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A note on strong solutions to the Stokes system

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Abstract We give an alternative and quite simple proof of existence of $W^{2,q}$-$W^{1,q}$-strong solutions for the Stokes system, endowed with Dirichlet boundary conditions in a bounded smooth domain.

Keywords Stokes system, strong solutions, existence.

Mathematics Subject Classification (2000) 35Q30

1 Introduction

The aim of this note is to give a rather elementary proof of existence, uniqueness, and data dependence for strong solutions to the Stokes system with Dirichlet boundary conditions. Let $\Omega \subset \mathbf{R}^3$ be an open bounded set with a $C^{1,1}$ boundary $\partial\Omega$ and let $u = (u_1, u_2, u_3)$ and π denote the unknown velocity and pressure, respectively. We will use customary Lebesgue $L^q(\Omega)$ and Sobolev $W^{k,q}(\Omega)$ spaces (see e.g. Brezis [6]) and we define the following spaces: $V_q := (W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))^3$, with norm $\|u\|_{V_q} = \|u\|_{W^{2,q}}$, and also $M_q := \{f \in W^{1,q}(\Omega) : \int_\Omega f\, dx = 0\}$, with norm $\|f\|_{M_q} = \|\nabla f\|_{L^q}$, which is equivalent to that in $W^{1,q}(\Omega)$ thanks to the Poincaré inequality. We also set $X_q := V_q \times M_q$. By adapting techniques from Beirão da Veiga [4], we will give an alternative proof of the following well-known result, fundamental in the theory of the Navier-Stokes equations.

Theorem 1 *Let be given $0 < \nu \leq 1$, $f \in (L^q(\Omega))^3$, and $g \in M_q$, for some $1 < q < \infty$. Then, there exists a unique solution $(u, \pi) \in X_q$ to the (non homogeneous) Stokes problem*

$$\begin{cases} -\nu \Delta u + \nabla \pi = f & \text{in } \Omega, \\ \nabla \cdot u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

and there exists $C = C(q, \Omega) > 0$ such that

$$\nu \|u\|_{W^{2,q}} + \|\pi\|_{W^{1,q}} \leq C(\|f\|_{L^q} + \|g\|_{W^{1,q}}). \tag{2}$$

To prove Theorem 1 we will re-cast some standard tools in a new way. We recall that Theorem 1 dates back to Cattabriga [7] and to the announcement in Solonnikov [15] (the complete proofs of Solonnikov result appeared in the Russian version of Ladyžhenskaya book [13]) and the proofs are based on accurate analysis of hydrodynamic potentials. See also Vorovich and Yudovich [19] for $q > 6/5$; a different approach is presented in Amrouche and Girault [3], with the aid of vector potentials. As usual, the case $q = 2$ can be handled without potential theory, see Solonnikov and Ščadilov [17], Constantin and Foias [8], and a recent overview about the history of the problem can be also found in Galdi [9, Ch. IV].

Dedicated to Hugo Beirão da Veiga on the occasion of his 70th birthday

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Moreover, Beirão da Veiga [4] introduced for the case $q = 2$ an elegant, ingenious, and extremely simple approach, which is based essentially only on $W^{2,2}(\Omega)$-estimates for the solution of scalar Poisson problems. Inspired by the latter reference (especially [4, § 4]) we will use the same technique to decouple the equation for the velocity from that for the pressure. Nevertheless, the extension to the non-Hilbertian setting and also a more general derivation of the *a priori* estimates are new (Our proof can also be easily extended to any space dimension $d \geq 2$ and to a non-zero velocity boundary datum).

2 Preliminary results and a priori estimates

One of the pre-requisites is the Banach contraction/fixed-point theorem, and it is interesting to note how following [4] a technique typical of non-linear functional analysis is used also in a linear context. The other pre-requisite is the following estimate for the scalar Poisson problem, see Simader and Sohr [14] for a proof, also in very general domains.

