsilon \Delta u_\varepsilon - \frac{\Delta u_\varepsilon}{(\mu + |\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}} - (p-2) \frac{(\nabla u_\varepsilon \otimes \nabla u_\varepsilon) D^2 u_\varepsilon}{(\mu + |\nabla u_\varepsilon|^2)^{\frac{4-p}{2}}} = f \quad \text{a.e. in } E. \tag{3.6}$$

Multiplying equation (3.6) by $(\mu + |\nabla u_\varepsilon|^2)^{\frac{2-p}{2}}$, and taking the $L^2(E)$-norm of both sides, we get

$$\begin{aligned}
\|\Delta u_\varepsilon\| &\le \|\Delta u_\varepsilon (1 + \varepsilon (\mu + |\nabla u_\varepsilon|^2)^{\frac{2-p}{2}})\| \\
&\le (2-p) \|D^2 u_\varepsilon\| + \|(\mu + |\nabla u_\varepsilon|^2)^{\frac{2-p}{2}} f\| \\
&\le (2-p) \|D^2 u_\varepsilon\| + \|f |\nabla u_\varepsilon|^{2-p}\| + \mu^{\frac{2-p}{2}} \|f\|.
\end{aligned} \tag{3.7}$$

In order to estimate the L^2-norm of $D^2 u_\varepsilon$ we use Lemma 2.1, which yields

$$\|D^2 u_\varepsilon\|_{2,E} \le \|\Delta u_\varepsilon\|_{2,E} + C \|\nabla u_\varepsilon\|_{2,\Gamma_1}, \tag{3.8}$$

and the constant C depends on the geometry of Γ_1 but not on $|E|$. To estimate the boundary term we consider a fixed neighborhood $\widetilde{E} \subset E$ of Γ_1, and we make use of Gagliardo-Nirenberg's inequality, and then of Young's inequality, to get,

$$\begin{aligned}
C \|\nabla u_\varepsilon\|_{2,\Gamma_1} \le C \|\nabla u_\varepsilon\|_{2,\partial \widetilde{E}} &\le C c_2 \left(\|\nabla u_\varepsilon\|_{2,\widetilde{E}} + \|\nabla u_\varepsilon\|_{2,\widetilde{E}}^{\frac{1}{2}} \|D^2 u_\varepsilon\|_{2,\widetilde{E}}^{\frac{1}{2}} \right) \\
&\le \sigma \|D^2 u_\varepsilon\|_{2,\widetilde{E}} + C c_2 \left(\frac{C c_2}{4\sigma} + 1 \right) \|\nabla u_\varepsilon\|_{2,\widetilde{E}}
\end{aligned} \tag{3.9}$$

for any $\sigma > 0$ and the constants, here and in the following inequalities, do not depend on $|E|$. Employing once again Gagliardo-Nirenberg's inequality with $a := \frac{n(2-p)}{2n+2p-np}$, and successively Young's inequality, we get

$$\begin{aligned}
\|\nabla u_\varepsilon\|_{2,\widetilde{E}} &\le c_3(\widetilde{E}) \left(\|D^2 u_\varepsilon\|_{2,\widetilde{E}}^a \|\nabla u_\varepsilon\|_{p,\widetilde{E}}^{1-a} + \|\nabla u_\varepsilon\|_{p,\widetilde{E}} \right) \\
&\le \delta \|D^2 u_\varepsilon\|_{2,\widetilde{E}} + c(\delta) \|\nabla u_\varepsilon\|_{p,\widetilde{E}}.
\end{aligned} \tag{3.10}$$

Substituting estimate (3.10) in (3.9), choosing δ small enough and $\sigma \in (0, \frac{1}{2})$, by (3.8), we get

$$\|D^2 u_\varepsilon\|_{2,E} \le \frac{1}{1-2\sigma}\|\Delta u_\varepsilon\|_{2,E} + c(\sigma)\|\nabla u_\varepsilon\|_{p,E}. \tag{3.11}$$

