Platonic polyhedra, periodic orbits and chaotic motions in the N-body problem with non-Newtonian forces.

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Abstract

We consider the N-body problem with interaction potential $U_\alpha = \frac{1}{|x_i - x_j|^\alpha}$ for $\alpha > 1$. We assume that the particles have all the same mass and that N is the order $|\mathcal{R}|$ of the rotation group $\mathcal{R}$ of one of the five Platonic polyhedra. We study motions that, up to a relabeling of the N particles, are invariant under $\mathcal{R}$. By variational techniques we prove the existence of periodic and chaotic motions.

1 Introduction

In a previous paper with Piero Negrini [10] we focused on the rotation groups $\mathcal{T}, \mathcal{O}, \mathcal{I}$[1] of the Platonic polyhedra and, for $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$, we studied periodic motions of $N = |\mathcal{R}|$ equal particles $\mathbf{u} = \{u_R\}_{R\in\mathcal{R}}$ that, at each time t, satisfy the symmetry condition

$$u_R(t) = Ru_I(t),\ R \in \mathcal{R},\ t \in \mathbb{R}, \tag{1.1}$$

that is, for each $t \in \mathbb{R}$ the configuration of the system coincides with the orbit $\{Ru_I(t)\}_{R\in\mathcal{R}}$, under the group $\mathcal{R}$, of the particle u_I associated to the identity $I \in \mathcal{R}$. By condition (1.1) u_I determines the motion of the whole system, therefore we refer to u_I as to the motion of the *generating particle*.

From (1.1) we have that

$$u_{R_1}(t) = u_{R_2}(t) \ \Leftrightarrow\ u_I(t) = R_1^{-1}R_2u_I(t).$$

Therefore the system has a collision at time t if and only if $u_I(t)$ belongs to the rotation axis $a(R)$ of some $R \in \mathcal{R} \setminus \{I\}$. It follows that a motion of the system is free of collisions if and only if

$$u_I(\mathbb{R}) \cap \Gamma = \emptyset, \quad \text{where } \ \Gamma = \cup_{R\in\mathcal{R}\setminus\{I\}} a(R). \tag{1.2}$$

Each u_I which is periodic and satisfies (1.2) belongs to a well defined free homotopy class in $\mathbb{R}^3 \setminus \Gamma$ represented by u_I. In [10] we proved the existence of periodic motions of $|\mathcal{R}|$ unit masses subject to Newtonian interaction for u_I in various homotopy classes of $\mathbb{R}^3 \setminus \Gamma$. To state the result proved in [10] we recall how the free homotopy classes of $\mathbb{R}^3 \setminus \Gamma$ can be coded by periodic sequences of vertexes of an Archimedean polyhedron naturally associated to $\mathcal{R}$. We denote by $\mathbb{S}^2$ the unit sphere in $\mathbb{R}^3$ and let $\mathcal{P} = \Gamma \cap \mathbb{S}^2$ be the set of *poles* of $\mathcal{R}$.

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[1]$\mathcal{T}$ the rotation group of the Tetrahedron of order 12, $\mathcal{O}$ the rotation group of Hexahedron and Octahedron of order 24 and $\mathcal{I}$ the group of Icosahedron and Dodecahedron of order 60.

The poles are the vertexes of a tessellation of $\mathbb{S}^2$ with $2|\mathcal{R}|$ congruent rectangular spherical triangles (cfr. Fig. 1). Let τ be one of these triangles and let s be the side of τ opposite to the vertex $p \in \mathcal{P}$ of τ corresponding to the right angle. There is a choice of $q \in s$ such that q and the corresponding points q_i of the three triangles τ_i, $i = 1, 2, 3$ having the vertex p in common with τ are the vertexes of a square. For this particular choice of $q \in s$, a special case of the classical Wythoff construction [7], the convex hull $\mathrm{co}(\{Rq\}_{R\in\mathcal{R}})$ of the orbit of q under $\mathcal{R}$ is an Archimedean polyhedron $\mathcal{Q}_{\mathcal{R}}$. $\mathcal{Q}_{\mathcal{R}}$ has $|\mathcal{R}|$ vertexes, $2|\mathcal{R}|$ equal edges, $\sharp\mathcal{P} = |\mathcal{R}| + 2$ faces, and the axis of each face is one of the axes $a(R)$ of some $R \in \mathcal{R} \setminus \{I\}$. $\mathbb{R}^3 \setminus \Gamma$ is homotopically equivalent to $\mathbb{S}^2 \setminus \mathcal{P}$ and therefore to the union $\mathcal{L}_{\mathcal{R}}$ of the edges of $\mathcal{Q}_{\mathcal{R}}$.

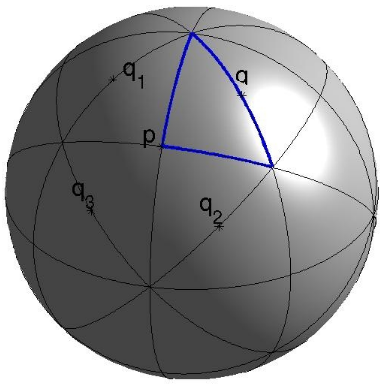

Figure 1: Tessellation of $\mathbb{S}^2$. $\mathcal{R} = \mathcal{O}$.

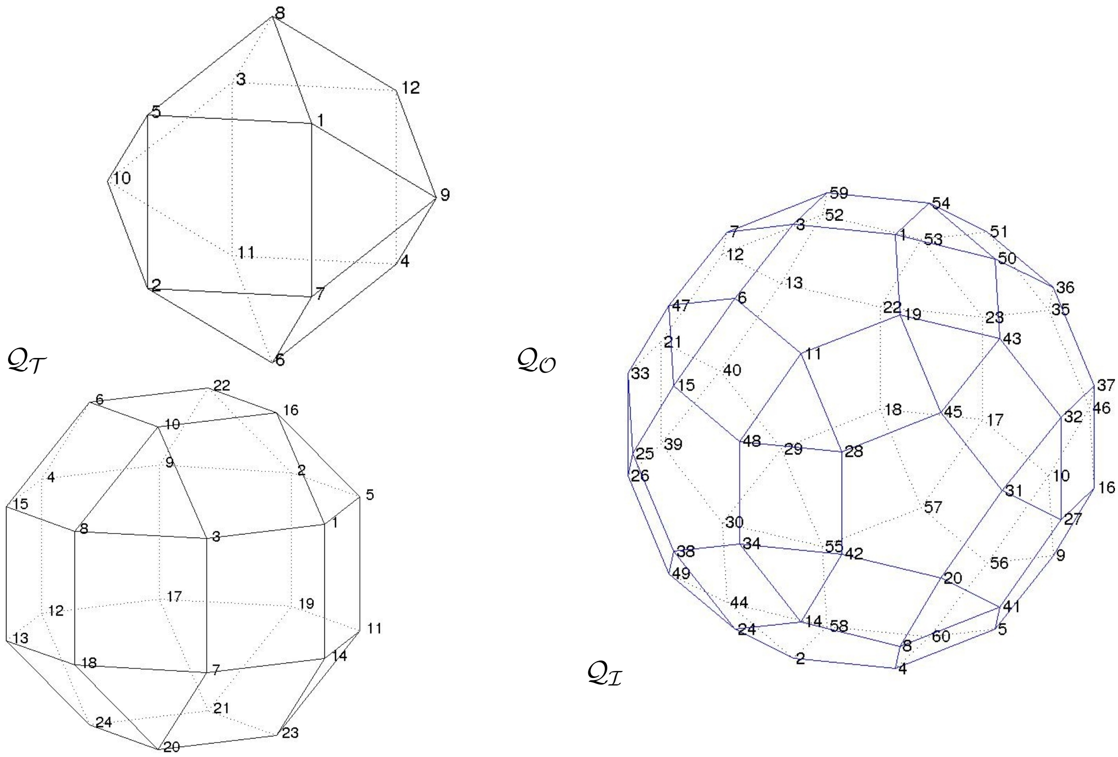

Figure 2: The Archimedean polyhedra $\mathcal{Q}_{\mathcal{T}}, \mathcal{Q}_{\mathcal{O}}, \mathcal{Q}_{\mathcal{I}}$.

Consider the set of sequences $\nu = \{\nu_k\}_{k=0}^{K}$ of vertexes of $\mathcal{Q}_{\mathcal{R}}$ that satisfy the conditions

(i) $\nu_K = \nu_0$

(ii) for each $k \in \{0, \dots, K-1\}$ the segment $[\nu_k, \nu_{k+1}]$ coincides with one of the edges of $\mathcal{Q}_{\mathcal{R}}$.

(iii) $\nu_{k-1} \neq \nu_{k+1}$, for $k \in \{0, \dots, K-1\}$, with $\nu_{-1} = \nu_{K-1}$.

Two such sequences ν and ν' are said to be equivalent if

$$K = K', \tag{1.3}$$

and the periodic extensions $\tilde{\nu}$, $\tilde{\nu}'$ of ν, ν' coincide up to translation. The homotopic equivalence of $\mathbb{R}^3 \setminus \Gamma$ and $\mathcal{L}_{\mathcal{R}}$ implies that the free homotopy classes of $\mathbb{R}^3 \setminus \Gamma$ are in one to one correspondence with the equivalence classes of sequences ν that satisfy (i),(ii),(iii). With reference to the numbering of vertexes of $\mathcal{Q}_{\mathcal{R}}$ in Fig. 2 the result proved in [10] is

Theorem 1.1. *For each sequence ν listed in Table 1 there exists a T-periodic solution $\mathbf{u}_* = \{u_{*R}\}_{R\in\mathcal{R}}$ of the classical Newtonian N-body problem. Moreover $\mathbf{u}_*$ satisfies the symmetry condition (1.1) and u_{*I} is a minimizer of the action integral*

$$\mathcal{A}(u_I) = \frac{1}{2}\int_0^T \Big(|\dot{u}_I|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_I|} \Big)\, dt,$$

in the set of T-periodic H^1 maps in the homotopy class determined by ν.

Table 1: The sequences are given with reference to the enumeration of vertexes of $\mathcal{Q}_{\mathcal{R}}$ in Figure 2.

$\mathcal{R}$	ν
$\mathcal{T}$	$\nu^1 = [1, 8, 5, 2, 6, 7, 1]$
	$\nu^2 = [2, 7, 9, 12, 3, 10, 2]$
	$\nu^3 = [1, 8, 5, 10, 2, 6, 7, 9, 1]$
	$\nu^4 = [1, 7, 6, 4, 9, 1, 5, 2, 7, 9, 12, 8, 1]$
	$\nu^5 = [1, 5, 8, 12, 3, 10, 11, 4, 6, 2, 7, 9, 1]$
$\mathcal{O}$	$\nu^1 = [1, 14, 23, 11, 5, 16, 1]$
	$\nu^2 = [1, 16, 10, 3, 7, 20, 23, 14, 1]$
	$\nu^3 = [1, 5, 16, 1, 3, 7, 18, 20, 7, 14, 1]$
	$\nu^4 = [16, 10, 3, 8, 18, 20, 23, 14, 11, 5, 16]$
	$\nu^5 = [1, 16, 10, 8, 18, 7, 3, 10, 6, 15, 8, 3, 1]$
	$\nu^6 = [9, 22, 6, 15, 8, 3, 7, 20, 23, 11, 19, 17, 9]$
	$\nu^7 = [11, 5, 2, 22, 16, 10, 6, 15, 8, 18, 13, 24, 20, 23, 21, 19, 11]$
	$\nu^8 = [4, 15, 8, 10, 3, 7, 18, 20, 23, 14, 11, 19, 21, 17, 9, 2, 22, 6, 4]$
	$\nu^9 = [1, 5, 16, 1, 3, 7, 14, 23, 11, 14, 1, 3, 7, 18, 20, 7, 14, 1, 3, 10, 8, 3, 7, 14, 1]$
$\mathcal{I}$	$\nu^1 = [11, 48, 34, 14, 42, 28, 11]$
	$\nu^2 = [11, 48, 34, 42, 28, 11, 6, 15, 48, 28, 45, 19, 11]$
	$\nu^3 = [11, 19, 43, 50, 1, 3, 7, 47, 6, 11, 28, 45, 19, 1, 54, 59, 3, 6, 15, 48, 11]$
	$\nu^4 = [1, 54, 59, 3, 6, 11, 28, 42, 14, 24, 2, 58, 60, 4, 8, 14, 34, 48, 11, 19, 1]$
	$\nu^5 = [15, 48, 34, 42, 28, 45, 31, 32, 43, 50, 36, 51, 54, 59, 52, 12, 7, 47, 33, 25, 15]$
	$\nu^6 = [26, 49, 38, 34, 14, 8, 20, 31, 27, 16, 37, 36, 35, 23, 53, 52, 13, 40, 21, 33, 26]$
	$\nu^7 = [26, 49, 24, 38, 34, 42, 14, 8, 41, 20, 31, 32, 27, 16, 46, 37, 36, 51, 35, 23, 22,$ $53, 52, 12, 13, 40, 39, 21, 33, 25, 26]$
	$\nu^8 = [11, 19, 43, 50, 54, 59, 3, 6, 11, 19, 1, 54, 59, 7, 47, 6, 11, 19, 1, 3, 7, 47, 15,$ $48, 11, 19, 1, 3, 6, 15, 48, 28, 45, 19, 1, 3, 6, 11, 28, 45, 43, 50, 1, 3, 6, 11]$

For animations of some of the motions in Theorem 1.1 click here:[2]
$(\mathcal{T}, \nu_4)$, $(\mathcal{O}, \nu_7)$, $(\mathcal{I}, \nu_3)$.