Lemma 1 *Let Ω be as in Theorem 1 and let be given $\mathcal{F} \in L^r(\Omega)$, for $1 < r < \infty$. Then, there exists $C_r = C(r, \Omega) > 0$ such that*

$$\|\Phi\|_{W^{2,r}} \leq C_r \|\mathcal{F}\|_{L^r}, \tag{3}$$

where Φ is the unique strong solution of the Dirichlet problem

$$\begin{cases} -\Delta\Phi = \mathcal{F} & \text{in } \Omega, \\ \Phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Remark 1 The estimate (3) is classical, at least for a bounded smooth domain, and follows from the theory of singular integrals. For $r = 2$, estimate (3) can be proved by Nirenberg's translations method, without resorting to Calderón-Zygmund estimates, see for instance the clear presentation in [6, §9].

To construct strong solutions to the Stokes system, we approximate –for $\lambda \in]0, 1]$– the Stokes system (1) by the "*penalty method*" [18] (called also "*numerical regularization*" [10]) with the system

$$\begin{cases} -\nu\Delta u + \nabla\pi = f & \text{in } \Omega, \\ \lambda\,\pi + \nabla\cdot u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4}$$

A peculiar use of system (4) comes from [4], but recall that approximation by the penalized system (4) has been also previously used in a different way in [3, §5]. As in [4], we further approximate system (4) by the following family of systems indexed by the parameter $t \in [0, 1]$

$$\begin{cases} -\nu\Delta u + t\,\nabla\pi = f & \text{in } \Omega, \\ \lambda\,\pi + t\,\nabla\cdot u = g & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

For $t = 1$ system (5) reduces to (4), while for $t = 0$ we have the two decoupled problems (observe also that each component u_i of the velocity solves a scalar Poisson problem)

$$\begin{cases} -\nu\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \lambda\,\pi = g \quad \text{in } \Omega.$$

For $t = 0$, by using Lemma 1 it follows that there exists a unique solution $(u, \pi) \in X_q$ to (5), such that

$$\nu\|u\|_{W^{2,q}} + \|\pi\|_{W^{1,q}} \leq C_q\|f\|_{L^q} + \frac{1}{\lambda}\|g\|_{W^{1,q}},$$

and this estimate cannot be uniform in λ, since π is obtained simply as $\frac{g(x)}{\lambda}$.

As first step (cf. [4, § 4]) we show that problem (5) is uniquely solvable for all $t \in [0, t_0]$, for some $t_0 \in]0, 1]$, which possibly depends on λ. To this end we consider the following problem: Given $(v, p) \in X_q$ find $(u, \pi) \in X_q$ such that

$$\begin{cases} -\nu\Delta u = f - t\,\nabla p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \text{and} \qquad \lambda\,\pi = g - t\,\nabla\cdot v \quad \text{in } \Omega. \tag{6}$$

Lemma 2 *There exists $t_0 > 0$ such that the map $X_q \ni (v,p) \mapsto (u,\pi) \in X_q$ is a strict contraction for all $t \in [0,t_0]$. Then, for all $t \in [0,t_0]$ there exists a unique solution $(u,\pi) \in X_q$ of (5), satisfying the following estimate:*

$$\|u\|_{W^{2,q}} + \|\nabla\pi\|_{L^q} \leq \frac{2C_q}{\nu}\|f\|_{L^q} + \frac{2}{\lambda}\|\nabla g\|_{L^q}.$$

Proof Let be given $(v_i, p_i) \in X_q$, for $i = 1,2$. We denote by (u_i, π_i) the corresponding solutions to problem (6). By using the equations satisfied by $(u_1 - u_2, \pi_1 - \pi_2)$ we obtain

$$\nu\|u_1 - u_2\|_{W^{2,q}} + \|\nabla\pi_1 - \nabla\pi_2\|_{L^q} \leq t\, C_q\|\nabla p_1 - \nabla p_2\|_{L^q} + \frac{t}{\lambda}\|v_1 - v_2\|_{W^{2,q}}.$$

Consequently, if

$$t \leq t_0 := \min\left\{\frac{\nu}{2C_q}, \frac{\lambda}{2}\right\}, \tag{7}$$

the map $(v,p) \mapsto (u,\pi)$ is a strict contraction, and its unique fixed point $(u,\pi) \in X_q$ satisfies

$$\begin{cases} -\nu\Delta u = f - t\,\nabla\pi & \text{in } \Omega, \\ \lambda\,\pi = g - t\,\nabla\cdot u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

