RIGOROUS APPROXIMATION OF STATIONARY MEASURES FOR ITERATED FUNCTION SYSTEMS

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Abstract. We study the problem of the rigorous computation of the stationary measure of an IFS described by a stochastic mixture of two or more dynamical systems which are either all uniformly expanding on the interval, either all contractive.

In the expanding case, the associated transfer operators satisfy a Lasota-Yorke inequality, and we compute rigorously the approximations in the L^1 norm. The rigorous computation requires a computer-aided proof of the contraction of the transfer operators for the maps, and we show that this property propagates to the transfer operators of the IFS.

In the contractive case we perform a rigorous approximation of the stationary measure in the Wasserstein-Kantorovich distance, using the same functional analytic approach.

We show that a finite computation can produce a realistic computation of all contraction rates for the whole parameter space. We conclude with a description of the implementation and numerical experiments.

1. Introduction

The reliable simulation and forecasting of the statistical properties of a chaotic dynamical model is a difficult and important task. We investigate in the direction of the rigorous approximation of the stationary measure of random dynamical systems. By rigorous approximation we mean a computation for wich the result is mathematically certified up an explicitly given error of approximation. We show how this can be achieved in the approximation of the stationary measure for some class of Iterated Function System.

An *iterated function system* (IFS) is the datum of a space X and a finite collection of transformations $T_i : X \to X$ for $i \in I$, plus a set of positive parameters (probabilities) p_i for $i \in I$, summing to 1. Iteratively, one of the maps T_i is applied on the set X, chosen with probability p_i independently of the previous steps.

In this article we study the problem of computing effectively (with rigorous error) the stationary (invariant) measure of an iterated function systems.

We will use the functional analytic approach, assuming the space X to be equipped with a σ-algebra (which we assume to be preserved by the inverse of the T_i maps), and considering the transfer operator acting on Borel signed measures. For a transformation T the corresponding transfer operator L acting on

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probability measures is defined as

$$(L\mu)(A) = \mu(T^{-1}(A)),$$

and when μ represents a probability distribution in X, then $L\mu$ represents the probability distribution on X after one application of the transformation T.

Let us consider an IFS constructed with the maps T_i having associated transfer operators L_i and probabilities p_i. The transfer operator associated to such an IFS is defined by

$$\sum_{i \in I} p_i L_i.$$

A fixed point for this transfer operator is called *stationary measure* for the IFS (see [KL06, MR09]).

We point out that we do not assume the maps T_i to be contractive, but on the other hand we will need the operators L_i to be contractions in a suitable sense on a suitable space. This will be illustrated by the two main cases, but we will develop this machinery abstractly and specialize to a particular space only when necessary.

Let us introduce some notation: on a suitable space of measures $\mathcal{B}$, we will denote by $V_{\mathcal{B}}$ (or just V when no confusion is possible) the set of *zero-average measures*, that is

$$V_{\mathcal{B}} = \left\{ \mu \in \mathcal{B} : \int_X 1 d\mu = 0 \right\}.$$

When working on the interval, we denote by $\| \cdot \|_{BV}$ the norm on measures defined as

$$\|\mu\|_{BV} = \sup_{\phi \in C^1 : \|\phi\|_\infty = 1} \int_X \phi' d\mu,$$

the measures having finite norm are absolutely continuous with bounded variation density (see [Liv04]).

We will abuse of notation extending the L^1 norm to signed measures as

$$\|\mu\|_{L^1} = \mu^+(X) + \mu^-(X) \tag{1}$$

for a Hahn decomposition $\mu = \mu^+ - \mu^-$ [Hal13], it coincides with the L^1 norm of the density of μ is absolutely continuous.

- The first case we consider is the case of uniformly expanding transformations in the interval, in such a case for a transformation T which is piecewise C^2 we have a Lasota-Yorke inequality with respect to the BV and L^1 norms

  $$\|L\mu\|_{BV} \leq \lambda \|\mu\|_{BV} + B\|\mu\|_{L^1} \tag{2}$$

  for explicit constants $\lambda < 1$ and B. If the system has a unique absolutely continuous invariant measure, then the iterates of L eventually contracts V_{BV} and the invariant measure is approximated iterating L on some absolutely continuous probability measure. It is possible to find a suitable finite rank approximation of L satisfying the same property (Equation 2) and having a unique fixed point which is close to the unique invariant measure of L.
- The second case is the classic case of contracting maps, on a generic bounded subset of $\mathbb{R}^n$. In such a case the transfer operator is a contraction in the dual space of Lipschitz function, that is, the space of measures with respect to the Wasserstein-Kantorovich norm.

There has been a recent surge of interest on the topic of computation applied to dynamical systems, arising from the availability of high performance computing, and looking forward to explore up to which extent it is possible to analyze a dynamical system in an automated way, computing long time invariants and more in general any rigorous information that can be useful for computer-aided proofs.

All software developed in this work is publicly available as free software, as detailed in section 6.

2. Summary

We present here an outline of the results. Our main aim is using the *a priori* knowledge about the IFS to reduce to the minimum the amount of checking necessary, in particular we will show how most checking can be done once and for all at the level of the maps T_i, and later be used to compute the invariant measure for all the values of the $\{p_i\}$.

The computational verification that an operator is a contraction on a high-dimensional vector space is a very time-consuming check, so it is desirable to avoid having to repeat such checks for all the values of the parameters $\{p_i\}$.

In Section 3 we work out the abstract approximation strategy, that can be used in both the expanding and contractive case.

In Section 4 we prove that contraction properties are preserved for nearby operators, and that that proving contraction on the operators L_i is indeed enough to get a contraction of the IFS for any choice of the probabilities. In practice the estimation of the contraction rate obtained without checking can be quite pessimistic, but still, it can be used to have a usable estimate, as it will be shown in an example.

In Section 5 we demonstrate the application of the above results in the case of the L^1 approximation of the invariant measure for IFS formed by uniformly expanding transformations of the interval.

In Section 8 we treat the more classical case of contracting transformations, approximating the invariant measure in the Wasserstein-Kantorovich metric.