[2]see also [11].

Special motions satisfying (1.1) were considered in [8], [12]. Many other interesting motions of the N-body problem subjected to various kinds of symmetry and topological constraints were discovered in the last ten years. See for instance [3], [4], [6], [9], [13], [15], [16], [17], [14] and the references therein.

In this paper we consider interaction potentials of the form

$$U_\alpha(r) = \frac{1}{r^\alpha}, \qquad \alpha > 0, r > 0 \text{the inter-particle distance,}$$

which includes as a special case the Newtonian potential ($\alpha = 1$).

Here we concentrate on the case $\alpha > 1$. Simple computations indicate that the contribution of collisions to the action integral increases with α and diverges to $+\infty$ as $\alpha \nearrow 2$. For example, if we consider two unit masses moving on a line and colliding in the center of mass 0 at $t = 0$, from the conservation of energy we have

$$\dot{x}^2 = \frac{1}{2^\alpha x^\alpha} + h \tag{1.4}$$

where x is the distance from 0. Since the constant h has a negligible effect on the behavior of x near $x = 0$ we set $h = 0$ in (1.4), which then yields $x(t) = \frac{1}{2}[(2+\alpha)t]^{\frac{2}{2+\alpha}}$. It follows

$$\int_0^1 2\big(\dot{x}^2 + \frac{1}{2^\alpha x^\alpha}\big)dt = \frac{4}{2-\alpha}(2+\alpha)^{\frac{2-\alpha}{2+\alpha}} \to +\infty \qquad \text{as } \alpha \nearrow 2. \tag{1.5}$$

Therefore we can expect that for each free homotopy class of $\mathbb{R}^3 \setminus \Gamma$ which is *tied* to Γ (see (1.6)) there is a critical value $\alpha_{cr} < 2$ of α such that, if u_{*I} is a minimizer of the action

$$\mathcal{A}_\alpha(u_I) = \frac{1}{2}\int_0^T \Big(|\dot{u}_I|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_I|^\alpha}\Big)\, dt,$$

then u_{*I} is smooth for $\alpha > \alpha_{cr}$, while has collisions for $\alpha \leq \alpha_{cr}$. We prove

Theorem 1.2. *For each sequence ν that satisfies conditions* (i),(ii),(iii) *above and*

$$\nu \not\subset \overline{\mathcal{F}}, \ \textit{for all the faces } \mathcal{F} \textit{ of } \mathcal{Q}_\mathcal{R}, \tag{1.6}$$

there exists $\alpha_\nu < 2$ such that for each $\alpha > \alpha_\nu$ there is a classical solution $\mathbf{u}_*^\alpha = \{u_{*R}^\alpha\}_{R\in\mathcal{R}}$ of the N-body problem with interaction potential U_α. Moreover $\mathbf{u}_*^\alpha$ satisfies the symmetry condition (1.1) and u_{*I}^α is a minimizer of the action integral

$$\mathcal{A}_\alpha(u_I) = \frac{1}{2}\int_0^T \Big(|\dot{u}_I|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_I|^\alpha}\Big)dt,$$

in the set $H_T^1(\mathbb{R};\mathbb{R}^3)$ of T-periodic maps in the homotopy class determined by ν.

For animations of orbits in Theorem 1.2 for the sequences in Table 2 click here:

$(\mathcal{O}, \nu_1)$, $(\mathcal{O}, \nu_2)$, $(\mathcal{I}, \nu_1)$.

Condition (1.6) ensures that the orbit of $u_I \in \mathcal{H}_\nu^1$ is tied to Γ in the sense that does not winds around a single axis of rotation in Γ. This implies the geometric inequality

$$\|u_I\|_\infty \leq c_0 \int_0^T |\dot{u}_I| dt \tag{1.7}$$

Table 2: The sequences refer to the enumeration of vertexes of $\mathcal{Q}_{\mathcal{R}}$ in Figure 2.

$\mathcal{R}$	ν
$\mathcal{O}$	$\nu_1 = [10, 3, 1, 16, 10, 3, 7, 14, 1, 3, 8, 18, 7, 3, 8, 15, 13, 18, 8, 10, 6, 15, 8, 10, 16, 22, 6, 10]$
	$\nu_2 = [1, 16, 10, 3, 1, 16, 10, 3, 1, 14, 7, 3, 1, 14, 7, 20, 23, 14, 7, 20, 23, 14, 7, 3, 1, 14, 7, 3, 1]$
$\mathcal{I}$	$\nu_1 = [1, 54, 50, 1, 3, 7, 59, 3, 6, 15, 47, 6, 11, 28, 48, 11, 19, 43, 45, 19, 1]$

for some constant $c_0 > 0$ independent of $\mathcal{R} \in \{\mathcal{T}, \mathcal{O}, \mathcal{I}\}$ and of ν satisfying (1.6).

It is natural to investigate the asymptotic behavior for $\alpha \to +\infty$ of the motions given in Theorem 1.2. We analyze this question in Section 3. In particular we show that there exists an asymptotic shape of the trajectory of u_I which is determined by a geometric minimization with unilateral constraints, see Theorem 3.2.

Since if $\alpha < 2$ collisions give a finite contribution to the action integral, we may conjecture that, for fixed $\alpha < 2$, the sequences ν corresponding to free homotopy classes that contain a smooth minimizer of $\mathcal{A}_\alpha$ have a bounded *length*

$$K < K_\alpha,$$

where $K_\alpha > 0$ is determined by α. Indeed it can be expected that to minimize $\mathcal{A}_\alpha$, if K is too large, it may be more convenient to *crash* part of the orbit into a collision rather than to keep wandering around the axes of the rotations in $\mathcal{R}$.

For $\alpha \geq 2$, the so called *strong force* case, there are no collisions and therefore no upper bound for the length K of ν should be expected. This suggests the existence of *aperiodic* orbits of *infinite* length obtained as limit of sequences of periodic orbits corresponding to sequences ν^j of length K_j converging to $+\infty$. Our next theorem states that this is indeed the case. To state the theorem we utilize a different way of coding the free homotopy classes of $\mathbb{R}^3 \setminus \Gamma$. We have already remarked that $\mathbb{R}^3 \setminus \Gamma$ is homotopically equivalent to $\mathbb{S}^2 \setminus \mathcal{P}$ and therefore to the plane Π_Q punctured with $Q = \sharp\mathcal{P} - 1$ distinct points.
Π_Q is homotopically equivalent to the union Σ_Q of Q copies $\mathbb{S}^1_1, \ldots, \mathbb{S}^1_Q$ of $\mathbb{S}^1$ with a common point O.

Each free homotopy class of Σ_Q determines, up to translation, a periodic sequence $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$ with $2Q$ symbols

$$\omega_i \in \{\sigma_1, \ldots, \sigma_Q, \sigma_1^{-1}, \ldots, \sigma_Q^{-1}\},$$

where σ_i corresponds to traveling $\mathbb{S}^1_i$ from O and back to O in a preassigned positive direction and σ_i^{-1} to the inverse path. We assume that ω is reduced in the sense that it does not contain expressions like $\sigma_i\sigma_i^{-1}$ corresponding to loops homotopic to O. Beside ω, each free homotopy class of Σ_Q uniquely determines a number $M \in \mathbb{N}$, which coincides with one of the periods of ω. Indeed the same ω can be associated to different homotopy classes that can be distinguished by the period attributed to ω. For example we can consider a closed loop that coincides with $\mathbb{S}^1_1$ and a loop that describes $\mathbb{S}^1_1$ more than once, say $M > 1$ times before closing up. All these loops have the same ω but are topologically distinct and can be classified by the value of M. In conclusion: each free homotopy class of Σ_Q determines (an equivalence class of) ω and a particular period M of ω and viceversa. We let Ω be the set of these sequences with the associated period and define a distance in Ω by setting

$$d(\omega, \omega') = |f(M) - f(M')| + \min_{h\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \frac{\delta(\omega_{i+h}, \omega'_i)}{2^{|i|}}, \tag{1.8}$$

where $f : [0, +\infty) \to \mathbb{R}$, $f(0) = 0$, is a strictly increasing bounded function and δ is the discrete metric. We let $\hat{\Omega}$ be the completion of Ω with respect to the metric (1.8).

We need to characterize the subset $\Omega_0 \subset \Omega$ of the sequences corresponding to orbits that *coil* around just one of the axes in Γ. We identify Π_Q with the stereographic projection of $\mathbb{S}^2 \setminus \mathcal{P}$, from some $p_0 \in \mathcal{P}$, on the plane tangent to $\mathbb{S}^2$ at $-p_0$.

If we consider the constant sequence $\omega = \{\omega_i\}_{i\in\mathbb{Z}}$, with $\omega_i = \tilde{\sigma}\ \forall i$ for some $\tilde{\sigma} \in \{\sigma_1, \dots, \sigma_Q, \sigma_1^{-1}, \dots, \sigma_Q^{-1}\}$, then $\omega \in \Omega_0$ and corresponds to an orbit winding around the axis a_p through some $p \in \mathcal{P} \setminus \{p_0\}$. The remaining $\omega \in \Omega_0$ correspond to orbits winding around a_{p_0}. These are the sequences ω which have $M = nQ$ for some $n \geq 1$, and satisfy (up to translations) $\omega_i = \sigma_i$ for $i = 1, \dots, Q$, or $\omega_i = \sigma_i^{-1}$ for $i = 1, \dots, Q$ and $\omega_{i+jQ} = \omega_i$ for $j = 1, \dots, n-1$.

For each $\omega \in \Omega$ we let $\mathcal{H}^1_\omega$ be the set of $H^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^3)$ periodic maps in the free homotopy class determined by ω. Given $\alpha > 0$ and $H \in \mathbb{R}$, for each H^1_{loc} periodic map $v_I : \mathbb{R} \to \mathbb{R}^3$ we define

$$\mathcal{A}_{\alpha,H}(v_I) = \int_0^{T_{v_I}} \Big(H + \frac{1}{2} \Big[|\dot{v}_I|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)v_I|^\alpha} \Big] \Big)\, dt. \tag{1.9}$$

Notice that in (1.9) the period T_{v_I} is not a fixed constant but it is allowed to depend on v_I. We are now in the position to state our main result.

