

COMPUTING INVARIANTS OF SEMI-LOG-CANONICAL SURFACES

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ABSTRACT. We describe some general methods to compute fundamental groups, (co)homology, and irregularity of semi-log-canonical surfaces.

As an application, we show that there are exactly two irregular Gorenstein stable surfaces with $K^2 = 1$, which have $\chi(X) = 0$ and $\text{Pic}^0(X) = \mathbb{C}^*$ but different homotopy type.

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1. INTRODUCTION

Whenever one finds a new way to construct interesting varieties, one is subsequently presented with the task of computing algebraic and topological invariants. We address in this paper the case of surfaces with semi-log-canonical (slc) singularities.

Semi-log-canonical surfaces with ample canonical divisor are called stable and their moduli space is a natural compactification of the Gieseker moduli space of canonical models of surfaces of general type (see [Kol12, Kol15]). Indeed, this was one of the motivations for the introduction of this class of singularities by Kollár and Shepherd-Barron in [KSB88].

Here we start by giving a general method to compute (co)homology and fundamental group of a variety (or complex space) from a birational modification (cf. Section 3); in the applications this can be, for instance, the normalisation or a partial resolution. Then we explain how to compute the irregularity $q = h^1(\mathcal{O}_X)$ and also the automorphism group of a non-normal slc surface.

Our motivation for these results comes from our work in progress on Gorenstein stable surfaces with $K^2 = 1$ begun in [FPR14]. More precisely, since minimal surfaces of general type with $K^2 = 1$ are known to be regular, we wished to decide whether the same is true in the stable case. In the smooth case regularity is proven via a covering trick: if $q > 0$ there exist étale coverings of arbitrary degree and these would violate the Noether-inequality ([Bom73, Lem. 14]).

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For stable surfaces, even Gorenstein, this argument does not work because the slope of the Noether line is 1 [LR13]. Still, the situation is remarkably similar:

Theorem A — *There are exactly two irregular Gorenstein stable surfaces with $K^2 = 1$. They both have $\chi(X) = 0$ and $q(X) = 1$, the same normalisation ($= \mathbb{P}^2$), the same integral homology and Picard group, but they have different homotopy type.*

Each one of these surfaces forms a connected component of the moduli space of stable (not necessarily Gorenstein) surfaces with $K^2 = 1$.

Since smooth surfaces of general type have $\chi(X) > 0$, the irregular surfaces we find cannot be smoothable. A natural question raised by Theorem A is whether there are non-Gorenstein stable surfaces with $K^2 = 1$ and $\chi \leq 0$.

Our proof consists of two steps: first in Section 4.A we prove by “classical” methods (restriction to a canonical curve) that if $\chi(X) > 0$ and $K^2 = 1$ then $q(X) = 0$. By the results of [FPR14], this implies that an irregular surface X with $K^2 = 1$ has $\chi(X) = 0$ and is a projective plane glued to itself along four lines. Secondly, in Section 4.B we classify Gorenstein stable surfaces constructed from the projective plane glued to itself along four lines, showing in particular that there are exactly two with $\chi(X) = 0$. The topological and algebraic invariants of these surfaces are computed in the last section 4.C, using the methods previously devised.

In a forthcoming work we will study the Gorenstein stable surfaces with $K^2 = 1$ and $\chi(X) > 0$ (hence $q(X) = 0$) and their moduli spaces.

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2. SEMI-LOG-CANONICAL SURFACES

In this section we recall briefly the facts about semi-log-canonical surfaces that we use later. Let X be a *demi-normal* surface, that is, X satisfies Serre’s condition S_2 and at each point of codimension one either it is regular or it has an ordinary double point. We denote by $\pi: \bar{X} \rightarrow X$ the normalisation of X . Observe that X is not assumed irreducible; in particular, \bar{X} is possibly disconnected. The conductor ideal $\mathcal{H}om_{\mathcal{O}_X}(\pi_*\mathcal{O}_{\bar{X}}, \mathcal{O}_X)$ is an ideal sheaf in both \mathcal{O}_X and $\mathcal{O}_{\bar{X}}$ and as such defines subschemes $\bar{D} \subset X$ and $D \subset \bar{X}$, both reduced and pure of codimension 1; we often refer to D as the *non-normal locus* of X .

Definition 2.1 — The demi-normal surface X is said to have *semi-log-canonical (slc) singularities* if it satisfies the following conditions:

- (i) The canonical divisor K_X is \mathbb{Q} -Cartier.
- (ii) The pair (\bar{X}, \bar{D}) has log-canonical (lc) singularities (cf. [KM98, §4.1]).

It is called a *stable* surface if in addition K_X is ample. In that case we define the *geometric genus* of X to be $p_g(X) = h^0(X, K_X) = h^2(X, \mathcal{O}_X)$ and the *irregularity*

as $q(X) = h^1(X, K_X) = h^1(X, \mathcal{O}_X)$ (cf. [LR14, Lem. 3.3] for the last equality). We denote by $\chi(X) = \chi(X, \mathcal{O}_X)$ the holomorphic Euler-characteristic.

A Gorenstein stable surface is a stable surface such that K_X is a Cartier divisor.

2.A. Kollár's gluing principle. Let X be a demi-normal surface. Since X has at most double points in codimension one, the map $\pi: \bar{D} \rightarrow D$ on the conductor divisors is generically a double cover and thus induces a rational involution on \bar{D} . Normalising the conductor loci we get an honest involution $\tau: \bar{D}^\nu \rightarrow \bar{D}^\nu$ such that $D^\nu = \bar{D}^\nu/\tau$, where \bar{D}^ν, D^ν are the normalizations of \bar{D} , resp. D .

Theorem 2.2 ([Kol13, Thm. 5.13]) — *Associating to a stable surface X the triple $(\bar{X}, \bar{D}, \tau: \bar{D}^\nu \rightarrow \bar{D}^\nu)$ induces a one-to-one correspondence*

$$\left\{ \begin{array}{c} \text{stable} \\ \text{surfaces} \end{array} \right\} \leftrightarrow \left\{ (\bar{X}, \bar{D}, \tau) \left| \begin{array}{l} (\bar{X}, \bar{D}) \text{ log-canonical pair with} \\ K_{\bar{X}} + \bar{D} \text{ ample,} \\ \tau: \bar{D}^\nu \rightarrow \bar{D}^\nu \text{ involution s.th.} \\ \text{Diff}_{\bar{D}^\nu}(0) \text{ is } \tau\text{-invariant.} \end{array} \right. \right\}.$$

Addendum: *In the above correspondence the surface X is Gorenstein if and only if $K_{\bar{X}} + \bar{D}$ is Cartier and the involution $\tau: \bar{D}^\nu \rightarrow \bar{D}^\nu$ induces a fixed-point free involution on the preimages of the nodes of \bar{D} .*

For the definition of the different see for example [Kol13, 5.11]. The addendum has been proved in [FPR14, §3.1]. It is important that the involution τ acts only the normalization \bar{D}^ν and not on \bar{D} itself; as an example consider the union of the three coordinate planes in \mathbb{C}^3 .

An important consequence, which allows to understand the geometry of stable surfaces from the normalisation, is that

$$(2.3) \quad \begin{array}{ccccc} \bar{X} & \xleftarrow{\bar{\iota}} & \bar{D} & \xleftarrow{\bar{\nu}} & \bar{D}^\nu \\ \downarrow \pi & & \downarrow \pi & & \downarrow / \tau \\ X & \xleftarrow{\iota} & D & \xleftarrow{\nu} & D^\nu \end{array}$$

is a pushout diagram.