3. Strategy for rigorous estimation

We describe here the general strategy for the rigorous computation of the invariant measure of dynamical system, see [GN14] for more details (in particular about systems satisfying a Lasota-Yorke inequality).

Let L be a transfer operator acting on a Banach space $\mathcal{B}$ of Borel measures and having fixed point μ. We will use a finite-rank projection $\pi_\delta : \mathcal{B} \to \mathcal{B}$ (describing a finite dimensional approximation of measures), and denote by L_δ the approximated operator $\pi_\delta L \pi_\delta$, and by μ_δ its fixed point. While performing a computation, L_δ is computable with rigorous error and representable in a suitable basis as a matrix of floating point numbers on computer, and μ_δ can be rigorously approximated as a vector.

The following Theorem can be used to compute rigorously the error of approximating μ by μ_δ with respect to the norm $\|\cdot\|_{\mathcal{B}}$, it is essentially Theorem 1 of [GN14].

Theorem 3.1. *Suppose that*

(1) $\|(L_\delta - L)\mu\|_{\mathcal{B}} < \infty$,
(2) $\exists N$ *such that* $\|L_\delta^N\|_{\mathcal{B}\to\mathcal{B}} \leq \alpha < 1$,

(3) $\|L_\delta^i\|_{\mathcal{B}\to\mathcal{B}} \leq C_i$ *for* $i = 0, \ldots, N-1$.

Then

$$\|\mu_\delta - \mu\|_{\mathcal{B}} \leq \|(L_\delta - L)\mu\|_{\mathcal{B}} \cdot \frac{1}{1-\alpha} \sum_{i=0}^{N-1} C_i$$

Proof. See [GN14]. □

It easily follows

Corollary 3.2. *If L_δ is a contraction and $\|L_\delta\|_{\mathcal{B}} \leq \gamma < 1$, then*

$$\|\mu_\delta - \mu\|_{\mathcal{B}} \leq \frac{1}{1-\gamma}\|(L_\delta - L)\mu\|_{\mathcal{B}}.$$

Remark 3.3. Both Theorem 3.1 and Corollary 3.2 are still valid when $\|\cdot\|_{\mathcal{B}}$ is only defined on the space of measures of zero average. This is very useful for working with Wasserstein-Kantorovich distance, which only defines a norm on measures with zero average.

Item 1 in the theorem requires an estimation that depends on the properties of the invariant density f and on the goodness of the approximation of L by L_δ.

In practice, this can be done using a stronger norm $\mathcal{B}'$ that can be used to estimate the invariant measure μ, and such that $\|L_\delta - L\|_{\mathcal{B}'\to\mathcal{B}}$ can be made arbitrarily small, so that

$$\|L_\delta f - Lf\|_{\mathcal{B}} \leq \|L_\delta - L\|_{\mathcal{B}'\to\mathcal{B}} \cdot \|\mu\|_{\mathcal{B}'}.$$

Such hypotheses are available when a Lasota-Yorke inequality involving $\mathcal{B}'$ and $\mathcal{B}$ is satisfied.

Remark 3.4. Note that in this estimation it was important the use of the regularity of μ in an auxiliary norm. In general we have no hope of making $\|L_\delta - L\|_{\mathcal{B}\to\mathcal{B}}$ arbitrarily small, as L_δ has finite rank.

Items 2 and 3 of Theorem 3.1 can be verified computationally, but their estimation is neither trivial nor rapid, so we developed a strategy that permits to give a priori bound for these items for any combination of the $\{p_i\}$ once we have estimated items 2 and 3 for some specific choices of the parameters; next section explains this strategy.

Informally, the algorithm can be described as follows:

- Input the maps $\{T_i\}$, the probabilities $\{p_i\}$, and the partition.
- For each map T_i compute the matrix P_i approximating $L_{i,\delta}$.
- Compute L_δ as linear combination of the $\{L_{i,\delta}\}$ with the $\{p_i\}$ as coefficients.
- Compute the approximated fixed point μ_δ of L_δ up to some required approximation ϵ_1.
- Compute an estimation for $\|L_\delta\mu - L\mu\|_{\mathcal{B}}$ up to some error ϵ_2, as required by 1 of Theorem 3.1.
- Compute N such that item 2 of Theorem 3.1 is verified, in practice we compute the smallest N such that $\|(P_i|_V)^N\|_{\mathcal{B}\to\mathcal{B}} \leq \frac{1}{2}$ (alternatively estimate such N via Theorem 4.2).
- Estimate the C_i of 3 of Theorem 3.1 (for $\mathcal{B} = L^1$ they are all ≤ 1, so this step can be skipped).
- If all computations end successfully, output μ_δ and the error rigorously estimated via Theorem 3.1.

We will illustrate the algorithm in particular cases in Sections 7 and 9.

4. Decay time estimations

In order to make a bounded number of checks valid for the whole parameter space, we need the contraction of the (discretized) operator on the zero-average subspace to be preserved in a neighborhood. We will work with a general norm $\|\cdot\|$, that we will take to be L^1 in the expanding case, and Wasserstein norm in the contractive case.

Assume n to be such that $\|L^n\| < 1/4$. In our application we will assume L to be the discretized transfer operator restricted to the space of zero-average functions. Assume M be a nearby (discretized) operator, in the sense that the operator norm $\|L - M\|$ is small. Then

Proposition 4.1. *Assume $\|L^n\| < \frac{1}{4}$, and that $\|L^i\| < C$ for each i. If M is another operator such that $\|M - L\| < \frac{1}{n4C^2}$, and such that $\|M^i\| < C$ for each i, then $\|M^n\| \leq \frac{1}{2}$.*

Proof. Indeed:

$$\begin{aligned}
\|M^n\| &= \|L^n + (M^n - L^n)\| \\
&\leq \|L^n\| + \|M^n - L^n\| \\
&\leq \|L^n\| + \sum_{i=0}^{n-1} \|M^i(M-L)L^{n-i-1}\| \\
&\leq \|L^n\| + \sum_{i=0}^{n-1} \|M^i\| \cdot \|M-L\| \cdot \|L^{n-i-1}\| \\
&\leq \|L^n\| + nC^2 \cdot \|M-L\| \\
&\leq \frac{1}{2}.
\end{aligned}$$

□

The above proposition ensures that for each contracting operator, all nearby operators are also contractions. The space describing all possibile probabilities is compact, and consequently assuming all combinations of the operators L_i to be contractions (taking a power if necessary), we could use compactness to prove it in a finite number of steps for a sufficiently fine grid of possibilities, each step granting the contraction in a neighborhood.