Theorem 1.3. *Let $\alpha > 2$ and $H > 0$ be fixed. Then*

(I) For each $\omega \in \Omega \setminus \Omega_0$ there is a periodic solution $\mathbf{u}_*^{\alpha,H} = \{u_{*R}^{\alpha,H}\}_{R\in\mathcal{R}}$ of the N-body problem with interaction potential U_α. Moreover $\mathbf{u}_*^{\alpha,H}$ satisfies (1.1) and $u_{*I}^{\alpha,H} \in \mathcal{H}^1_\omega$ lies on the surface of constant energy

$$|\dot{v}_I|^2 - \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)v_I|^\alpha} = 2H, \tag{1.10}$$

and minimizes $\mathcal{A}_{\alpha,H}$ on $\mathcal{H}^1_\omega$.

(II) Given a sequence $\{\omega^j\}_{j\in\mathbb{N}} \subset \Omega\setminus\Omega_0$ and the corresponding sequence $\{\mathbf{u}_*^{\alpha,H,j}\}_{j\in\mathbb{N}}$ from (I), there exist a subsequence $\{j_h\}_{h\in\mathbb{N}}$, $\hat{\omega} \in \hat{\Omega}$ and a C^∞ bounded solution $\hat{\mathbf{u}}_*^{\alpha,H}$ of the N-body problem such that

$$\hat{\mathbf{u}}_*^{\alpha,H}(t) = \lim_{h\to\infty} \mathbf{u}_*^{\alpha,H,j_h}(t), \tag{1.11}$$

locally in $C^{2+\frac{1}{2}}$, and

$$\hat{\omega} = \lim_{h\to\infty} \omega^{j_h}. \tag{1.12}$$

Moreover $\hat{\mathbf{u}}_*^{\alpha,H}$ satisfies (1.1) and $\hat{u}_{*I}^{\alpha,H}$ lies on the surface of constant energy (1.10).

(III) Given $\hat{\omega} \in \hat{\Omega} \setminus \Omega_0$ there exist a sequence $\{\omega^j\}_{j\in\mathbb{N}} \subset \Omega \setminus \Omega_0$ and a bounded solution $\hat{\mathbf{u}}_*^{\alpha,H}$ of the N-body problem such that $\omega^j \to \hat{\omega}$ and $\mathbf{u}_*^{\alpha,H,j} \to \hat{\mathbf{u}}_*^{\alpha,H}$, where $\mathbf{u}_*^{\alpha,H,j}$ is the map associated to ω^j by (I).

In the following we drop the subscript I and we write simply u, v in place of u_I, v_I.

The paper is organized as follows. In Section 2 we prove Theorem 1.2, in Section 3 we investigate the asymptotic behavior of the motions in Theorem 1.2 for $\alpha \to +\infty$. In Section 4 we prove Theorem 1.3.

2 The proof of Theorem 1.2

Given a sequence ν that satisfies (i), (ii), (iii), we let $\mathcal{H}^1_\nu$ be the set of T-periodic maps $u \in H^1_{\rm loc}(\mathbb{R};\mathbb{R}^3)$ in the free homotopy class of $\mathbb{R}^3 \setminus \Gamma$ determined by ν. We prove Theorem 1.2 by a variational technique: we show that if ν satisfies (i), (ii), (iii) and also condition (1.6), then the action integral

$$\mathcal{A}_\alpha(u) = \frac{1}{2}\int_0^T \Big(|\dot{u}|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha}\Big)dt$$

possesses a minimizer u^α_* in $\mathcal{H}^1_\nu$. We then show that there is $\alpha_\nu < 2$ such that $\alpha > \alpha_\nu$ implies u^α_* is free of collisions and therefore is in the interior of $\mathcal{H}^1_\nu$ and thus a smooth solution of the Euler-Lagrange equations. The possibility that, when $\alpha < 2$, a minimizer of the action presents collisions is one of the main obstructions to the variational approach to the existence of periodic motions for the N-body problem. We refer to [1], [5] and [9] for various results on the problem of collisions.

Lemma 2.1. *Given a sequence ν that satisfies* (i), (ii), (iii) *there is a constant $A_\nu > 0$ with the property that for each $\alpha \in (0,+\infty)$ there exists a map $\bar{u}^\alpha \in \mathcal{H}^1_\nu$ such that*

$$\mathcal{A}_\alpha(\bar{u}^\alpha) \leq A_\nu. \tag{2.1}$$

Proof. We define the orbit of $\bar{u}$ to be the orbit O_ν defined by ν on $\mathcal{L}_\mathcal{R}$ (see Section 1) and assume that the generating particle moves on O_ν with constant speed. We can choose $\lambda > 1$ such that $\lambda|(R-I)\bar{u}| > 1$ for $R \in \mathcal{R}\setminus\{I\}$. Set $\bar{u}^\alpha = \lambda\bar{u}$. Then the action $\mathcal{A}_\alpha(\bar{u}^\alpha)$ is well defined and bounded in $\alpha \in (0,+\infty)$.

□

Remark. From (2.1) and Hölder's inequality it follows an L^∞ bound independent of α:

$$\|u\|_\infty \leq c_0 T^{1/2} A_\nu^{1/2},$$

which is valid for each $u \in \mathcal{H}^1_\nu$ satisfying $\mathcal{A}_\alpha(u) \leq A_\nu$.

Lemma 2.2. *Assume that ν satisfies* (i), (ii), (iii) *and condition (1.6). Then $\mathcal{A}_\alpha$ is coercive on $\mathcal{H}^1_\nu$.*

Proof. It follows immediately from (1.7) and Hölder's inequality. □

Lemma 2.3. *Assume that ν satisfies* (i), (ii), (iii) *and let A_ν be as in Lemma 2.1. Then there exists $\alpha_\nu < 2$ such that*

$$\alpha > \alpha_\nu,\ \ u \in \mathcal{H}^1_\nu,\ \ \textit{and}\ \ \mathcal{A}_\alpha(u) \leq A_\nu \ \ \Rightarrow\ \ u(\mathbb{R})\cap\Gamma = \emptyset.$$

Proof. Let $r = \min_{R\in\mathcal{R}\setminus\{I\}}\{|(R-I)u|\}$ and $\bar{r} = \max_{t\in[0,T)} r(t)$. Then

$$A_\nu \geq \mathcal{A}_\alpha(u) \geq \frac{1}{2}\int_0^T \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha}dt \geq \int_0^T \frac{1}{2r^\alpha}dt \geq \frac{T}{2\bar{r}^\alpha},$$

and therefore

$$\bar{r} \geq \Big(\frac{T}{2A_\nu}\Big)^{\frac{1}{\alpha}}. \tag{2.2}$$

Moreover we have

$$\mathcal{A}_\alpha(u) \geq \int_0^T \sqrt{\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha}}|\dot{u}|dt = \int_0^L \sqrt{\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha}}ds, \tag{2.3}$$

where s is arc-length and $L = \int_0^T |\dot{u}|dt$. Now we observe that from

$$\Big|\frac{dr}{ds}\Big| = \Big|\frac{d}{ds}\min_{R\in\mathcal{R}\setminus\{I\}}|(R-I)u|\Big| \leq \max_{R\in\mathcal{R}\setminus\{I\}}\Big|\frac{d}{ds}|(R-I)u|\Big|, \qquad \text{a.e.}$$

$$\begin{aligned}\Big|\frac{d}{ds}|(R-I)u|\Big| &= \Big|\langle (R-I)\frac{du}{ds}, \frac{(R-I)u}{|(R-I)u|}\rangle\Big| \leq \Big|(R-I)\frac{du}{ds}\Big| \\ &\leq \max_{\substack{R\in\mathcal{R}\setminus\{I\}\\ \eta\in\mathbb{S}^2}} |(R-I)\eta| =: k_{\mathcal{R}}, \qquad \text{a.e.}\end{aligned}$$

where $k_{\mathcal{R}}$ depends on the group $\mathcal{R}$. It follows that

$$\Big|\frac{dr}{ds}\Big| \leq k_{\mathcal{R}}. \tag{2.4}$$

Assume that there is an arc of length $\bar{s} > 0$ such that $r(0) = 0$ and $r(\bar{s}) = \bar{r}$. Then, from (2.3), the definition of r and (2.4) we have

$$A_\nu \geq \int_0^{\bar{s}} \sqrt{\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha}}ds \geq \int_0^{\bar{s}} r^{-\frac{\alpha}{2}}ds \geq \frac{1}{k_{\mathcal{R}}}\int_0^{\bar{r}} r^{-\frac{\alpha}{2}}dr.$$

This and (2.2) imply

$$\frac{2^{\frac{3\alpha-2}{2\alpha}}}{k_{\mathcal{R}}(2-\alpha)}\frac{T^{\frac{2-\alpha}{2\alpha}}}{A_\nu^{\frac{2+\alpha}{2\alpha}}} \leq 1. \tag{2.5}$$

Inequality (2.5) is violated for $\alpha > \alpha_\nu$, for some $\alpha_\nu < 2$ that can be estimated from (2.5). □

We can now complete the proof of Theorem 1.2. Lemma 2.1 and Lemma 2.2 and standard arguments of variational calculus show that, provided ν satisfies (i), (ii), (iii) and (1.6), there exists $u_*^\alpha \in \mathcal{H}_\nu^1$ that satisfies

$$\mathcal{A}_\alpha(u_*) = \min_{u\in\mathcal{H}_\nu^1} \mathcal{A}_\alpha(u).$$

Lemma 2.3 implies that there exists $\alpha_\nu < 2$ such that, for $\alpha > \alpha_\nu$, u_*^α is collision free and therefore, by elliptic regularity, is a smooth solution of the equation of motion

$$\ddot{u} = \alpha \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{(R-I)u}{|(R-I)u|^{\alpha+2}}. \tag{2.6}$$

The proof of Theorem 1.2 is complete.

3 Asymptotic behavior of minimizers for $\alpha \to +\infty$

In this section we fix a sequence ν as in Theorem 1.2. We discuss the asymptotic behavior for $\alpha \to +\infty$ of the solution u_*^α of the N-body problem determined in that theorem. If $\alpha \gg 1$ the force of attraction between particles is very small (very large) when the inter-particle distance is larger than 1 (smaller than 1). This observation suggests that

1) the limit behavior of minimizers u_*^α for $\alpha \to +\infty$ is constrained to the sub-region $\mathcal{Y} \subset \mathbb{R}^3$ of the configuration space defined by

$$\mathcal{Y} = \{x \in \mathbb{R}^3 : |(R - I)x| \geq 1 \ \forall R \in \mathcal{R} \setminus \{I\}\} .$$

Indeed violating this condition generates, if $\alpha \gg 1$, a large contribution of the potential term to the action integral;

2) in the limit $\alpha \to +\infty$ the trajectory of the minimizer u_*^α tends to the shortest path compatible with condition $|(R - I)x| \geq 1$, $\forall R \in \mathcal{R} \setminus \{I\}$ and with the topological constraint defined by ν.

For $\alpha \to +\infty$ the inter-particle attraction should act as a *unilateral holonomic constraint* and the limit motion should be a kind of *geodesic motion* with constant kinetic energy. Then minimizing the action should be equivalent to minimizing the length of the trajectory of the generating particle.

For each $p \in \mathcal{P}$ let $Cyl_p \subset \mathbb{R}^3$ be the open cylinder with axis the line through O and P and radius $r_p = \frac{1}{2\sin(\pi/o_p)}$, where o_p is the order of the pole p.[3] Then we have

$$\mathcal{Y} = \mathbb{R}^3 \setminus \bigcup_{p \in \mathcal{P}} Cyl_p .$$

We first characterize the asymptotic behavior of the action $\mathcal{A}_\alpha$ in the sense of Γ-convergence, see [2].