Remark 2.4 — Consider diagram (2.3) for any demi-normal surface X , not necessarily stable. By [Kol13, Prop. 5.3] and proof thereof, X has the following universal property: given any finite surjective morphism $f: \bar{X} \rightarrow Y$ that induces a τ -invariant map $\bar{D}^\nu \rightarrow Y$, there is a unique morphism $g: X \rightarrow Y$ such that $f = g \circ \pi$.

3. COMPUTING INVARIANTS

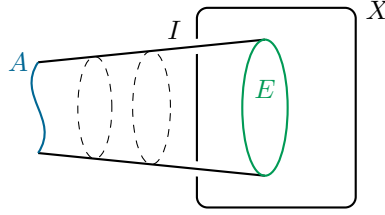
In this section we illustrate some methods for the computation of invariants of non-normal surfaces. We apply these results to stable Gorenstein surfaces with $K^2 = 1$ in Section 4.

3.A. Topology. First we compute the fundamental group and (co)homology of a non-normal surface in terms of its normalisation.

The proof of the following result is a synthesis of some conversations with Stefan Bauer, Kai-Uwe Bux, Michael Lönne, and Hanno von Bodecker. It is probably well known to experts.

Proposition 3.1 — *Let $\pi: \bar{X} \rightarrow X$ be a holomorphic map of compact complex analytic spaces. Assume A is a closed analytic subspace of X such that, if we set $E = \pi^{-1}A$, the map $\pi: \bar{X} \setminus E \rightarrow X \setminus A$ is an isomorphism.*

Let M be the double mapping cylinder of $\pi: E \rightarrow A$ and the inclusion $\iota: E \rightarrow \bar{X}$, that is, $M = A \cup_{\pi} E \times I \cup_{\iota} \bar{X}$ where we glue $E \times \{0\}$ to A via π and $E \times \{1\}$ to \bar{X} via ι :



Then the natural map $M \rightarrow X$ is a homotopy equivalence.

Proof. By [Loj64] (see also [BHPV04, Thm. I.8.8]) the subspace E is a neighbourhood deformation retract in \bar{X} . This implies [May99, Ch. 6, § 4] that the inclusion $E \hookrightarrow \bar{X}$ is a cofibration and we conclude by [May99, Ch. 10, § 7, Lemma on p. 78]. \square

We formulate the application of the above result only in the case we use, that is, for the normalisation of a demi-normal surface.

Corollary 3.2 — *Let X be a demi-normal surface and let D , \bar{D} , and \bar{X} be as in (2.3). Then:*

(i) *There is a Mayer-Vietoris exact sequence for homology*

$$\longrightarrow H_i(\bar{D}, \mathbb{Z}) \xrightarrow{(\pi_*, \bar{\iota}_*)} H_i(D, \mathbb{Z}) \oplus H_i(\bar{X}, \mathbb{Z}) \xrightarrow{\iota_* - \pi_*} H_i(X, \mathbb{Z}) \longrightarrow$$

(ii) *Suppose in addition that \bar{D} is connected. Then*

$$\pi_1(X) \cong \pi_1(D) \star_{\pi_1(\bar{D})} \pi_1(\bar{X}),$$

where \star denotes amalgamated product of groups.

(iii) *Suppose that \bar{D} is connected and $\pi_1(D)$ and $\pi_1(\bar{X})$ are abelian. Then*

$$\pi_1(X) \cong H_1(D, \mathbb{Z}) \star_{H_1(\bar{D}, \mathbb{Z})} H_1(\bar{X}, \mathbb{Z}).$$

Proof. We apply Proposition 3.1 to $\pi: \bar{X} \rightarrow X$ with $E = \bar{D}$ and $A = D$. Choose a base point $x_0 \in \bar{D}$ and take as a base point on the homotopy model $Z \cup_{\bar{D}} \bar{X}$ the point $(x_0, 1/2)$ on the mapping cylinder. The open set $\{(x, t) \in Z \mid t > 1/4\}$ and the complement of $\{(x, t) \in Z \mid t \leq 3/4\}$ cover $Z \cup_{\bar{D}} \bar{X}$. Applying Mayer-Vietoris, respectively Seifert-van Kampen, to this decomposition gives the claimed result. \square

Remark 3.3 — In the category of complex algebraic varieties the cohomology groups carry a mixed Hodge structure; the compatibility of the Mayer-Vietoris sequence with this additional structure is proven in [PS08, 5.37].

3.B. Automorphisms. Let X be a demi-normal surface and let $\sigma \in \text{Aut}(X)$. Then σ induces an automorphism $\bar{\sigma}$ of \bar{X} such that $\bar{\sigma}(\bar{D}) = \bar{D}$; we denote by $\bar{\sigma}^\nu$ the automorphism of \bar{D}^ν induced by the restriction of $\bar{\sigma}$. Clearly τ and $\bar{\sigma}^\nu$ commute. We denote by $\text{Aut}(\bar{X}, \bar{D}, \tau)$ the subgroup of $\text{Aut}(\bar{X})$ consisting of the automorphisms ψ such that $\psi(\bar{D}) = \bar{D}$ and the induced automorphism ψ^ν of \bar{D}^ν commutes with τ . The map $\text{Aut}(X) \rightarrow \text{Aut}(\bar{X}, \bar{D}, \tau)$ defined by $\sigma \rightarrow \bar{\sigma}$ is clearly an injective homomorphism. In addition, one has:

Lemma 3.4 — *Let X be a demi-normal surface obtained from a triple (\bar{X}, \bar{D}, τ) . Then $\text{Aut}(X) \rightarrow \text{Aut}(\bar{X}, \bar{D}, \tau)$ is an isomorphism.*

Proof. It suffices to prove that the map is surjective, namely that given an automorphism $\psi \in \text{Aut}(\bar{X}, \bar{D}, \tau)$ there exists $\sigma \in \text{Aut}(X)$ such that $\bar{\sigma} = \psi$. Applying Remark 2.4 to the finite surjective map $f = \pi \circ \psi: \bar{X} \rightarrow X$ one obtains $\sigma: X \rightarrow X$ such that $\bar{\sigma} = \psi$. By general nonsense σ is indeed an automorphism. \square

3.C. Irregularity. Let X be an slc surface, not necessarily Gorenstein. While it is easy to compute $\chi(X)$ from the normalisation, the calculation of the irregularity (or geometric genus) is more subtle: the irregularity can either drop or increase in the normalisation (see [LR13, Sect. 5.3.1] for some examples.)

In this section, we give an algorithm to compute the irregularity in concrete examples, at least as long as the normalisation has irregularity $q = 0$; the reason for considering here only slc surfaces is that in our computations we exploit the classification of slc surface singularities.