However, we have that in the IFS case all operators are convex combinations of contractions (in a certain number of steps), and this information can be used at once to bound the contraction time of a combination. We start working with the combination of two operators. We remark that a combination of any finite number of operator can be seen as obtained taking inductively a convex combination of two operators that are contractions, so it is possible to use the theorem that follows to work with any finite number of operators.

Assume L_0 and L_1 to be operators on a Banach space, and for a sequence $\omega = (\omega_1, \omega_2, \ldots, \omega_k)$ with $\omega_i \in \{0,1\}$, denote $L^\omega = L_{\omega_1} L_{\omega_2} \ldots L_{\omega_k}$. We also denote by $|\omega|$ its length k, and by $|\omega|_0$, $|\omega|_1$ the number of occurrences of 0 or 1 in ω, respectively. Assume that

$$\forall\omega\|L^\omega\| < C, \qquad \|L_0^{n_0}\| < \frac{1}{2C}, \qquad \|L_1^{n_1}\| < \frac{1}{2C}.$$

for some $C > 0$ and n_0, n_1 big enough.

Theorem 4.2. *Let $p \in [0,1]$, then*

$$\left\|(pL_0 + (1-p)L_1)^M\right\| < \frac{1}{2}$$

for all M satisfying the lower bound

$$M \geq N - 1 + N \frac{\log 2C}{-\log\left(1 - \frac{p^{n_0}}{2} - \frac{(1-p)^{n_1}}{2}\right)}.$$

for $N = \max\{n_0, n_1\}$.

Proof. Let's expand the M-th power in all possible compositions of L_0 and L_1, indexed by all words ω of length M:

$$L^M = (pL_0 + (1-p)L_1)^M = \sum_{\omega : |\omega| = M} p^{|\omega|_0}(1-p)^{|\omega|_1} L^{\omega}.$$

Estimating the norm of L^{ω} for a given ω, we can start with the uniform estimation C, and for each occurrence of 0^{n_0} or 1^{n_1} in ω we can account a contribution of an extra factor $\frac{1}{2C} \cdot C = \frac{1}{2}$ to the estimate. That is, $\|L^{\omega}\| \leq C \cdot 2^{-H(\omega)}$, where $H(\omega)$ denotes the number of occurrences of either 0^{n_0} or 1^{n_1} in ω. Consequently, the norm in the claim can be estimated with

$$S(M) = C \cdot \sum_{\omega : |\omega| = M} p^{|\omega|_0}(1-p)^{|\omega|_1} 2^{-H(\omega)}.$$

To estimate $S(M)$, we will proceed by induction on M and denote by $S_0(M)$ and $S_1(M)$ the same sum restricted to the ω satisfying $\omega_1 = 0$ or 1 respectively. Decomposing the sum depending on the biggest number of initial 0's or 1's in ω, we have

$$\begin{aligned} S(M) \leq & \sum_{i=1}^{n_0 - 1} p^i S_1(M-i) + \frac{1}{2} p^{n_0} S(M - n_0) \\ & + \sum_{i=1}^{n_1 - 1} (1-p)^i S_0(M-i) + \frac{1}{2}(1-p)^{n_1} S(M - n_1). \end{aligned}$$

Considering that for each $i > 1$

$$S_0(i) < pS(i-1), \qquad S_1(i) < (1-p)S(i-1),$$

we can estimate

$$\begin{aligned} S(M) \leq & \sum_{i=1}^{n_0 - 1} p^i (1-p) S(M-i-1) + \frac{1}{2} p^{n_0} S(M - n_0) \\ & + \sum_{i=1}^{n_1 - 1} p(1-p)^i S(M-i-1) + \frac{1}{2}(1-p)^{n_1} S(M - n_1). \end{aligned}$$

We have now a sequence $S(M)$ which satisfies a recurrence inequality. It is natural to compare it with the sequence satisfying the exact recurrence (with equality), which will provide an upper bound on $S(M)$.

A recurrence where each next element is defined as a positive combination of previous terms can be estimated using the powers of the unique positive real root of the characteristic polynomial. This technique is standard in the theory of linear recurrence sequences, and based on the following very simple idea: if the powers of the real number α satisfy a linear equation

$$\alpha^n = \sum_{i=0}^{n-1} c_i \alpha^i$$

with positive coefficients c_i (such α annihilates the polynomial, $X^n - c_{n-1}X^{n-1}\cdots - c_0$, called characteristic polynomial of the recurrence), then whenever positive real numbers $x_0, \dots, x_n$ satisfy

$$x_n = \sum_{i=0}^{n-1} c_i x_i$$

and satisfy $x_i \leq K\alpha^i$ for $0 \leq i < n$ and some constant K, then $x_n \leq K\alpha^n$. Iterating and using $K\alpha, K\alpha^2, \dots$ in the role of K, if the x_i are defined by recurrence for $i > n$, that is for each $k > n$ we have

$$x_k = \sum_{i=0}^{n-1} c_i x_{k-n+i},$$

we obtain that $x_i \leq C\alpha^i$ for each $i \geq 0$.

Let $N = \max\{n_0, n_1\}$. In our case, the characteristic polynomial is

$$X^N = \sum_{i=1}^{n_0-1} p^i(1-p)X^{N-i} + \frac{1}{2}p^{n_0}X^{N-n_0} + \sum_{i=1}^{n_1-1} p(1-p)^i X^{N-i} + \frac{1}{2}(1-p)^{n_1}X^{N-n_1},$$

and such an equation has a real root which is < 1 by intermediate value, because the LHS is smaller than the RHS for $X = 0$, but becomes bigger for $X = 1$.

