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On the canonical ring of curves and surfaces

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Abstract Let C be a curve (possibly non reduced or reducible) lying on a smooth algebraic surface. We show that the canonical ring $R(C,\omega_C) = \bigoplus_{k\geq 0} H^0(C,{\omega_C}^{\otimes k})$ is generated in degree 1 if C is numerically 4-connected, not hyperelliptic and even (i.e. with ω_C of even degree on every component).

As a corollary we show that on a smooth algebraic surface of general type with $p_g(S)\geq 1$ and $q(S)=0$ the canonical ring $R(S,K_S)$ is generated in degree ≤ 3 if there exists a curve $C \in |K_S|$ numerically 3-connected and not hyperelliptic.

keyword: algebraic curve, surface of general type, canonical ring, pluricanonical embedding.

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1 Introduction

Let C be a curve (possibly non reduced or reducible) lying on a smooth algebraic surface S and let ω_C be the dualizing sheaf of C. The purpose of this paper is to analyze the canonical ring of C, that is, the graded ring

$$R(C,\omega_C) = \bigoplus_{k\geq 0} H^0(C,{\omega_C}^{\otimes k})$$

under some suitable assumptions on the curve C.

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The rationale of our analysis stems from several aspects of the theory of algebraic surfaces.

The first such aspect is the analysis of surface's fibrations and the study of their applications to surface's geography. Indeed, given a genus g fibration $f : S \to B$ over a smooth curve B, an important tool in this analysis is the *relative canonical algebra* $R(f) = \bigoplus_{n\geq 0} f_*(\omega_{S/B}^{\otimes n})$.

In recent years the importance of $R(f)$ has become clear (see Reid's unpublished note [20]) and a way to understand its behavior consists in studying the canonical ring of every fibre of f. More specifically, denoting by $C = f^{-1}(P)$ the scheme theoretic fibre over a point $p \in B$, the local structure around P of the *relative canonical algebra* can be understood via the canonical algebra of C, since the reduction modulo $\mathscr{M}_P$ of the stalk at P of the *relative canonical algebra* is nothing but $R(C, \omega_C)$ (see [20, §1]). Mendes Lopes in [17] studied the cases where the genus g of the fibre is $g \leq 3$ whereas in [14] and [11] it is shown that for every $g \geq 3$, $R(C, \omega_C)$ is generated in degree ≤ 4 if every fibre is numerically connected and in degree ≤ 3 if furthermore there are no multiple fibres.

More recently Catanese and Pignatelli in [7] illustrated two structure theorems for fibration of low genus using a detailed description of the relative canonical algebra. In particular they showed an interesting characterization of isomorphism classes of relatively minimal fibration of genus 2 and of relatively minimal fibrations of genus 3 with fibres numerically 2-connected and not hyperelliptic (see [7, Thms. 4.13, 7.13]).

Finally, as shown in [6], the study of invertible sheaves on curves possibly reducible or non reduced is rich in implications in the cases where Bertini's theorem does not hold or simply if one needs to consider every curve contained in a given linear system. For instance, one can acquire information on the canonical ring of a surface of general type simply by taking its restriction to an effective canonical divisor $C \in |K_S|$ (not necessarily irreducible, neither reduced) and considering the canonical ring $R(C, \omega_C)$ (see Thm. 12 below).

In this paper we analyze the canonical ring of C when the curve C is m-connected and *even*, and we show an application to the study of the canonical ring of an algebraic surface of general type.

For a curve C lying on a smooth algebraic surface S, being *m-connected* means that $C_1 \cdot C_2 \geq m$ for every effective decomposition $C = C_1 + C_2$, (where $C_1 \cdot C_2$ denotes their intersection number as divisors on S). If C is 1-connected usually C is said to be numerically connected. The definition goes back to Franchetta (cf. [10]) and has many relevant implications. For instance in [6, §3] (cf. also the papers [9], Appendix and [18]) it is shown that if the curve C is 1-connected then $h^0(C, \mathscr{O}_C) = 1$, if C is 2-connected then the system $|\omega_C|$ is base point free, whereas if C is 3-connected and not honestly hyperelliptic (i.e., a finite double cover of $\mathbb{P}^1$ induced by the canonical morphism) then ω_C is very ample.