We keep the notations from Diagram (2.3). We define a sheaf \mathcal{Q} via the following diagram:

$$(3.5) \quad \begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}_D & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_D & \longrightarrow & 0 \\ & & \parallel & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_* \mathcal{O}_{\bar{X}}(-\bar{D}) & \longrightarrow & \pi_* \mathcal{O}_{\bar{X}} & \longrightarrow & \pi_* \mathcal{O}_{\bar{D}} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{Q} & = & \mathcal{Q} & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Note that the two rows of the diagram give the formula

$$(3.6) \quad \chi(X) = \chi(\bar{X}) - \chi(\bar{D}) + \chi(D).$$

We wish to use the second column to compute also the irregularity $q(X)$, so we need to understand \mathcal{Q} . First note that \mathcal{Q} is a torsion-free sheaf on the curve D , because \bar{D} is a curve without embedded points. We now compare the third column with the analogous sequence on the normalisations. Recall that $\pi: \bar{D}^\nu \rightarrow D^\nu$ is a double cover with involution τ , so $\pi_* \mathcal{O}_{\bar{D}^\nu} = \mathcal{O}_{D^\nu} \oplus \mathcal{L}^{-1}$ where $\mathcal{L}^{\otimes 2}$ is the line bundle associated

with the branch locus. Then there is a diagram with exact rows and columns

$$(3.7) \quad \begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_D & \longrightarrow & \nu_* \mathcal{O}_{D^\nu} & \longrightarrow & \nu_* \mathcal{O}_{D^\nu} / \mathcal{O}_D \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \pi_* \mathcal{O}_{\bar{D}} & \longrightarrow & \pi_* \bar{\nu}_* \mathcal{O}_{\bar{D}^\nu} & \longrightarrow & \pi_* (\bar{\nu}_* \mathcal{O}_{\bar{D}^\nu} / \mathcal{O}_{\bar{D}}) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathcal{Q} & \longrightarrow & \nu_* \mathcal{L}^{-1} & \longrightarrow & \nu_* \mathcal{L}^{-1} / \mathcal{Q} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Indeed, the first and second row, respectively column, are easily seen to be exact. Let \mathcal{K} be the kernel of $\mathcal{Q} \rightarrow \nu_* \mathcal{L}^{-1}$. By the snake Lemma it injects into $\nu_* \mathcal{O}_{D^\nu} / \mathcal{O}_D$ and thus is supported at finitely many points. On the other hand it is included in the torsion-free sheaf on \mathcal{Q} and thus $\mathcal{K} = 0$ and also the third row, respectively column, is exact.

We need to introduce some further notation locally analytically at a point $p \in D$. If p is semi-log-terminal then $\pi^{-1}(p)$ does not contain nodal points of \bar{D} (cf. [KSB88, Thm. 4.23]) and thus $\mathcal{Q} \cong \nu_* \mathcal{L}^{-1}$ at p by the exactness of the third column.

So assume that $p \in D$ is a degenerate cusp or a quotient thereof. For the details of the following discussion we refer to [LR14, Sect. 4.1]. Let $q_1, \dots, q_\mu \in \bar{D}$ be the preimages of p . If q_i is a nodal point of \bar{D} , then we denote its two preimages in \bar{D}^ν by r_i and s_i .

If p is a degenerate cusp then all the q_i are nodal points of \bar{D} and we can order the q_i in such a way, that $\tau(s_i) = r_{i+1}$ where the index is computed modulo μ . We illustrate this in Figure 1 for $\mu = 3$.

If p is a quotient of a degenerate cusp, then one can order the q_i in such a way that $q_2, \dots, q_{\mu-1}$ are nodes and τ acts as above for $\mu = 2, \dots, \mu - 2$. The *end-points* q_1 and q_μ can be of two types: either a nodal point of \bar{D} which is the image of a fixed point of the involution τ or a dihedral point of \bar{X} which is a smooth point of \bar{D} . If q_1 is a smooth point of \bar{D} , then its preimage in \bar{D}^ν is mapped to r_2 by τ , while if it is a node with preimages r_1, s_1 then s_1 is mapped to r_2 by τ while s_1 is fixed by τ . The situation for q_μ is similar.

Lemma 3.8 — *With the above notations, let $p \in D$ be a point. Then*

(i) *The length of the sheaf $\nu_* \mathcal{L}^{-1} / \mathcal{Q}$ at p is*

$$l_p(\nu_* \mathcal{L}^{-1} / \mathcal{Q}) = \begin{cases} 1 & \text{if } p \text{ is a degenerate cusp of } X, \\ 0 & \text{else.} \end{cases}$$

(ii) *If p is a degenerate cusp, then the linear form $\varphi_p: \mathcal{O}_{\bar{D}^\nu, (\pi \circ \bar{\nu})^{-1}(p)} \rightarrow \mathbb{C}$ defined by*

$$f \mapsto \sum_{i=1}^{\mu} (f(r_i) - f(s_i)),$$

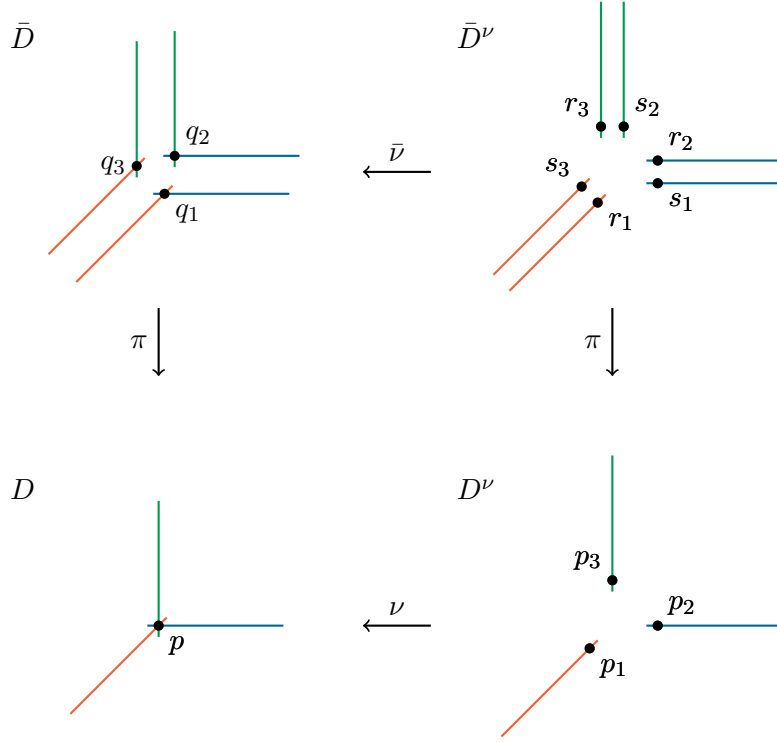


FIGURE 1. Local notation at a degenerate cusp p with $\mu = 3$: $\bar{\nu}(r_i) = \bar{\nu}(s_i) = q_i$ and $\tau(s_i) = r_{i+1}$.

induces an isomorphism

$$\varphi_p: (\nu_* \mathcal{L}^{-1}/\mathcal{Q})_p \cong \left(\frac{\pi_* (\bar{\nu}_* \mathcal{O}_{\bar{D}^\nu}/\mathcal{O}_{\bar{D}})}{\nu_* \mathcal{O}_{D^\nu}/\mathcal{O}_D} \right)_p \xrightarrow{\cong} \mathbb{C}.$$