Definition. Let $\mathcal{X}$ be a metric space. We say that a sequence $\{\mathcal{F}_\epsilon\}_{\epsilon>0}$, $\mathcal{F}_\epsilon : \mathcal{X} \to \mathbb{R}\cup\{+\infty\}$ Γ-converges in $\mathcal{X}$ to $\mathcal{F} : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ if for each $x \in \mathcal{X}$

(LB) for every sequence $\{x_\epsilon\}$ that converges to x as $\epsilon \to 0$

$$\liminf_{\epsilon\to 0} \mathcal{F}_\epsilon(x_\epsilon) \geq \mathcal{F}(x); \tag{3.1}$$

(UB) there exists a sequence $\{x_\epsilon\}$ converging to x as $\epsilon \to 0$ such that

$$\limsup_{\epsilon\to 0} \mathcal{F}_\epsilon(x_\epsilon) \leq \mathcal{F}(x). \tag{3.2}$$

The functional $\mathcal{F}$ is called the Γ-limit of $\{\mathcal{F}_\epsilon\}$ and we write

$$\Gamma \overset{\mathcal{X}}{-} \lim_{\epsilon\to 0} \mathcal{F}_\epsilon = \mathcal{F}.$$

[3] r_p is the radius of the circumcircle of a regular polygon with $o(p)$ sides of unitary length.

Theorem 3.1. *Let $\mathcal{L}^2_\nu$ be the L^2-closure of $\mathcal{H}^1_\nu$, and let $\mathcal{A}_\alpha : \mathcal{L}^2_\nu \to \mathbb{R} \cup \{+\infty\}$, $\mathcal{A}_\infty : \mathcal{L}^2_\nu \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$\mathcal{A}_\alpha(u) = \begin{cases} \frac{1}{2}\int_0^T (|\dot{u}|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha})dt, & u \in \mathcal{H}^1_\nu, \\ +\infty, & u \in \mathcal{L}^2_\nu \setminus \mathcal{H}^1_\nu, \end{cases}$$

$$\mathcal{A}_\infty(u) = \begin{cases} \frac{1}{2}\int_0^T |\dot{u}|^2 dt, & u \in \mathcal{H}^1_\nu,\ u(\mathbb{R}) \cap (\cup_{p\in\mathcal{P}} \mathrm{Cyl}_p) = \emptyset \\ +\infty, & u \in \mathcal{L}^2_\nu \setminus \mathcal{H}^1_\nu \ \text{ or } u(\mathbb{R}) \cap (\cup_{p\in\mathcal{P}} \mathrm{Cyl}_p) \neq \emptyset \end{cases}$$

Then

$$\Gamma \overset{L^2}{-} \lim_{\alpha\to+\infty} \mathcal{A}_\alpha = \mathcal{A}_\infty$$

Proof. We apply Definition 3 with $\mathcal{X} = \mathcal{L}^2_\nu$ and $\epsilon = 1/\alpha$. Fix $u \in \mathcal{L}^2_\nu$.

Proof of (LB). Let $u^\alpha \overset{L^2}{\to} u$. If $\liminf_{\alpha\to+\infty} \mathcal{A}_\alpha(u^\alpha) = +\infty$ there is nothing to prove. Therefore we assume $\liminf_{\alpha\to+\infty} \mathcal{A}_\alpha(u^\alpha) = A_\infty < +\infty$. It follows that there is a sequence u^{α_j} such that

i) $\lim_{\alpha\to+\infty} \mathcal{A}_\alpha(u^{\alpha_j}) = A_\infty$;

ii) u^{α_j} converges weakly in H^1 to u;

iii) u^{α_j} converges to u also in L^∞.

By the lower semicontinuity of the L^2 norm we have

$$A_\infty \geq \liminf_{j\to\infty} \frac{1}{2}\int_0^T |\dot{u}^{\alpha_j}|^2 dt \geq \frac{1}{2}\int_0^T |\dot{u}|^2 dt = \mathcal{A}_\infty(u).$$

This concludes the proof of (LB).

Proof of (UB). If $\mathcal{A}_\infty(u) = +\infty$ then there is nothing to prove. Assume $\mathcal{A}_\infty(u) < +\infty$ and consider the sequence

$$u^\alpha = (1 + \frac{1}{\sqrt{\alpha}})u. \tag{3.3}$$

Then from

$$\lim_{\alpha\to+\infty} \frac{1}{2}\int_0^T (1+\frac{1}{\sqrt{\alpha}})^2 |\dot{u}|^2 dt = \mathcal{A}_\infty(u),$$
$$\lim_{\alpha\to+\infty} \frac{1}{(1+\frac{1}{\sqrt{\alpha}})^\alpha} \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha} dt = 0$$

it follows that

$$\lim_{\alpha\to+\infty} \mathcal{A}_\alpha(u^\alpha) = \mathcal{A}_\infty(u).$$

This concludes the proofs of (UB) and of the theorem.

□

As far as the limit behavior of the minimizers u^α_* for $\alpha \to +\infty$ is concerned we have

Theorem 3.2. *Let $\{\alpha_j\}_{j\in\mathbb{N}}$ be a sequence that converges to $+\infty$. Then there is a subsequence $\{u_*^{\alpha_{j_h}}\}_{h\in\mathbb{N}}$ such that*

$$\lim_{h\to\infty} \|u_*^{\alpha_{j_h}} - u_*^\infty\|_{L^\infty(\mathbb{R};\mathbb{R}^3)} = 0,$$

where u_*^∞ is a minimizer of $\mathcal{A}_\infty$ in $\mathcal{H}^1_\nu$. Moreover

$$\lim_{h\to\infty} \mathcal{A}_{\alpha_{j_h}}(u_*^{\alpha_{j_h}}) = \mathcal{A}_\infty(u_*^\infty). \tag{3.4}$$

Proof. Let $\bar{u}_*^\infty \in \mathcal{H}^1_\nu$ be a minimizer of $\mathcal{A}_\infty$ and set

$$\bar{u}^\alpha = (1 + \frac{1}{\sqrt{\alpha}})\bar{u}_*^\infty. \tag{3.5}$$

As in the proof of Theorem 3.1 we have

$$\lim_{\alpha\to+\infty} \mathcal{A}_\alpha(\bar{u}^\alpha) = \mathcal{A}_\infty(\bar{u}_*^\infty), \tag{3.6}$$

therefore

$$\mathcal{A}_\alpha(\bar{u}^\alpha) \leq \overline{A}$$

for some $\overline{A}$ independent of $\alpha \in [2, +\infty)$. To conclude the proof we need the following lemmas.

Lemma 3.3. *Given $\delta \in (0,1)$ there is $\alpha_\delta > 0$ such that for $\alpha > \alpha_\delta$, given $R \in \mathcal{R} \setminus \{I\}$ and $u_I \in \mathcal{H}^1_\nu$ with $\mathcal{A}_\alpha(u_I) \leq \overline{A}$, we have*

$$|(R-I)u_I(\bar{t})| > 1 - \delta,$$

for some $\bar{t} \in [0,T)$ depending on α, R, u.

Proof. Assume that $|(R-I)u_I(t)| \leq 1-\delta$ for all $t \in [0,T)$. Then

$$\frac{T}{(1-\delta)^\alpha} \leq \int_0^T \frac{1}{|(R-I)u_I|^\alpha} dt \leq \mathcal{A}_\alpha(u_I) \leq \overline{A}$$

which is not true for α large enough.

□

Lemma 3.4. *Given $\delta \in (0,1)$ there is $\alpha_\delta > 0$ such that for $\alpha > \alpha_\delta$ we have*

$$\min_{t\in[0,T]} |(R-I)u_I(t)| > 1 - \delta \tag{3.7}$$

for each $R \in \mathcal{R} \setminus \{I\}$ and for each $u_I \in \mathcal{H}^1_\nu$ such that $\mathcal{A}_\alpha(u_I) \leq \overline{A}$.

Proof. Fix $\delta > 0$. From Lemma 3.3 there exist $\alpha_{\delta/2} > 0$ and $\bar{t}$ such that

$$|(R-I)u_I(\bar{t})| > 1 - \delta/2$$

for each $\alpha > \alpha_{\delta/2}$. If (3.7) does not hold there are t_1, t_2, $0 < t_2 - t_1 < T$ such that

$$\begin{cases} |(R-I)u_I(t_1)| = 1-\delta, \quad |(R-I)u_I(t_2)| = 1 - \dfrac{\delta}{2}, \\ 1-\delta \leq |(R-I)u_I(t)| \leq 1 - \dfrac{\delta}{2}, \quad \text{for all } t \in [t_1, t_2]. \end{cases}$$

We have

$$\frac{\tau}{(1-\delta/2)^{\alpha}} \leq \sum_{R\in\mathcal{R}\setminus\{I\}} \int_0^T \frac{1}{|(R-I)u_I|^{\alpha}}\,dt, \tag{3.8}$$

where $\tau = t_2 - t_1$. On the other hand

$$\begin{aligned}\frac{\delta}{2} &= |(R-I)u_I(t_2)| - |(R-I)u_I(t_1)| \leq \|R-I\||u_I(t_2)-u_I(t_1)| \\ &\leq 2|u_I(t_2)-u_I(t_1)| \leq 2\int_0^{\tau} |\dot{u}|dt \leq 2\Big(\tau\int_0^T |\dot{u}|^2 dt\Big)^{1/2}\end{aligned}$$

From this and (3.8) it follows

$$\frac{1}{2}\frac{\tau}{(1-\delta/2)^{\alpha}} + \frac{\delta^2}{32\tau} \leq \mathcal{A}_{\alpha}(u) \leq \overline{A}. \tag{3.9}$$

Therefore

$$\min_{\tau\in[0,T]}\Big(\frac{1}{2}\frac{\tau}{(1-\delta/2)^{\alpha}} + \frac{\delta^2}{32\tau}\Big) = \frac{\delta}{8(1-\delta/2)^{\alpha/2}} \leq \overline{A},$$

which is not true for α large enough.

□

For a minimizer $u_*^{\alpha} \in \mathcal{H}_{\nu}^1$ we have

$$\mathcal{A}_{\alpha}(u_*^{\alpha}) \leq \mathcal{A}_{\alpha}(\bar{u}^{\alpha}) \leq \overline{A} \tag{3.10}$$

From this and the remark after Lemma 2.1 it follows that the family $\{u_*^{\alpha}\}_{\alpha}$ is bounded in H_T^1. Therefore there exists a sequence $\alpha_j \to +\infty$, and $v^{\infty} \in \mathcal{H}_{\nu}^1$ such that $\{u_*^{\alpha_j}\}_j$ weakly converges in H_T^1 to v^{∞} . We can also assume that $\{u_*^{\alpha_j}\}_j$ converges to v^{∞} in L^{∞}.
From Lemma 3.4 the map v^{∞} satisfies the constraint

$$v^{\infty}(\mathbb{R}) \cap (\cup_{p\in\mathcal{P}}\mathrm{Cyl}_p) = \emptyset.$$

From the lower semicontinuity of the L^2-norm we have

$$\liminf_{j\to\infty} \mathcal{A}_{\alpha}(u_*^{\alpha_j}) \geq \liminf_{j\to\infty} \frac{1}{2}\int_0^T |\dot{u}_*^{\alpha_j}|^2 dt \geq \frac{1}{2}\int_0^T |\dot{v}_I^{\infty}|^2 dt = \mathcal{A}_{\infty}(v^{\infty}). \tag{3.11}$$

On the other hand, from (3.10) and (3.6) it follows that

$$\limsup_{j\to\infty} \mathcal{A}_{\alpha}(u_*^{\alpha_j}) \leq \limsup_{j\to\infty} \mathcal{A}_{\alpha}(\bar{u}^{\alpha_j}) = \mathcal{A}_{\infty}(\bar{u}_*^{\infty}) \tag{3.12}$$

It follows that

$$\mathcal{A}_{\infty}(v^{\infty}) \leq \mathcal{A}_{\infty}(\bar{u}_*^{\infty}) \tag{3.13}$$

therefore in (3.13) the equality holds and v^{∞} is a minimizer of $\mathcal{A}_{\infty}$. Moreover, from (3.11), (3.12) follows the existence of the limit (3.4). Therefore we take $u_*^{\infty} = v^{\infty}$. The proof of Theorem 3.2 is concluded.

□

For animations of orbits in Theorem 3.2 for the sequences in Table 3 click here:

$(\mathcal{O}, \nu_1)$, $(\mathcal{O}, \nu_2)$.

Table 3: The sequences refer to the enumeration of vertexes of $\mathcal{Q}_{\mathcal{R}}$ in Figure 2.