Such a root α should satisfy

$$\begin{aligned}
\alpha^N \leq & \sum_{i=1}^{n_0-1} p^i(1-p) + \frac{1}{2}p^{n_0} + \sum_{i=1}^{n_1-1} p(1-p)^i + \frac{1}{2}(1-p)^{n_1} \\
= & \left[\sum_{i=1}^{n_0-1} p^i(1-p) + p^{n_0}\right] - \frac{1}{2}p^{n_0} + \left[\sum_{i=1}^{n_1-1} p(1-p)^i + (1-p)^{n_1}\right] - \frac{1}{2}(1-p)^{n_1} \\
= & 1 - \frac{1}{2}p^{n_0} - \frac{1}{2}(1-p)^{n_1},
\end{aligned}$$

observing that the sums telescopize to p and $1-p$. Consequently, we have

$$\alpha \leq \left(1 - \frac{1}{2}p^{n_0} - \frac{1}{2}(1-p)^{n_1}\right)^{1/N}.$$

Furthermore, we know that $S(i) \leq C < C\alpha^{i-N+1}$ for $0 \leq i < N$, and this implies that $S(M) < C\alpha^{M-N+1}$ for all M, applying the above reasoning. Consequently $S(M) < \frac{1}{2}$ whenever $\alpha^{M-N+1} \leq \frac{1}{2C}$, and it will be sufficient that

$$(M - N - 1)log(\alpha) \leq -log(2C).$$

Dividing by $log(\alpha)$ (which is negative) and taking into account the estimation we have for α we obtain the result whenever M satisfies the inequality

$$M \geq N - 1 + N\frac{\log 2C}{-\log\left(1 - \frac{p^{n_0}}{2} - \frac{(1-p)^{n_1}}{2}\right)}. \qquad \square$$

5. Estimation of the error in the uniformly expanding case

In this section we explain how to estimate Item 1 in the uniformly expanding case, in particular we recall what is a Lasota-Yorke inequality and how, if such an inequality is satisfied by all the dynamics T_i such an inequality can be proved for the convex combination of their transfer operators.

Let X be the either unit interval $[0,1]$, either S^1 (which we still identify to the unit interval with the additional identification $0 = 1$).

Definition 5.1. We say that a map T on the interval X is *piecewise expanding* if X can be partitioned in finite set of intervals where T is C^2 and $|T'| > 2$, and furthermore $\frac{T''}{(T')^2}$ is bounded.

We will work with transfer operators on measures satisfying a Lasota-Yorke inequality

$$\|L^N \mu\|_{\mathcal{B}'} \leq \lambda_1^N \|\mu\|_{\mathcal{B}'} + B_1 \|\mu\|_{\mathcal{B}}.$$

Note that if all the transfer operators of the single maps in an IFS satisfy a Lasota-Yorke inequality, then the transfer operator of the IFS also satisfy such an inequality, as formalized in the following proposition.

Proposition 5.2. *Assume the operators L_i to satisfy the inequality*

$$\|L_i f\|_{\mathcal{B}'} \leq \lambda_i \|f\|_{\mathcal{B}'} + B_i \|f\|_{\mathcal{B}},$$

for $i = 1, \ldots, k$ then the convex combination $L = \sum_{i=1}^k p_i L_i$ satisfies

$$\|Lf\|_{\mathcal{B}'} \leq \sum_{i=1}^k p_i \lambda_i \|f\|_{\mathcal{B}'} + \sum_{i=1}^k p_i B_i \|f\|_{\mathcal{B}}.$$

The proof is straightforward.

Remark 5.3. Please note that the fact that all the L_i satisfy such inequalities is a sufficient condition for the operator L to satisfy such an inequality: when all L_i satisfy such an inequality, then a convex combination also does. On the other hand, this condition is not necessary. When some L_i do not satisfy a Lasota-Yorke inequality, a combination may still satisfy such inequality for a suitable choice of the p_i. An interesting case could be with one operator being the transfer operator of an irrational rotation on the circle, and other operators satisfying a Lasota-Yorke inequality (see a concrete example in Section 7).

Remark 5.4. Note also that if

$$\|L\mu\|_{\mathcal{B}'} \leq \lambda \|\mu\|_{\mathcal{B}'} + B \|\mu\|_{\mathcal{B}}.$$

then applying the inequality iteratively we have

$$\|L^N \mu\|_{\mathcal{B}'} \leq \lambda^N \|\mu\|_{\mathcal{B}'} + \frac{CB}{1-\lambda} \|\mu\|_{\mathcal{B}}.$$

for each N, where C is an upper bound for $\|L^i\|_{\mathcal{B} \to \mathcal{B}}$ for $0 < i < N$ (note that $C = 1$ for $\mathcal{B} = L^1$). Hence the Lasota-Yorke can always be recovered from the $N = 1$ case, up to replacing B with a bigger constant.

We cite here a few results on the existence of such inequality in the case of uniformly expanding maps of the interval.

Theorem 5.5. *Assume T to be continuous and piecewise expanding in the interval X, then its transfer operator on measures of bounded variation satisfies for each N*

$$\|L^N\mu\|_{BV} \leq \lambda^N\|\mu\|_{BV} + B\|\mu\|_{L^1}$$

where

$$\lambda \leq 2\cdot\left\|\frac{1}{T}\right\|_\infty, \qquad B \leq 2\cdot\left\|\frac{T''}{(T')^2}\right\|_\infty.$$

For proof, see [GN14, Theorem 7 and Remark 9].

Since now on, L will be assumed to be an operator obtained considering a convex combination $L = \sum_{i=1}^k p_i L_i$, and satisfying

$$\|L^N\mu\|_{BV} \leq \lambda^N\|\mu\|_{BV} + B\|\mu\|_{L^1}.$$

Remark 5.6. An invariant probability measure of an operator satisfying such a Lasota-Yorke inequality has bounded variation and satisfies $\|\mu\|_{BV} \leq B$.