Proof. In our chosen notation, by diagram (3.7) we can identify

$$(\nu_* \mathcal{L}^{-1}/\mathcal{Q})_p \cong \left(\frac{\pi_* (\bar{\nu}_* \mathcal{O}_{\bar{D}^\nu}/\mathcal{O}_{\bar{D}})}{\nu_* \mathcal{O}_{D^\nu}/\mathcal{O}_D} \right)_p = \frac{\bigoplus_i (\mathcal{O}_{r_i} \oplus \mathcal{O}_{s_i})/\mathcal{O}_{q_i}}{(\bigoplus_i \mathcal{O}_{p_i})/\mathcal{O}_p}.$$

The first part follows from a simple dimension count. For the second part we only need to note that the given function is non-zero and descends to the quotient. \square

Proposition 3.9 — *Let X be a connected slc surface with degenerate cusps p^1, \dots, p^k . At each p^i choose an isomorphism φ_{p^i} as in Lemma 3.8. Choose a basis f_1, \dots, f_l of $H^0(\nu_* \mathcal{L}^{-1}) = H^0(\mathcal{L}^{-1}) \subset H^0(\mathcal{O}_{\bar{D}^\nu})$, viewed as the τ -anti-invariant functions on \bar{D}^ν , and consider the $k \times l$ -matrix $M = (\varphi_{p^i}(f_j))_{ij}$. Then*

$$h^0(\mathcal{Q}) = \dim \ker M.$$

In particular, if the normalisation \bar{X} is the disjoint union of m surfaces with irregularity $q = 0$ then

$$q(X) = h^0(\mathcal{Q}) - (m - 1) = \dim \ker M - m + 1.$$

Proof. By Lemma 3.8 the matrix M represents the map $H^0(\nu_* \mathcal{L}^{-1}) \rightarrow H^0(\nu_* \mathcal{L}^{-1}/\mathcal{Q})$ in the long exact cohomology sequence associated with the third row of (3.7), which proves the first part. The second part follows from the second column of (3.5). \square

Note that the basis of $H^0(\mathcal{L}^{-1}) \subset H^0(\mathcal{O}_{\bar{D}^\nu})$ required in the above proposition is easy to choose. If a component $C \subset \bar{D}^\nu$ is fixed by τ , then any τ -anti-invariant function on C vanishes. If $\tau(C) = C' \neq C$ then the only τ -anti-invariant functions on $C \sqcup C'$ are locally constant of the form $\lambda(1 \sqcup -1)$ for $\lambda \in \mathbb{C}$.

A sample computation in an explicit example is carried out in the proof of Proposition 4.10.

4. GORENSTEIN STABLE SURFACES WITH $K^2 = 1$

In this section we prove Theorem A, stated in the Introduction. To this end we first identify possible invariants of irregular surfaces and then, using results from [FPR14], we describe the two irregular Gorenstein surfaces with $K_X^2 = 1$. The last two sections are devoted to a detailed study of these, first from a topological and then from a moduli perspective.

Along the way, we give the complete classification of stable Gorenstein surfaces with normalisation $(\mathbb{P}^2, 4 \text{ lines})$.

4.A. Classical methods to prove regularity. Let X be a stable Gorenstein surface with $K_X^2 = 1$. For $p_g(X) > 0$ we look at the canonical curves:

Lemma 4.1 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$ and let $C \in |K_X|$ be a canonical curve.*

Then C is a reduced and irreducible Gorenstein curve with $p_a(C) = 2$, not contained in the non-normal locus.

Proof. By assumption $\mathcal{O}_X(C) = \omega_X$ is a line bundle, so C is a Gorenstein curve and by adjunction we have $p_a(C) = 2$. Since $K_X C = 1$ and K_X is an ample Cartier divisor, the curve C is reduced and irreducible. Since no component of the non-normal locus is Cartier and C is reduced, C cannot be contained in the non-normal locus. \square

Proposition 4.2 — *Let X be a Gorenstein stable surface with $K_X^2 = 1$. Then $q(X) > 0$ if and only if $\chi(X) = 0$.*

Proof. Clearly a surface with $\chi(X) = 0$ is irregular. For the other direction, let X be a Gorenstein stable surface with $K_X^2 = 1$. We have $\chi(X) \geq 0$ by [FPR14, Thm.3.6], hence it is enough to show that for $\chi(X) > 0$ one has $q(X) = 0$.

So assume by contradiction that $\chi(X) > 0$ and $q(X) > 0$. Let $0 \neq \eta \in \text{Pic}^0(X)$; since $h^2(K_X + \eta) = h^0(-\eta) = 0$, it follows that $h^0(K_X + \eta) \geq \chi(K_X + \eta) = \chi(X) > 0$. Since $\chi(X) > 0$, the linear system $|K_X|$ is nonempty. Pick $C \in |K_X|$: since $h^0(\eta) = 0$, there is an injection $H^0(K_X + \eta) \hookrightarrow H^0((K_X + \eta)|_C)$. In particular $h^0((K_X + \eta)|_C) \geq h^0(K_X + \eta) \geq 1$. But $(K_X + \eta)|_C$ is an invertible sheaf of degree 1 on C , which is an irreducible curve of genus 2 (see Lemma 4.1), so by Riemann-Roch $h^0((K_X + \eta)|_C) \leq 1$. Indeed, assume otherwise: then the linear system $|K_X + \eta|_C$ would induce a rational map $f: C \rightarrow \mathbb{P}^1$ of degree 1. Composing with the normalisation of C we get a contradiction.

Therefore we conclude that $h^0((K_X + \eta)|_C) = h^0(K_X + \eta) = \chi(X) = 1$. For $\eta \neq 0$, if we denote by C_η the only curve in $|K_X + \eta|$ then C_η intersects C transversely at a point P_η which is smooth for C .

In addition, by the generalized Kodaira vanishing (cf. [LR14, Thm. 3.1]) one has $H^1(\mathcal{O}_X(-C)) = 0$, hence $H^1(\mathcal{O}_X) \rightarrow H^1(\mathcal{O}_C)$ is an injection and the homomorphism of algebraic groups $\text{Pic}^0(X) \rightarrow \text{Pic}^0(C)$ has finite kernel. It follows that the map

$\eta \rightarrow P_\eta$, $\eta \in \text{Pic}^0(C) \setminus \{0\}$, is not constant. So the system $|2K_X|_C$ contains the infinitely many divisors $P_\eta + P_{-\eta}$. Since $h^0(K_C) = 2$, it follows that the restriction map $H^0(2K_X) \rightarrow H^0(2K_X|_C) = H^0(K_C)$ is surjective.

By the generalized Kodaira vanishing and Serre duality we have $H^1(2K_X) = 0$ and the adjunction sequence $0 \rightarrow K_X \rightarrow 2K_X \rightarrow K_C \rightarrow 0$ gives an exact sequence:

$$H^0(2K_X) \rightarrow H^0(K_C) \rightarrow H^1(K_X) \rightarrow 0,$$

and therefore $q(X) = h^1(K_X) = 0$, contradicting the assumptions. \square

Corollary 4.3 — *Let X be an irregular Gorenstein stable surface with $K^2 = 1$. Then X is non-normal and the normalisation of X is $(\mathbb{P}^2, 4 \text{ general lines})$.*

Proof. By Proposition 4.2 we have $\chi(X) = 0$. By [FPR14, Thm. 3.6] X is non-normal and its normalisation is \mathbb{P}^2 with conductor a stable quartic. We have $\chi(D) = \chi(X) - \chi(\bar{X}) + \chi(\bar{D}) = -3$, e.g., by (3.6). So in this case the inequality (ii) of [FPR14, Lem. 3.5] is an equality, and by ibidem \bar{D} has rational components and 6 nodes. It follows that \bar{D} is the union of four lines in general position. \square

4.B. Interlude: Gorenstein stable surfaces from four lines in the plane.