$\mathcal{R}$	ν
$\mathcal{O}$	$\nu_1 = [3, 10, 8, 3, 1, 5, 16, 1, 14, 23, 11, 14, 7, 18, 20, 7, 3]$
	$\nu_2 = [3, 10, 8, 3, 1, 14, 7, 3, 1, 5, 16, 1, 14, 7, 3, 1, 14, 23, 11, 14, 7, 3, 1, 14, 7, 18, 20, 7, 3, 1, 14, 7, 3]$

4 The proof of Theorem 1.3

4.1 Minimizing the action in a given homotopy class

In the following lemmas we set $A^{\omega}_{\alpha,H} = \mathcal{A}_{\alpha,H}(v)$ for a test function $v \in \mathcal{H}^1_{\omega}$ that satisfies (1.10).

Lemma 4.1. *Let $\alpha > 2, H > 0$ and $\omega \in \Omega \setminus \Omega_0$ be given. Then there exist $r^{\omega}_{\alpha} > 0$ and $T^{\omega}_{\alpha} > \tau^{\omega}_{\alpha} > 0$ such that each $u \in \mathcal{H}^1_{\omega}$ with $\mathcal{A}_{\alpha,H}(u) \leq A^{\omega}_{\alpha,H}$ satisfies*

$$d(u(\mathbb{R}), \Gamma) > r^{\omega}_{\alpha}. \tag{4.1}$$

$$\tau^{\omega}_{\alpha} \leq T_u \leq T^{\omega}_{\alpha}. \tag{4.2}$$

Proof. Let

$$r = \min_{R \in \mathcal{R}\setminus\{I\}} |(R - I)u|.$$

Then from the fact that the Lagrangian is the sum of two non-negative terms we have

$$\begin{aligned}
\mathcal{A}_{\alpha,H}(u) = \quad & \int_0^{T_u} \Big[H + \frac{1}{2}\Big(|\dot{u}|^2 + \sum_{R \in \mathcal{R}\setminus\{I\}} \frac{1}{|(R - I)u|^{\alpha}}\Big)\Big] dt \\
& \geq \sqrt{2} \int_0^{T_u} \sqrt{H + \frac{1}{2} \sum_{R \in \mathcal{R}\setminus\{I\}} \frac{1}{|(R - I)u|^{\alpha}}} |\dot{u}| dt \\
& \geq \sqrt{2} \int_0^{T_u} \sqrt{H + \frac{1}{2r^{\alpha}}} |\dot{u}| dt > \int_0^{T_u} r^{-\frac{\alpha}{2}} |\dot{u}| dt.
\end{aligned} \tag{4.3}$$

Set

$$\bar{r} = \inf_{[0,T_u]} r.$$

Assume $\bar{r} = 0$. Then, for each $r_0 > 0$ small enough, there is an interval $[\tau, \tau']$ such that $\frac{r_0}{2} \leq r \leq r_0$, for $t \in [\tau, \tau']$, with $r = r_0$ for $t = \tau$ and $r = \frac{r_0}{2}$ for $t = \tau'$. Therefore, in this case, if we set $L_0 = \int_{\tau}^{\tau'} |\dot{u}| dt$, (4.3) implies

$$\mathcal{A}_{\alpha,H}(u) \geq \int_{\tau}^{\tau'} r^{-\frac{\alpha}{2}} |\dot{u}| dt = \int_0^{L_0} r^{-\frac{\alpha}{2}} ds \geq \frac{1}{k_{\mathcal{R}}} \int_{\frac{r_0}{2}}^{r_0} r^{-\frac{\alpha}{2}} dr, \tag{4.4}$$

where s is arc-length and we have used (2.4).

Since $\alpha > 2$ the last integral in (4.4) diverges to $+\infty$ as $r_0 \to 0^+$ in contradiction with the bound $\mathcal{A}_{\alpha,H}(u) \leq A^{\omega}_{\alpha,H}$. It follows that $\bar{r} > 0$. Since $\omega \in \Omega \setminus \Omega_0$ the length

$L_u = \int_0^{T_u} |\dot{u}|dt$ of the orbit of u has a lower bound of the form $L_u \geq c_\omega \bar{r}$ for some $c_\omega > 0$. By utilizing this and (4.3) we get

$$A^\omega_{\alpha,H} \geq \int_0^{L_u} r^{-\frac{\alpha}{2}}\, ds \geq \frac{1}{k_{\mathcal{R}}} \int_{\bar{r}}^{(1+c_\omega)\bar{r}} r^{-\frac{\alpha}{2}} dr = \frac{2}{k_{\mathcal{R}}(\alpha-2)} \frac{[(1+c_\omega)^{\frac{\alpha-2}{2}} - 1]}{(1+c_\omega)^{\frac{\alpha-2}{2}}} \frac{1}{\bar{r}^{\frac{\alpha-2}{2}}}.$$

This inequality shows the existence of a lower bound $r^\omega_\alpha > 0$ for $\bar{r}$ and establishes (4.1). To prove (4.2) we observe that, by Hölder's inequality, we obtain

$$A^\omega_{\alpha,H} \geq \int_0^{T_u} \frac{1}{2}|\dot{u}|^2 dt \geq \frac{1}{2}\frac{c^2_\omega (r^\omega_\alpha)^2}{T_u}, \tag{4.5}$$

that is $T_u \geq \frac{1}{2}\frac{c^2_\omega (r^\omega_\alpha)^2}{A^\omega_{\alpha,H}}$. The other inequality follows from

$$A^\omega_{\alpha,H} \geq HT_u. \tag{4.6}$$

The proof is concluded. □

Lemma 4.2. *Assume $\alpha > 0$, $H > 0$ and $\omega \in \Omega \setminus \Omega_0$. Then $\mathcal{A}_{\alpha,H}$ is coercive on $\mathcal{H}^1_\omega$, that is*

$$\mathcal{A}_{\alpha,H}(u) \geq c_H \|u\|_{L^\infty(\mathbb{R};\mathbb{R}^3)}, \quad u \in \mathcal{H}^1_\omega \tag{4.7}$$

for some constant $c_H > 0$ independent of ω and u.

Proof. From (1.7) it follows

$$\frac{\sqrt{2}}{c_0} H^{\frac{1}{2}} \|u\|_{L^\infty} \leq \frac{1}{2c_0^2}\frac{\|u\|^2_{L^\infty}}{T_u} + HT_u \leq \int_0^{T_u} \Big[H + \frac{1}{2}|\dot{u}|^2\Big] dt \leq \mathcal{A}_{\alpha,H}(u).$$

This concludes the proof. □

Lemma 4.3. *Assume $\alpha > 2$ and $H > 0$. Then for each $\omega \in \Omega \setminus \Omega_0$ there exists $u^{\alpha,H}_* \in \mathcal{H}^1_\omega$ such that*

$$\mathcal{A}_{\alpha,H}(u^{\alpha,H}_*) = \inf_{u_i \in \mathcal{H}^1_\omega} \mathcal{A}_{\alpha,H}(u). \tag{4.8}$$

Proof. Let $\{u^j\} \subset \mathcal{H}^1_\omega$ be a minimizing sequence and let T_j be the minimal period of u^j. We can assume

$$\mathcal{A}_{\alpha,H}(u^j) \leq A^\omega_{\alpha,H}\,, \qquad j \in \mathbb{N} \tag{4.9}$$

and, up to subsequences, we can also assume

(i) The sequence $\{T_j\}$ is monotone and

$$\tau_\omega \leq \lim_{j\to\infty} T_j = T \leq T_\omega. \tag{4.10}$$

where we have also made use of Lemma 4.1

(ii) There exists a T-periodic map $u_*^{\alpha,H} \in C^{0,\frac{1}{2}}(\mathbb{R};\mathbb{R}^3)$ such that

$$\lim_{j\to\infty} u^j(t) = u_*^{\alpha,H}(t), \qquad t \in \mathbb{R}$$

uniformly on compact sets. This follows from Ascoli-Arzelà's theorem after observing that (4.9) and (4.7) in Lemma 4.2 imply

$$\|u^j\|_{L^\infty} \le \mathrm{C}A^\omega_{\alpha,H}\,, \qquad \text{for } j \in \mathbb{N},\ C = 1/c_H \tag{4.11}$$

while (4.9) and Hölder's inequality yield

$$|u^j(t_1) - u^j(t_2)| \le |t_1 - t_2|^{\frac{1}{2}} \left| \int_{t_1}^{t_2} |\dot{u}^j|^2 dt \right|^{\frac{1}{2}} \le |t_1 - t_2|^{\frac{1}{2}} \sqrt{2A^\omega_{\alpha,H}}.$$

(iii) From (4.9) and (4.11) it follows that the sequence $\{u^j\}$ is uniformly bounded in $H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^3)$. Therefore we can assume that $u_*^{\alpha,H}$ is the weak H^1-limit of u^j as $j \to \infty$ and conclude that $u_*^{\alpha,H}$ is a H^1 map:

$$u_*^{\alpha,H} \in \mathcal{H}^1_\omega.$$

To prove (4.8) we distinguish two cases:

a) $T_{j+1} \le T_j$. In this case it results

$$\mathcal{A}_{\alpha,H}(u^j) \ge \int_0^T \Big[H + \frac{1}{2}\Big(|\dot{u}^j|^2 + \sum_{R \in \mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u^j|^\alpha} \Big) \Big]\, dt \tag{4.12}$$

and we have

$$\lim_{j\to\infty} \int_0^T \sum_{R \in \mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u^j|^\alpha}\, dt = \int_0^T \sum_{R \in \mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u_*^{\alpha,H}|^\alpha}\, dt \tag{4.13}$$

since $u^j \to u_*^{\alpha,H}$ uniformly in compacts and

$$\liminf_{j\to\infty} \int_0^T |\dot{u}^j|^2 dt \ge \int_0^T |\dot{u}_*^{\alpha,H}|^2 dt \tag{4.14}$$

by lower semi-continuity of the L^2 norm with respect to weak convergence. This concludes the proof in case a).

b) $T_{j+1} \ge T_j$. Define

$$v^j(t) = \begin{cases} u^j(t), & \text{for } t \in [0, T_j], \\ u^j(T_j), & \text{for } t \in (T_j, T]. \end{cases}$$

Since the sequence $\{v^j\}$ is bounded in $H^1([0,T];\mathbb{R}^3)$ we can assume that $v^j \rightharpoonup u_*^{\alpha,H}$ weakly in $H^1([0,T];\mathbb{R}^3)$. This and the uniform convergence of v^j to $u_*^{\alpha,H}$ imply

$$\begin{aligned} \liminf_{j\to\infty} \mathcal{A}_{\alpha,H}(v^j) &= \liminf_{j\to\infty} \int_0^T \Big[H + \frac{1}{2}\Big(|\dot{v}^j|^2 + \sum_{R \in \mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)v^j|^\alpha} \Big) \Big]\, dt \\ &\ge \mathcal{A}_{\alpha,H}(u_*^{\alpha,H}). \end{aligned}$$

This and the identity

$$\begin{aligned}\mathcal{A}_{\alpha,H}(v^j) &= \int_0^T \frac{1}{2}\Big[H + \Big(|\dot{v}^j|^2 + \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)v^j|^\alpha}\Big)\Big]\, dt \\ = \quad & \mathcal{A}_{\alpha,H}(u^j) + \Big[H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u^j(T_j)|^\alpha}\Big](T - T_j)\end{aligned}$$

conclude the proof.

