

Womersley Flow of Generalized Newtonian Liquid

Giovanni P. Galdi

University of Pittsburgh,
630 Benedum Engineering Hall, Pittsburgh, PA 15621, USA
(galdi@pitt.edu)

Carlo R. Grisanti

Dipartimento di Matematica, Università di Pisa,
Via Buonarroti 1/c, 56127 Pisa, Italy
(grisanti@dma.unipi.it)

(MS received ;)

We show that in an infinite straight pipe of arbitrary (sufficiently smooth) cross section, a generalized non-Newtonian liquid admits one and only one fully developed time-periodic flow (*Womersley flow*), either when the flow rate (Problem 1) or the axial pressure gradient (Problem 2) is prescribed in analogous time-periodic fashion. In addition, we show that the relevant solution depends continuously upon the data in appropriate norms. As is well known from the Newtonian counterpart of the problem [3, 8], the latter is pivotal for the analysis of flow in a general unbounded pipe system with cylindrical outlets (*Leray problem*). It is also worth remarking that Problem 1 possesses an intrinsic interest from both mathematical and physical viewpoints, in that it constitutes a (nonlinear) inverse problem with a significant bearing on several applications, including blood flow modeling in large arteries.

1. Introduction

The motion of a viscous liquid in a bent tube or in a pipe system is among the classical problems in fluid dynamics; see, e.g., [2, Chapter 4]. Because of their basic role in applied science, particular importance is given to flows of this kind when the field variables (velocity and pressure) are independent of time (steady-state) or, more generally, time-periodic. Noteworthy applications of time-periodic and/or pulsatile¹ pipe flow can be found in the modeling of blood flow in large arteries [14, 15]. Other significant applications regard mass transportation in pipes by a pulsatile pumping driving mechanism [19].

Despite its undoubted relevance, a rigorous and systematic mathematical study of the flow of a viscous liquid in an unbounded pipe system (by the mathematical community often referred to as *Leray problem*) has begun only a few decades ago with the work of Amick [1], Ladyzhenskaya & Solonnikov [11], and Pileckas [13], for liquids described by the Navier-Stokes equations (Newtonian liquids). It must be remarked, however, that these papers deal with the steady-state case only, whereas the more general time-periodic case was virtually left untouched until recently, when it was finally analyzed to a full extent first by Beirão da Veiga [3] and, successively, Galdi & Robertson [10]; see also [8, Chapter I].

¹Namely, a time-periodic flow superimposed to a steady-state one.

As is well known, the crucial ingredient in the study of these motions is the knowledge of the mathematical properties of the so called *fully developed flow* (in the following referred to as FDF) [3, 10, 8]. We recall that FDF occurs in an infinite pipe of constant cross-section, Σ , and is characterized by the property that the velocity field is parallel to the axis of the pipe, and depends only on the cross-sectional variables and time. In fact, the solution to the general Leray problem is found as a “perturbation” around the appropriate FDF, and for this reason continuous dependence on the data of FDF in appropriate norms is of the utmost importance. In the case when the liquid is Newtonian (described by the Navier-Stokes equations), time-periodic FDF is often referred to as *Womersley flow*, after J.R. Womersley who, in a hemodynamics context, first proved the existence of such a motion when Σ is a circle and the time-periodic driving mechanism (the axial pressure gradient, that is) has a finite number of active modes [20].² In this regard, we observe that FDF can be driven either by assigning the axial pressure gradient, Γ (which we call Problem 2), or else the total flow-rate, α , through the cross-section (Problem 1). Problem 2 is a *direct one* and it is readily solvable. For example, in the classical (Newtonian) Navier-Stokes liquid model, it reduces to find time-periodic solutions to the standard heat equation with a *prescribed* time-periodic forcing term Γ . Of course, once the solution is found, α is determined as well. On the other hand, Problem 1 is of *inverse type*, and, as such, less trivial. Actually, even for the simpler Newtonian liquid, it can be viewed as a heat-conducting problem where the distribution of temperature *and* the forcing term must be determined only from the knowledge of the average temperature value, α , on Σ .

The main objective of this paper is to prove existence, uniqueness and continuous data dependence of solutions to both Problems 1 and 2 in the case of generalized Newtonian liquid, thus extending, in particular, Womersley’s results to more general liquid models.

When dealing with such models, the problem of determining time-periodic fully developed flow is more complicated than in the Navier-Stokes case, due to the circumstance that the relevant equations are no longer linear. As a consequence, the methods used in [3, 10] to prove existence, uniqueness and continuous data dependence of time-periodic fully developed flow no longer apply, and we have to resort to a different approach. In this respect, we wish to observe that, even though the more challenging problem is when the flow-rate is prescribed (i.e., Problem 1), also the direct problem, namely, when the pressure gradient is assigned (Problem 2), presents some difficulty related to the nonlinear character of the equation. However, since the technical details in the proofs are similar, we shall mainly focus on the resolution of Problem 1 (see theorems 5.1 and 6.1), whereas we limit ourselves to state the main results for Problem 2 (see theorems 7.2 and 7.3), and briefly sketch its proof.

Our results are proved for a generalized Newtonian liquid satisfying very general and standard assumptions (see Section 2). In particular, in Section 5 we shall treat the “shear thinning case”, namely, when the shear viscosity of the liquid is a decreasing function of the magnitude of the stretching tensor, while in Section 6 we

²A similar problem, but in different setting, was already considered by T. Sestl [17], with the aim of explaining the so-called Richardson annular effect, which consists of the presence of a velocity overshoot near to the wall of the pipe.

investigate the same problem for a “shear thickening” liquid, where, instead, the viscosity is an increasing function.

The approach we follow relies upon the following two points. In the first place, we begin to suitably reformulate Problem 1, in such a way that it becomes equivalent to an appropriate *direct* problem (Section 4). This being established, we then provide the existence of time-periodic weak solutions by coupling the Galerkin method with the theory of monotone operators (basically, the “Minty’s trick”).³ Though based on classical tools, the implementation of our approach to existence (namely, the proof of appropriate uniform estimates on the approximating solutions) requires, however, a further effort. In fact, we are able to show that the weak solutions we obtain have the additional properties of being differentiable in time, and of having enstrophy that is uniformly bounded in time. These properties are then used to prove *continuous dependence upon the prescribed flow rate* in appropriate norm. As emphasized a few times earlier on, the latter is pivotal for the resolution of the (time-periodic) Leray problem in an unbounded pipe system with cylindrical outlet extending to infinity, a question that will be the object of future work.

The plan of the paper is the following. After introducing, in Section 2, the main notation along with the basic assumptions on the Cauchy stress tensor, in Section 3 we collect a number of preparatory and known results that will be needed further on. In Section 4 we reformulate Problem 1 in such a way that it becomes equivalent to a direct problem, and give the definition of corresponding time-periodic weak solutions. Successively, in Section 5, we prove the existence of unique weak solutions to Problem 1, along with their continuous data dependence in the shear-thinning case. The same results for the shear-thickening case are proved in Section 6. Finally, in Section 7, we briefly treat the well posedness of Problem 2.

2. Notation

By Σ we denote a connected, bounded subset of the plane \mathbb{R}^2 , with Lipschitz boundary. *All functions considered throughout are time periodic with period T .* Consequently, we shall restrict our attention to the time-interval $[0, T]$. We set

$$C_{0,per}^\infty = \{ \phi \in C_0^\infty(\Sigma \times [0, T]) : \phi(x, 0) = \phi(x, T) \forall x \in \Sigma \}$$

$$\mathcal{V} = \left\{ \psi \in C_0^\infty(\Sigma) : \int_\Sigma \psi(x) dx = 0 \right\}$$

$$\mathcal{V}_T = \left\{ \phi \in C_{0,per}^\infty : \int_\Sigma \phi(x, t) dx = 0, \forall t \in [0, T] \right\}$$

$$W_p = \left\{ w \in W_0^{1,p}(\Sigma) : \int_\Sigma w(x) = 0 \right\} \quad p \geq 1$$

For any $p > 1$, $p' = \frac{p}{p-1}$ will be the conjugate exponent of p . With the symbol v' we will denote differentiation with respect to the time variable t . We will also

³The reader interested to a general overview on parabolic evolution equations involving monotone operators, is referred to the classical literature on the topic [12, Chapitre 2], [6, Chapter 3].

make use of the space of distributions on the time interval $(0, T)$ with values in a Banach space X , indicated by $\mathfrak{D}'(0, T; X)$.

With (\cdot, \cdot) we will denote the usual L^2 scalar product with respect to the space variable

$$(\phi, \psi) = \int_{\Sigma} \phi(x)\psi(x) dx,$$

and with $\langle \cdot, \cdot \rangle$ the duality pairing between Banach spaces.