For the term $\|f|\nabla u_\varepsilon|^{2-p}\|$ in (3.7) we distinguish between $n = 2$ and $n \ge 3$. Let be $n \ge 3$. By applying Hölder's inequality with exponents $\frac{\widehat{r}}{2}$, $n/(n-2)(2-p)$, we have

$$\|f\,|\nabla\, u_\varepsilon|^{2-p}\| \le \|\, f\|_{\widehat{r}}\, \|\, \nabla\, u_\varepsilon\, \|^{2-p}_{\frac{2n}{n-2}}\,. \tag{3.12}$$

If $n = 2$, since $\widehat{r} \in (2, \frac{2}{p-1})$, we can find $r > 2$ such that $\widehat{r} = \frac{2r}{r-2(2-p)}$. Hence, we apply Hölder's inequality with exponents $\frac{\widehat{r}}{2}$, $\frac{r}{2(2-p)}$, and we obtain

$$\|f\,|\nabla\, u_\varepsilon|^{2-p}\| \;\le \|\, f\|_{\widehat{r}}\, \|\, \nabla\, u_\varepsilon\, \|^{2-p}_{r}\,. \tag{3.13}$$

We set $\overline{r} = \frac{2n}{n-2}$, for $n \ge 3$, and $\overline{r} = r > n$, for $n = 2$. In any case we can apply the Sobolev embedding theorem, and obtain

$$\|\, \nabla\, u_\varepsilon\|_{\overline{r}} \le c(\partial E)(\|D^2 u_\varepsilon\| + \|\nabla u_\varepsilon\|), \tag{3.14}$$

where the constant c depends only on the cone determining the cone property of E^*. Interpolating L^2 between L^p and $L^{\overline{r}}$ and using Young's inequality we get that, for any $\delta > 0$,

$$\|\nabla u_\varepsilon\|_{2,E} \le \delta\|\nabla u_\varepsilon\|_{\overline{r},E} + c(\delta)\|\nabla u_\varepsilon\|_{p,E}. \tag{3.15}$$

If we choose a suitable δ in (3.15), and we replace (3.15) in (3.14), we get

$$\|\, \nabla\, u_\varepsilon\|_{\overline{r}} \le c(\|D^2 u_\varepsilon\| + \|\nabla u_\varepsilon\|_p). \tag{3.16}$$

Hence, by applying Young's inequality, we finally get, for any $\eta > 0$,

$$\|f|\nabla u_\varepsilon|^{2-p}\| \le \|f\|_{\widehat{r}}\|\nabla u_\varepsilon\|^{2-p}_{\overline{r}} \le \eta\|D^2 u_\varepsilon\| + c\|f\|_{\widehat{r}}^{\frac{1}{p-1}} + c\|\nabla u_\varepsilon\|_p. \tag{3.17}$$

Therefore, by using estimate (3.17) and (3.11) in (3.7), we obtain, for any $\sigma \in (0, \frac{p-1}{2})$ and $\eta \in (0, p-1-2\sigma)$,

$$\left(1 - \frac{(2-p)}{1-2\sigma} - \frac{\eta}{1-2\sigma}\right)\|\Delta u_\varepsilon\| \le c\|\nabla u_\varepsilon\|_p + c\|f\|_{\widehat{r}}^{\frac{1}{p-1}} + \mu^{\frac{2-p}{2}}\|f\|. \tag{3.18}$$

Hence, by (3.11) and (3.18) we get

$$\|D^2 u_\varepsilon\|_{2,E} \le c\,(\|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}}|E|^{\frac{n}{p}} + \|f\|_{\widehat{r}}^{\frac{1}{p-1}} + \mu^{\frac{2-p}{2}}\|f\|), \tag{3.19}$$

for any $\varepsilon > 0$, where $c = c(p)$. Therefore, with the aid of Poincaré's inequality, we get that

$$\|u_\varepsilon\|_{2,2,E} \le c(E)(\|f\|_{-1,p'}^{\frac{1}{p-1}} + \mu^{\frac{1}{2}}|E|^{\frac{n}{p}} + \|f\|_{\widehat{r}}^{\frac{1}{p-1}} + \mu^{\frac{2-p}{2}}\|f\|),$$