□

Lemma 4.4. *Assume $\alpha > 2$, $H > 0$ and $\omega \in \Omega \setminus \Omega_0$ and let $u_*^{\alpha,H} \in \mathcal{H}^1_\omega$ be the minimizer of $\mathcal{A}_{\alpha,H}|_{\mathcal{H}^1_\omega}$ in Lemma 4.3. Then $u_*^{\alpha,H} \in C^\infty(\mathbb{R};\mathbb{R}^3)$ and $u_*^{\alpha,H}$ satisfies (1.10).*

Proof. From (4.1) in Lemma 4.1 $u_*^{\alpha,H}(\mathbb{R})$ has a positive distance from the singular set Γ. This implies that we can regard the potential

$$\mathcal{U}_\alpha(z) = \sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)z|^\alpha}$$

as a C^∞ bounded function. Using this and the fact that, being $u_*^{\alpha,H}$ a minimizer of $\mathcal{A}_{\alpha,H}|_{\mathcal{H}^1_\omega}$, the first variation of $\mathcal{A}_{\alpha,H}$ at $u_*^{\alpha,H}$ in the class of $\mathcal{H}^1_\omega$ maps of period $T = T_{u_*^{\alpha,H}}$ vanishes we deduce that $u_*^{\alpha,H} \in C^\infty(\mathbb{R},\mathbb{R}^3)$ by elliptic regularity. Then it follows that $u_*^{\alpha,H}$ satisfies the equation of motion and therefore also (1.10) for some value H_* of the constant H. We now show that H_* is exactly the constant H in the definition of $\mathcal{A}_{\alpha,H}$. We temporarily write u for $u_*^{\alpha,H}$ and let $v : \mathbb{R}\times(-s_0,s_0) \to \mathbb{R}^3$, $T : (-s_0,s_0) \to \mathbb{R}$ be smooth maps defined in a neighborhood $(-s_0,s_0)$ of $s = 0$. We assume that

1. For each $s \in (-s_0,s_0)$, $v(\cdot,s)$ is periodic with period $T(s)$.

2. $T(0) = T$ and $v(\cdot,0) = u$.

Set

$$\mathcal{A}(s) = \mathcal{A}_{\alpha,H}(v(\cdot,s)) = \int_0^{T(s)} \Big(H + \frac{1}{2}\big[|\dot{v}(t,s)|^2 + \mathcal{U}_\alpha(v(t,s))\big]\Big)dt.$$

Then $\mathcal{A}(s)$ is a smooth function in $(-s_0,s_0)$ and the minimality of v implies that $\frac{d}{ds}\mathcal{A}(s)|_{s=0} = 0$. Differentiating the identity

$$v(t+T(s),s) = v(t,s),\ t\in\mathbb{R},\ s\in(-s_0,s_0)$$

with respect to s yields

$$\dot{v}(t+T(s),s)T_s(s) + v_s(t+T(s),s) = v_s(t,s).$$

If we set $s = 0$ in this identity we obtain

$$\dot{u}(t+T)T_s(0) + v_s(t+T,0) = v_s(t,0). \tag{4.15}$$

Differentiating $\mathcal{A}(s)$ yields

$$\begin{aligned}\frac{d}{ds}\mathcal{A}(s) \quad = \quad & \int_0^{T(s)} (\langle\dot{v},\dot{v}_s\rangle + \frac{1}{2}\langle\mathcal{U}_{\alpha,z}(v),v_s\rangle)dt \\ & + \Big(H + \frac{1}{2}\big[|\dot{v}(T(s),s)|^2 + \mathcal{U}_\alpha(v(T(s),s))\big]\Big)T_s(s),\end{aligned}$$

which for $s = 0$ becomes

$$\begin{aligned}\frac{d}{ds}\mathcal{A}(s)|_{s=0} &= \int_0^T (\langle \dot{u}, \dot{v}_s\rangle + \frac{1}{2}\langle \mathcal{U}_{\alpha,z}(u), v_s\rangle)dt \\ &\quad + \Big(H + \frac{1}{2}\big[|\dot{u}(T)|^2 + \mathcal{U}_\alpha(u(T))\big]\Big)T_s(0) \\ &= \int_0^T \langle -\ddot{u} + \frac{1}{2}\mathcal{U}_{\alpha,z}(u), v_s\rangle dt + \langle \dot{u}, v_s\rangle|_{t=0}^{t=T} \\ &\quad + \Big(H + \frac{1}{2}\big[|\dot{u}(T)|^2 + \mathcal{U}_\alpha(u(T))\big]\Big)T_s(0). \end{aligned} \tag{4.16}$$

Note that

$$\begin{aligned}\langle \dot{u}, v_s\rangle|_{t=0}^{t=T} &= \langle \dot{u}(T), v_s(T,0)\rangle - \langle \dot{u}(0), v_s(0,0)\rangle \\ &= \langle \dot{u}(T), v_s(T,0) - v_s(0,0)\rangle = -|\dot{u}(T)|^2 T_s(0),\end{aligned}$$

where we have used (4.15) for $t = 0$. From this and (4.16) it follows

$$\begin{aligned}\frac{d}{ds}\mathcal{A}(s)\Big|_{s=0} &= \int_0^T \langle -\ddot{u} + \frac{1}{2}\mathcal{U}_{\alpha,z}(u), v_s\rangle dt \\ &\quad + \Big(H + \frac{1}{2}\big[-|\dot{u}(T)|^2 + \mathcal{U}_\alpha(u(T))\big]\Big)T_s(0).\end{aligned} \tag{4.17}$$

Since the minimality of u implies $\frac{d}{ds}\mathcal{A}(s)\big|_{s=0} = 0$ for any choice of the map v introduced above, we can regard v_s and $T_s(0)$ as arbitrary quantities and deduce from (4.17) that

$$\frac{1}{2}|\dot{u}(T)|^2 = H + \frac{1}{2}\mathcal{U}_\alpha(u(T))$$

which, since the choice of the initial time $t = 0$ is arbitrary, concludes the proof. □

4.2 L^∞ bounds independent of $\omega \in \Omega \setminus \Omega_0$.

From Lemma 4.4 we have that, provided $\alpha > 2$ and $H > 0$, for each $\omega \in \Omega \setminus \Omega_0$ the minimizer $u_*^{\alpha,H} \in \mathcal{H}_\omega^1$ given by Lemma 4.3 is a smooth map that satisfies (1.10). Since $u_*^{\alpha,H}$ minimizes $\mathcal{A}_{\alpha,H}$ on the whole of $\mathcal{H}_\omega^1$, $u_*^{\alpha,H}$ is obviously a minimizer of $\mathcal{A}_{\alpha,H}$ in the subset $\mathcal{H}_{\omega,H}^1$ of $\mathcal{H}_\omega^1$ of the maps that satisfy (1.10). On the basis of Lemma 4.1 we can also assume that maps in $\mathcal{H}_{\omega,H}^1$ satisfy (4.1). For $u \in \mathcal{H}_{\omega,H}^1$, using (1.10) we can write $\mathcal{A}_{\alpha,H}$ in Jacobi form:

$$\begin{aligned}\mathcal{A}(u) &= \sqrt{2}\int_0^{T_u} \sqrt{H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha}}|\dot{u}|dt \\ &= \sqrt{2}\int_0^{L_u} \sqrt{H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}} \frac{1}{|(R-I)u|^\alpha}}ds\end{aligned}$$

where we have simply written $\mathcal{A}$ instead of $\mathcal{A}_{\alpha,H}$ and set $L_u = \int_0^{T_u}|\dot{u}|dt$.

Given a pole $p \in \mathcal{P}$ and $z \in \mathbb{R}^3$ we set $z = z^p + z^{p\perp}$ where $z^p = \langle z, p\rangle p$ and $z^{p\perp} = z - \langle z, p\rangle p$.

Proposition 4.5. *Assume $\alpha > 2$, $H > 0$ and $\omega \in \Omega \setminus \Omega_0$. Then there exist $\bar{r} > 0$ and $\bar{\rho} > 0$ independent of $\omega \in \Omega \setminus \Omega_0$ such that if $u \in \mathcal{H}^1_{\omega,H}$ satisfies*

$$u(\mathbb{R}) \cap (\cup_{p\in\mathcal{P}}\{z \in \mathbb{R}^3 : |z^{p\perp}| < \bar{\rho}\} \cup B_{\bar{r}}) \neq \emptyset,$$

then there exists $v \in \mathcal{H}^1_{\omega,H}$ such that

$$\begin{cases} v(\mathbb{R}) \cap (\cup_{p\in\mathcal{P}}\{z \in \mathbb{R}^3 : |z^{p\perp}| < \bar{\rho}\} \cup B_{\bar{r}}) = \emptyset, \\ \mathcal{A}(v) < \mathcal{A}(u). \end{cases} \tag{4.18}$$

Proof. For each $z \in \mathbb{S}^2$ the map $r \mapsto r\sqrt{H + \frac{1}{2r^\alpha}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)z|^\alpha}}$ is strictly decreasing in $(0, r(z))$, and strictly increasing in $(r(z), +\infty)$, where

$$r(z) = (\frac{\alpha-2}{4H}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)z|^\alpha})^{\frac{1}{\alpha}}. \tag{4.19}$$

Let $\bar{r} = \min_{z\in\mathbb{S}^2} r(z)$ and define $w \in \mathcal{H}^1_{\omega,H}$ by setting

$$w(s) = \begin{cases} u(s), & \text{if } |u(s)| \geq \bar{r}, \\ \\ \bar{r}\frac{u(s)}{|u(s)|}, & \text{if } |u(s)| < \bar{r}\,, \end{cases} \tag{4.20}$$

where s is arc-length along the orbit of u. Note that, for $|u(s)| < \bar{r}$,

$$\Big|\frac{dw}{ds}(s)\Big| = \frac{\bar{r}}{|u(s)|}\Big|\frac{du}{ds}(s) - \langle\frac{du}{ds}(s), \frac{u(s)}{|u(s)|}\rangle\frac{u(s)}{|u(s)|}\Big| \leq \frac{\bar{r}}{|u(s)|},$$

where we have used $|\frac{du}{ds}(s)| = 1$. Therefore, if $s \in \mathbb{R}$ is such that $|u(s)| < \bar{r}$ we have

$$\sqrt{H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)w(s)|^\alpha}}|\frac{dw}{ds}(s)| \leq \sqrt{H + \frac{1}{2\bar{r}^\alpha}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)\frac{u(s)}{|u(s)|}|^\alpha}}\frac{\bar{r}}{|u(s)|}$$
$$< \sqrt{H + \frac{1}{2|u(s)|^\alpha}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)\frac{u(s)}{|u(s)|}|^\alpha}}\frac{|u(s)|}{|u(s)|} = \sqrt{H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)u(s)|^\alpha}}.$$

These inequalities and (4.20) imply

$$\mathcal{A}(w) < \mathcal{A}(u).$$

To complete the proof of Proposition 4.5 we need the following Lemmas.

Lemma 4.6. *Let $H > 0$ and $\bar{\zeta} > 0$ be fixed. Given $p \in \mathcal{P}, \zeta \geq \bar{\zeta}$ and a unit vector n orthogonal to p, let $\varphi_{p,n,\zeta}$ be the map defined by*

$$\rho \overset{\varphi_{p,n,\zeta}}{\mapsto} \rho\sqrt{H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)(\zeta p + \rho n)|^\alpha}}.$$

Then there is $\bar{\rho} > 0$ such that $\varphi_{p,n,\zeta}$ is strictly decreasing in $(0, \bar{\rho}]$. Moreover $\bar{\rho}$ can be chosen independent of $p \in \mathcal{P}, \zeta$ and n.