For any tensors $A, B \in \mathbb{R}^{3 \times 3}$ we set

$$A : B = A_{ij}B_{ij} \quad \text{and} \quad |A| = |A : A|^{1/2}.$$

The nonlinear part of the viscosity term will be described by a function $\mathbf{S} : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ of the form

$$\mathbf{S}(A) = h(|A|^2)A \tag{2.1}$$

with

$$h : [0, +\infty) \rightarrow [0, +\infty), \quad h \in C((0, +\infty)). \tag{2.2}$$

We require \mathbf{S} to be continuous and to satisfy suitable coercivity, growth and monotonicity conditions (see [8] II.(2.9) - II.(2.11)). Precisely, for any $A, B \in \mathbb{R}^{3 \times 3}$

$$\mathbf{S}(A) : A \geq \kappa_1 |A|^p - k_2, \quad \kappa_1 > 0, \quad k_2 \in \mathbb{R}, \quad p > 1 \tag{2.3}$$

$$|\mathbf{S}(A)| \leq \kappa_3 (|A|^{p-1} + 1), \quad \kappa_3 > 0, \quad p > 1 \tag{2.4}$$

$$(\mathbf{S}(A) - \mathbf{S}(B)) : (A - B) \geq 0 \quad \forall A, B. \tag{2.5}$$

For any vector valued function $u : \mathbb{R}^3 \rightarrow \mathbb{R}^3$

$$Du = \frac{1}{2}(\nabla u + (\nabla u)^T)$$

denotes its symmetric gradient.

Throughout the paper $\alpha(t)$ will be a scalar function in $W^{1,2}((0, T))$, and χ a fixed function such that

$$\chi \in C_0^\infty(\Sigma), \quad \int_{\Sigma} \chi(x) dx = 1. \tag{2.6}$$

Finally, for sake of simplicity, we shall not relabel subsequences.

3. Auxiliary results

In this section we collect some basic properties concerning the functions spaces we will use. They are well known, especially in the framework of fluid mechanics problems. However, they are usually formulated for divergence free functions. Instead, we will need them in a formulation involving time-periodic functions with zero average.

Lemma 3.1. *For any $p > 1$, W_p is a reflexive and separable Banach space. Moreover W_p is continuously and densely embedded in $L^2(\Sigma)$ for any $p \geq 1$.*

Proof. The proof follows easily because W_p is a closed subspace of $W_0^{1,p}(\Sigma)$ and from the Sobolev embedding theorem since $n=2$. \square

Lemma 3.2. *If $u \in L^p(0, T; W_p(\Sigma))$ and $u' \in L^p(0, T; W_p'(\Sigma))$ then, up to a subset of zero Lebesgue measure in $[0, T]$, $u \in C([0, T]; L^2(\Sigma))$.*

Proof. See [12, Remarque II.1.2]. \square

Lemma 3.3. *For any $p \geq 1$ there exists a sequence $\{V_k\}$ of finite dimensional subspaces of W_p such that $V_k \subset V_{k+1}$ for any $k \in \mathbb{N}$ and $\bigcup_{k \in \mathbb{N}} V_k$ is dense in W_p .*

Proof. The proof follows straightforward from the separability of W_p . \square

4. The fully developed periodic flow

In this section we derive the relevant equations of a fully developed time-periodic flow of a generalized Newtonian liquid model specified by the properties (2.1)-(2.5), and will provide the corresponding weak (distributional) formulation.

Let $\Omega = \mathbb{R} \times \Sigma$ be the cylindrical pipe of infinite length occupied by the fluid. We choose the coordinate system in such a way that Σ lies in the $\{x_2, x_3\}$ plane. The generic time-periodic motion of the liquid is then characterized by velocity field V and pressure π satisfying the following set of equations

$$\begin{cases} V' + V \cdot \nabla V = \mu_0 \Delta V + \nabla \cdot \mathbf{S}(DV) - \nabla \pi & \text{in } \Omega \times [0, T] \\ V(x, t) = 0 & \text{in } \partial\Omega \times [0, T] \\ \int_{\Sigma} V(\cdot, t) \cdot n \, d\sigma = \alpha(t) & \text{in } [0, T] \\ \nabla \cdot V = 0 & \text{in } \Omega \times [0, T] \\ V(\cdot, 0) = V(\cdot, T), \quad \pi(\cdot, 0) = \pi(\cdot, T) \end{cases} \quad (4.1)$$

where n is a unit normal vector to the (cross-section) Σ , and $\alpha(t) \in W^{1,2}$ is the prescribed T -periodic flow rate.

In accordance with the definition of fully developed flow, the velocity profile has to be invariant by translations along the axis $\mathbf{a} \parallel x_1$ of the pipe and directed along it, while the pressure gradient is parallel to \mathbf{a} and may depend on time only. Namely,

$$V(x, t) = v(x_2, x_3, t)e_1, \quad \pi(x, t) = -\Gamma(t)x_1. \quad (4.2)$$

With these restrictions, the convective term $V \cdot \nabla V$ vanishes identically, while the zero-divergence constraint is always satisfied. As far as the nonlinear contribution to the dissipative term, we observe that the only non-zero components of the symmetric gradient DV are

$$(DV)_{21} = (DV)_{12} = \frac{1}{2}\partial_2 v, \quad (DV)_{31} = (DV)_{13} = \frac{1}{2}\partial_3 v.$$

As a result,

$$|DV| = \frac{1}{\sqrt{2}}|\tilde{\nabla}v|$$

where the tilde denotes differentiation with respect to the variables x_2 and x_3 only. This, in turn implies by a straightforward calculation that the only non-zero component of the vector $\nabla \cdot \mathbf{S}(DV)$ is the first one, so that setting

$$h(s) = \frac{1}{2} h\left(\frac{s}{2}\right) \quad (4.3)$$

we get at once

$$[\nabla \cdot \mathbf{S}(DV)]_1 = \tilde{\nabla} \cdot (h(|\tilde{\nabla} v|^2) \tilde{\nabla} v).$$

Now we can define a new function $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$S(a) = h(|a|^2)a. \quad (4.4)$$

It is not difficult to verify that the function S has the same properties of \mathbf{S} . Indeed, by (2.3), (2.4) and (2.5) we have that, for any $a, b \in \mathbb{R}^2$

$$S(a) \cdot a \geq \kappa_1 2^{-p/2} |a|^p - k_2 := k_1 |a|^p - k_2, \quad (4.5)$$

$$|S(a)| \leq \frac{\kappa_3}{\sqrt{2}} \left(|a|^{p-1} 2^{\frac{1-p}{2}} + 1 \right) \leq \frac{\kappa_3}{\sqrt{2}} (|a|^{p-1} + 1) := k_3 (|a|^{p-1} + 1), \quad (4.6)$$

$$(S(a) - S(b)) : (a - b) \geq 0 \quad \forall a, b. \quad (4.7)$$

In the following lemma we prove some elementary, but crucial, properties related to the coercivity and the growth of S which will be useful in the following sections.

Lemma 4.1. *Let be $p > 1$ and h as in (4.3). Setting $H(s) = \int_0^s h(\tau) d\tau$ for any $s \geq 0$, it results*

$$0 \leq H(s) \leq 2k_3 \left(\frac{s^{\frac{p}{2}}}{p} + s^{\frac{1}{2}} \right). \quad (4.8)$$

Moreover, if $k_2 = 0$ in the coercivity condition (2.3), then

$$H(s) \geq \frac{2k_1}{p} s^{\frac{p}{2}}.$$

Proof. We remark that for any $t > 0$, setting $a = \begin{pmatrix} \sqrt{t} \\ 0 \end{pmatrix}$ in inequality (4.7), we get that

$$0 \leq h(t) \leq k_3 \left(t^{\frac{p-2}{2}} + t^{-\frac{1}{2}} \right).$$

Since $p > 1$ the function H is well defined and $H'(s) = h(s)$ for any $s > 0$. Moreover, integrating the above inequality it readily follows that, for any $s \geq 0$

$$0 \leq H(s) \leq 2k_3 \left(\frac{s^{\frac{p}{2}}}{p} + s^{\frac{1}{2}} \right).$$

Concerning the estimate from below if $k_2 = 0$, by (4.5)

$$S(a) \cdot a \geq k_1 |a|^p$$

which, with the above choice for a , provides

$$h(t) \geq k_1 t^{\frac{p-2}{2}}$$

and

$$H(s) \geq \int_0^s k_1 \tau^{\frac{p-2}{2}} d\tau = \frac{2k_1}{p} s^{\frac{p}{2}}.$$

□

For the sake of simplicity, and with a little abuse of notation, in the rest of the paper we will drop the superscript \sim , and set $x = (x_2, x_3)$.

Thus, from all the above, we conclude that fully developed time-periodic flow of our shear-thinning model is governed by the following equations

$$\begin{cases} v' = \mu_0 \Delta v + \nabla \cdot S(\nabla v) + \Gamma(t) & \text{in } \Sigma \times [0, T] \\ v(x, t) = 0 & \text{on } \partial\Sigma \times [0, T] \\ \int_{\Sigma} v(x, t) dx = \alpha(t) & \text{in } [0, T] \\ v(\cdot, 0) = v(\cdot, T), \quad \Gamma(0) = \Gamma(T), \end{cases} \quad (4.9)$$

where v and Γ are the scalar functions defined in (4.2), whereas S is a vector function satisfying conditions (4.5)-(4.7).