*Note that when $E = \Omega_R$, the cone property is uniform with respect to R.

uniformly in ε. It follows that we can find a function $u_E \in W^{2,2}(E) \cap W_0^{1,2}(E)$ such that, up to subsequences, $u_\varepsilon \rightharpoonup u_E$ weakly in $W^{2,2}(E)$. Moreover, by the Relich-Kondrachov embedding theorem, we can suppose that $\nabla u_\varepsilon \to \nabla u_E$ almost everywhere in E. Since the sequence $\{(\mu + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon\}$ is bounded in $L^{p'}(E)$, Lemma I.1.3 in [20] allows us to infer the weak convergence

$$(\mu + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon \rightharpoonup (\mu + |\nabla u_E|^2)^{\frac{p-2}{2}} \nabla u_E \qquad \text{in } L^{p'}(E).$$

This is enough to ensure that u_E is a solution of (3.1) and satisfies estimates (3.2) and (3.3). □

In the above proposition we have taken care to get all the constants independent of $|E|$. Further, as previously remarked, if $E = \Omega_R$ the constants depending on ∂E actually depend on $\partial\Omega$. Hence, from the above Proposition we immediately obtain the following result.

Corollary 3.1. *Let $p \in (1, 2)$ and let $f \in C^\infty(\overline{\Omega}_R)$, for a fixed R. Further assume that*

$$\mu^{\frac{1}{2}} R^{\frac{n}{p}} = R^\alpha, \qquad \text{for some } \alpha < 0. \tag{3.20}$$

Then, there exists a solution $u_R := u_E \in W_0^{1,2}(\Omega_R) \cap W^{2,2}(\Omega_R)$ of system (3.1). Moreover, setting

$$\Lambda := c\,(\|f\|_{-1,p'}^{\frac{1}{p-1}} + R^\alpha + \|f\|_{\hat{r}}^{\frac{1}{p-1}} + \mu^{\frac{2-p}{2}} \|f\|), \tag{3.21}$$

the following estimates hold

$$\|\nabla u_R\|_{p,\Omega_R} \le c \left(\|f\|_{-1,p'}^{\frac{1}{p-1}} + R^\alpha \right), \tag{3.22}$$

$$\|D^2 u_R\|_{2,\Omega_R} \le \Lambda, \tag{3.23}$$

with c independent of R.

Proof of Theorem 1.2. We start with the case of Ω exterior.

Firstly we assume $f \in C_0^\infty(\overline{\Omega})$. Then there exists a $\overline{R}$ such that for any $R > \overline{R}$, $f \in C^\infty(\overline{\Omega}_R)$. Further we assume that μ and R satisfy (3.20). Then, from Corollary 3.1, for any fixed R (and μ), there exists a solution u_R of system (3.1). The solution u of (1.1) corresponding to $f \in C_0^\infty(\overline{\Omega})$ can be obtained as limit of the sequence of solutions $\{u_R\}$, letting R go to infinity †. We omit the proof of this convergence, as it will be completely given in the last section (Proof of Theorem 1.1, Step 1). The estimates on ∇u and $D^2 u$ follow from (3.22) and (3.23), respectively, and the lower semicontinuity.

Let us remove the extra assumption on f. Therefore let $f \in L^{\hat{r}}(\Omega) \cap (\widehat{W}^{1,p}(\Omega))'$. In this case the result can be obtained by approximating f in the norms of $L^{\hat{r}}(\Omega)$ and $(\widehat{W}^{1,p}(\Omega))'$ throughout a suitable sequence. We omit the details and refer to the last section (Proof of Theorem 1.1, Step 2).

† We point out that as R goes to infinity the parameter μ goes to zero following the behavior given in (3.20).

Let us consider the case of a bounded domain E. For any $\mu > 0$, let $\{u_E^\mu\}$ the sequence of solutions, obtained in Proposition 3.1. The solution u of (1.1) in E can be obtained as limit of this sequence as $\mu \to 0$. The proof is an easy adaptation of the proof of convergence of the sequence $\{u_R\}$, as $R \to \infty$ (see the last section, Proof of Theorem 1.1, Step 1). The estimates on ∇u and $D^2 u$ follow from (3.2) and (3.3), respectively, and the lower semicontinuity.