Motivated by Corollary 4.3, we classify here Gorenstein stable surfaces with normalisation $(\mathbb{P}^2, 4 \text{ lines})$. We take Kollár's approach (cf. Section 2.A) to the classification of stable surfaces, as obtained from an lc pair (\bar{X}, \bar{D}) by gluing \bar{X} along \bar{D} via an involution τ of the normalization \bar{D}^ν of \bar{D} .

We take $\bar{X} = \mathbb{P}^2$ and $\bar{D} = L_1 + \dots + L_4$, where $L_1 = \{x_0 = 0\}$, $L_2 = \{x_1 = 0\}$, $L_3 = \{x_2 = 0\}$ and $L_4 = \{x_0 + x_1 + x_2 = 0\}$, and classify all the stable Gorenstein surfaces that arise from this lc pair; as a byproduct we obtain the complete classification of the stable Gorenstein surfaces with $K_X^2 = 1$ and $\chi(X) = 0$ (Proposition 4.12).

Denote by $P_{(ij)} \in \bar{D}$ the intersection point of L_i and L_j . The normalization of \bar{D} is $\bar{D}^\nu = \bigsqcup L_i$: we denote by P_{ij} the point of $L_i \subset \bar{D}^\nu$ that maps to $P_{(ij)}$, so that each component of \bar{D}^ν contains three marked points. Recall (cf. Addendum to Thm. 2.2) that an involuton τ of \bar{D}^ν gives rise to a Gorenstein stable surface if and only if it induces a fixed point free involution of the marked points. Since every component of \bar{D}^ν contains three such points, τ cannot preserve any of the L_i , so we may assume that it maps L_1 to L_2 and L_3 to L_4 . Then τ is uniquely determined by two bijections

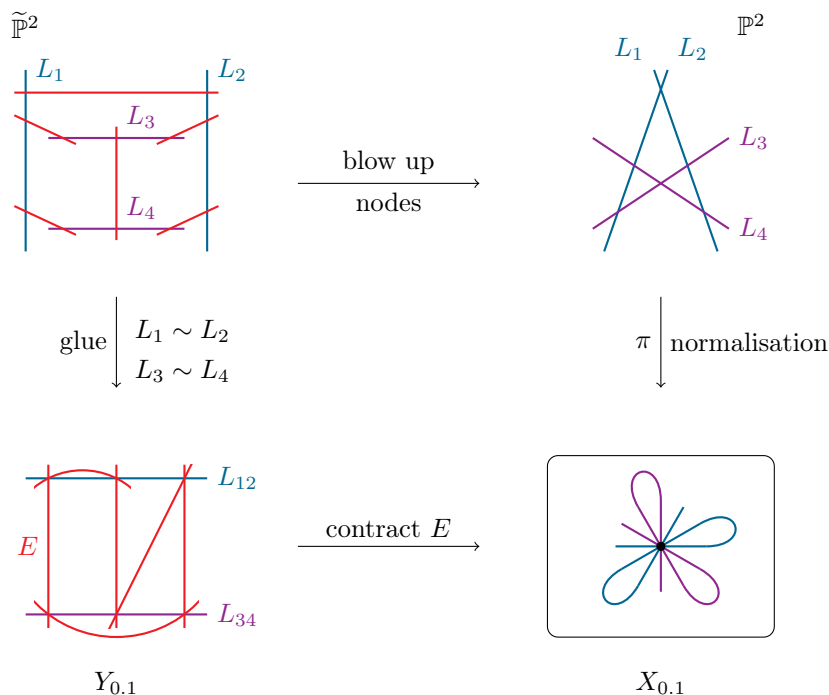
$$\begin{aligned} \varphi_{12}: \{P_{12}, P_{13}, P_{14}\} &\rightarrow \{P_{21}, P_{23}, P_{24}\}, \\ \varphi_{34}: \{P_{31}, P_{32}, P_{34}\} &\rightarrow \{P_{41}, P_{42}, P_{43}\}. \end{aligned}$$

The set of possible choices of $(\varphi_{12}, \varphi_{34})$ can be identified with $S_3 \times S_3$.

Example 4.4 — To visualise what is going on it is helpful to take a log-resolution of (\bar{X}, \bar{D}) , that is, blow up the intersection points of the four lines and then glue the strict transform of \bar{D} , which is \bar{D}^ν . On the resulting surface one can contract the exceptional curves to degenerate cusps. In technical terms, we construct the minimal semi-resolution and then pass to the canonical model.

Let us illustrate this procedure in an explicit example: consider the involution given by

$$\varphi_{12} = \begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{23} & P_{21} & P_{24} \end{pmatrix}, \quad \varphi_{34} = \begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{41} & P_{43} & P_{42} \end{pmatrix}.$$

FIGURE 2. Construction of $X_{0,1}$

Glueing the strict transforms of the lines in the blow up of \mathbb{P}^2 gives a semi-smooth surface $Y_{0,1}$; contracting the exceptional curves on $Y_{0,1}$ yields the stable surface $X_{0,1}$. This is illustrated in Figure 2.

We see that the non-normal locus D has two components, each a rational curve with a triple point. At the intersection point of the curves, the surface $X_{0,1}$ has a degenerate cusp P , which locally look like the cone over a circle of six independent lines in projective space, that is, locally analytically $(X_{0,1}, P)$ is isomorphic to the vertex of the cone in \mathbb{A}^6 given by equations $\{z_i z_j \mid i - j \neq -1, 0, 1 \pmod{6}\}$. Thus blowing up the degenerate cusp P results in the semi-resolution $Y_{0,1}$, the exceptional divisor E being a cycle of six smooth rational curves.

We will study this surface and its cousin $X_{0,2}$ more in detail in the next section.

Proposition 4.5 — *Let X be a Gorenstein stable surface such that the normalization \bar{X} of X is \mathbb{P}^2 and the double locus $\bar{D} \subset \bar{X}$ is the union of four lines. Then X is isomorphic to one (and only one) of the surfaces corresponding to the involutions listed in Table 1 on page 16.*

Proof. The curve \bar{D} is nodal since (\bar{X}, \bar{D}) is lc, so it consists of four lines in general position and we may assume without loss of generality that $D = L_1 + \dots + L_4$ and that τ interchanges L_1 with L_2 and L_3 with L_4 . Every permutation of L_1, \dots, L_4 is induced by an element of $\text{Aut}(\mathbb{P}^2)$, so the automorphism group of (\bar{X}, \bar{D}) can be identified with S_4 ; our choice of which lines should be interchanged by the involution τ reduces the symmetry group to D_4 , generated by the involutions (12), (34) and (13)(24). The symmetry group D_4 acts on these choices by permutation of the indices; by Lemma 3.4 the stabiliser is the automorphism group of the corresponding stable surface.