Objective of this paper is to prove that for an arbitrarily given time-periodic function α in a suitable class, problem (4.9) has one and only one corresponding time-periodic distributional solution (v, Γ) . To this end, we begin to give a weak formulation of (4.9), where $(4.9)_1$ is projected on the space of zero mean value functions. In this way, we eliminate the pressure gradient from the equation and reduce the problem to finding only the unknown velocity field; see remark 4.3.

Definition 4.2. *We say that v is a weak solution of the problem (4.9) if $v \in L^2(0, T; W_0^{1,2}(\Sigma)) \cap L^p(0, T; W_0^{1,p}(\Sigma)) \cap L^\infty(0, T; L^2(\Sigma))$ and*

$$\begin{cases} \int_0^T (v, \phi') - (\mu_0 \nabla v + S(\nabla v), \nabla \phi) dt = 0 \quad \forall \phi \in \mathcal{V}_T \\ \int_{\Sigma} v(x, t) dx = \alpha(t) & \text{for a.e. } t \in [0, T] \\ v(x, 0) = v(x, T) & \text{for a.e. } x \in \Sigma \end{cases} \quad (4.10)$$

Remark 4.3. *If v is a weak solution then, if $1 < p < 2$*

$$v' = \mu_0 \Delta v + \nabla \cdot S(\nabla v) \quad \text{in } \mathfrak{D}'(0, T; W_2')$$

and $v' \in L^2(0, T; W_2')$. In the case $p > 2$ we have that

$$v' = \mu_0 \Delta v + \nabla \cdot S(\nabla v) \quad \text{in } \mathfrak{D}'(0, T; W_p')$$

and $v' \in L^{p'}(0, T; W_p')$. However, in both cases, by lemma 3.2, v can be modified on a set of null Lebesgue measure in $[0, T]$ in such a way that $v \in C([0, T]; L^2(\Sigma))$. In principle, this is not enough to deduce the existence of the pressure gradient, even in

the distribution sense. However, if v is slightly smoother, say, $v' \in L^2(0, T; W_0^{-1,2})$ if $1 < p < 2$ or $v' \in L^{p'}(0, T; W_0^{-1,p'})$ if $p > 2$, then there exists a time-periodic function $\Gamma \in L^2(0, T)$, respectively $\Gamma \in L^{p'}(0, T)$, such that

$$\int_0^T (v, \psi') - (\mu_0 \nabla v + S(\nabla v), \nabla \psi) + (\Gamma, \psi) dt = 0 \quad \forall \psi \in C_{0,per}^\infty. \quad (4.11)$$

In fact, let $\psi \in C_{0,per}^\infty$ and set $\phi = \psi - \chi \int_\Sigma \psi$. Replacing such a ϕ in (4.10)₁, and taking into account the periodicity of v and ψ we show, after simple manipulation, that (4.11) holds with

$$\Gamma := \langle v', \chi \rangle + (\mu_0 \nabla v + S(\nabla v), \nabla \chi), \quad (4.12)$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between $W_0^{1,p}(\Sigma)$ and $W^{-1,p'}(\Sigma)$. It is easy to see that Γ is independent of the particular choice of χ . In fact, let χ_1 be another such a function, and let θ be arbitrary in $C_0^\infty((0, T))$. Then, we may take $\phi = \theta(\chi - \chi_1)$ in (4.10)₁ and show, by the arbitrariness of θ , and with $\xi := \chi - \chi_1$

$$\int_{t_0}^t (\mu_0 \nabla v + S(\nabla v), \nabla \xi) d\tau = -\langle v(t) - v(t_0), \xi \rangle, \quad 0 < t_0 \leq t < T,$$

from which, after differentiation with respect to t , the desired property follows. Clearly, if v is even more regular, from (4.11) we may prove that (v, Γ) obeys (4.9)₁ in the ordinary sense.

Next, we wish to make a suitable lift of the flux condition (4.9)₃. Actually, setting

$$u(x, t) = v(x, t) - \alpha(t)\chi(x), \quad (4.13)$$

we find $\int_\Sigma u(x, t) dx = 0$ for any $t \in [0, T]$ and that (u, Γ) solves the problem

$$\begin{cases} u' = \mu_0 \Delta u + \nabla \cdot S(\nabla u + \alpha \nabla \chi) + \mu_0 \alpha \Delta \chi - \alpha' \chi + \Gamma(t) & \text{in } \Sigma \times [0, T] \\ u(x, t) = 0 & \text{on } \partial \Sigma \times [0, T] \\ u(x, 0) = u(x, T) \end{cases} \quad (4.14)$$

if and only if (v, Γ) solves (4.9). Therefore, in analogy with definition 4.2, we give the following

Definition 4.4. We say that u is a weak solution of the problem (4.14) if $u \in L^2(0, T; W_2) \cap L^p(0, T; W_p) \cap L^\infty(0, T; L^2(\Sigma))$ and

$$\begin{cases} \int_0^T (u, \phi') - (\mu_0 \nabla u + S(\nabla u + \alpha \nabla \chi) + \mu_0 \alpha \nabla \chi, \nabla \phi) - (\alpha' \chi, \phi) dt = 0 \quad \forall \phi \in \mathcal{V}_T \\ u(x, 0) = u(x, T) \end{cases} \quad \text{for a.e. } x \in \Sigma$$

Remark 4.5. v is a weak solution of the problem (4.9) if and only if u is a weak solution of the problem (4.14).

Remark 4.6. If u is a weak solution of the problem (4.14) then, by remark 4.3, we may redefine u on a set of zero Lebesgue measure in $[0, T]$ so that $u \in C([0, T]; L^2(\Sigma))$. This makes clear the way in which the periodicity condition (4.14)₃ should to be understood.

In order to deal with the above problem, for any $t \in [0, T]$ we define the following functionals

$$A(t, \cdot) : W_2 \longrightarrow W'_2, \quad A(t, v) = -\mu_0 \Delta v - \nabla \cdot S(\nabla v + \alpha(t) \nabla \chi), \quad 1 < p < 2, \quad (4.15)$$

$$B(t, \cdot) : W_p \longrightarrow W'_p, \quad B(t, v) = -\mu_0 \Delta v - \nabla \cdot S(\nabla v + \alpha(t) \nabla \chi), \quad p > 2. \quad (4.16)$$

Proposition 4.7. A and B are strictly monotone, coercive and satisfy the following growth conditions uniformly for any $t \in [0, T]$

$$\|A(t, v)\|_{W'_2} \leq ck_3 \left(\|\alpha\|_{1,2} \|\nabla \chi\|_2 + |\Sigma|^{1/2} \right) + (ck_3 + \mu_0) \|\nabla v\|_2, \quad (4.17)$$

$$\|B(t, v)\|_{W'_p} \leq c(\mu_0 + k_3) \|\nabla v\|_p^{p-1} + c(\mu_0 + k_3) |\Sigma|^{\frac{p-1}{p}} + ck_3 \|\alpha\|_{1,2}^{p-1} \|\nabla \chi\|_p^{p-1}. \quad (4.18)$$

Proof. Let be $v_1, v_2 \in W_2$. We have, by (4.7)

$$\langle A(t, v_1) - A(t, v_2), v_1 - v_2 \rangle \geq \mu_0 \|\nabla v_1 - \nabla v_2\|_2^2$$

hence A is strictly monotone. The same assertion holds true for B .

By (4.7) with $b = 0$ we get that $S(\nabla v + \alpha(t) \nabla \chi), \nabla v + \alpha(t) \nabla \chi \geq 0$ and by (4.6), if $1 < p < 2$

$$\begin{aligned} |(S(\nabla v + \alpha \nabla \chi), \alpha \nabla \chi)| &\leq \int_{\Sigma} k_3 (|\nabla v + \alpha \nabla \chi|^{p-1} + 1) |\alpha \nabla \chi| dx \\ &\leq k_3 \left(\|\nabla v + \alpha \nabla \chi\|_2^{p-1} \|\alpha \nabla \chi\|_{\frac{2}{3-2p}} + \|\alpha\|_{\infty} \|\nabla \chi\|_1 \right) \\ &\leq k_3 \left(\|\alpha\|_{\infty} \|\nabla \chi\|_{\frac{2}{3-2p}} \|\nabla v\|_2^{p-1} + \|\alpha\|_{\infty}^p \|\nabla \chi\|_2^{p-1} \|\nabla \chi\|_{\frac{2}{3-2p}} + \|\alpha\|_{\infty} \|\nabla \chi\|_1 \right) \\ &\leq c(1 + \|\nabla \chi\|_{\infty}^2) \left(c(\epsilon) \|\alpha\|_{\infty}^{\frac{2}{3-2p}} + \epsilon \|\nabla v\|_2^2 + \|\alpha\|_{\infty}^p + \|\alpha\|_{\infty} \right) \\ &\leq \epsilon c(1 + \|\nabla \chi\|_{\infty}^2) \|\nabla v\|_2^2 + \|\alpha\|_{\infty} \left(c(\epsilon) \|\alpha\|_{\infty}^{\frac{p-1}{3-2p}} + \|\alpha\|_{\infty}^{p-1} + 1 \right) (1 + \|\nabla \chi\|_{\infty}^2). \end{aligned}$$