By uniqueness, the solution coincides with the unique weak solution of (1.1), for which we refer to [20, §2.2]. □

4 High regularity for solutions of the approximating system

We begin this section with a regularity result for the non singular approximating problem. The method is based on classical elliptic estimates, hence the proof will only be sketched.

Proposition 4.1. *Let $p \in (1,2)$, $\mu > 0$ and $f \in C^\infty(\overline{E})$. Then the solution u of the problem*

$$(4.1)\qquad \begin{aligned} -\nabla \cdot \left(\left(\mu + |\nabla u|^2\right)^{\frac{p-2}{2}} \nabla u \right) &= f \quad \textit{in } E \\ u &= 0 \quad \textit{on } \partial E \end{aligned}$$

has second derivatives in $L^r(E)$, for any $r \in [2, \infty)$.

Proof. The solution u_E of system (3.1) belongs to $W^{2,2}(E)$, by Proposition 3.1, and it clearly coincides with the unique solution u of system (4.1). From [7, Theorem 2] we also know that Du is Hölder continuous in $\overline{E}$. Let us consider the following system in the unknown v

$$\begin{aligned} -\Delta v - (p-2) \frac{\nabla u \otimes \nabla u}{\mu + |\nabla u|^2} D^2 v = f \left(\mu + |\nabla u|^2\right)^{\frac{p-2}{2}} &\quad \text{in } E \\ v = 0 &\quad \text{on } \partial E. \end{aligned}$$

The above system is linear, the coefficients are uniformly continuous and satisfy the Legendre-Hadamard condition, hence any $W^{2,2}$ solution has second derivatives in $L^r(E)$ (see [3],[24],[16]). Since u solves the system, we immediately get the result. □

Our aim is to get an explicit dependence on R of $\|D^2 u_R\|_q$, $q > 2$, with u_R given in Corollary 3.1.

Let us set $\rho = \frac{2}{3} R_0$ and define

$$(4.2)\qquad p(r) := 2 - \frac{1}{K_3(r)},$$

with $K_3(r)$ introduced in Lemma 2.3.

Proposition 4.2. *Let $f \in C^\infty(\overline{\Omega}_R)$ and u_R be the solution obtained in Corollary 3.1. For any $r > n$ and $p \in (p(r), 2)$, the following estimate holds*

$$(4.3)\qquad \|D^2 u_R\|_r \le \mu^{\frac{2-p}{2}} \|f\|_r + c\|f\|_r^{\frac{1}{p-1}} + c\left(\|f\|_{-1,p'}^{\frac{1}{p-1}} + R^\alpha \right),$$

with α as in (3.20).

Proof. We remark that, by Proposition 4.1, $u_R \in W^{2,q}(\Omega_R)$ for any $q \geq 1$. Multiplying equation (3.1) by $(\mu + |\nabla u_R|^2)^{\frac{2-p}{2}}$, and taking the L^r-norm of both sides, we have

$$\begin{aligned} \|\Delta u_R\|_r &\leq (2-p) \left\| \frac{\nabla u_R \otimes \nabla u_R}{\mu + |\nabla u_R|^2} D^2 u_R \right\|_r + \|f(\mu + |\nabla u_R|^2)^{\frac{2-p}{2}}\|_r \\ &\leq (2-p)\|D^2 u_R\|_r + \mu^{\frac{2-p}{2}} \|f\|_r + \|f\|_r \|\nabla u_R\|_\infty^{2-p} . \end{aligned} \tag{4.4}$$