Keeping the usual notation for effective divisors on smooth surfaces, i.e., writing C as $\sum_{i=1}^{s} n_i \Gamma_i$ (where the Γ_i's are the irreducible components of C and for every i n_i denotes the multiplicity of Γ_i in C), the second condition can be illustrated by the following definition.

Definition *Let $C = \sum_{i=1}^{s} n_i \Gamma_i$ be a curve contained in a smooth algebraic surface. C is said to be* even *if $\deg(\omega_{C|\Gamma_i})$ is even for every irreducible $\Gamma_i \subset C$ (that is, $\Gamma_i \cdot (C - \Gamma_i)$ even for every $i = 1, ..., s$.)*

We note that an even curve has no decomposition $C = A + B$ with $A \cdot B$ an odd integer.

Even curves appear for instance when considering the canonical system $|K_S|$ for a surface S of general type. Indeed, by adjunction, for every curve $C \in |K_S|$ we have $|(2K_S)_{|C}| = |K_C|$, that is, every curve in the canonical system is even.

The main result of this paper is a generalization to even curves of the classical Theorem of Noether and Enriques on the degree of the generators of the graded ring $R(C,\omega_C)$:

Theorem 11 *Let C be an even 4-connected curve contained in a smooth algebraic surface. If $p_a(C) \geq 3$ and C is not honestly hyperelliptic then $R(C,\omega_C)$ is generated in degree 1.*

Following the notations of [19], this result can be rephrased by saying that ω_C is normally generated on C. In this case the embedded curve $\varphi_{|\omega_C|}(C) \subset \mathbb{P}^{p_a(C)-1}$ is arithmetically Cohen–Macaulay.

The proof of Theorem 11 is based on the ideas adopted by Mumford in [19] and on the results obtained in [11] for adjoint divisors, via a detailed analysis of the possible decompositions of the given curve C.

As a corollary we obtain a bound on the degree of the generators of the canonical ring of a surface of general type.

If S is a smooth algebraic surface and K_S a canonical divisor, the canonical ring of S is the graded algebra

$$R(S,K_S) = \bigoplus_{k\geq 0} H^0(S,{K_S}^{\otimes k})$$

In [8] a detailed analysis of $R(S,K_S)$ is presented in the most interesting case where S is of general type and there are given bounds (depending on the invariants $p_g(S) := h^0(S,K_S)$, $q := h^1(S,\mathscr{O}_S)$, and K_S^2) on the degree of elements of $R(S,K_S)$ forming a minimal system of homogeneous generators. Furthermore it is shown that for small values of p_g some exceptions do occur, depending substantially on the numerical connectedness of the curves in the linear system $|K_S|$. In particular [8, §4] presents examples of surfaces of general type with K_S not 3-connected whose canonical ring is not generated in degree ≤ 3 and it is conjectured that the 3-connectedness of the canonical divisor K_S should imply the generation of $R(S,K_S)$ in degree 1,2,3, at least in the case $q = 0$.

Here we show that this is the case. We remark that Konno in [15] has obtained analogous results, giving a degree bound for primitive generators and relations of the canonical ring of a minimal surface of general type with $|2K_S|$ free or with $p_g(S) := h^0(S,K_S) \geq 2$, $K_S^2 \geq 3$ and $q := h^1(S,\mathscr{O}_S) = 0$ (see also [16] for the analysis of the fixed part of the canonical system of a surface of general type via the study of the relative canonical algebra).

Our result, obtained essentially by restriction to a curve $C \in |K_S|$, is the following

Theorem 12 *Let S be a surface of general type with $p_g(S) := h^0(S,K_S) \geq 1$ and $q := h^1(S,\mathscr{O}_S) = 0$. Assume that there exists a curve $C \in |K_S|$ such that C is numerically 3-connected and not honestly hyperelliptic. Then the canonical ring of S is generated in degree ≤ 3.*

The paper is organized as follows: in §2 some useful background results are illustrated; in §3 we introduce the notion of disconnecting component; in §4 we prove Thm. 11; in §5 we give the proof of Thm. 12.