We can now describe the approximation strategy. To obtain an estimate in L^1 we can take as approximation operator π_δ the conditional expectation operator with respect to a uniform partition F_δ of X in intervals of size δ:

$$\pi_\delta(\mu) = \mathbf{E}(\mu|F_\delta).$$

The approximated operator $L_\delta = \pi_\delta L \pi_\delta$ is known as the *Ulam approximation* of L.

Proposition 5.7. *If L_δ is defined as above then*

$$\|L_\delta - L\|_{BV\to L^1} \leq 2\delta.$$

For proof, we refer to Lemma 10 of [GN14] which proves the inequality for the transfer operator of associated to a map (as each of the L_i is). Thus being L obtained as a convex combination we have

$$\|L_\delta - L\|_{BV\to L^1} \leq \sum_i p_i\|L_{i,\delta} - L_i\|_{BV\to L^1} \leq \sum_i p_i 2\delta = 2\delta.$$

Since as observed in Remark 5.6 we have an estimate of μ_{BV}, thanks to the above proposition we have an estimation sufficient for the item 1 of Theorem 3.1.

Remark 5.8. As pointed out in [BB11, GN14], similar results hold for the pair of norms *Lip* and L^∞, in the place of the pair BV, L^1. It is hence possible to obtain similarly a rigorous computation of the invariant density in the L^∞ norm, using essentially the same strategy. The same extension can also be done in the IFS case.

6. Some remarks on implementation

The topics in this Section apply equally to the experiments in Sections 7 and 9. The code used in our experiments is available at

https://bitbucket.org/fph/compinvmeas-python.

To certify the numerics, we use the interval arithmetics libraries which are available with the SAGE Mathematics Software [Dev15].

The assembly of the matrix in the piecewise expanding case is done by using interval arithmetics Newton methods while the estimates for the contraction rate are made with double precision arithmetics with rigorous (and conservative) bounds on the iteration error.

In the contractive case the Newton method is not necessary, and we can rigorously assemble the approximated operator directly, using interval arithmetic to keep track of possible numerical errors.

7. Implementation in some piecewise expanding examples

Let's consider the smooth dynamical systems on $[0,1]$ given by

$$T_2(x) = 4x + 0.01 \cdot sin(16\pi x), \qquad T_2(x) = 5x + 0.03 \cdot sin(5\pi x).$$

Then in our case we have (not iterated) Lasota-Yorke constants that are

$$\left[\begin{array}{l} \lambda(T_1) = 0.333924 \\ B(T_1) = 11.26927 \end{array}\right. \qquad \left[\begin{array}{l} \lambda(T_2) = 0.246455 \\ B(T_2) = 1.798453 \end{array}\right.$$

and consider the sistems defined via the maps T_1, T_2 where the probabilities are selected putting

$$p_1 = 0.1, 0.3, 0.5, 0.7, 0.9$$

and $p_2 = 1 - p_1$. The constants λ, B we obtain by Theorem 5.5, as well as an estimate on the BV norm of the stationary measure, are respectively given by the pairs

p_1	0.1	0.3	0.5	0.7	0.9
λ	0.255202	0.272696	0.290190	0.307683	0.325177
B	2.74553	4.63969	6.53386	8.42802	10.32219
$\|\mu\|_{BV}$	3.68628	6.37931	9.20508	12.17366	15.29615

We select $\delta = 2^{-16}$, obtaining

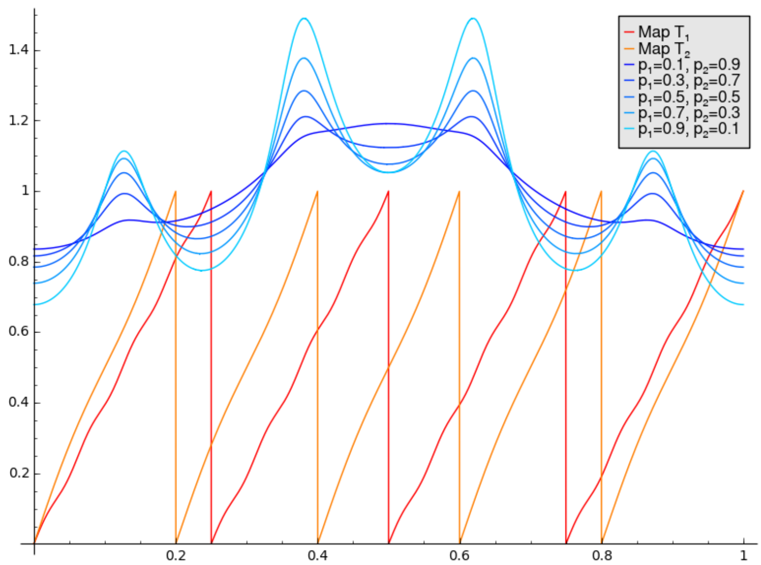

Figure 1. The stationary measures in the expanding examples.

$$\|L - L_\delta\|_{BV \to L^1} \leq 2^{-15}$$

from Prop. 5.7. We prove computationally that the Ulam matrices of T_1, T_2 contract to $\alpha = \frac{1}{2}$ in 9 and 8 steps respectively, and similarly compute the contracting rate for the IFS transfer operator. Considering that in Theorem 3.1 all C_i are 1 we can estimate the L_1 error as

$$\|f - f_\delta\|_{L^1} \leq \|L - L_\delta\|_{BV \to L^1} \cdot \|\mu\|_{BV} \cdot \frac{N}{1-\alpha},$$

N being such that $\|L_\delta^N\| < \alpha$ (contraction rate). As we will show later, we can also give an error estimation depending on an *a priori* estimation of the decay time. The estimation obtained in this way is rather pessimistic, but is obtained only from the contraction time for the operators associated to T_1 and T_2, that is, skipping the more numerically intensive computation.

p_1	0.1	0.3	0.5	0.7	0.9
N (contraction rate)	8	7	7	8	9
L^1 error	0.00180	0.00272	0.00393	0.00594	0.00840
A priori N (contraction rate)	34	222	2135	314	37
A priori L^1 error	0.00766	0.0865	1.200	0.233	0.0345

In the table the contraction rate is the N such that $\|L_\delta^N\| < \alpha$, while the error is $\|f - f_{\delta,c}\|_{L^1}$ ($f_{\delta,c}$ begin the computed approximation of f_δ, that is, a rigorous estimate of the numerical error has been added). The *a priori* equivalents are obtained via Theorem 4.2 rather than via an expensive computation; except for two central values, the a priori error could already be considered acceptable.