For the last term on the right-hand side, we reason as follows. By employing the Sobolev embedding theorem, the convexity of the norm, and then Young's inequality we have

$$\begin{aligned} \|\nabla u_R\|_\infty \leq c(\|D^2 u_R\|_r + \|\nabla u_R\|_r) &\leq c(\|D^2 u_R\|_r + \|\nabla u_R\|_\infty^\theta \|\nabla u_R\|_p^{1-\theta}) \\ &\leq c\|D^2 u_R\|_r + \delta \|\nabla u_R\|_\infty + c(\delta)\|\nabla u_R\|_p , \end{aligned}$$

with $\theta = \frac{r-p}{r}$. We remark once again that the constant c depends on the cone property of Ω, hence not on R. Choosing a small $\delta > 0$ we get,

$$\|\nabla u_R\|_\infty \leq c(\|D^2 u_R\|_r + \|\nabla u_R\|_p) .$$

Therefore, by applying Lemma 2.3 and Young's inequality, we obtain

$$\begin{aligned} \|f\|_r \|\nabla u_R\|_\infty^{2-p} &\leq c\, \|f\|_r (\|\Delta u_R\|_r + c(R_0)\|\nabla u_R\|_p)^{2-p} \\ &\leq \varepsilon \|\Delta u_R\|_r + c(\varepsilon)\|f\|_r^{\frac{1}{p-1}} + c(R_0)\|f\|_r \|\nabla u_R\|_p^{2-p} . \end{aligned}$$

Inserting this estimate in (4.4), we get

$$\begin{aligned} \|\Delta u_R\|_r \leq\ & (2-p)\|D^2 u_R\|_r + \varepsilon \|\Delta u_R\|_r + \mu^{\frac{2-p}{2}} \|f\|_r + c(\varepsilon)\|f\|_r^{\frac{1}{p-1}} \\ & + c(R_0)\|f\|_r \|\nabla u_R\|_p^{2-p} . \end{aligned} \tag{4.5}$$

By using Lemma 2.3, we get

$$\begin{aligned} (1 - (2-p)K_3(r) - \varepsilon)\|\Delta u_R\|_r &\leq c(R_0)\|\nabla u_R\|_p + \mu^{\frac{2-p}{2}} \|f\|_r \\ &\quad + c(\varepsilon)\|f\|_r^{\frac{1}{p-1}} + c\|f\|_r \|\nabla u_R\|_p^{2-p} , \end{aligned}$$

whence, applying Young's inequality, recalling estimate (3.22) and the assumption on p, we get (4.3).

□

5 Proof of Theorem 1.1

Step 1: $f \in C_0^\infty(\overline{\Omega})$.

Let R and μ be as in (3.20) and $p > p(r)$ defined in (4.2). First of all we observe that, from Corollary 3.1, for any R such that $supp\, f \subset \overline{\Omega}_R$,

$$\|\nabla u_R\|_{p,\Omega_R} \leq c \left(\|f\|_{-1,p'}^{\frac{1}{p-1}} + R^\alpha \right) . \tag{5.1}$$

Moreover, by Proposition 4.2, $\|D^2u_R\|_r$ satisfies (4.3). Let us show that, in the limit as R goes to ∞, u_R tends to a function u, which is the high-regular solution of (1.1), and, from (5.1), (4.3), and the lower-semicontinuity, satisfies

$$\|\nabla u\|_{p,\Omega} \le c\|f\|_{-1,p'}^{\frac{1}{p-1}}, \tag{5.2}$$

$$\|D^2u\|_{r,\Omega} \le c\|f\|_{-1,p'}^{\frac{1}{p-1}} + c\|f\|_r^{\frac{1}{p-1}}. \tag{5.3}$$

Let us fix $l \in \mathbb{N},\ l > 2R_0$. Considering integer values of R we have that, by (5.1), the sequence $\{u_R\}$ is bounded in $L^{\frac{np}{n-p}}(\Omega_l)$, $\{\nabla u_R\}$ is bounded in $L^p(\Omega_l)$, and by (4.3) $\{D^2u_R\}$ is bounded in $L^r(\Omega_l)$. Hence we can extract a subsequence $\{u_R^l\}$ from $\{u_R\}$, and find a function u^l such that

$$u_R^l \rightharpoonup u^l \quad \text{weakly in } L^{\frac{np}{n-p}}(\Omega_l),$$
$$\nabla u_R^l \rightharpoonup \nabla u^l \quad \text{weakly in } L^p(\Omega_l),$$
$$D^2u_R^l \rightharpoonup D^2u^l \quad \text{weakly in } L^r(\Omega_l).$$