By using again (3) this shows that, for all $t \in [0, t_0]$,

$$\|u\|_{W^{2,q}} + \|\nabla\pi\|_{L^q} \leq \frac{C_q}{\nu}\|f\|_{L^q} + \frac{t\,C_q}{\nu}\|\nabla\pi\|_{L^q} + \frac{1}{\lambda}\|g\|_{W^{1,q}} + \frac{t}{\lambda}\|u\|_{W^{2,q}},$$

and by using the bound on t_0 from (7) we get

$$\|u\|_{W^{2,q}} + \|\nabla\pi\|_{L^q} \leq \frac{C_q}{\nu}\|f\|_{L^q} + \frac{1}{\lambda}\|g\|_{W^{1,q}} + \frac{1}{2}\left(\|\nabla\pi\|_{L^q} + \|u\|_{W^{2,q}}\right),$$

concluding the proof.

Observe that estimates are still not uniform as $\lambda \to 0^+$. We will overcome this by proper *a priori* estimates on strong solutions of system (5).

2.1 A priori estimates

In order to further apply the contraction argument for all $t \in [0,1]$, we need some *a priori* estimates on solutions to (5). In [4, Eq. (4.3)] very precise estimates are obtained for $q = 2$ by the translation method. The same approach seems not replicable here, and to obtain suitable estimates we will verify that systems (1), (4), and (5) are all elliptic in the sense of Agmon, Douglis, and Nirenberg [2] (denoted ADN later on). We recall that the theory developed in [1,2] shows estimates, *provided that* the solution exists in appropriate spaces. Generally speaking, the most difficult part is that of showing such existence and one can refer to Temam [18, Ch. I], for a proof of existence for the two dimensional Stokes system.

The system (1) **is ADN.** We will mainly check that the boundary conditions in the Stokes system (1) are complementing [2, § 2]. This result seems very classical even if the available references (e.g. [18, Ch 1,§ 2], [3, § 4], and [16, § 3]) are not detailed enough for a non-expert reader and also for our purposes of proving similar estimates for the modified systems (4) and (5). On the other hand, full details on several first order formulations of (1) can be found in Bochev and Gunzburger [5, Appendix D] and for related systems see also Kozono and Yanagisawa [12]. For the reader's convenience, but also to clarify some delicate points about the algebraic conditions, we start by a meticulous inspection of system (1). For simplicity we assume $\nu = 1$, since estimates for different values of the viscosity can be obtained by scaling of the variables $U_i = u_i$ for $i = 1,2,3$, and $U_4 = \pi$. For any $P \in \Omega$ and $\Xi = (\xi_1, \xi_2, \xi_3) \in \mathbf{R}^3$ the matrix $l_{ij}(P,\Xi) = l_{ij}(\Xi)$, $i,j = 1,\dots,4$, associated with the system is the following:

$$l_{ij}(\Xi) := \begin{pmatrix} -|\Xi|^2 & 0 & 0 & \xi_1 \\ 0 & -|\Xi|^2 & 0 & \xi_2 \\ 0 & 0 & -|\Xi|^2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & 0 \end{pmatrix}.$$

With the weights $s_1 = s_2 = s_3 = 0$, and $s_4 = -1$; $t_1 = t_2 = t_3 = 2$, and $t_4 = 1$, it turns out that the principal part is $l'_{ij}(\Xi) = l_{ij}(\Xi)$ and that

$$|L(\Xi)| = |\det l'_{ij}(\Xi)| = |\Xi|^6,$$

showing that the system is *uniformly elliptic*, with constant of ellipticity $A = 1$, see [2, Eq. (1.7)]. (In space dimension $d \geq 3$ there is no need to check the *supplementary condition on* L, see [2, p. 39].) The boundary conditions associated to any $P \in \partial\Omega$ are expressed by the matrix $B_{hj}(P, \Xi) = B_{hj}(\Xi)$, for $h = 1, 2, 3$ and $j = 1, 2, 3, 4$

$$B_{hj}(\Xi) := \begin{pmatrix} 1\,0\,0\,0 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \end{pmatrix},$$

and, by choosing the weights $r_1 = r_2 = r_3 = 0$, and $r_4 = 1$, it turns out that $B'_{hj}(\Xi) = B_{hj}(\Xi)$.