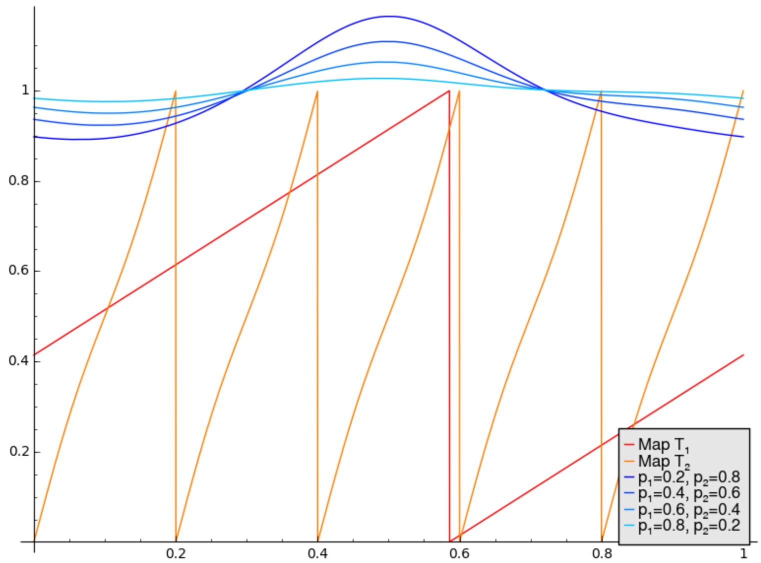

FIGURE 2. Examples with rotation and expanding map.

We conclude with an example of IFS formed replacing the T^1 map above with an irrational rotation by $\sqrt{2}$. Such a map satisfies a (trivial) Lasota-Yorke with $\lambda = 1$ and $B = 0$, but for each non-trivial value of the p_i the transfer operator of the corresponding IFS satisfied a non-trivial Lasota-Yorke (with $\lambda < 1$). Taking $\delta = 2^{-14}$, we can compute the stationary measure up to the error specified below.

p_1	0.2	0.4	0.6	0.8
λ	0.39716	0.54787	0.69858	0.84929
B	1.43876	1.07907	0.71938	0.35969
N (contraction rate)	8	9	12	20
L^1 error	0.0011687	0.0013139	0.0017506	0.0029155

8. Contractive Maps

In the case of contractive maps, the strategy is similar but the functional spaces are totally different. In fact, the transfer operator turns out not to be a contraction when applied to spaces of regular absolutely continuous measures like BV or C^1.

A space where the transfer operator attached to a contractive map is contractive is the dual of Lipschitz, that is the space of measures having finite norm

$$\|\mu\|_W = \sup_{\phi \in C^0(X): Lip(\phi) \leq 1} \int_X \phi d\mu,$$

where we denote by $Lip(\phi)$ the Lipschitz constant of ϕ (with respect to some distance on X). We remark that μ has to be a zero-average measure for $\|\mu\|_W$ to be finite; if $\mu = \mu_1 - \mu_2$ for positive measures μ_1 and μ_2, $\|\mu\|_W$ is also known as the Wasserstein-Kantorovich distance (well known in Transportation Theory, and also known as earth-moving distance) of μ_1 to μ_2.

Let T be a contraction with contraction rate α and L be the corresponding transfer operator. Then for each ϕ satisfying $Lip(\phi) \leq 1$ we have

$$\begin{aligned} \int_X \phi(x) dL\mu(x) &= \int_X \phi(T(x)) d\mu(x) \\ &= \alpha \int_X \frac{\phi(T(x))}{\alpha} d\mu(x) \leq \alpha \|\mu\|_W \end{aligned}$$

observing that

$$Lip\left(\frac{\phi(T(x))}{\alpha}\right) \leq 1.$$

This proves that the operator L satisfies

$$\|L\mu\|_W \leq \alpha \|\mu\|_W \tag{3}$$

for each zero-average Borel measure $\mu \in V$.

Remark 8.1. Since now on we will just assume L to be a contraction, i.e. satisfying the (3). Since we just proved that the transfer operator associated to a single contractive map satisfied the (3), this will also be true for the transfer operator of an IFS as its transfer operator is obtained as a convex combination of the operators.

Assume now X be a bounded domain in $\mathbb{R}^n$ equipped with the "Manhattan" (L^1) distance

$$|x|_M = \sum_{i=0}^{n} |x_i|,$$

with respect to which we will assume our maps to be contractions (and that we will use to evaluate the $\|\cdot\|_W$ norm). We will define a projection on the space of measures that is a contraction in the $\|\cdot\|_W$ distance.

We will discretize spatially the bounded measures in R^n working one dimension at a time, depending on a parameter δ determining the coarseness of the discretization. Assume our domain X to be contained in a parallelepiped

$$\Pi = [P_1, Q_1] \times [P_2, Q_2] \times \cdots \times [P_n, Q_n],$$

and let's also assume by convenience that each size $Q_k - P_k$ is an integer multiple of δ, $N_k\delta$ say. We will put

$$p_{k,i} = P_k + i\delta$$

for $1 \le k \le n$ and $0 \le i \le N_k$.