Considering the set Ω_{l+1} we can use the same procedure and find another function defined on Ω_{l+1}. By the uniqueness of the weak limit, the new function coincides with the previous one on Ω_l. By applying the Cantor diagonalization method, we construct a subsequence, still denoted by $\{u_R\}$, converging to u in the whole Ω. We want to show that the limit function u solves (1.1) in Ω. We recall that u_R is a solution of the problem (3.1) which, using identity (3.20), we write as follows

$$-\nabla\cdot\left(\left(R^\beta + |\nabla u_R|^2\right)^{\frac{p-2}{2}}\nabla u_R\right) = f \qquad \text{in } \Omega_R, \tag{5.4}$$

$\beta = 2(\alpha - \frac{n}{p})$. For a fixed $\varphi \in C_0^\infty(\Omega)$ we can find $\overline{R} \in \mathbb{N}$ such that the support of φ is contained in $\Omega_{\overline{R}}$. Multiplying (5.4) by φ and considering that φ is null outside $\Omega_{\overline{R}}$ we have

$$\int_\Omega f\cdot\varphi\,dx = \int_{\Omega_{\overline{R}}}\left(R^\beta + |\nabla u_R|^2\right)^{\frac{p-2}{2}}\nabla u_R\cdot\nabla\varphi\,dx. \tag{5.5}$$

Since the domain $\Omega_{\overline{R}}$ is bounded, by (4.3) and Rellich-Kondrachov's theorem we can extract a subsequence (not relabeled) such that

$$\nabla u_R \longrightarrow \nabla u \quad \text{a.e. in } \Omega_{\overline{R}}.$$

It follows that

$$\left(R^\beta + |\nabla u_R(x)|^2\right)^{\frac{p-2}{2}}\nabla u_R(x) \longrightarrow |\nabla u(x)|^{p-2}\nabla u(x) \quad \text{a.e. in } \Omega_{\overline{R}}.$$

Since by (5.1) the sequence $\left\{\left(R^\beta + |\nabla u_R|^2\right)^{\frac{p-2}{2}}\nabla u_R\right\}$ is bounded in $L^{p'}(\Omega_{\overline{R}})$ we can apply [20, Lemma I.1.3] and conclude that

$$\left(R^\beta + |\nabla u_R|^2\right)^{\frac{p-2}{2}}\nabla u_R \rightharpoonup |\nabla u|^{p-2}\nabla u \quad \text{weakly in } L^{p'}(\Omega_{\overline{R}}).$$

Passing to the limit in (5.5) we have that

$$\int_\Omega f \cdot \varphi \, dx = \int_{\Omega_{\overline{R}}} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dx.$$

Since the choice of φ is arbitrary, we have proved that u is a solution of problem (1.1). To prove estimates (5.2) and (5.3), we fix a bounded domain $\Omega' \subset \Omega$ and, for R large enough we can write (5.1) and (4.3) on Ω'. Letting R go to infinity we get (5.2) and (5.3) on Ω', and, for the arbitrariness of Ω', on Ω.

Step 2: $f \in L^r(\Omega) \cap (\widehat{W}^{1,p}(\Omega))'$.

Set $\zeta^\xi := 1 - \chi_2^\xi(x)$ and

$$G_{\epsilon,\xi}(x) := J_\epsilon(f \, \zeta^\xi)(x),$$

where J_ϵ is a Friedrich's mollifier. Note that, by the assumptions on f, $G_{\epsilon,\xi} \in C_0^\infty(\overline{\Omega})$,

$$\|G_{\epsilon,\xi}\|_r \le \|f\|_r \, , \text{ and } \lim_{\epsilon \to 0} \|G_{\epsilon,\xi} - f\zeta^\xi\|_r = 0 \, . \tag{5.6}$$