Next, we check that the boundary conditions are *complementing*. We first evaluate the *adjoint* matrix $L^{jk}(\Xi) := (l')^{-1}_{jk}(\Xi)L(\Xi)$ (namely the transpose of the co-factor matrix of $l'_{ij}(\Xi)$, see [2, Eq. (3.2)])

$$L^{jk}(\Xi) = |\xi|^2 \begin{pmatrix} \xi_2^2 + \xi_3^2 & -\xi_1\,\xi_2 & -\xi_1\,\xi_3 & -\xi_1\,|\xi|^2 \\ -\xi_1\,\xi_2 & \xi_1^2 + \xi_3^2 & -\xi_2\,\xi_3 & -\xi_2\,|\xi|^2 \\ -\xi_1\,\xi_3 & -\,\xi_2\,\xi_3 & \xi_1^2 + \xi_2^2 & -\xi_3\,|\xi|^2 \\ -\xi_1\,|\xi|^2 & -\xi_2\,|\xi|^2 & -\xi_3\,|\xi|^2 & -|\xi|^4 \end{pmatrix}.$$

As in [2, Sec. 2], taking the exterior normal unit vector n at $P \in \partial\Omega$ and a tangent vector $\Xi \neq 0$, we have to check that the rows of the matrix $\sum_{j=1}^4 B'_{hj}(\Xi + \tau\, n)L^{jk}(\Xi + \tau\, n)$ are linearly independent (as polynomials in the complex variable τ), modulo the polynomial $M^+(\Xi, \tau)$ defined below by (8).

Without loss of generality, by means of translations and orthonormal transformations (the equations are invariant by these transformations), we can take $P = (0, 0, 0)$ $n = (0, 0, -1)$ and $\Xi = (|\Xi|, 0, 0)$. We also observe that since Ξ and n are orthogonal $|\Xi + \tau n|^2 = (\Xi + \tau n) \cdot (\Xi + \tau n) = |\Xi|^2 + \tau^2$, and the equation $L(\Xi + \tau\, n) = 0$ has three roots with positive imaginary part, all equal to $\tau^+ = i|\Xi|$, hence

$$M^+(\Xi, \tau) := (\tau - i|\Xi|)^3. \tag{8}$$

In this way we get

$$\sum_{j=1}^{4} B'_{hj}(\Xi + \tau n)L^{jk}(\Xi + \tau n) = (|\Xi|^2 + \tau^2) \begin{pmatrix} \tau^2 & 0 & \tau|\Xi| & -|\Xi|(|\Xi|^2 + \tau^2) \\ 0 & \Xi^2 + \tau^2 & 0 & 0 \\ \tau\,|\Xi| & 0 & |\Xi|^2 & \tau\,|\Xi|^2 \end{pmatrix},$$

If there exists constants C_1, C_2, C_3 such that $\sum_{h=1}^3 \sum_{j=1}^4 C_h B'_{hj}(\Xi + \tau n)L^{jk}(\Xi + \tau n) \equiv 0$ modulo $M^+(\Xi, \tau)$, and observing that the second column has only one non-vanishing term, then necessarily

$$C_2(|\Xi|^2 + \tau^2)^2 \equiv 0 \pmod{M^+(\Xi, \tau)}.$$

Since the polynomial (in τ) $(|\Xi|^2 + \tau^2)^2$ has only two roots equal to $i|\Xi|$, then $C_2 = 0$, because of the definition (8). Moreover, by considering the third column, we will also have to satisfy

$$(|\Xi|^2 + \tau^2)(C_1\tau|\Xi| + C_3|\Xi|^2) \equiv 0 \qquad \pmod{M^+(\Xi, \tau)},$$

but again this is possible only if $C_1 = C_3 = 0$, because on the left-hand side we can have at most two roots equal to $i|\Xi|$, one coming from the term $(|\Xi|^2 + \tau^2)$ and the other one existing for a suitable choice of the parameters C_1, C_2 in the first order polynomial $C_1\tau|\Xi| + C_3|\Xi|^2$. This shows that $C_1 = C_3 = 0$, proving that the rows are linearly independent, hence that the complementary condition is satisfied. This proves that *provided that* there exists a solution $(u, \pi) \in X_q$, then it satisfies the estimate (2).