For each set $A \subseteq \Pi$ denote by $A_{i,k}$ the set

$$A_{i,k} = \left\{(x_1, \dots, x_n) \in A : x_k = p_{k,i}\right\},$$

obtained *slicing* A at the i-th sample of the k-th dimension, and let $A_{i,k,\delta}$ be the δ-thickening along the slicing direction, that is,

$$A_{i,k,\delta} = \left\{(x_1, \dots, x_n) : |x_k = p_{k,i}| \le \delta \wedge (x_1, \dots, x_{k-1}, p_{k,i}, x_{k+1}, \dots, x_n) \in A\right\}.$$

Let now

$$h_{i,k}(x) = \begin{cases} 1 - \frac{1}{\delta}|x_k - p_{k,i}| & \text{if } |x_k - p_{k,i}| \le \delta, \\ 0 & \text{in any other case.} \end{cases}$$

We define the k-th projection $\pi_{\delta,k}$ as defined, for each measure μ, by

$$(\pi_{\delta,k}\mu)(A) = \sum_{i=0}^{N_k} \int_{A_{i,k,\delta}} h_{i,k} d\mu.$$

Intuitively, $\pi_{\delta,k}$ can be viewed as the operation of moving "sliding the k-th coordinate" all the mass to the affine planes of equations $x_k = p_{k,i}$, using the functions $h_{k,i}$ to spread linearly the contribution from each point to the nearby planes.

We put $\pi_\delta = \pi_{\delta,1} \cdots \pi_{\delta,n}$ (these operators obviously commute), and such π_δ can be easily described as

$$\pi_\delta \mu = \sum_{p=(p_{1,i_1}, \dots, p_{n,i_n})} \delta_p \cdot \int h_p d\mu$$

where for such given point of the grid

$$h_p = \prod_{j=1}^{n} h_{j,p_{j,i_j}}.$$

We prove the following proposition.

Proposition 8.2. *If* $\|\mu\|_W \le 1$*, then* $\|\pi_{\delta,k}\mu\|_W \le 1$.

Proof. We need to prove that $\int \phi d\pi_{\delta,k}\mu \le 1$ for each admissible function ϕ. We will describe a suitable linearization $\tilde{\phi}$ of ϕ that on one hand will satisfy

$$\int \phi d\pi_{\delta,k}\mu = \int \tilde{\phi} d\pi_{\delta,k}\mu, \tag{4}$$

and on the other hand will have Lipschitz constant ≤ 1 and satisfy

$$\int \tilde{\phi} d\pi_{\delta,k}\mu = \int \tilde{\phi} d\mu. \tag{5}$$

Given ϕ, we put

$$\tilde{\phi}(x) = \sum_{i=0}^{N_k} h_{k,i}(x)\phi\big((x_1, \dots, x_{k-1}, p_{k,i}, x_{k+1}, \dots, x_n)\big).$$

On the points x such that x_k is equal to some $p_{k,i}$ the $\tilde{\phi}$ is equal to ϕ, so the (4) is clearly satisfied.

To prove the (5), we can just check it on the μ that are δ_y for some $y = (y_1, \dots, y_n)$ and reason by density. Note that

$$\pi_{\delta,k}\delta_y = \sum_{i=0}^{N_k} h_{k,i}(y)\delta_{(y_1,\dots,y_{k-1},p_{k,i},y_{k+1},\dots,y_n)}, \tag{6}$$

so the LHS of the (5) turns out to be equal to

$$\tilde{\phi}(y) = \int\int \tilde{\phi} d\delta_y.$$

I remains to prove that $\tilde{\phi}$ also has Lipschitz constant ≤ 1 (with respect to the Manhattan distance), but let y, z be two points and put $w = (y_1, \dots, y_{k-1}, z_k, y_{k+1}, \dots, y_n)$. We have

$$\left|\tilde{\phi}(y) - \tilde{\phi}(z)\right| \leq \left|\tilde{\phi}(y) - \tilde{\phi}(w)\right| + \left|\tilde{\phi}(w) - \tilde{\phi}(z)\right|$$

and note that on the segment from y to w $\tilde{\phi}$ is piecewise linear with slope ≤ 1, while $\tilde{\phi}(w) - \tilde{\phi}(z)$ is obtained as convex combination of quantities that are all $\leq |w-z|_M$. Consequently

$$\left|\tilde{\phi}(y) - \tilde{\phi}(z)\right| \leq \|y - z\|_M.$$

We conclude that $\int \phi d\pi_{\delta,k}\mu \leq 1$. □

We are left with the problem of estimating the distance between the fixed points of L and L_δ, where L is the transfer operator of the IFS (or more in general any operator satisfying $\|L\mu\|_W < \alpha\|\mu\|_W$).

Proposition 8.3. *If L and L_δ are defined as above we have*

$$\|L - L_\delta\|_{L^1\to W} \leq (\alpha + 1)\frac{n\delta}{2}.$$

Proof. Recall the definition of $\|\mu\|_{L^1}$ in equation (1), we start observing that

$$\|1 - \pi_{\delta,k}\|_{L^1\to W} \leq \frac{\delta}{2}.$$

When $\pi_{\delta,k}$ is applied to an atomic measure $\mu = \delta_y$ the mass will be split by $\pi_{\delta,k}$ in at most two atoms at distance $\delta\lambda$ and $\delta(1 - \lambda)$ and mass $1 - \lambda$ and λ, and consequently

$$\|\mu - \pi_{\delta,k}\mu\|_W \leq 2\delta\lambda(1 - \lambda) \leq \frac{\delta}{2}$$

taking the maximum over all $\lambda \in [0, 1]$. This inequality holds when μ is an atomic measure $\mu = \delta_y$, and extends to the case of μ being a finite convex combination of such measures (by linearity of $\pi_{\delta,k}$). Such measures are dense in the space of all probability measures with respect to the $\|\cdot\|_W$ distance, hence this inequality hold for all probabilities measures μ.