2 Notation and preliminary results

2.1 Notation

We work over an algebraically closed field $\mathbb{K}$ of characteristic ≥ 0.

Throughout this paper S will be a smooth algebraic surface over $\mathbb{K}$ and C will be a curve lying on S (possibly reducible and non reduced). Therefore C will be written (as a divisor on S) as $C=\sum_{i=1}^{s} n_i\Gamma_i$, where the Γ_i's are the irreducible components of C and the n_i's are the multiplicities. A subcurve $B\subseteq C$ will mean a curve $\sum m_i\Gamma_i$, with $0\le m_i\le n_i$ for every i.

By abuse of notation if $B\subset C$ is a subcurve of C, $C-B$ denotes the curve A such that $C=A+B$ as divisors on S.

C is said to be *m-connected* if for every decomposition $C=A+B$ one has $A\cdot B\ge m$.

C is said to be *numerically connected* if it is 1-connected.

ω_C denotes the dualizing sheaf of C (see [13], Chap. III, §7), and $p_a(C)$ the arithmetic genus of C, $p_a(C)=1-\chi(\mathcal{O}_C)$.

If $G\subset C$ is a proper subcurve of C we denote by $H^0(G,\omega_C)$ the space of sections of $\omega_{C|G}$.

Let $\mathscr{F}$ be an invertible sheaf on C.

If $G\subset C$ is a proper subcurve of C then $\mathscr{F}_{|G}$ denotes its restriction to G.

For each i the natural inclusion map $\varepsilon_i:\Gamma_i\to C$ induces a map $\varepsilon_i^*:\mathscr{F}\to\mathscr{F}_{|\Gamma_i}$. We denote by $d_i=\deg(\mathscr{F}_{|\Gamma_i})=\deg_{\Gamma_i}\mathscr{F}$ the degree of $\mathscr{F}$ on each irreducible component, and by $\mathbf{d}:=(d_1,...,d_s)$ the multidegree of $\mathscr{F}$ on C. If $B=\sum m_i\Gamma_i$ is a subcurve of C, by $\mathbf{d}_B$ we mean the multidegree of $\mathscr{F}_{|B}$.

C is said to be *even* if $\deg_{\Gamma_i}\omega_C$ is even for every irreducible $\Gamma_i\subset C$.

Similarly, if $\mathscr{H}$ is an invertible sheaf on C, then $\mathscr{H}$ is said to be *even* if $\deg_{\Gamma_i}\mathscr{H}$ is even for every irreducible $\Gamma_i\subset C$.

By $\mathrm{Pic}^{\mathbf{d}}(C)$ we denote the Picard scheme which parametrizes the classes of invertible sheaves of multidegree $\mathbf{d}=(d_1,\dots,d_s)$ (see [11]).

We recall that for every $\mathbf{d}=(d_1,...,d_s)$ there is an isomorphism $\mathrm{Pic}^{\mathbf{d}}(C)\cong\mathrm{Pic}^{\mathbf{0}}(C)$ and furthermore $\dim\mathrm{Pic}^{\mathbf{0}}(C)=h^1(C,\mathcal{O}_C)$ (cf. e.g. [2]).

Concerning the Picard group of C and the Picard group of a subcurve $B\subset C$ we have

$$\mathrm{Pic}^{\mathbf{d}}(C)\twoheadrightarrow\mathrm{Pic}^{\mathbf{d}_B}(B)\quad\forall\mathbf{d}$$

(see [11, Rem. 2.1]).

An invertible sheaf $\mathscr{F}$ is said to be nef if $d_i\ge 0$ for every i. Two invertible sheaves $\mathscr{F}$, $\mathscr{F}'$ are said to be numerically equivalent on C (notation: $\mathscr{F}\overset{\text{num}}{\sim}\mathscr{F}'$) if $\deg_{\Gamma_i}\mathscr{F}=\deg_{\Gamma_i}\mathscr{F}'$ for every $\Gamma_i\subseteq C$.