Proof. let $C_p \subset \mathcal{R}$ be the cyclic subgroup of the rotations that leave p fixed. For $R \in C_p$ it results

$$(R-I)(\zeta p + \rho n) = \rho(R-I)n, \ \Rightarrow \sum_{R \in C_p \setminus \{I\}} \frac{1}{|(R-I)(\zeta p + \rho n)|^\alpha} = \frac{c_p}{\rho^\alpha},$$

where, denoting by o_p the order of C_p, we have

$$c_p = \sum_{R \in C_p \setminus \{I\}} \frac{1}{|(R-I)n|^\alpha} = \sum_{j=1}^{o_p - 1} \frac{1}{|2\sin(\frac{j\pi}{o_p})|^\alpha}.$$

The condition $\varphi'_{p,n,\zeta}(\rho) < 0$ is equivalent to

$$\frac{\alpha-2}{4}\frac{c_p}{\rho^\alpha} > H + \frac{1}{2}\sum_{R \in \mathcal{R} \setminus C_p} \frac{1}{|(R-I)(\zeta p + \rho n)|^\alpha} - \frac{\alpha\rho}{4}\sum_{R \in \mathcal{R} \setminus C_p} \frac{\langle (R-I)n, (R-I)(\zeta p + \rho n)\rangle}{|(R-I)(\zeta p + \rho n)|^{\alpha+2}}. \tag{4.21}$$

Let

$$d = \min_{\substack{p,q \in \mathcal{P} \\ p \neq q}} |p - q|$$

and let $\bar{\rho}' > 0$ be a fixed number which is small with respect to $d\bar{\zeta}$. Set

$$K = \max_{p,n,\zeta \geq \bar{\zeta}, \rho \leq \bar{\rho}'} \Big| \frac{1}{2}\sum_{R \in \mathcal{R} \setminus C_p} \frac{1}{|(R-I)(\zeta p + \rho n)|^\alpha} - \frac{\alpha\rho}{4}\sum_{R \in \mathcal{R} \setminus C_p} \frac{\langle (R-I)n, (R-I)(\zeta p + \rho n)\rangle}{|(R-I)(\zeta p + \rho n)|^{\alpha+2}} \Big|. \tag{4.22}$$

Let $c = \min_{p \in \mathcal{P}} c_p$ and let $\bar{\rho} > 0$ be a number smaller than $\min\{\bar{\rho}', \rho_0\}$ where ρ_0 is the root of the equation

$$\frac{(\alpha-2)}{4}\frac{c}{\rho^\alpha} = H + K. \tag{4.23}$$

Then $\rho \in (0, \bar{\rho}]$ is a sufficient condition in order that (4.21) holds and therefore in order that $\varphi'_{p,n,\zeta}(\rho) < 0$. □

Lemma 4.7. *Let $\bar{r}$ be as before and $\bar{\rho}$ be the number corresponding to $\bar{\zeta} = \bar{r}/2$ defined in Lemma 4.6. Assume that $u \in \mathcal{H}^1_{\omega,H}$ satisfies*

$$\begin{cases} u(\mathbb{R}) \cap B_{\bar{r}} = \emptyset, \\ u(\mathbb{R}) \cap (\cup_{p \in \mathcal{P}}\{z \in \mathbb{R}^3 : |z^{p\perp}| < \bar{\rho}\}) \neq \emptyset. \end{cases} \tag{4.24}$$

Then there exists $v \in \mathcal{H}^1_{\omega,H}$ such that

$$\begin{cases} v(\mathbb{R}) \cap (\cup_{p \in \mathcal{P}}\{z \in \mathbb{R}^3 : |z^{p\perp}| < \bar{\rho}\} \cup B_{\bar{r}}) = \emptyset \\ \mathcal{A}(v) < \mathcal{A}(u). \end{cases} \tag{4.25}$$

Proof. Define $v \in \mathcal{H}^1_{\omega,H}$ by

$$v(s) = \begin{cases} u(s), & \text{if } u(s) \notin \cup_{p \in \mathcal{P}}\{z \in \mathbb{R}^3 : |z^{p\perp}| < \bar{\rho}\} \\ u^p(s) + \bar{\rho}\frac{u^{p\perp}(s)}{|u^{p\perp}(s)|}, & \text{for } |u^{p\perp}(s)| < \bar{\rho}. \end{cases}$$

For $|u^{p\perp}(s)| < \bar{\rho}$ we have

$$\begin{aligned}\frac{dv^{p\perp}}{ds} &= \frac{\bar{\rho}}{|u^{p\perp}|}\Big(\frac{du^{p\perp}}{ds} - \langle\frac{du^{p\perp}}{ds}, \frac{u^{p\perp}}{|u^{p\perp}|}\rangle\frac{u^{p\perp}}{|u^{p\perp}|}\Big) \\ &\Rightarrow \Big|\frac{dv^{p\perp}}{ds}\Big| \leq \frac{\bar{\rho}}{|u^{p\perp}|}\Big|\frac{du^{p\perp}}{ds}\Big|. \end{aligned} \tag{4.26}$$

Set

$$\phi_p(x) = \sqrt{H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)(u^p+x)|^\alpha}}$$

where $x \in \mathbb{R}^3$ satisfies $\langle x, p\rangle = 0$. From (4.26) and Lemma 4.6 it follows, for $|u^{p\perp}(s)| < \bar{\rho}$,

$$\begin{aligned}\phi_p(u^{p\perp})\Big|\frac{du^{p\perp}}{ds}\Big| &\geq \phi_p(u^{p\perp})\frac{|u^{p\perp}|}{\bar{\rho}}\Big|\frac{dv^{p\perp}}{ds}\Big| \\ &> \phi_p\Big(\bar{\rho}\frac{u^{p\perp}}{|u^{p\perp}|}\Big)\Big|\frac{dv^{p\perp}}{ds}\Big| = \phi_p(v^{p\perp})\Big|\frac{dv^{p\perp}}{ds}\Big|. \end{aligned} \tag{4.27}$$

We also have, for $|u^{p\perp}(s)| < \bar{\rho}$,

$$\phi_p(u^{p\perp})\Big|\frac{du^p}{ds}\Big| > \phi_p\Big(\bar{\rho}\frac{u^{p\perp}}{|u^{p\perp}|}\Big)\Big|\frac{du^p}{ds}\Big| = \phi_p(v^{p\perp})\Big|\frac{dv^p}{ds}\Big|, \tag{4.28}$$

where we have used that $|x| < \bar{\rho} \Rightarrow \phi_p(\bar{\rho}\frac{x}{|x|}) < \phi_p(x)$.

Inequalities (4.27) and (4.28) imply that, for each s such that $u(s) \in \cup_{p\in\mathcal{P}}\{z \in \mathbb{R}^3 : |z^{p\perp}| < \bar{\rho}\}$, we have

$$\sqrt{H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)u|^\alpha}}\,|\frac{du}{ds}| > \sqrt{H + \frac{1}{2}\sum_{R\in\mathcal{R}\setminus\{I\}}\frac{1}{|(R-I)v|^\alpha}}\,|\frac{dv}{ds}|.$$

This concludes the proof of the lemma. □

The map $w \in \mathcal{H}^1_{\omega,H}$, constructed in the first part of the proof, satisfies $w(\mathbb{R}) \cap B_{\bar{r}} = \emptyset$. Therefore Lemma 4.7 with $u = w$ yields a map v such that (4.18) holds. The proof of Proposition 4.5 is complete.

□

We now show that minimizing orbits are contained in a fixed ball.

Proposition 4.8. *Assume $\alpha > 2$ and $H > 0$. Then there exists $r_0 > 0$ independent of $\omega \in \Omega \setminus \Omega_0$ such that*

$$u \in \mathcal{H}^1_{\omega,H} \quad \textit{and} \quad u(\mathbb{R}) \setminus \overline{B_{r_0}} \neq \emptyset \tag{4.29}$$

imply the existence of $v \in \mathcal{H}^1_{\omega,H}$ which satisfies

$$v(\mathbb{R}) \subset \overline{B_{r_0}} \qquad \textit{and} \qquad \mathcal{A}(v) < \mathcal{A}(u).$$

Proof. Let $\bar{r}$ and $\bar{\rho}$ be as in Proposition 4.5. Define

$$\Sigma = \{z \in \mathbb{R}^3 : |z| \geq r(\frac{z}{|z|})\},$$

with $r(z)$ as in (4.19), and let

$$\mathcal{D}_{r,\rho} = \mathbb{R}^3 \setminus [(\cup_{p\in\mathcal{P}}\{z : |z^{p\perp}| < \rho\}) \cup B_r].$$

Fix $\bar{\rho}_1 \in (0, \bar{\rho})$ and let

$$D_r = \mathcal{D}_{r,\bar{\rho}_1}$$

for $r \geq \bar{r}$. From (4.19) it follows that there exists $\bar{r}_1 > \bar{r}$ such that

$$D_r \subset \Sigma \qquad \text{for all } \ r \geq \bar{r}_1.$$

From Proposition 4.5 we can assume

$$u(\mathbb{R}) \subset \overline{D}, \tag{4.30}$$

where

$$\overline{D} = \mathcal{D}_{\bar{r},\bar{\rho}}. \tag{4.31}$$

Define $\varphi_0 > 0$ by

$$\sin\varphi_0 = \frac{\bar{\rho}_1}{\bar{r}_1}.$$

For $0 < \varphi \leq \varphi_0$ let

$$\mathcal{C}_\varphi = \cup_{p\in\mathcal{P}}\{z \in \mathbb{R}^3 : \frac{|z^{p\perp}|}{|z|} < \sin\varphi\}$$

and let

$$\phi(u) = \sup\{\varphi > 0 : u(\mathbb{R}) \cap (\mathcal{C}_\varphi \cup B_{\bar{r}}) = \emptyset\}.$$

From (4.30) if follows that $\phi(u) > 0$. For $0 < \varphi \leq \varphi_0$ let

$$r_\varphi = \frac{\bar{\rho}_1}{\sin\varphi}$$

and define (see Figure 3)

$$\begin{aligned} E_\varphi &= \mathbb{R}^3 \setminus (\mathcal{C}_\varphi \cup B_{r_\varphi}), \\ F_\varphi &= E_\varphi \cup (\overline{D} \setminus D_{r_\varphi}). \end{aligned}$$

Observe that it results

$$E_\varphi \subset \Sigma, \ \text{ for all } \ \varphi \leq \varphi_0. \tag{4.32}$$

Assume that $\varphi \in (0, \varphi_0)$ and $w \in \mathcal{H}^1_{\omega,H}$, with $w(\mathbb{R}) \subset \overline{D}$, be such that

$$\begin{aligned} &w(\mathbb{R}) \subset F_\varphi, \\ &w(\mathbb{R}) \setminus \overline{B_{r_\varphi}} \neq \emptyset. \end{aligned}$$

Let $\hat{w} \in \mathcal{H}^1_{\omega,H}$ be the map defined by

$$\hat{w}(s) = \begin{cases} w(s), & \text{if } \ |w(s)| \leq r_\varphi, \\ \\ r_\varphi \frac{w(s)}{|w(s)|}, & \text{if } \ |w(s)| > r_\varphi. \end{cases} \tag{4.33}$$

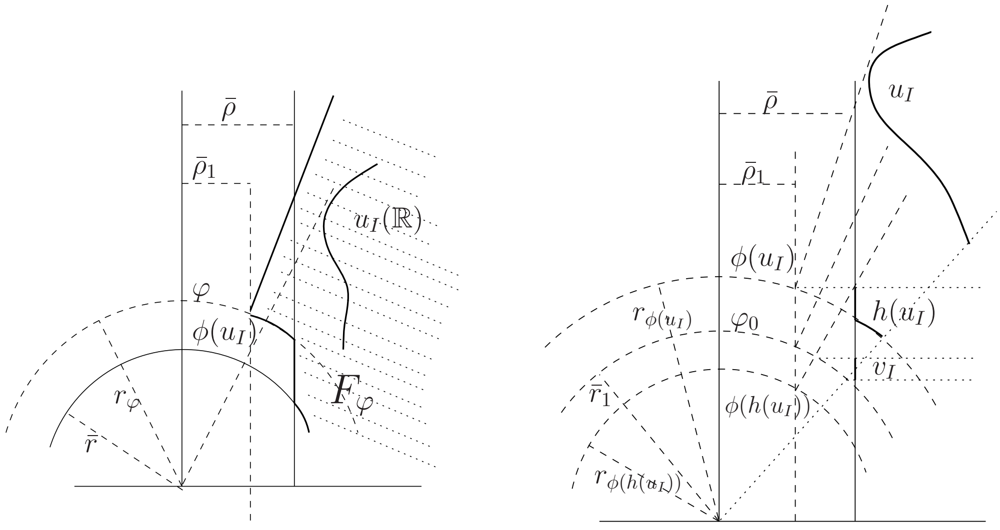

Figure 3: Left: the set F_φ and the angles φ, $\phi(u)$. Right: the map h.