Further, since by Hardy's inequality

$$\|\nabla \zeta^\xi \, J_\epsilon \psi\|_p \le c \left\| \frac{J_\epsilon \psi}{\xi} \right\|_{L^p(\xi \le |x| \le 2\xi)} \le c \|\nabla \, J_\epsilon \psi\|_{L^p(|x| \ge \xi)} \, ,$$

with c independent of ξ, the following ϵ, ξ-uniform bound holds: for any $\psi \in \widehat{W}^{1,p}(\Omega)$

$$\begin{aligned} |\langle G_{\epsilon,\xi}, \psi \rangle| \quad &= |\langle f, \zeta^\xi J_\epsilon \psi \rangle| \le \|f\|_{-1,p'} \|\nabla(\zeta^\xi J_\epsilon \psi)\|_p \\ &\le c\|f\|_{-1,p'} \|\nabla J_\epsilon \psi\|_p \le c\|f\|_{-1,p'} \|\nabla \psi\|_p \, . \end{aligned}$$

Hence

$$\|G_{\epsilon,\xi}\|_{-1,p'} \le c\|f\|_{-1,p'} \, . \tag{5.7}$$

Let us consider system (1.1) where the right-hand side f is replaced by $G_{\epsilon,\xi}$. As, for any fixed ϵ, ξ, the function $G_{\epsilon,\xi}$ satisfies the assumptions of *Step 1*, there exists a corresponding function $u^{\epsilon,\xi}$, which is solution of system

$$\nabla \cdot (|\nabla u^{\epsilon,\xi}|^{p-2} \, \nabla u^{\epsilon,\xi}) = G_{\epsilon,\xi} \, , \quad \text{in } \Omega \, , \tag{5.8}$$

in the sense of Definition 1.1. Further, thanks to estimates (5.2) and (5.3) together with (5.6) and (5.7), it satisfies

$$\|\nabla u^{\epsilon,\xi}\|_p \le c \, \|f\|_{-1,p'}^{\frac{1}{p-1}} \, . \tag{5.9}$$

$$\|D^2 u^{\epsilon,\xi}\|_r \le c \, (\|f\|_{-1,p'}^{\frac{1}{p-1}} + \|f\|_r^{\frac{1}{p-1}}), \tag{5.10}$$

with a constant c independent of ϵ, ξ. We want to pass to the limits as $\epsilon \to 0$ and then as $\xi \to \infty$. We just sketch one of these convergences, since the proofs are quite the same. We pass to the limit as $\epsilon \to 0$. The ϵ-uniform bounds

(5.9) and (5.10) ensure the existence of a subsequence, that we do not relabel, and a function u^ξ, such that, in the limit as $\epsilon \to 0$, $D^2u^{\epsilon,\xi} \rightharpoonup D^2u^\xi$ in $L^r(\Omega)$, $\nabla u^{\epsilon,\xi} \rightharpoonup \nabla u^\xi$ in $L^p(\Omega)$, $|\nabla u^{\epsilon,\xi}|^{p-2}\nabla u^{\epsilon,\xi} \rightharpoonup \Psi$ in $L^{p'}(\Omega)$. Hence, by Rellich-Kondrachov's theorem, there exists a further subsequence, depending on σ, such that

$$\nabla u^{\epsilon,\xi} \to \nabla u^\xi \quad \text{in} \quad L^p(B_\sigma)\,,$$

$$|\nabla u^{\epsilon,\xi}|^{p-2}\nabla u^{\epsilon,\xi} \to |\nabla u^\xi|^{p-2}\nabla u^\xi \quad \text{a.e. in} \quad B_\sigma\,.$$

Since this last subsequence weakly converges to Ψ in $L^{p'}(B_\sigma)$ too, we find that $\Psi = |\nabla u^\xi|^{2-p}\nabla u^\xi$, on each compact $B_\sigma \subset \Omega$, which ensures that

$$|\nabla u^{\epsilon,\xi}|^{p-2}\nabla u^{\epsilon,\xi} \rightharpoonup |\nabla u^\xi|^{p-2}\nabla u^\xi \quad \text{in} \quad L^{p'}(\Omega) \text{ as } \epsilon \to 0\,. \tag{5.11}$$

Hence u^ξ is high-regular solution of the limit problem and satisfies estimates (1.3) and (1.4) by lower semicontinuity.

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