Finally, a curve C is said to be *honestly hyperelliptic* if there exists a finite morphism $\psi\colon C\to\mathbb{P}^1$ of degree 2. In this case C is either irreducible, or of the form $C=\Gamma_1+\Gamma_2$ with $p_a(\Gamma_i)=0$ and $\Gamma_1\cdot\Gamma_2=p_a(C)+1$ (see [6, §3] for a detailed treatment).

2.2 General divisors of low degree

Let $C=\sum_{i=1}^{s} n_i\Gamma_i$ be a curve lying on a smooth algebraic surface S. An invertible sheaf on C of multidegree $\mathbf{d}=(d_1,...,d_s)$ is said to be "general" if the corresponding class in the Picard scheme $\mathrm{Pic}^{\mathbf{d}}(C)$ is in general position, i.e., if it lies in the complementary of a proper closed subscheme (see [11] for details).

We recall two vanishing results for general invertible sheaves of low degree.

Theorem 21 ([11, Thms. 3.1, 3.2]) *(i) If $\mathscr{F}$ is a "general" invertible sheaf such that $\deg_B \mathscr{F} \geq p_a(B)$ for every subcurve $B \subseteq C$, then $H^1(C,\mathscr{F}) = 0$.*

(ii) If $\mathscr{F}$ is a "general" invertible sheaf such that $\deg_B \mathscr{F} \geq p_a(B)+1$ for every subcurve $B \subseteq C$, then the linear system $|\mathscr{F}|$ is base point free.

In particular we obtain the following

Proposition 22 *Let $C = \sum_{i=1}^s n_i\Gamma_i$ be a 1-connected curve contained in a smooth algebraic surface, and consider a proper subcurve $B \subsetneq C$. Let $\mathbf{d} = (d_1,...,d_s) \in \mathbb{Z}^s$ be such that $d_i \geq \frac{1}{2}\deg_{\Gamma_i}\omega_C \ \forall\, i = 1,...,s$.*

Then for a "general" invertible sheaf $\mathscr{F}$ in $\mathrm{Pic}^{\mathbf{d}_B}(B)$:

(i) $H^1(B,\mathscr{F}) = 0$;

(ii) $|\mathscr{F}_{|B}|$ is a base point free system on B if C is 3- connected.

Considering the case where C is an even 4-connected curve we obtain

Corollary 23 *Let $C = \sum_{i=1}^s n_i\Gamma_i$ be a 4-connected even curve contained in a smooth algebraic surface.*

For every $i = 1,\dots,s$, let $d_i = \frac{1}{2}\deg_{\Gamma_i}\omega_C$ and let $\mathbf{d} = (d_1,...,d_s) \in \mathbb{Z}^s$.

Let $B \subsetneq C$ be a proper subcurve of C and consider a a "general" invertible sheaf $\mathscr{F}$ in $\mathrm{Pic}^{\mathbf{d}_B}(B)$ (i.e., with an abuse of notation we can write $\mathscr{F} \overset{\mathrm{num}}{\sim} \frac{1}{2}\omega_{C|_B}$).

Then $H^1(B,\mathscr{F}) = 0$ and $|\mathscr{F}_{|B}|$ is a base point free system.

2.3 Koszul cohomology groups of algebraic curves

Let $C = \sum_{i=1}^s n_i\Gamma_i$ be a curve lying on a smooth algebraic surface S and let $\mathscr{H}$, $\mathscr{F}$ be invertible sheaves on C. Consider a subspace $W \subseteq H^0(C,\mathscr{F})$ which yields a base point free system of projective dimension r.

The Koszul groups $\mathscr{K}_{p,q}(C,W,\mathscr{H},\mathscr{F})$ are defined as the cohomology at the middle of the complex

$$\bigwedge^{p+1} W \otimes H^0(\mathscr{H}\otimes\mathscr{F}^{q-1}) \longrightarrow \bigwedge^{p} W \otimes H^0(\mathscr{H}\otimes\mathscr{F}^{q}) \longrightarrow \bigwedge^{p-1} W \otimes H^0(\mathscr{H}\otimes\mathscr{F}^{q+1})$$

If $W = H^0(C,\mathscr{F})$ they are usually denoted by $\mathscr{K}_{p,q}(C,\mathscr{H},\mathscr{F})$, while if $\mathscr{H} \cong \mathscr{O}_C$ the usual notation is $\mathscr{K}_{p,q}(C,\mathscr{F})$ (see [12] for the definition and main results).