The system (4) **is ADN.** Let us consider, for $\lambda \in]0, 1]$, system (4). The matrix associated is now the following

$$l_{ij}(\Xi) := \begin{pmatrix} -|\xi|^2 & 0 & 0 & \xi_1 \\ 0 & -|\xi|^2 & 0 & \xi_2 \\ 0 & 0 & -|\xi|^2 & \xi_3 \\ \xi_1 & \xi_2 & \xi_3 & \lambda \end{pmatrix}.$$

With the same weights as before $l'_{ij}(\Xi) = l_{ij}(\Xi)$ and

$$|L(\Xi)| = |\det l'_{ij}(\Xi)| = |\Xi|^6(1 + \lambda),$$

hence the system is still *uniformly elliptic*, with constant of ellipticity $A = 2$. Since $\lambda > 0$, then $M^+(\Xi, \tau)$ is the same as in (8). The adjoint matrix $L^{jk}(\Xi)$ is now:

$$L^{jk}(\Xi) = |\xi|^2 \begin{pmatrix} \xi_2^2 + \xi_3^2 + |\xi|^2 \lambda & -\xi_1 \xi_2 & -\xi_1 \xi_3 & -\xi_1 |\xi|^2 \\ -\xi_1 \xi_2 & \xi_1^2 + \xi_3^2 + |\xi|^2 \lambda & -\xi_2 \xi_3 & -\xi_2 |\xi|^2 \\ -\xi_1 \xi_3 & -\xi_2 \xi_3 & \xi_1^2 + \xi_2^2 + |\xi|^2 \lambda - \xi_3 |\xi|^2 \\ -\xi_1 |\xi|^2 & -\xi_2 |\xi|^2 & -\xi_3 |\xi|^2 & -|\xi|^4 \end{pmatrix}.$$

We take the same vectors Ξ and n as before, the matrix $B_{hj}(\Xi)$ is unchanged as well, and it follows:

$$\sum_{j=1}^{4} B'_{hj}(\Xi + \tau n) L^{jk}(\Xi + \tau n)$$
$$= (|\Xi|^2 + \tau^2) \cdot \begin{pmatrix} \tau^2 + \lambda (|\Xi|^2 + \tau^2) & 0 & |\Xi| \, \tau & -|\Xi| \, (|\Xi|^2 + \tau^2) \\ 0 & (1 + \lambda)(|\Xi|^2 + \tau^2) & 0 & 0 \\ |\Xi| \, \tau & 0 & \Xi^2 + \lambda (|\Xi|^2 + \tau^2) & \tau \, |\Xi|^2 \end{pmatrix}.$$

We prove that the rows are linearly independent. In fact C_2 must be zero exactly by the same argument as before. Next if $(|\Xi|^2 + \tau^2)\big(C_1 \tau |\Xi| + C_3(\Xi^2 + \lambda (|\Xi|^2 + \tau^2))\big) \equiv 0$, then

$$C_1 \tau |\Xi| + C_3(\Xi^2 + \lambda (|\Xi|^2 + \tau^2)) \equiv (\tau - i|\Xi|)^2 \qquad \big(\text{mod } M^+(\Xi, \tau)\big),$$

which cannot hold for any choice of C_1, C_3, since the left hand side should have two coincident roots equal to $i|\Xi|$ and this is impossible. This shows that *provided that* $(u, \pi) \in X_q$ is a solution to (4), then (2) holds true (This result is stated without proof also in [3, §5]). In particular, estimate (2) is independent of λ since either the ellipticity constant or the roots of $L(\Xi + \tau n)$ do not depend on it. As in [1, 2] the representation's formula (by means of the Poisson kernel and simple-wave expansions from [11]) shows that the constant depend mainly on the ellipticity constant, together with the order of the operator, the space dimension, and the domain Ω, see [1, p. 652].