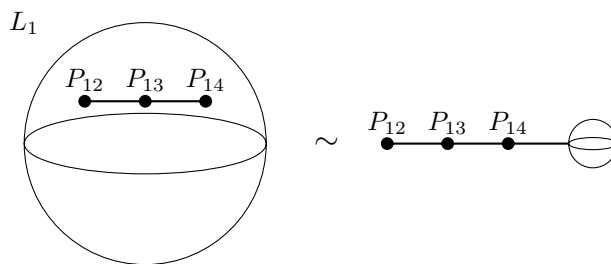


FIGURE 3. Homotopically equivalent model of L_1 with three marked points

In Table 1 we give a representative for each orbit, together with some information on the stable surface X constructed from the triple (\bar{X}, \bar{D}, τ) . Given this data one can prove that the list is complete and without redundancies in the following way: first check that the stabilisers are correct and then use the orbit stabiliser theorem to show that the given orbits fill up $S_3 \times S_3$ if they are disjoint. Apart from case 1.2 and 1.3, where an explicit computation is needed, no two orbits can be equal because either the stabilisers are not conjugate in D_4 or the structure of the resulting degenerate cusps is different.

Let μ be the number of degenerate cusps of X . Then by (3.6) and [FPR14, Lem. 3.5] we have $\chi(X) = \chi(\bar{X}) - \chi(\bar{D}) + \chi(D) = \mu - 1$.

The irregularity $q(X)$ vanishes if $\chi(X) \geq 1$ by Proposition 4.2. The fact that $q(X) = 1$ if $\chi(X) = 0$ will be proved below in Proposition 4.10. \square

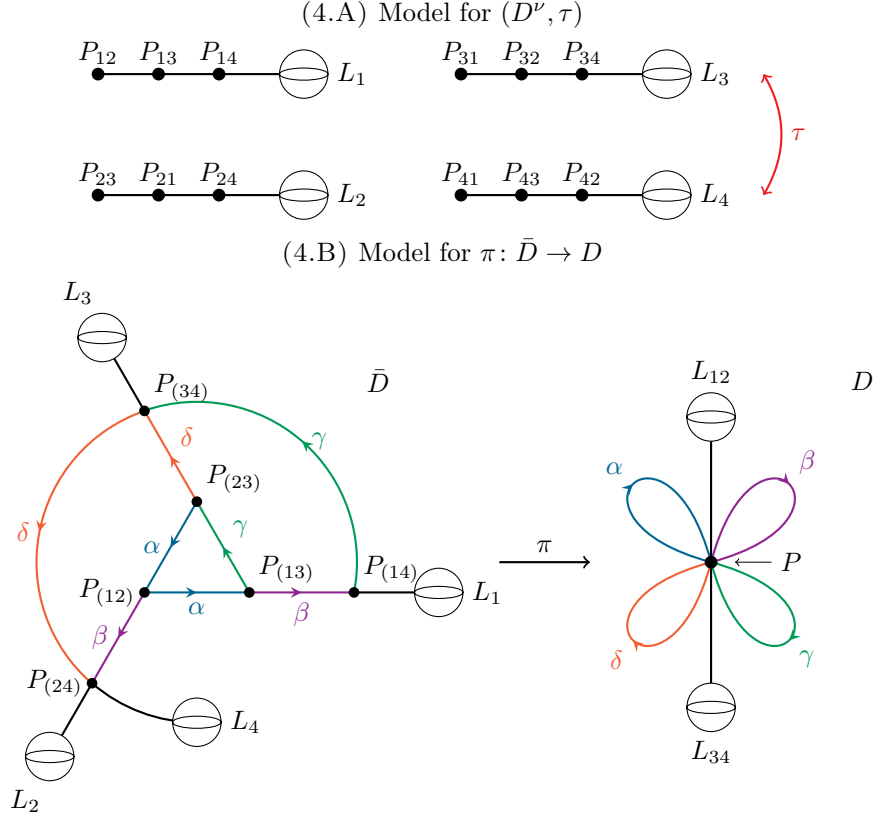
4.C. Invariants of irregular Gorenstein stable surfaces with $K^2 = 1$. By Corollary 4.3 and Proposition 4.5 there are exactly two irregular Gorenstein stable surfaces with $K^2 = 1$, which we called $X_{0.1}$ and $X_{0.2}$. We now illustrate the techniques of Section 3 to compute their integral homology, fundamental group, irregularity and Picard group.

Proposition 4.6 — *The surfaces $X_{0.1}$ and $X_{0.2}$ have the following topological invariants:*

- (i) $\pi_1(X_{0.1}) \cong \langle A, B \mid A^{-1}B^{-1}A^2B^2 \rangle$ and $\pi_1(X_{0.2}) \cong \langle A, B \mid AB^{-1}A^2B^2 \rangle$ and these two groups are not isomorphic.
- (ii) $H_i(X_{0.1}, \mathbb{Z}) = H_i(X_{0.2}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 1, 2, 4 \\ \mathbb{Z}^2 & i = 3 \end{cases}$

The fact that the two groups in (i) are not isomorphic was explained to us by Kai-Uwe Bux.

Proof. We continue to use the notation from (2.3). In order to make explicit computations, we choose a homotopy-equivalent model for $\pi: \bar{D} \rightarrow D$: each line L_i is topologically a sphere with three marked points, the intersection points. Choosing an order for these three points, this space is homotopy equivalent to the 1-point-union of an interval with three marked points and a sphere as in Figure 3. Note that this and the following are *real* pictures. Doing this for all four lines we may choose the order of the points so that the action of τ on \bar{D}^ν is compatible with our model, as in Figures 4.A and 5.A. Glueing the four components back together we get a model for \bar{D} , while first taking the quotient by the involution and then glueing gives a model

FIGURE 4. Homotopy-equivalent models in case $X_{0,1}$

for D ; this, together with the map π , is shown in Figure 4.B for $X_{0,1}$ and in Figure 5.B for $X_{0,2}$.

(i) By Corollary 3.2 we have

$$\pi_1(X_{0,1}) = \pi_1(D) \star_{\pi_1(\bar{D})} \pi_1(\mathbb{P}^2) \cong \frac{\pi_1(D)}{\langle \pi_1(\bar{D}) \rangle}$$

We choose as base points $P \in D$ and $P_{(23)} \in \bar{D}$ (cf. Figures 4.B and 5.B). We read off our model that $\pi_1(D, P)$ is the free group generated by the loops $\alpha, \beta, \gamma, \delta$ depicted in Figure 4.B in the case of $X_{0,1}$ and in Figure 5.B in the case of $X_{0,2}$.