We point out that the multiplication map

$$W \otimes H^0(C,\mathscr{H}) \to H^0(C,\mathscr{F}\otimes\mathscr{H})$$

is surjective iff $\mathscr{K}_{0,1}(C,W,\mathscr{H},\mathscr{F}) = 0$ and the ring $R(C,\mathscr{F}) = \bigoplus_{k\geq 0} H^0(C,\mathscr{F}^{\otimes k})$ is generated in degree 1 if and only if $\mathscr{K}_{0,q}(C,\mathscr{F}) = 0 \ \forall\, q \geq 1$. Moreover if $\mathscr{F}$ is very ample and $R(C,\mathscr{F})$ is generated in degree 1, then, identifying C with its image in $\mathbb{P}^r \cong \mathbb{P}(H^0(\mathscr{F})^\vee)$, $\mathscr{K}_{1,1}(C,\mathscr{F}) \cong I_2(C,\mathbb{P}^r)$, the space of quadrics in $\mathbb{P}^r$ vanishing on C (see [12]).

For our analysis the main applications of Koszul cohomology are the following propositions (see [11, §1], [14, §1] for further details on curves lying on smooth surfaces).

Proposition 24 (Duality) *Let $\mathscr{F}$, $\mathscr{H}$ be invertible sheaves on C and assume $W \subseteq H^0(C,\mathscr{F})$ to be a subspace of $\dim = r+1$ which yields a base point free system. Then*

$$\mathscr{K}_{p,q}(C,W,\mathscr{H},\mathscr{F}) \overset{\mathrm{d}}{=} \mathscr{K}_{r-p-1,2-q}(C,W,\omega_C\otimes\mathscr{H}^{-1},\mathscr{F})$$

(where $\overset{\mathrm{d}}{=}$ means duality of vector space).

For a proof see [11, Prop. 1.4]. Following the ideas outlined in [14, Lemma 1.2.2] we have a slight generalization of Green's H^0-Lemma.

Proposition 25 (H^0-Lemma) *Let C be 1-connected and let $\mathscr{F}$, $\mathscr{H}$ be invertible sheaves on C and assume $W \subseteq H^0(C,\mathscr{F})$ to be a subspace of* $\dim = r+1$ *which yields a base point free system. If either*

(i) $H^1(C,\mathscr{H}\otimes\mathscr{F}^{-1}) = 0$,
or
(ii) C is numerically connected, $\omega_C \cong \mathscr{H}\otimes\mathscr{F}^{-1}$ and $r \geq 2$,
or
(iii) C is numerically connected, $h^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F}) \leq r-1$ and there exists a reduced subcurve $B \subseteq C$ such that:
- *$W \cong W_{|B}$,*
- *$H^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F}) \hookrightarrow H^0(B,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F})$,*
- *every non–zero section of $H^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F})$ does not vanish identically on any component of B;*

then $\mathscr{K}_{0,1}(C,W,\mathscr{H},\mathscr{F}) = 0$, that is, the multiplication map

$$W\otimes H^0(C,\mathscr{H}) \to H^0(C,\mathscr{F}\otimes\mathscr{H})$$

is surjective.

Proof. By duality we need to prove that $\mathscr{K}_{r-1,1}(C,W,\omega_C\otimes\mathscr{H}^{-1},\mathscr{F}) = 0$. With this aim let $\{s_0,\ldots,s_r\}$ be a basis for W and let $\alpha = \sum s_{i_1}\wedge s_{i_2}\wedge\ldots\wedge s_{i_{r-1}}\otimes\alpha_{i_1i_2\ldots i_{r-1}} \in \bigwedge^{r-1}W\otimes H^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F})$ be an element in the Kernel of the Koszul map $d_{r-1,1}$.