The system (5) **is ADN.** For $\lambda \in]0, 1]$ and $t \in [0, 1]$ the matrix associated with the system (5) is

$$l_{ij}(\Xi) := \begin{pmatrix} -|\xi|^2 & 0 & 0 & \xi_1 t \\ 0 & -|\xi|^2 & 0 & \xi_2 t \\ 0 & 0 & -|\xi|^2 & \xi_3 t \\ \xi_1 t & \xi_2 t & \xi_3 t & \lambda \end{pmatrix}.$$

With the same weights as before $l'_{ij}(\Xi) = l_{ij}(\Xi)$ and $|L(\Xi)| = |\det l'_{ij}(\Xi)| = |\Xi|^6 (t^2 + \lambda)$, hence the system is uniformly elliptic, for fixed $t, \lambda \in]0, 1]$, but the constant of ellipticity is *not uniform* with respect to $t, \lambda \in]0, 1]$. We recall that later on we will need to use the *a priori* estimates only for $t \in [t_0, 1]$, where $t_0 > 0$ is that from Lemma 2 and consequently

$$|L(\Xi)| = |\det l'_{ij}(\Xi)| \geq |\Xi|^6 \, t_0^2 \qquad \forall \, \lambda \in]0, 1] \text{ and } \forall \, t \in [t_0, 1].$$

Hence, with this restricted set of parameters the system is *uniformly elliptic* with constant $A = \max \{2, 1/t_0^2\}$. The adjoint matrix $L^{jk}(\Xi)$ is

$$L^{jk}(\Xi) = |\xi|^2 \begin{pmatrix} (\xi_2^2 + \xi_3^2) t^2 + |\xi|^2 \lambda & -\xi_1 \xi_2 t^2 & -\xi_1 \xi_3 t^2 & -\xi_1 \, t |\xi|^2 \\ -\xi_1 \xi_2 t^2 & (\xi_1^2 + \xi_3^2) t^2 + |\xi|^2 \lambda & -\xi_2 \xi_3 t^2 & -\xi_2 |\xi|^2 t \\ -\xi_1 \xi_3 t^2 & -\xi_2 \xi_3 t^2 & (\xi_1^2 + \xi_2^2) t^2 + |\xi|^2 \lambda - \xi_3 |\xi|^2 t \\ -\xi_1 |\xi|^2 t & -\xi_2 |\xi|^2 t & -\xi_3 |\xi|^2 t & -|\xi|^4 \end{pmatrix}.$$

The complementing conditions with the same matrix $B_{hj}(\Xi)$ concerns the rows of

$$\sum_{j=1}^{4} B'_{hj}(\Xi + \tau n) L^{jk}(\Xi + \tau n)$$
$$= (|\Xi|^2 + \tau^2) \cdot \begin{pmatrix} t^2 \tau^2 + \lambda (|\Xi|^2 + \tau^2) & 0 & t^2 |\Xi| \, \tau & -t |\Xi| \, (|\Xi|^2 + \tau^2) \\ 0 & (t^2 + \lambda)(|\Xi|^2 + \tau^2) & 0 & 0 \\ t^2 |\Xi| \, \tau & 0 & t^2 \Xi^2 + \lambda (|\Xi|^2 + \tau^2) & t \, \tau \, |\Xi|^2 \end{pmatrix}.$$

Exactly the same calculations as before show that the rows are linearly independent modulo $M^+(\Xi,\tau)$ (still the same as in (8)). This proves the following result: *Provided that* $(u,\pi)\in X_q$ *is a solution to* (5), for some $t\in[t_0,1]$, then there exists $C(t_0,q,\Omega)>0$ such that

$$\nu\,\|u\|_{W^{2,q}}+\|\nabla\pi\|_{L^q}\le C(t_0,q,\Omega)(\|f\|_{L^q}+\|\nabla g\|_{L^q}). \tag{9}$$

Remark 2 The reader should be warned that the constant t_0 depends on λ, hence it may look that (9) is not suitable to get uniform estimates from the regularization/approximation by means of system (5). This is overcome by first fixing some $\lambda>0$ and allowing the parameter t to span $[t_0,1]$. Next, having proved existence of a unique solution to (5) for $t=1$ (which turns out to be penalized Stokes system (4)), we can use the *a priori* estimates for (4), which are independent of λ.