If $X = X_{0,1}$, then the image of $\pi_1(\bar{D}, P_{(23)})$ is generated by

$$(4.7) \quad \alpha^2\gamma, \delta\gamma^{-1}\beta^{-1}\gamma, \alpha\beta\delta^{-2}.$$

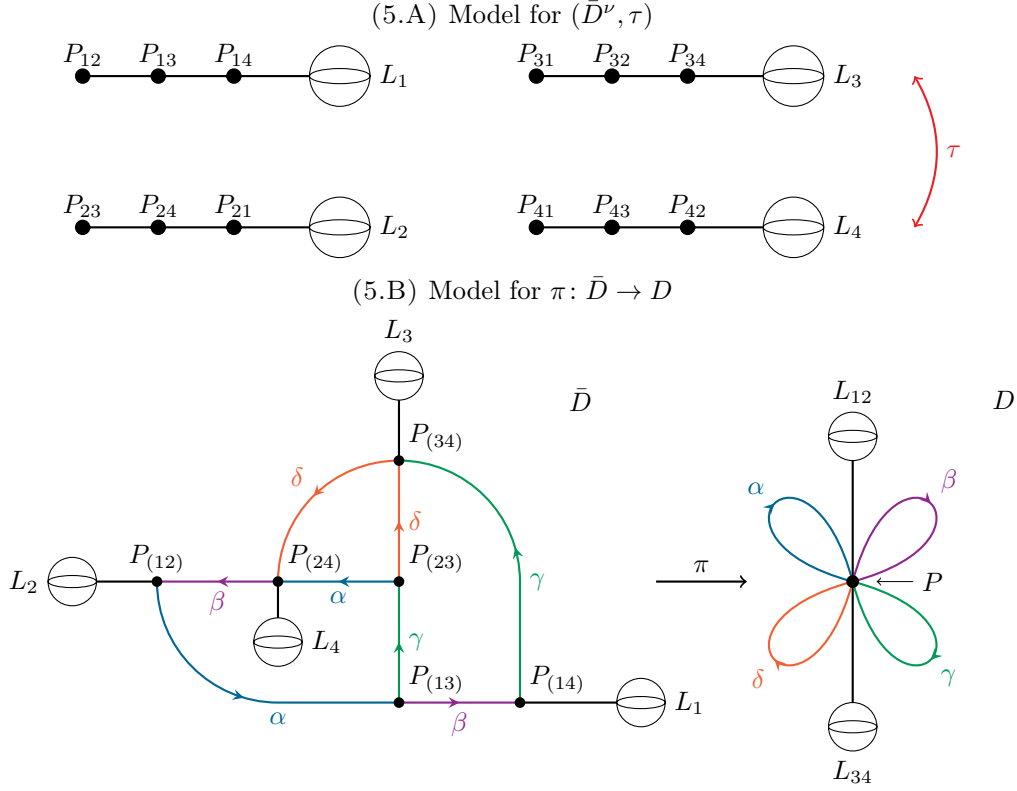
Solving for $\gamma = \alpha^{-2}$ and $\delta = \gamma^{-1}\beta\gamma$, one sees that the quotient group is generated by (the classes of) α and β with the relation:

$$\alpha\beta\alpha^2\beta^{-2}\alpha^{-2}.$$

Denoting by A the class of α and by B the class of $\alpha^{-2}\beta^{-1}\alpha^2$ one gets the presentation given in the statement.

For $X_{0,2}$ the image of $\pi_1(\bar{D}, P_{(23)})$ is generated by:

$$(4.8) \quad \delta^2\alpha^{-1}, \alpha\beta\alpha\gamma, \delta\gamma^{-1}\beta^{-1}\gamma$$


 FIGURE 5. Homotopy-equivalent models in case $X_{0,2}$

Solving for $\alpha = \delta^2$ and $\beta = \gamma\delta\gamma^{-1}$, one sees that the quotient group is generated by (the classes of) γ and δ with the relation:

$$\delta^2\gamma\delta\gamma^{-1}\delta^2\gamma.$$

Denoting by A the class of δ and by B the class of $\delta^2\gamma$ the relation becomes $BAB^{-1}A^2B$ and one gets the presentation given in the statement after conjugation with B .

To prove that the two groups are not isomorphic, we show that $\pi_1(X_{0,1})$ admits a surjective isomorphism onto the group A_4 , while $\pi_1(X_{0,2})$ does not.

The homomorphism $\psi: \pi_1(X_{0,1}) \rightarrow A_4$ is defined by $A \mapsto (234)$ and $B \mapsto (123)$: it is easy to check that $\psi(A^{-1}B^{-1}A^2B^2) = 1$.

On the other hand, assume for contradiction that $\psi: \pi_1(X_{0,2}) \rightarrow A_4$ is a surjective homomorphism and set $a = \psi(A)$ and $b = \psi(B)$; the equality:

$$(4.9) \quad ab^{-1}a^2b^2 = 1,$$

holds in A_4 . Note that since ψ is surjective at least one between a and b has order 3. If a and b both have order 3, (4.9) gives $a^{-1}ba = b^{-1}$, and therefore the subgroup $\langle b \rangle$ is normal in A_4 , a contradiction. The remaining cases can be excluded by similar (easier) arguments.

(ii) Let $X = X_{0,j}$ for $j = 1, 2$. We apply the Mayer-Vietoris sequence from Corollary 3.2 and get $\pi_*: H_4(\mathbb{P}^2, \mathbb{Z}) \cong H_4(X, \mathbb{Z})$ and a long exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & H_3(X, \mathbb{Z}) & \longrightarrow & & & \\
& & & & & & \\
& \longrightarrow & \underbrace{H_2(\bar{D}, \mathbb{Z})}_{=\mathbb{Z}\langle [L_1], \dots, [L_4] \rangle} & \xrightarrow{N} & \underbrace{H_2(D, \mathbb{Z})}_{=\mathbb{Z}\langle [L_{12}], [L_{34}] \rangle} \oplus \underbrace{H_2(\mathbb{P}^2, \mathbb{Z})}_{=\mathbb{Z}\langle [L_i] \rangle} & \longrightarrow & H_2(X, \mathbb{Z}) \longrightarrow \\
& & & & & & \\
& \longrightarrow & H_1(\bar{D}, \mathbb{Z}) & \xrightarrow{M_j} & H_1(D, \mathbb{Z}) \oplus \underbrace{H_1(\mathbb{P}^2, \mathbb{Z})}_{=0} & \longrightarrow & H_1(X, \mathbb{Z}) \longrightarrow 0,
\end{array}$$

where

$$N = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

in the given bases.

We computed the image of π_* on fundamental groups in (4.7) and (4.8) and we get, by abelianisation, matrix representations

$$M_1 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -2 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}.$$

In both cases the map is injective with cokernel isomorphic to \mathbb{Z} . Thus the homology is as claimed. \square

Proposition 4.10 — *Let X be a Gorenstein stable surface with $K^2 = 1$ and $\chi(X) = 0$. Then $q(X) = 1$, and the exact sequence for $\text{Pic}(X)$ is:*

$$0 \rightarrow \text{Pic}^0(X) \cong \mathbb{C}^* \rightarrow \text{Pic}(X) \rightarrow H^2(X, \mathbb{Z}) \cong \mathbb{Z} \rightarrow 0.$$

Proof. By Proposition 4.12, X is one of the surfaces $X_{0,1}$ or $X_{0,2}$ of Table 1. In both cases, to compute the irregularity q we follow the method of Proposition 3.9. Let us first consider $X_{0,1}$. We need to establish notation as in Section 3.C for the points of \bar{D}^ν that map to the unique degenerate cusp $P \in X_{0,1}$. Let

$$\begin{array}{ll}
s_1 = P_{12}, & r_2 = \tau(s_1) = P_{23}, \\
s_2 = P_{32}, & r_3 = \tau(s_2) = P_{43}, \\
s_3 = P_{34}, & r_4 = \tau(s_3) = P_{42}, \\
s_4 = P_{24}, & r_5 = \tau(s_4) = P_{14}, \\
s_5 = P_{41}, & r_6 = \tau(s_5) = P_{31}, \\
s_6 = P_{13}, & r_1 = \tau(s_6) = P_{21},
\end{array}$$

and define a basis f_1, f_2 of τ -anti-invariant functions on \bar{D}^ν by

$$\begin{array}{ll}
f_1(P_{1*}) = 1 = -f_1(P_{2*}), & f_1(P_{3*}) = f_1(P_{4*}) = 0, \\
f_2(P_{3*}) = 1 = -f_2(P_{4*}), & f_2(P_{2*}) = f_2(P_{1*}) = 0.
\end{array}$$

Then the matrix M from Proposition 3.9 is $M = (-2, -2)$ and thus $q(X) = 1$ and $p_g(X) = \chi(X) - 1 + q(X) = 0$.