Applying this estimate for all dimensions we obtain that

$$\|1 - \pi_\delta\|_{L^1 \to W} \leq \frac{n\delta}{2},$$

and it follows that

$$\begin{aligned}\|L - L_\delta\|_{L^1 \to W} &\leq \|L(1 - \pi_\delta)\| + \|(\pi_\delta - 1)L\pi_\delta\| \\ &\leq \|L\|_W \cdot \|1 - \pi_\delta\|_{L^1 \to W} + \|1 - \pi_\delta\|_{L^1 \to W} \cdot \|L\pi_\delta\|_{L^1} \\ &\leq (\alpha + 1)\frac{n\delta}{2}.\end{aligned}$$

□

Since an invariant probability measure has L^1 norm equal to 1, we have that

$$\|L\mu - L_\delta\mu\|_W \leq (\alpha + 1)\frac{n\delta}{2}, \tag{7}$$

and as consequence of Corollay 3.2 we have

$$\|\mu - \mu_\delta\|_W \leq \frac{(1+\alpha)n\delta}{2(1-\alpha)}.$$

This gives an estimation that can be applied be applied to obtain 1 of Theorem 3.1, and is our main ingredient in the estimation of the error for the approximation of the stationary measure in this kind of systems.

9. An example in the contractive case

In the contactive case each map has an approximated transfer operator $L_\delta = \pi_\delta L \pi_\delta$ (repesentable as a matrix) that can be computed very easily. We take the image of a δ_x for x in the grid (an atom in x) via the map and approximate the $\delta_{f(x)}$ obtained (which will not be aligned to the grid) to the measure $\pi_\delta(\delta_{f(x)})$ supported on the grid using equation (6).

We take all δ_x for x in the grid as a basis of the finite dimensional space of measures supported on the grid, and being $L_\delta(\delta_x) = \pi_\delta(\delta_{f(x)})$ we obtained the expression of $L_\delta(\delta_x)$ as combination of elements of the basis. The matrix P obtained represents the operator L_δ in this basis, and the decay time of L_δ (or of P) can be estimated by the contraction rate of L via to Prop. 8.2.

The IFS also has a transfer operator, obtained as the linear combination of the transfer operator of the maps, and is also contractive being a convex combination of contractive operators, (its decay time can obtained as combination of the decay times of the maps). The obtained matrix is then iterated to approximate the fixed point of L_δ, the goodness of the approximation is then a consequence of its proven decay time.

The informal description of the algorithm is the same as explained in Section 3, with the difference that the decay time N is estimated directly from the contraction rate of the operator L. The matrix P representing (an approximation) of L_δ is built as explained in the beginning of the section.

We made an example in the following case: the maps $T_1, \ldots, T_4$ are defined on the square $[0,1] \times [0,1]$ as

- $T1$: scaling by 0.4 around $(0.6, 0.2)$ with rotation by $\pi/6$,
- $T2$: scaling by 0.6 around $(0.05, 0.2)$ with rotation by $-\pi/30$,
- $T3$: scaling by 0.5 around $(0.95, 0.95)$,
- $T4$: scaling by 0.45 around $(0.1, 0.9)$.

We took the probabilities

$$p_1 = 0.18, \quad p_2 = 0.22, \quad p_3 = 0.3, \quad p_4 = 0.3,$$

and a grid of size $2^{10} \times 2^{10}$, so that $\delta = 2^{-10}$. It turns out that the contraction rate α is at most 0.659430, so the error can be estimated by

$$\|f - f_\delta\|_W \leq \frac{(1+\alpha)n\delta}{2(1-\alpha)} \leq 0.0047583.$$

Here is an image of the computed invariant density.

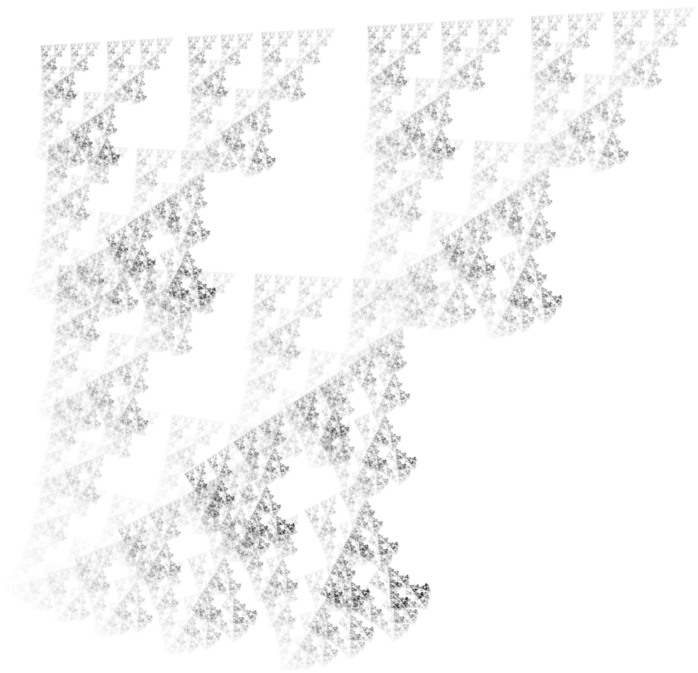

FIGURE 3. The stationary measure of the contractive example.

References

[BB11] Wael Bahsoun and Christopher Bose, *Invariant densities and escape rates: rigorous and computable approximations in the $l\infty$-norm*, Nonlinear Analysis: Theory, Methods & Applications **74** (2011), no. 13, 4481–4495.

[Dev15] The Sage Developers, *Sage Mathematics Software (Version 6.7)*, 2015, `http://www.sagemath.org`.

[GN14] Stefano Galatolo and Isaia Nisoli, *An elementary approach to rigorous approximation of invariant measures*, SIAM Journal on Applied Dynamical Systems **13** (2014), no. 2, 958–985.

[Hal13] Paul R Halmos, *Measure theory*, vol. 18, Springer, 2013.

[KL06] Yuri Kifer and Pei-Dong Liu, *Random dynamics, handbook of dynamical systems, vol. 1b, 379-499*, 2006.

[Liv04] Carlangelo Liverani, *Invariant measures and their properties. a functional analytic point of view*, Dynamical systems. Part II (2004), 185–237.

[MR09] Philippe Marie and Jerome Rousseau, *Recurrence for random dynamical systems*, arXiv preprint arXiv:0906.4847 (2009).