Choosing $\epsilon = \frac{\mu_0}{2c(1 + \|\nabla \chi\|_{\infty}^2)}$ we get the following coercivity for A

$$\langle A(t, v), v \rangle \geq \frac{\mu_0}{2} \|\nabla v\|_2^2 - c \|\alpha\|_{1,2} \left(\|\alpha\|_{1,2}^{\frac{p-1}{3-2p}} + \|\alpha\|_{1,2}^{p-1} + 1 \right) (1 + \|\nabla \chi\|_{\infty}^2). \quad (4.19)$$

In a similar way in the case $p > 2$ we have, for arbitrary $\epsilon > 0$

$$\begin{aligned} |(S(\nabla v + \alpha \nabla \chi), \alpha \nabla \chi)| &\leq k_3 \left(\|\nabla v + \alpha(t) \nabla \chi\|_p^{p-1} \|\alpha \nabla \chi\|_p + \|\alpha \nabla \chi\|_1 \right) \\ &\leq \epsilon k_3 \|\nabla v + \alpha(t) \nabla \chi\|_p^p + c(\epsilon) \|\alpha(t) \nabla \chi\|_p^p + k_3 \|\alpha(t) \nabla \chi\|_1. \end{aligned} \quad (4.20)$$

Hence, by (4.5) and choosing $\epsilon = \frac{k_1}{2k_3}$ in (4.20) we have

$$\begin{aligned} \langle B(t, v), v \rangle &= \mu_0 \|\nabla v\|_2^2 + (S(\nabla v + \alpha \nabla \chi), \nabla v + \alpha \nabla \chi) - (S(\nabla v + \alpha(t) \nabla \chi), \alpha \nabla \chi) \\ &\geq \mu_0 \|\nabla v\|_2^2 + \frac{k_1}{2} \|\nabla v + \alpha \nabla \chi\|_p^p - k_2 |\Sigma| - c \|\alpha\|_{1,2} (1 + \|\nabla \chi\|_p^p) \end{aligned} \quad (4.21)$$

that proves the coercivity also for B .

Concerning the growth of A by (4.7) and since $1 < p < 2$ we have

$$\begin{aligned} \|S(\nabla v + \alpha(t) \nabla \chi)\|_2^2 &\leq k_3^2 \int_{\Sigma} (|\nabla v + \alpha(t) \nabla \chi|^{p-1} + 1)^2 dx \\ &\leq ck_3^2 (|\Sigma| + \|\nabla v\|_2^2 + \|\alpha\|_{1,2}^2 \|\nabla \chi\|_2^2). \end{aligned}$$

Since

$$\|A(t, v)\|_{W'_2} \leq \mu_0 \|\nabla v\|_2 + \|S(\nabla v + \alpha(t) \nabla \chi)\|_2$$

we get (4.17).

Finally the growth of B is controlled, thanks to (4.6), in the following way

$$\begin{aligned} \|B(t, v)\|_{W'_p} &\leq \mu_0 \|\nabla v\|_{p'} + \|S(\nabla v + \alpha \nabla \chi)\|_{p'} \\ &\leq c\mu_0 \left(\|\nabla v\|_p^{p-1} + |\Sigma|^{\frac{p-1}{p}} \right) + ck_3 \left(\|\nabla v\|_p^{p-1} + \|\alpha\|_{1,2}^{p-1} \|\nabla \chi\|_p^{p-1} + |\Sigma|^{\frac{p-1}{p}} \right) \end{aligned}$$

□

5. The Shear-Thinning case

The main goal of this section is to show the following result.

Theorem 5.1. *Let $\Sigma \subset \mathbb{R}^2$ be a Lipschitz domain, $\mu_0 > 0$, $1 < p < 2$ and $\alpha \in W^{1,2}((0, T))$ such that $\alpha(0) = \alpha(T)$. If S satisfies (4.5), (4.6) and (4.7), then there exist a unique pair of functions (v, Γ) with*

$$v \in L^\infty(0, T; W_0^{1,2}(\Sigma)), \quad v' \in L^2(\Sigma \times (0, T)), \quad \Gamma \in L^2((0, T))$$

such that v is the weak solution of the problem (4.9), in the sense of definition 4.2, and (v, Γ) satisfies (4.11). Moreover there exists a constant $K(\alpha)$ depending also on $\Sigma, T, k_1, k_3, \mu_0, p$ with $K(\alpha) \rightarrow 0$ as $\|\alpha\|_{1,2} \rightarrow 0$ such that

$$\sup_{t \in [0, T]} \|\nabla v(\cdot, t)\|_2^2 dt + \int_0^T \|v'(\cdot, t)\|_2^2 dt \leq K(\alpha) \quad (5.1)$$

$$\|\Gamma\|_2^2 \leq c(1 + K(\alpha)) \quad (5.2)$$

and the map $\Phi : W^{1,2}((0, T)) \rightarrow L^2(\Sigma \times (0, T)) \times L^2((0, T))$ defined by $\Phi(\alpha) = (v, \Gamma)$ is continuous, strongly in its first component and weakly in the second one.

Proof. We will prove the above theorem using the Faedo-Galerkin approximation. Let $\{V_k\}$ be the sequence of Lemma 3.3 with $p = 2$. For any fixed $k \in \mathbb{N}$ and $\psi \in V_k$, let us consider the following problem

$$\begin{cases} (v'(t), \phi) + \langle A(t, v(t)), \phi \rangle = (\mu_0 \alpha(t) \Delta \chi - \alpha'(t) \chi, \phi) & \text{a.e. in } [0, T], \quad \forall \phi \in V_k \\ v(0) = \psi & \text{in } \Sigma \end{cases} \quad (5.3)$$

where A is defined in (4.15). By Proposition 4.7 and [16, Lemma 8.26] we can find a solution $u_k := v \in L^\infty(0, T; V_k)$ of the above problem such that $u'_k \in L^2(0, T; V_k)$. Choosing $\phi = u_k(t)$ in (5.3)₁, by the coercivity condition (4.19) and Poincaré inequality, we get

$$\begin{aligned} \frac{d}{dt} \|u_k\|_2^2 &\leq -\mu_0 \|\nabla u_k\|_2^2 + c \|\alpha\|_{1,2} \left(\|\alpha\|_{1,2}^{\frac{p-1}{3-p}} + \|\alpha\|_{1,2}^{p-1} + 1 \right) (1 + \|\nabla \chi\|_\infty^2) \\ &\quad + (\mu_0 \|\alpha\|_\infty \|\Delta \chi\|_2 + \|\alpha'\|_2 \|\chi\|_\infty) \|u_k\|_2 \\ &\leq -\frac{\mu_0}{2} \|\nabla u_k\|_2^2 + M \|\alpha\|_{1,2} \end{aligned} \quad (5.4)$$

with

$$M := c \left(\left(\|\alpha\|_{1,2}^{\frac{p-1}{3-p}} + \|\alpha\|_{1,2}^{p-1} + 1 \right) (1 + \|\nabla \chi\|_\infty^2) + \|\alpha\|_{1,2} \left(\mu_0 \|\Delta \chi\|_2^2 + \frac{\|\chi\|_\infty^2}{\mu_0} \right) \right).$$

Setting $Y_k(t) := \|u_k(t)\|_2^2$, by Poincaré inequality and (5.4) we have

$$Y'_k(t) \leq M \|\alpha\|_{1,2} - c\mu_0 Y_k(t).$$

By simple manipulation of the previous differential inequality, we infer

$$Y_k(t) \leq Y_k(0) e^{-c\mu_0 t} + \frac{M \|\alpha\|_{1,2}}{c\mu_0} (1 - e^{-c\mu_0 t}), \quad \forall t \in [0, T], \quad (5.5)$$

which furnishes,

$$Y_k(0) \leq R_* \implies Y_k(t) \leq R_*, \quad \forall t \in [0, T] \quad (5.6)$$

provided

$$R_* := \frac{M \|\alpha\|_{1,2}}{c\mu_0}. \quad (5.7)$$

Let us set

$$B_k = \{\phi \in V_k : \|\phi\|_2 \leq R_*\}.$$

Thanks to (5.6) we can define a map $\mathbb{P}_k : B_k \rightarrow B_k$ by means of $\mathbb{P}_k(\psi) = u_k(T)$. Let us show that \mathbb{P}_k is continuous. Indeed if we choose another initial data $\tilde{\psi} \in B_k$ and we call \tilde{u}_k the corresponding solution, we have

$$\frac{1}{2} \frac{d}{dt} \|u_k(t) - \tilde{u}_k(t)\|_2^2 \leq -\mu_0 \|\nabla u_k(t) - \nabla \tilde{u}_k(t)\|_2^2 \leq 0$$

hence

$$\|\mathbb{P}_k(\psi) - \mathbb{P}_k(\tilde{\psi})\|_2 \leq \|\psi - \tilde{\psi}\|_2.$$