From this definition and the inclusion (4.32) it follows

$$\mathcal{A}(\hat w) < \mathcal{A}(w).$$

Finally define $\tilde w \in \mathcal{H}^1_{\omega,H}$ by setting

$$\tilde w(s) = \begin{cases} \hat w(s), & \text{if } |\hat w^{p\perp}(s)| \geq \bar\rho, \\ \\ \hat w^p(s) + \bar\rho \frac{\hat w^{p\perp}(s)}{|\hat w^{p\perp}(s)|}, & \text{if } |\hat w^{p\perp}(s)| < \bar\rho. \end{cases} \tag{4.34}$$

From this definition and Proposition 4.5 it follows

$$\mathcal{A}(\tilde w) < \mathcal{A}(\hat w) < \mathcal{A}(w).$$

Note also that from (4.33) and (4.34) it follows

$$\sin\phi(\tilde w) > \frac{\bar\rho}{r^*_\varphi},$$
$$\tilde w(\mathbb{R}) \subset \overline{B_{r^*_\varphi}}.$$

where

$$r^*_\varphi = \sqrt{r_\varphi^2 - \bar\rho_1^2 + \bar\rho^2}.$$

In conclusion the map $w \stackrel{h}{\to} \tilde w$ constructed by means of (4.33), (4.34) has the following properties

$$\left.\begin{array}{l} w(\mathbb{R}) \subset F_\varphi, \\ w(\mathbb{R}) \setminus \overline{B_{r_\varphi}} \neq \emptyset, \end{array}\right\} \Rightarrow \left\{\begin{array}{l} \sin\phi(h(w)) > \frac{\bar\rho}{r^*_\varphi}, \\ h(w)(\mathbb{R}) \subset \overline{D} \cap \overline{B_{r^*_\varphi}} \\ \mathcal{A}(h(w)) < \mathcal{A}(w), \end{array}\right. \tag{4.35}$$

where we have used (4.30). Consider first the case $\phi(u) \geq \varphi_0$. Then we have $u(\mathbb{R}) \subset F_{\varphi_0}$ and assumption (4.29) implies $u(\mathbb{R}) \setminus \overline{B_{\bar r_1}} \neq \emptyset$. Take $\phi = \phi_0$. Then (4.35) implies that in this case to prove the proposition we can take

$$v = h(u) \qquad \text{and} \qquad r_0 = r^*_{\varphi_0}.$$

Assume now that $\phi(u) < \varphi_0$. In this case we define $w^1 = h(u),\ w^k = h(w^{k-1}), k = 2, \ldots;\ \varphi_1 = \phi(u),\ \varphi_k = \phi(h(w^{k-1})),\ k = 2, \ldots$ From the inequality for $\phi(h(w))$ in (4.35) it follows that there is a first value k_0 of k such that $\phi(w^{k_0}) \geq \varphi_0$ and we are back in the previous case and we can take

$$\begin{aligned} v &= h^{k_0+1}(u)\ , \\ r_0 &= r^*_{\varphi_0}. \end{aligned} \tag{4.36}$$

This concludes the proof. □

4.3 Convergence of sequences of orbits and conclusion of the proof of Theorem 1.3

From each $\omega \in \Omega \setminus \Omega_0$, Lemma 4.3 yields a minimizer $u_*^{\alpha,H}$ of $\mathcal{A}_{\alpha,H}|_{\mathcal{H}^1_\omega}$ and, by Lemma 4.4, $u_*^{\alpha,H}$ satisfies (1.10). This proves (I).
From Proposition 4.5, Lemma 4.7 and Proposition 4.8 the minimizer $u_*^{\alpha,H} \in \mathcal{H}^1_\omega$ satisfies

$$u_*^{\alpha,H}(\mathbb{R}) \subset \overline{B}_{R_0} \cap \overline{D}, \tag{4.37}$$

with $\overline{D}$ defined as in (4.31). This implies that there are constants $c, C > 0$, independent of $\omega \in \Omega \setminus \Omega_0$ such that

$$c \leq \sum_{R \in \mathcal{R} \setminus \{I\}} \frac{1}{|(R - I)u_*^{\alpha,H}|^\alpha} \leq C.$$

Therefore Lemma 4.4 and (1.10) imply that, independently of $\omega \in \Omega \setminus \Omega_0$, we have

$$c' \leq |\dot{u}_*^{\alpha,H}| \leq C' \tag{4.38}$$

for some constants $c', C' > 0$. By differentiating the equation of motion (2.6) and using (4.38) and (4.37) we obtain that the third derivative of $u_*^{\alpha,H}$ is bounded by a constant $\overline{C} > 0$, which is again independent of $\omega \in \Omega \setminus \Omega_0$. Given a sequence $\{\omega^j\} \subset \Omega \setminus \Omega_0$, Lemma 4.3 yields a minimizer $u_*^{\alpha,H,j} \in \mathcal{H}^1_{\omega^j}$ of $\mathcal{A}_{\alpha,H}|_{\mathcal{H}^1_{\omega^j}}$. The bound on the third derivative of $u_*^{\alpha,H,j}$ established above implies the existence of a subsequence $\{j_h\}$ such that u_*^{α,H,j_h} converges locally in the $C^{2+\frac{1}{2}}$ sense to a map $\hat{u}_*^{\alpha,H}$ which, via (1.1), defines a solution $\hat{u}_*^{\alpha,H}$ of the equation of motion. To complete the proof of (II) let M^j be the period associated to ω^j. We can assume that $\{M^{j_h}\}$ converges:

$$\lim_{h \to \infty} M^{j_h} = M^\infty \in \mathbb{N} \cup \{+\infty\}.$$

If $M^\infty < +\infty$, since the number of periodic sequences with period M^∞ is finite, by passing to a subsequence if necessary, we can assume that there is $\omega \in \Omega \setminus \Omega_0$ with period M^∞ such that

$$\omega^{j_h} = \omega, \quad \text{for all} \ \ h \in \mathbb{N}$$

and therefore

$$\lim_{h \to \infty} \omega^{j_h} = \omega.$$

Now suppose that $M^\infty = +\infty$. Then, arguing as before, we see that we can associate to each $n \in \mathbb{N}$ a subsequence $\{j_h^n\}_{h \in \mathbb{N}}$ such that

$$\{j_h^{n+1}\}_{h \in \mathbb{N}} \subset \{j_h^n\}_{h \in \mathbb{N}}, \quad \text{for} \ \ n \in \mathbb{N},$$

$$M^{j_h^n} \geq n, \ \text{for} \ \ h \in \mathbb{N},$$

$$\text{for} \ \ h \in \mathbb{N} : \ \omega_i^{j_h^n} = \tilde{\sigma}_i, \ i \in \{-n, \ldots, 0, \ldots, n\}$$

for some $\tilde{\sigma}_i \in \{\sigma_1, \ldots, \sigma_Q, \sigma_1^{-1}, \ldots, \sigma_Q^{-1}\}$. We claim that $\hat{\omega}^n = \omega^{j_n^n}$, $n \in \mathbb{N}$, is a Cauchy sequence. Indeed assuming $n \geq m$ we have

$$d(\hat{\omega}^m, \hat{\omega}^n) = |f(M^{j_m^m}) - f(M^{j_n^n})| + \sum_{i \in \mathbb{Z}} \frac{\delta(\hat{\omega}_i^m, \hat{\omega}_i^n)}{2^{|i|}} \leq |f(m) - f(+\infty)| + \sum_{|i|>m} \frac{1}{2^{|i|}}, \quad (4.39)$$

where we have used the implication

$$m \leq M^{j_m^m},\ M^{j_n^n} < +\infty \quad \Rightarrow \quad |f(M^{j_m^m}) - f(M^{j_n^n})| \leq |f(m) - f(+\infty)|.$$

From (4.39) it follows that there exists $\hat{\omega} \in \hat{\Omega}$ such that

$$\lim_{n \to \infty} \hat{\omega}^n = \hat{\omega}.$$

This shows that the subsequence $\{j_h\}$ can be chosen so that (1.12) holds and concludes the proof of (II).

To prove (III) let $\hat{\omega} \in \hat{\Omega} \setminus \Omega_0$ be given. If $\hat{\omega}$ is periodic, then (III) holds trivially with the constant sequence $\omega^j = \hat{\omega}$, $j \in \mathbb{N}$, and $\hat{u}_*^{\alpha,H}$ the map defined by (1.1) via the minimizer $\hat{u}_*^{\alpha,H} \in \mathcal{H}^1_{\hat{\omega}}$ given by Lemma 4.3. If $\hat{\omega}$ is not periodic we associate to $\hat{\omega}$ the sequence $\{\omega^j\} \subset \Omega \setminus \Omega_0$ where ω^j is the periodic extension of period $M^j = 2j + 1$ of the sequence $\{\hat{\omega}_i\}_{|i| \leq j}$. The sequence $\{\omega^j\}$ converges to $\hat{\omega}$ and, by passing to a subsequence if necessary, we can also assume that the sequence of minimizers $u_*^{\alpha,H,j} \in \mathcal{H}^1_{\omega^j}$ given by Lemma 4.3 converges to a map $\hat{u}_*^{\alpha,H}$ that via (1.1) determines a solution $\hat{u}_*^{\alpha,H}$ of the equation of motion. This completes the proof of Theorem 1.3.

References

[1] Barutello, V., Ferrario, D. L., Terracini, S.: *On the singularities of generalized solutions to n-body-type problems*, Int. Math. Res. Not., Art. ID rnn 069, 1–78 (2008)

[2] Braides, A.: *Γ-convergence for Beginners*, Oxford University Press (2005)

[3] Chen, K. C.: *Binary Decomposition for Planar N-body Problems and Symmetric Periodic Solutions*, Arch. Rational Mech. Anal. **170**, 247–276 (2003)

[4] Chenciner, A.: *Action minimizing solutions of the Newtonian n-body problem: from homology to symmetry*, ICM 2002, Beijing (2002)

[5] Chenciner, A.: *Symmetries and "simple" solutions of the classical N-body problem*, ICMP03 (2003)

[6] Chenciner, A., Venturelli, A.: *Minima de l'intégrale d'action du Problème newtonien de 4 corps de masses égales dans $\mathbb{R}^3$: orbites "hip–hop"*, Cel. Mech. Dyn. Ast. **77**, 139–152 (2000)

[7] Cundy, H., Rollett A. P.: *Mathematical Models*, Oxford at The Clarendon Press 1954

[8] Ferrario, D. L.: *Transitive decomposition of symmetry groups for the n-body problem.*, Adv. Math. **213/2**, 763–784 (2007)

[9] Ferrario, D., Terracini, S.: *On the existence of collisionless equivariant minimizers for the classical n-body problem*, Invent. Math. **155**, 305–362 (2004)

[10] G. Fusco, G. F. Gronchi, P. Negrini. Platonic polyhedra, topological constraints and periodic solutions of the classical N-body problem. *Invent. Math.* **185** (2011), 283–332.

[11] `http://adams.dm.unipi.it/~gronchi/nbody.html`

[12] Moore, C., Nauenberg, M.: *New Periodic Orbits for the n–Body Problem*, preprint, arXiv:math/0511219v1 (2008)

[13] Montgomery, R.: *The n-body problem, the braid group and action minimizing periodic solutions*, Nonlinearity **11**, 363–376 (1998)

[14] Ouyang, T., Xie, Z.: *A new variational method with SPBC and many stable choreographic solutions of the Newtonian 4body problem*, preprint (2013)

[15] Simó, C.: *New Families of Solutions in N–Body Problems*, Proceedings of the third European Congress of Mathematics, Casacuberta et al. edits, *Progress in Mathematics* **201**, 101–115 (2001)

[16] Terracini, S. and Venturelli, A.: *Symmetric Trajectories for the $2N$-Body Problem with Equal Masses*, Arch. Rational Mech. Anal. **184**, 465–493 (2007)

[17] Venturelli, A.: *Une caracterisation variationelle des solutions de Lagrange du problème plan des trois corps* Comp. Rend. Acad. Sci. Paris, **332** Série I, 641-644 (2001)