We skip the analogous calculation for the second case.

Since $h^2(\mathcal{O}_X) = p_g(X) = 0$ for both $X = X_{0,1}$ and $X = X_{0,2}$, the exponential sequence gives:

$$0 \rightarrow H^1(X, \mathbb{Z}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^1(X, \mathcal{O}_X^*) \rightarrow H^2(X, \mathbb{Z}) \rightarrow 0,$$

The statement now follows since $h^1(\mathcal{O}_X) = 1$ and $H^i(X, \mathbb{Z}) \cong \mathbb{Z}$ for $i = 1, 2$ by Proposition 4.6. \square

Remark 4.11 — One could also deduce $q(X) = 1$ in a less elementary way: an slc surface X is semi-normal and thus by [Ale02, Thm. 4.1.7] its $\text{Pic}^0(X)$ has unipotent rank zero. In other words, it is an extension of multiplicative groups and an abelian variety and thus by the exponential sequence $q(X) \leq b_1(X)$. For the two surfaces at hand we know $q(X) > 0$ and $b_1(X) = 1$, thus $q(X) = 1$.

4.D. Moduli perspective on irregular stable surfaces with $K_X^2 = 1$ and the proof of Theorem A. We have proved in Proposition 4.2 and Corollary 4.3 that there are exactly two different irregular Gorenstein stable surfaces with $K_X^2 = 1$. The computation of their invariants in Section 4.C proves the first part of Theorem A, stated in the introduction.

The next result takes a moduli perspective on these surfaces and thereby proves the remaining part of Theorem A.

Proposition 4.12 — *The moduli space $\overline{\mathfrak{M}}_{1,0}^{(Gor)}$ of Gorenstein stable surfaces with $K_X^2 = 1$ and $\chi(X) = 0$ consists of two isolated points corresponding to the surfaces $X_{0,1}$ and $X_{0,2}$ in Table 1 on page 16; each point is a connected component of the moduli space of stable surfaces.*

Moreover, $\overline{\mathfrak{M}}_{1,0}^{(Gor)}$ coincides with the moduli space of irregular Gorenstein stable surfaces with $K^2 = 1$.

We do not know whether there are any irregular normal stable surfaces with $K^2 = 1$; of course, such surfaces would not be Gorenstein by Proposition 4.2. Note that stable surfaces have DuBois singularities [Kol13, Cor. 6.23], therefore the irregularity $h^1(\mathcal{O}_X)$ is constant in flat and proper families (see e.g. [Kol12, Prop. 20]). Thus an irregular surface cannot be the limit of regular ones and vice versa, i.e., the moduli space admits a finer decomposition according to geometric genus and irregularity.

Proof. The set-theoretical description of $\overline{\mathfrak{M}}_{1,0}^{(Gor)}$ has already been noted above. It remains to show that the subspace $\overline{\mathfrak{M}}_{1,0}^{(Gor)}$ is a union of connected components of the moduli space of stable surfaces.

The families considered in the construction of the moduli space [Kol12, Sect. 4] are in particular proper and flat and thus by [BH93, Cor. 3.3.15] the Gorenstein condition is open in the moduli space. Since $\overline{\mathfrak{M}}_{1,0}^{(Gor)}$ is finite, hence closed, our claim follows. \square

TABLE 1. Surfaces from four lines in the plane
(Notation as in Section 4.B)

surface	$\chi(X)$	φ_{12}	φ_{34}	degenerate cusps	$q(X)$	$\text{Aut}(X)$
$X_{3.1}$	3	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{21} & P_{24} & P_{23} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{42} & P_{41} & P_{43} \end{pmatrix}$	$\{P_{(12)}\}, \{P_{(34)}\}, \{P_{(13)}, P_{(24)}\}, \{P_{(23)}, P_{(14)}\}$	0	D_4
$X_{2.1}$	2	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{21} & P_{23} & P_{24} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{41} & P_{42} & P_{43} \end{pmatrix}$	$\{P_{(12)}\}, \{P_{(34)}\}, \{P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$	0	D_4
$X_{2.2}$	2	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{21} & P_{23} & P_{24} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{42} & P_{41} & P_{43} \end{pmatrix}$	$\{P_{(12)}\}, \{P_{(34)}\}, \{P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$	0	$\langle\langle(12), (34)\rangle\rangle$
$X_{2.3}$	2	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{23} & P_{24} & P_{21} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{42} & P_{41} & P_{43} \end{pmatrix}$	$\{P_{(12)}, P_{(23)}, P_{(14)}\}, \{P_{(13)}, P_{(24)}\}, \{P_{(34)}\}$	0	$\langle\langle(12)(34)\rangle\rangle$
$X_{1.1}$	1	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{21} & P_{23} & P_{24} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{41} & P_{43} & P_{42} \end{pmatrix}$	$\{P_{(12)}\}, \{P_{(34)}, P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$	0	$\langle\langle(34)\rangle\rangle$
$X_{1.2}$	1	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{21} & P_{23} & P_{24} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{42} & P_{43} & P_{41} \end{pmatrix}$	$\{P_{(12)}\}, \{P_{(34)}, P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$	0	$\langle\langle(12)(34)\rangle\rangle$
$X_{1.3}$	1	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{21} & P_{24} & P_{23} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{41} & P_{43} & P_{42} \end{pmatrix}$	$\{P_{(12)}\}, \{P_{(34)}, P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$	0	$\langle\langle(12)(34)\rangle\rangle$
$X_{1.4}$	1	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{23} & P_{24} & P_{21} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{42} & P_{43} & P_{41} \end{pmatrix}$	$\{P_{(12)}, P_{(34)}, P_{(14)}, P_{(23)}\}, \{P_{(13)}, P_{(24)}\}$	0	$\langle\langle(13)(24), (14)(23)\rangle\rangle$
$X_{1.5}$	1	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{23} & P_{24} & P_{21} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{43} & P_{41} & P_{42} \end{pmatrix}$	$\{P_{(12)}, P_{(23)}, P_{(14)}\}, \{P_{(13)}, P_{(24)}, P_{(34)}\}$	0	$\langle\langle(12)(13)(24)\rangle\rangle$
$X_{0.1}$	0	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{23} & P_{21} & P_{24} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{41} & P_{43} & P_{42} \end{pmatrix}$	$\{P_{(12)}, P_{(34)}, P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$	1	$\langle\langle(14)(23)\rangle\rangle$
$X_{0.2}$	0	$\begin{pmatrix} P_{12} & P_{13} & P_{14} \\ P_{23} & P_{24} & P_{21} \end{pmatrix}$	$\begin{pmatrix} P_{31} & P_{32} & P_{34} \\ P_{41} & P_{43} & P_{42} \end{pmatrix}$	$\{P_{(12)}, P_{(34)}, P_{(13)}, P_{(14)}, P_{(23)}, P_{(24)}\}$	1	$\{0\}$

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