Since B_k is a convex and compact set in a Banach space, we can apply Schauder theorem to get that \mathbb{P}_k has a fixed point ψ_k . From now on we will denote by u_k the solution of problem (5.3) with initial data ψ_k , which satisfies the periodicity condition $u_k(0) = u_k(T)$. Moreover, by (5.6) we get that

$$\sup_{t \in [0, T]} \|u_k(t)\|_2^2 \leq R_* \quad \forall k \in \mathbb{N} \quad (5.8)$$

and, integrating in time inequality (5.4), by the periodicity just obtained, we get also

$$\frac{\mu_0}{2} \int_0^T \|\nabla u_k(t)\|_2^2 dt \leq MT \|\alpha\|_{1,2}. \quad (5.9)$$

Let us consider the system (5.3) with $v = u_k$ and $\phi = tu'_k(t)$. We get

$$\begin{aligned} t\|u'_k(t)\|_2^2 &= -\frac{\mu_0 t}{2} \frac{d}{dt} \|\nabla u_k\|_2^2 - t(S(\nabla u_k(t) + \alpha(t)\nabla\chi), \nabla u'_k(t)) \\ &\quad + t(\mu_0\alpha(t)\Delta\chi - \alpha'(t)\chi, u'_k(t)). \end{aligned}$$

Using the notation of lemma 4.1 we have

$$\begin{aligned} &\frac{d}{dt} \int_{\Sigma} H(|\nabla u_k(t) + \alpha(t)\nabla\chi|^2) dx \\ &= 2 \int_{\Sigma} h(|\nabla u_k(t) + \alpha(t)\nabla\chi|^2) (\nabla u_k(t) + \alpha(t)\nabla\chi) \cdot (\nabla u'_k(t) + \alpha'(t)\nabla\chi) dx \\ &= 2(S(\nabla u_k(t) + \alpha(t)\nabla\chi), \nabla u'_k(t)) + 2(S(\nabla u_k(t) + \alpha(t)\nabla\chi, \alpha'(t)\nabla\chi)). \end{aligned}$$

Hence

$$\begin{aligned} t\|u'_k(t)\|_2^2 &+ \frac{\mu_0 t}{2} \frac{d}{dt} \|\nabla u_k\|_2^2 + \frac{t}{2} \frac{d}{dt} \int_{\Sigma} H(|\nabla u_k(t) + \alpha(t)\nabla\chi|^2) dx \\ &= t(S(\nabla u_k(t) + \alpha(t)\nabla\chi, \alpha'(t)\nabla\chi) + t(\mu_0\alpha(t)\Delta\chi - \alpha'(t)\chi, u'_k(t)) \\ &\leq t\|S(\nabla u_k(t) + \alpha(t)\nabla\chi)\|_{p'} \|\alpha'(t)\nabla\chi\|_p + \frac{t}{2} \|\mu_0\alpha(t)\Delta\chi - \alpha'(t)\chi\|_2^2 \\ &\quad + \frac{t}{2} \|u'_k(t)\|_2^2. \end{aligned}$$

Integrating by parts in t the above inequality, we get, for any $t \in [0, \frac{3}{2}T]$

$$\begin{aligned} &\int_0^t \frac{s}{2} \|u'_k(s)\|_2^2 ds + \frac{t}{2} \left(\mu_0 \|\nabla u_k(t)\|_2^2 + \int_{\Sigma} H(|\nabla u_k(t) + \alpha(t)\nabla\chi|^2) dx \right) \\ &\leq \frac{\mu_0}{2} \int_0^t \|\nabla u_k(s)\|_2^2 ds + \frac{1}{2} \int_0^t \int_{\Sigma} H(|\nabla u_k(s) + \alpha(s)\nabla\chi|^2) dx ds \\ &\quad + \frac{3}{2} T \int_0^t \|S(\nabla u_k(s) + \alpha(s)\nabla\chi)\|_{p'} \|\alpha'(s)\nabla\chi\|_p ds + c\|\alpha\|_{1,2}^2 \end{aligned} \quad (5.10)$$

Now we use (4.6), and (4.8) to get that, for any $t \in [\frac{T}{2}, \frac{3}{2}T]$ we have

$$\begin{aligned}
& \frac{T}{4} \int_{T/2}^t \|u'_k(s)\|_2^2 ds + \frac{T}{4} \mu_0 \|\nabla u_k(t)\|_2^2 \\
& \leq \mu_0 \|\nabla u_k\|_{L^2(\Sigma \times (0, T))}^2 + c \int_0^t \|\nabla u_k(s)\|_2 + \|\alpha(s) \nabla \chi\|_2 + \|\nabla u_k(s)\|_p^p ds \\
& \quad + c \int_0^t \|\alpha(s) \nabla \chi\|_p^p + \|\nabla u_k(s) + \alpha(s) \nabla \chi\|_p^{p-1} \|\alpha'(s) \nabla \chi\|_p ds \quad (5.11) \\
& \quad \quad \quad + c(\|\alpha\|_{1,2} + \|\alpha\|_{1,2}^2) \\
& \leq c \left(\|\nabla u_k\|_{L^2(\Sigma \times (0, T))}^2 + \|\nabla u_k\|_{L^2(\Sigma \times (0, T))} + \|\nabla u_k\|_{L^p(\Sigma \times (0, T))}^p \right. \\
& \quad \left. + \|\alpha\|_{1,2} \|\nabla u_k\|_{L^p(\Sigma \times (0, T))}^{p-1} + \|\alpha\|_{1,2} + \|\alpha\|_{1,2}^2 + \|\alpha\|_{1,2}^p \right).
\end{aligned}$$

Since $1 < p < 2$, and taking into account the periodicity of the functions u_k , by (5.9) we get that

$$\int_0^T \|u'_k(s)\|_2^2 ds + \sup_{t \in [0, T]} \|\nabla u_k(t)\|_2^2 \leq c \|\alpha\|_{1,2}^{\frac{1}{2}} \left(1 + \|\alpha\|_{1,2}^{\frac{3}{2}}\right) \equiv K(\alpha). \quad (5.12)$$

where $K(\alpha)$ depends also on $\Sigma, T, k_1, k_3, \mu_0, p$.

By (5.12) and (5.8) the sequence $\{u_k(0)\}$ is bounded in $W^{1,2}(\Sigma)$ hence, by the Rellich-Kondrachov theorem there exists $U \in L^2(\Sigma)$ such that, up to a subsequence, $u_k(0) \rightarrow U$ strongly in $L^2(\Sigma)$. Applying [16, Theorem 8.30] we get that

$$u_k \rightharpoonup u \quad \text{weakly in } L^2(0, T; W_2) \quad (5.13)$$

where u is a weak solution of problem (4.14). Moreover, by (5.12) we can select two subsequences and find two functions z and w such that

$$\begin{aligned}
\nabla u_k & \overset{*}{\rightharpoonup} z \quad \text{weakly } * \text{ in } L^\infty(0, T; L^2(\Sigma)) \\
u'_k & \rightharpoonup w \quad \text{weakly in } L^2(\Sigma \times [0, T]).
\end{aligned}$$

By (5.13) we have that $z = \nabla u$ and $w = u'$ hence

$$u \in L^\infty(0, T; W_0^{1,2}(\Sigma)), \quad u' \in L^2(\Sigma \times (0, T)), \quad (5.14)$$

$$u'_k \rightharpoonup u' \quad \text{weakly in } L^2(\Sigma \times [0, T]). \quad (5.15)$$

and, by (5.12),

$$\|u'\|_{L^2(\Sigma \times (0, T))}^2 + \sup_{t \in [0, T]} \|\nabla u(t)\|_2^2 \leq K(\alpha). \quad (5.16)$$

Coming back from the lifted solution u to the original problem we get that $v(x, t) = u(x, t) + \alpha(t)\chi(x)$ is a weak solution of (4.9) in the sense of definition 4.2. Moreover, by (5.14)

$$v \in L^\infty(0, T; W_0^{1,2}(\Sigma)), \quad v' \in L^2(\Sigma \times (0, T)). \quad (5.17)$$

In order to obtain the estimate (5.1) we need only to use (5.16), recall that $v = u + \alpha\chi$ and observe that the choice of χ is arbitrary and depends only on Σ . We remark that, by the definition of $K(\alpha)$ (see (5.12)) we get

$$\lim_{\|\alpha\|_{1,2} \rightarrow 0} K(\alpha) = 0.$$

In view of the fact that the weak solution v just constructed has $v' \in L^2(\Sigma \times [0, T])$, the existence of the associated pressure gradient Γ follows from remark 4.3. Moreover, integrating in time (4.12), by (5.1), we have

$$\|\Gamma\|_2^2 \leq \|\chi\|_\infty^2 \|v'\|_2^2 + T \|\nabla\chi\|_\infty^2 (\mu_0^2 \|\nabla v\|_\infty^2 + k_3^2 (|\Sigma| + \|\nabla v\|_\infty^2)) \leq c(1 + K(\alpha)). \quad (5.18)$$

The proof of the existence part is thus completed.