3 Proof of the main result

Following closely [4], we show that if we are able to solve (5) for some $\bar t\ge t_0$, then we are able to solve it for all $t\in[t_0,1]$.

Lemma 3 *Let us assume that the problem* (5) *is solvable in* X_q *for some* $\bar t\ge t_0>0$. *Then, there exists* $\delta>0$, *independent of* $\bar t$, *such that* (5) *is solvable in* X_q *for all* $t\in[\bar t,\bar t+\delta]$

Proof Let us consider, for given $(v,p)\in X_q$ and $\Delta t\ge 0$, the system

$$\begin{cases} -\nu\Delta u+\bar t\,\nabla\pi=f-\Delta t\ \nabla p & \text{in } \Omega,\\ \lambda\,\pi+\bar t\,\nabla\cdot u=g-\Delta t\ \nabla\cdot v & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases} \tag{10}$$

By hypothesis system (10) is uniquely solvable in X_q and, by using the *a priori* estimate (9), we show that there exists $\delta>0$ such that the map $(v,p)\mapsto(u,\pi)$ is a contraction if $0\le\Delta t\le\delta$. In fact, given $(v_i,p_i)\in X_q$, then $(u_i,\pi_i)\in X_q$ are solutions to (10), satisfying the estimate (9). From the equation for the difference, we get

$$\nu\|u_1-u_2\|_{W^{2,q}}+\|\nabla\pi_1-\nabla\pi_2\|_{L^q}\le C(t_0,q,\Omega)\,\delta\,\left(\|\nabla p_1-\nabla p_2\|_{L^q}+\|v_1-v_2\|_{W^{2,q}}\right).$$

By choosing

$$\delta:=\frac{\nu}{2C(t_0,q,\Omega)}$$

we have again a strict contraction, hence we can uniquely solve (5) for $t\in[\bar t,\bar t+\delta]$.

Proof (Proof of Theorem 1) Let be given $\lambda\in]0,1]$. By Lemma 2 there exists $t_0=t_0(\lambda)$, such that (5) is solvable for all $t\in[0,t_0]$. By Lemma 3 there exists $\delta=\delta(t_0)>0$, such that, we can solve (5) for $t\in[t_0,t_0+m\,\delta]$, for all $m\in\mathbf{N}$. This implies that with a *finite* number of steps we can solve problem (5) for all $t\in[0,1]$ (the number of steps depends on t_0, hence on λ). In particular, we constructed $(u_\lambda,\pi_\lambda)\in X_q$, the unique solution of (5) for $t=1$ (and it is denoted in this way to emphasize the λ-dependence). By using the *a priori* estimate (2) (which is valid since system (5) reduces to (4) when $t=1$), we obtain

$$\nu\|u_\lambda\|_{W^{2,q}}+\|\nabla\pi_\lambda\|_{L^q}\le c(q,\Omega)\left(\|f\|_{L^q}+\|\nabla g\|_{L^q}\right).$$

For $1<q<\infty$ the space X_q is reflexive, hence there exists a couple $(u,\pi)\in X_q$ and a positive sequence $\{\lambda_n\}_n$, such that $\lim_{n\to+\infty}\lambda_n=0$, with

$$(u_{\lambda_n},\pi_{\lambda_n})\rightharpoonup(u,\pi)\quad\text{weakly in } X_q.$$

By multiplying (5) by $(\phi,\psi)\in W_0^{1,q'}(\Omega)\times M_{q'}$, by integrating by parts, and by taking the limit over n, it follows that the couple $(u,\pi)\in X_q$ satisfies

$$\nu\int_\Omega\nabla u\nabla\phi\,dx-\int_\Omega\pi\,\nabla\cdot\phi\,dx+\int_\Omega\nabla\cdot u\,\psi\,dx=\int_\Omega f\phi\,dx+\int_\Omega g\,\psi\,dx$$

and $(u,\pi)\in X_q$ is then a weak solution to (1). By lower semi-continuity of the norm it also follows that (2) holds true. This ends the proof since (u,π) is a weak solution with the requested regularity, hence it is unique, see e.g. [9, Ch. IV.6].

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