Concerning uniqueness, we consider two solutions (v_1, Γ_1) , (v_2, Γ_2) and we set $w = v_1 - v_2$. A straightforward calculation and remark 4.3 show that w is a solution of the following distributional equation

$$w' = \mu_0 \Delta w + \nabla \cdot S(\nabla v_1) - \nabla \cdot S(\nabla v_2) \quad \text{in } \mathfrak{D}'(0, T; W_2'). \quad (5.19)$$

Let us apply the above identity to the function w , obtaining, by (4.7)

$$\langle w', w \rangle = -\mu_0 (\nabla w, \nabla w) + (S(\nabla v_1) - S(\nabla v_2), \nabla v_1 - \nabla v_2) \leq -\mu_0 \|\nabla w\|_2^2.$$

Now we recall that $w \in L^2(0, T; W_2)$, $w' \in L^2(0, T; W_2')$, $W_2 \subset L^2(\Sigma) \subset W_2'$, hence we have (see e.g. [18, lemma III.1.2])

$$\frac{1}{2} \frac{d}{dt} \|w\|_2^2 = \langle w', w \rangle \leq -\mu_0 \|\nabla w\|_2^2.$$

As a result, by the periodicity of w , $\int_0^T \|\nabla w(\cdot, t)\|_2^2 dt = 0$ that, observing that $w = 0$ on $\partial\Sigma$, gives $w(x, t) = 0$ for almost any $t \in [0, T]$ and $x \in \Sigma$. From (4.12) we also get that $\Gamma_1 = \Gamma_2$.

Finally we prove the continuity of the map that brings the flux α to the corresponding solution (v, Γ) . Let $\bar{\alpha} \in W^{1,2}((0, T))$ be a fixed flux and let $(\bar{v}, \bar{\Gamma})$ be the corresponding weak solution. We begin proving the strong continuity of the velocity v . By contradiction suppose there exist a number $\epsilon > 0$ and a sequence $\{\alpha_n\} \subset W^{1,2}((0, T))$ such that $\alpha_n \rightarrow \bar{\alpha}$ in $W^{1,2}((0, T))$ but the corresponding sequence of solutions $\{v_n\}$ satisfies

$$\int_0^T \int_\Sigma |v_n - \bar{v}|^2 dx dt > \epsilon \quad \forall n \in \mathbf{N}. \quad (5.20)$$

By remark 4.3 we have that

$$v_n' = \mu_0 \Delta v_n + \nabla \cdot S(\nabla v_n) \quad \text{in } \mathfrak{D}'(0, T; W_2'). \quad (5.21)$$

Using the estimate (5.1) we get that the sequence $\{v_n\}$ is bounded in $L^2(0, T; W_0^{1,2}(\Sigma))$ hence, up to a subsequence not relabeled, we can find \tilde{v} such that

$$v_n \rightharpoonup \tilde{v} \quad \text{weakly in } L^2(0, T; W_0^{1,2}(\Sigma)). \quad (5.22)$$

Now we can use the monotonicity to pass to the limit in equation (5.21) obtaining

$$\tilde{v}' = \mu_0 \Delta \tilde{v} + \nabla \cdot S(\nabla \tilde{v}) \quad \text{in } \mathfrak{D}'(0, T; W_2'). \quad (5.23)$$

For any $t \in [0, T]$ we set $\tilde{\alpha}(t) = \int_{\Sigma} \tilde{v}(x, t) dx$ and observe that $\tilde{\alpha} \in L^2((0, T))$. In particular we can consider the function $(x, t) \mapsto \bar{\alpha}(t) - \tilde{\alpha}(t)$ as an element of $L^2(0, T; W^{-1,2}(\Sigma))$ which can be applied to the function $v_n - \tilde{v} \in L^2(0, T; W_0^{1,2}(\Sigma))$. Hence

$$\lim_{n \rightarrow \infty} \int_0^T \langle \bar{\alpha}(t) - \tilde{\alpha}(t), v_n(t) - \tilde{v}(t) \rangle dt = \lim_{n \rightarrow \infty} \int_0^T (\bar{\alpha} - \tilde{\alpha})(\alpha_n - \tilde{\alpha}) dt = \|\bar{\alpha} - \tilde{\alpha}\|_2^2.$$

But, by (5.22) we have that

$$\lim_{n \rightarrow \infty} \int_0^T \langle \bar{\alpha}(t) - \tilde{\alpha}(t), v_n(t) - \tilde{v}(t) \rangle dt = 0$$

hence $\bar{\alpha} = \tilde{\alpha}$ almost everywhere in $[0, T]$. If we set $w = \tilde{v} - \bar{v}$ and we reason as in the proof of uniqueness, we get that

$$\tilde{v} = \bar{v}. \quad (5.24)$$

Since $K(\alpha)$ is continuous with respect to $\|\alpha\|_{1,2}$ (see (5.12)), by the strong convergence of α_n to $\bar{\alpha}$ in $W^{1,2}((0, T))$, we can conclude that

$$\|v_n'\|_{L^2(0, T; W^{-1,2}(\Sigma))} \leq \|v_n'\|_{L^2(0, T; L^2(\Sigma))} \leq K(\alpha_n) \leq cK(\bar{\alpha})$$

so that the sequence $\{v_n'\}$ is bounded in $L^2(0, T; W^{-1,2}(\Sigma))$. Since the sequence $\{v_n\}$ is bounded in $L^2(0, T; W^{1,2}(\Sigma))$, the injection $W^{1,2}(\Sigma) \subset L^2(\Sigma)$ is compact and the space $W_0^{1,2}(\Sigma)$ is reflexive, by [18, theorem III.2.1] we can extract from $\{v_n\}$ a subsequence strongly convergent in $L^2(0, T; L^2(\Sigma))$ to a function v^* . By (5.22) and (5.24) we get, a fortiori, that $v_n \rightharpoonup \bar{v}$ weakly in $L^2(0, T; L^2(\Sigma))$. By the uniqueness of the limit, we infer $v^* = \bar{v}$, and the strong convergence contradicts (5.20).

As far as the weak continuity of Γ is concerned, we write (4.12) for Γ_n , we multiply it by a generic periodic function $\gamma \in C^\infty((0, T))$ and we integrate in time. By (5.18) we have that, up to subsequences, $\{\Gamma_n\}$ converges weakly in $L^2((0, T))$ to a function $\tilde{\Gamma}$ hence

$$\lim_{n \rightarrow \infty} \int_0^T \Gamma_n \gamma dt = \int_0^T \tilde{\Gamma} \gamma dt.$$

The right-hand side, by monotonicity, converges to

$$\int_0^T \langle \bar{v}', \chi \gamma \rangle + (\mu_0 \nabla \bar{v} + S(\nabla \bar{v}), \nabla \chi \gamma) dt = \int_0^T \bar{\Gamma} \gamma dt$$

by uniqueness. Hence $\tilde{\Gamma} = \bar{\Gamma}$. □

We conclude with a regularity result for weak solutions, obtained by suitably combining theorem 5.1 with known results for quasilinear parabolic equations. Specifically, we have the following.

Theorem 5.2. *Let v be the solution of theorem 5.1, then*

$$v \in L^\infty(\Sigma \times [0, T]) \quad \text{and} \quad \|v\|_\infty \leq \tilde{K}(\alpha)$$

where the constant $\tilde{K}(\alpha)$ has the same properties of $K(\alpha)$ in theorem 5.1.

Proof. For any $t \in [0, T]$, we set

$$P(t) = \int_0^t \Gamma(s) ds \quad (5.25)$$

so that $P'(t) = \Gamma(t)$ and $P \in C([0, T])$. Moreover, let be

$$y(x, t) = v(x, t) + P(t)$$

hence, by (4.11),

$$y' = \mu_0 \Delta y + \nabla \cdot S(\nabla y) \quad \text{in } \mathcal{D}'(0, T; W^{-1,2}(\Sigma)).$$

We observe that

$$y(x, t) = v(x, t) + P(t) = P(t) \quad \forall x \in \partial\Sigma$$

and that $P \in L^\infty(\partial\Sigma \times [0, T])$. Moreover

$$y(x, 0) = v(x, 0) + P(0) = v(x, 0) \quad \forall x \in \Sigma$$

and that $v(\cdot, 0) \in L^2(\Sigma)$. Setting

$$a(\xi) = \mu_0 \xi + S(\xi), \quad \xi \in \mathbb{R}^2,$$

we find at once that y is a weak solution of the Dirichlet problem

$$\begin{cases} y' - \nabla \cdot a(\nabla y) = 0 & (x, t) \in \Sigma \times (0, T] \\ y(x, t) = P(t) & (x, t) \in \partial\Sigma \times (0, T) \\ y(x, 0) = v(x, 0) & x \in \Sigma \end{cases} \quad (5.26)$$

Since

$$a(\xi) \cdot \xi \geq \mu_0 |\xi|^2 + k_1 |\xi|^p - k_2 \geq \mu_0 |\xi|^2 - k_2,$$

and

$$|a(\xi)| \leq \mu_0 |\xi| + k_3 |\xi|^{p-1} + k_3 \leq (\mu_0 + k_3) |\xi| + 2k_3$$

we may deduce that (5.26) is a quasilinear parabolic problem with quadratic growth. We appeal to a result of boundedness for the solution of such a problem, as stated, for instance, in [5]. Roughly speaking, [5, theorem V.3.2] states that the $\|y(\cdot, t)\|_\infty$ is bounded, for any time $t > 0$ by the $L^\infty(\partial\Sigma \times [0, T])$ norm of the boundary datum and by the $L^2(\Sigma \times (0, t))$ norm of y . This estimate is no longer true in the limit $t \rightarrow 0$. However, we get rid of this problem by means of the periodicity nature of our solution. Let us remark that, since in our case y is both a sub and super solution, the upper bound in [5, theorem V.3.2] is actually a bound on $|y|$. This is

a consequence of the monotonicity of the principal part of the differential operator, as stated in [5, lemma II.1.2].

Let us estimate $P(t)$. By (5.25), (4.12) and (5.1), we have

$$\begin{aligned} |P(t)| &\leq \int_0^t \mu_0 \|\nabla v(s)\|_2 \|\nabla \chi\|_2 + \|S(\nabla v(s))\|_2 \|\nabla \chi\|_2 ds + (\|v(t)\|_2 + \|v(0)\|_2) \|\chi\|_2 \\ &\leq c (\|v\|_{L^2(0,T;W^{1,2}(\Sigma))} + \|v\|_{L^\infty(0,T;L^2(\Sigma))}) \leq cK(\alpha)^{\frac{1}{2}} \end{aligned}$$

for any $t \in [0, T]$. Now we apply [5, theorem V.3.2] to get an estimate for the L^∞ norm of y at time T

$$\begin{aligned} \sup_{x \in \Sigma} |y(x, T)| &\leq \sup_t |P(t)| + c \left(T + \frac{1}{T} \right)^{\frac{1}{2}} \left(\frac{1}{T} \int_0^T \int_\Sigma |y|^2 dx dt \right)^{\frac{1}{2}} \\ &\leq cK(\alpha)^{\frac{1}{2}} + c\|y\|_{L^2(\Sigma \times (0, T))} \\ &\leq cK(\alpha)^{\frac{1}{2}} + c(\|v\|_{L^2(\Sigma \times (0, T))} + c\|P\|_{L^\infty(0, T)}) \leq cK(\alpha)^{\frac{1}{2}} \end{aligned}$$

We remind that v is time-periodic hence

$$y(x, 0) = v(x, 0) = v(x, T) = y(x, T) - P(T)$$

and

$$\sup_{x \in \Sigma} |y(x, 0)| \leq \sup_x |y(x, T)| + |P(T)| \leq cK(\alpha)^{\frac{1}{2}}.$$

We apply once again [5, theorem V.3.2] to obtain the desired global bound

$$\begin{aligned} \sup_{x \in \Sigma} |v(x, t)| &\leq \sup_x |y(x, t)| + |P(t)| \\ &\leq \max\{\sup_t |P(t)|, \sup_x |y(x, 0)|\} + c \left(\int_0^t \int_\Sigma |y|^2 dx dt \right)^{\frac{1}{2}} \leq cK(\alpha)^{\frac{1}{2}} \equiv \tilde{K}(\alpha) \end{aligned} \tag{5.27}$$

uniformly in $t \in [0, T]$. \square

Remark 5.3. *We remark that we require the mere continuity of the shear viscosity S , allowing to deal even with the singular case $S(\nabla u) = |\nabla u|^{p-2} \nabla u$. In this respect we observe that for this specific tensor S , the growth condition (4.6) becomes simply $|S(a)| = |a|^{p-1}$ and (5.2) can be improved to $\|\Gamma\|_2^2 \leq K(\alpha)$ gaining the smallness of the pressure gradient for a small total flow-rate.*

6. The Shear-Thickening case

In this section we show how to get the same results stated so far, in the case $p > 2$ corresponding to the case of a shear-thickening liquid. The general scheme remains unvaried, hence we will point out only the changes to be made to the proof. The main theorem for the shear-thickening case is the following:

Theorem 6.1. *Let $\Sigma \subset \mathbb{R}^2$ be a Lipschitz domain, $\mu_0 > 0, p > 2$ and $\alpha \in W^{1,2}((0,T))$ such that $\alpha(0) = \alpha(T)$. If S satisfies (4.5), (4.6) and (4.7), then there exist a unique pair of functions (v, Γ) with*

$$v \in L^\infty(0, T; W_0^{1,2}(\Sigma)), \quad v' \in L^2(\Sigma \times (0, T)), \quad \Gamma \in L^{p'}(0, T)$$

such that v is the weak solution of the problem (4.9), in the sense of definition 4.2, and (v, Γ) satisfies (4.11). Moreover there exists a constant $K_1(\alpha)$ depending also on $\Sigma, k_1, k_3, \mu_0, p, T$ such that

$$\int_0^T \|\nabla v(t)\|_p^p dt + \int_0^T \|v'(t)\|_2^2 dt + \sup_{t \in (0, T)} \|\nabla v(t)\|_2^2 \leq K_1(\alpha) + ck_2 \quad (6.1)$$

$$\|\Gamma\|_{p'}^{p'} \leq c(1 + K(\alpha))$$

where $K_1(\alpha) \rightarrow 0$ as $\|\alpha\|_{1,2} \rightarrow 0$. Moreover the map $\Phi : W^{1,2}((0, T)) \rightarrow L^2(\Sigma \times (0, T)) \times L^{p'}((0, T))$ defined by $\Phi(\alpha) = (v, \Gamma)$ is continuous, strongly in its first component and weakly in the second one. Finally, if $k_2 = 0$ then $v \in L^\infty([0, T] \times \Sigma) \cap L^\infty(0, T; W_0^{1,p}(\Sigma))$ with

$$\|v(x, t)\|_\infty + \sup_{t \in [0, T]} \|\nabla u(t)\|_p^p \leq K_1(\alpha). \quad (6.2)$$

Proof. Let V_k be as in Lemma 3.3 with $p > 2$. In analogy with (5.3) we consider

$$\begin{cases} (v'(t), \phi) + \langle B(t, v(t)), \phi \rangle = (\mu_0 \alpha(t) \Delta \chi - \alpha'(t) \chi, \phi) & \text{a.e. in } [0, T], \quad \forall \phi \in V_k \\ v(0) = \psi & \text{in } \Sigma \end{cases} \quad (6.3)$$

with B defined in (4.16). We find a solution u_k of the above problem such that $u_k \in L^\infty(0, T; V_k)$, $u_k' \in L^{p'}(0, T; V_k)$. By the coercivity of B (4.21) follows the estimate

$$\begin{aligned} \frac{d}{dt} \|u_k\|_2^2 &\leq -\mu_0 \|\nabla u_k\|_2^2 - \frac{k_1}{2} \|\nabla u_k + \alpha \nabla \chi\|_p^p + c \|\alpha\|_{1,2} (1 + \|\nabla \chi\|_p^p) \\ &\quad + 2k_2 |\Sigma| + (\mu_0 \|\alpha\|_\infty \|\Delta \chi\|_2 + \|\alpha'\|_2 \|\chi\|_\infty) \|u_k\|_2 \\ &\leq -\frac{\mu_0}{2} \|\nabla u_k\|_2^2 - \frac{k_1}{2^p} \|\nabla u_k\|_p^p + M_1 \|\alpha\|_{1,2} + 2k_2 |\Sigma| \end{aligned} \quad (6.4)$$

with

$$M_1 = c \|\alpha\|_{1,2} \left(1 + \|\nabla \chi\|_p^p + \mu_0 \|\Delta \chi\|_2^2 + \frac{\|\chi\|_\infty}{\mu_0} + \|\alpha\|_{1,2}^{p-1} \|\nabla \chi\|_p^p \right).$$

The differential inequality (6.4) allows us to get that

$$\|u_k(0)\|_2^2 \leq R_* \implies \|u_k(t)\|_2^2 \leq R_* \quad \forall t \in [0, T]$$

where

$$R_* = \frac{M_1 \|\alpha\|_{1,2}}{c \mu_0} + 2k_2 |\Sigma|.$$

The Schauder fixed point theorem allows us to achieve the periodicity of the functions u_k and the uniform estimates, for any $k \in \mathbf{N}$

$$\sup_{t \in [0, T]} \|u_k(t)\|_2^2 \leq R_*$$

$$\frac{\mu_0}{2} \int_0^T \|\nabla u_k(t)\|_2^2 dt + \frac{k_1}{2^p} \int_0^T \|\nabla u_k(t)\|_p^p dt \leq (M_1 \|\alpha\|_{1,2} + 2k_2 |\Sigma|) T. \quad (6.5)$$

Testing the equation (6.3) with $tu_k(t)$ bring us to the estimate (5.11) which remains unchanged. Integrating in time, by the periodicity of the functions and (6.5) we have

$$\int_0^T \|u'_k(s)\|_2^2 ds + \sup_{t \in [0, T]} \|\nabla u_k(t)\|_2^2 \leq K_1(\alpha) + ck_2 \quad (6.6)$$

where $\lim_{\|\alpha\|_{1,2} \rightarrow 0} K_1(\alpha) = 0$.

If $k_2 = 0$, by lemma 4.1 we have that

$$\begin{aligned} \int_{\Sigma} H(|\nabla u_k(t) + \alpha(t)\nabla \chi|^2) dx &\geq \int_{\Sigma} \frac{2k_1}{p} |\nabla u_k(t) + \alpha(t)\nabla \chi|^p dx \\ &\geq \frac{2k_1}{p} (2^{1-p} \|\nabla u_k(t)\|_p^p - \|\alpha(t)\nabla \chi\|_p^p) \end{aligned}$$

Substituting this inequality in the left-hand side of (5.10) we can obtain the analogue of (5.11) and conclude that

$$\sup_{t \in [0, T]} \|\nabla u_k(t)\|_p^p \leq K_1(\alpha) \quad (6.7)$$

with a little abuse of notation, since $K_1(\alpha)$ is not exactly the same of (6.6) but it has the same properties. The remaining part of the proof follows the one of Theorem 5.1 with obvious changes. If $k_2 = 0$, the claim $v \in L^\infty(\Sigma \times [0, T])$ and the related bound (6.2), follow from estimate 6.7 and the Sobolev embedding theorem considering that $p > 2$. \square

Remark 6.2. *In the case $k_2 > 0$ it is still possible to get an L^∞ bound for v in $\Sigma \times [0, T]$. This result can be achieved via parabolic regularity. Indeed a result similar to theorem 5.2 holds true furnishing the following estimate*

$$\|v(t)\|_\infty \leq c \left(\|\alpha\|_{1,2}^{\frac{p-1}{p}} + k_2 \right).$$

7. The direct problem

In this closing section we show how to find the velocity field once the pressure gradient is prescribed. Despite the fact that the problem is simpler than the one treated in the previous section, we would like to address it for completeness, and also because it was the one originally studied by J.R. Womersley in the Newtonian case (see [20]).

The problem consists in solving the following set of equations

$$\begin{cases} u' - \mu_0 \Delta u - \nabla \cdot S(\nabla u) = \Gamma(t) & \text{in } \Sigma \times [0, T] \\ u(x, t) = 0 & \text{on } \partial\Sigma \times [0, T] \\ u(x, 0) = u(x, T) & \text{in } \Sigma \end{cases} \quad (7.1)$$

where Γ is a prescribed (time-periodic) function in $L^2((0, T))$ or in $L^p((0, T))$, respectively, according to whether we are in the shear-thinning or in the shear-thickening case. Due to the fact that, this time, the flow-rate is not prescribed, we do not need to project the equation on the set of zero mean value functions. Hence in the definition of weak solution we can use the test functions in the the class $C_{0,per}^\infty$.

Definition 7.1. *We say that u is a weak solution of the problem (7.1) if $u \in L^2(0, T; W_0^{1,2}(\Sigma)) \cap L^p(0, T; W_0^{1,p}(\Sigma)) \cap L^\infty(0, T; L^2(\Sigma))$ and*

$$\begin{cases} \int_0^T (u, \phi') - (\mu_0 \nabla u + S(\nabla u), \nabla \phi) + (\Gamma, \phi) dt = 0 & \forall \phi \in C_{0,per}^\infty \\ u(x, 0) = u(x, T) & \text{for a.e. } x \in \Sigma \end{cases} \quad (7.2)$$

The existence and uniqueness of the solution to the above problem for a shear-thinning liquid is contained in the following theorem

Theorem 7.2. *Let $\Sigma \subset \mathbb{R}^2$ be a Lipschitz domain, $\mu_0 > 0$, $1 < p < 2$ and $\Gamma \in L^2((0, T))$. If S satisfies (4.5), (4.6) and (4.7), then there exist a unique weak solution u of the problem (7.1) with*

$$u \in L^\infty(0, T; W_0^{1,2}(\Sigma)) \cap L^\infty(\Sigma \times [0, T]), \quad u' \in L^2(\Sigma \times (0, T))$$

and

$$\sup_{(x,t)} |u(x, t)| + \sup_t \|\nabla u(t)\|_2 + \int_0^T \|u'(t)\|_2^2 dt \leq K(\Gamma)$$

where K depends also on $\Sigma, T, k_1, k_3, \mu_0, p$ and $K(\Gamma) \rightarrow 0$ as $\|\Gamma\|_2 \rightarrow 0$.

Proof. The proof is along the same arguments used in the previous sections, hence it will be only sketched here. We use the Galerkin scheme of Theorem 5.1. For an arbitrary $\psi \in L^2(V_k)$ let u_k be the solution of the following problem

$$\begin{cases} (u'_k, \phi) - (\nabla \cdot (\mu_0 \nabla u_k + S(\nabla u_k)), \phi) = (\Gamma(t), \phi) & \text{a.e. in } [0, T], \forall \phi \in V_k \\ u_k(0) = \psi & \text{in } \Sigma \end{cases}$$

The existence is guaranteed by the monotonicity, coercivity and suitable growth condition of the operator $v \mapsto -\nabla \cdot (\mu_0 \nabla v + S(\nabla v))$ in $W_0^{1,2}(\Sigma)$. Testing the above equation with $\phi = u_k(t)$ we get

$$\frac{1}{2} \frac{d}{dt} \|u_k\|_2^2 + \frac{\mu_0}{2} \|\nabla u_k(t)\|_2^2 \leq \frac{c}{\mu_0} |\Sigma| |\Gamma(t)|^2. \quad (7.3)$$

Integrating in time and applying the Schauder fixed point theorem we get the periodicity of the solution along with the estimates

$$\|u_k(t)\|_2^2 \leq \frac{c}{\mu_0} |\Sigma| \|\Gamma\|_2^2 e^{c\mu_0 T} (1 - e^{-c\mu_0 T})^{-1} \quad \forall t \in [0, T],$$

$$\mu_0 \int_0^T \|\nabla u_k(t)\|_2^2 dt \leq \frac{c}{\mu_0} |\Sigma| \|\Gamma\|_2^2.$$

Testing the equation with $\phi = tu'_k(t)$ we obtain the estimate

$$t\|u'_k(t)\|_2^2 + \frac{\mu_0 t}{2} \frac{d}{dt} \|\nabla u_k(t)\|_2^2 + \frac{t}{2} \frac{d}{dt} \int_{\Sigma} H(|\nabla u_k(t)|^2) dx \leq \frac{t}{2} \|\Gamma\|_2^2 + \frac{t}{2} \|u'_k(t)\|_2^2$$

where H is defined in lemma 4.1. Integrating in time and employing the time periodicity, we conclude that

$$\int_0^T \|u'_k(t)\|_2^2 dt + \sup_t \|\nabla u_k(t)\|_2^2 \leq cK(\Gamma)$$

where $K(\Gamma) \rightarrow 0$ when $\|\Gamma\|_2 \rightarrow 0$.

Passing to the limit in the Galerkin sequence (see [16, Theorem 8.30]) we get the existence result. The uniqueness follows straightforward by the strict monotonicity and the time periodicity, like in theorem 5.1.

Finally, the uniform bound for $|u(x, t)|$ is a consequence of parabolic regularity, exactly as in theorem 5.2. \square

By suitably combining the proofs of theorem 6.1 and theorem 7.2, we may show the following result –analogous to theorem 7.2 in the shear-thickening case– whose proof will be, therefore, omitted.

Theorem 7.3. *Let $\Sigma \subset \mathbf{R}^2$ be a Lipschitz domain, $\mu_0 > 0, p > 2$ and $\Gamma \in L^{p'}((0, T))$. If S satisfies (4.5), (4.6) and (4.7), then there exists a unique weak solution of the problem (7.1) u with*

$$u \in L^\infty(0, T; W_0^{1,2}(\Sigma)) \cap L^p(0, T; W_0^{1,p}(\Sigma)) \cap L^\infty(\Sigma \times [0, T]), \quad u' \in L^2(\Sigma \times (0, T)).$$

Moreover there exists a constant $K_1(\Gamma)$ depending also on $\Sigma, k_1, k_3, \mu_0, p, T$ such that

$$\int_0^T \|u'(t)\|^2 dt + \int_0^T \|\nabla u(t)\|_p^p dt + \sup_t \|\nabla u(t)\|_2^2 + \sup_{x,t} |u(x, t)| \leq K_1(\Gamma) + ck_2$$

where $K_1(\Gamma) \rightarrow 0$ as $\|\Gamma\|_{p'} \rightarrow 0$. If $k_2 = 0$ then $u \in L^\infty(0, T; W_0^{1,p}(\Sigma))$ and

$$\sup_t \|\nabla u(t)\|_p^p \leq K_1(\Gamma). \quad (7.4)$$

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