In cases (i) obviously $\alpha = 0$ since by Serre duality $H^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F}) \cong H^1(C,\mathscr{H}\otimes\mathscr{F}^{-1}) = 0$.

In case (ii) $H^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F}) = H^0(C,\mathscr{O}_C) = \mathbb{K}$ by connectedness and we conclude similarly (see also [11, Prop. 1.5]).

In the latter case by our assumptions we can restrict to the curve B. Since B is reduced we can choose $r+1$ "sufficiently general points" on B so that $s_j(P_i) = \delta^i_j$. But then $\alpha \in \ker(d_{r-1,1})$ implies for every multiindex $\mathbf{I} = \{i_1,\ldots i_{r-2}\}$ the following equation (up to sign)

$$\alpha_{j_1i_1\ldots i_{r-2}}\cdot s_{j_1} + \alpha_{j_2i_1\ldots i_{r-2}}\cdot s_{j_2} + \alpha_{j_3i_1\ldots i_{r-2}}s_{j_3} = 0.$$

(where $\{i_1,\ldots i_{r-2}\}\cup\{j_1,j_2,j_3\} = \{0,\ldots,r+1\}$).

Evaluating at $P_j's$ and reindexing we get $\alpha_{i_1\ldots i_{r-1}}(P_{i_k}) = 0$ for $k = 1,\ldots,r-1$.

Let $\tilde{r} = h^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F})$. Since the $P_j's$ are in general position and every section of $H^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F})$ does not vanish identically on any component of B, we may assume that any $(\tilde{r}-1)$-tuple of points $P_{i_1},\ldots,P_{i_{r-1}}$ imposes independent conditions on $H^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F})$.

The proposition then follows by a dimension count since by assumption $\tilde{r} = h^0(C,\omega_C\otimes\mathscr{H}^{-1}\otimes\mathscr{F}) \leq h^0(C,\mathscr{F}) - 2 = r-1$.

□

In some particular cases we can obtain deeper results, which will turn out to be useful for our induction argument in the proof of Theorem 11.

Proposition 26 *Let C be either*

(i) an irreducible curve of arithmetic genus $p_a(C) \geq 1$;
or

(ii) $C = \Gamma_1 + \Gamma_2$, *with* Γ_i *irreducible and reduced rational curves (possibly* $\Gamma_1 = \Gamma_2$*) s.t.* $\Gamma_1 \cdot \Gamma_2 = p_a(C) + 1 \geq 2$.

Let $\mathscr{H} \overset{\text{num}}{\sim} \omega_C \otimes \mathscr{A}$ *be a very ample divisor on C s.t.* $\deg_C \mathscr{A} \geq 4$.

Then $\mathscr{K}_{0,1}(C, \mathscr{H}, \omega_C) = 0$, *that is* $H^0(C, \omega_C) \otimes H^0(C, \mathscr{H}) \twoheadrightarrow H^0(C, \omega_C \otimes \mathscr{H})$.

Proof. If $p_a(C) = 1$ then under our assumptions $\omega_C \cong \mathscr{O}_C$, whence the theorem follows easily.

If $p_a(C) \geq 2$ then by [6, Thms. 3.3, 3.4] $|\omega_C|$ is base point free and moreover it is very ample if C is not honestly hyperelliptic. We apply Prop. 25 with $\mathscr{F} = \omega_C$ and $W = H^0(\omega_C)$.

If C is irreducible and $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) = 0$ then the result follows by (i) of Prop. 25. If $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) \neq 0$ and $h^0(C, \mathscr{A}) = 0$ it follows by Riemann-Roch. In the remaining case we obtain $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) \leq h^0(C, \omega_C) - 2$ by Clifford's theorem since $\deg_C \mathscr{A} \geq 4$.

If $C = \Gamma_1 + \Gamma_2$ and $p_a(C) \geq 2$ we consider firstly the case where $\deg_{\Gamma_i} \mathscr{A} \geq -1$ for $i = 1, 2$. Under this assumption any non-zero section of $H^0(C, \omega_C \otimes \mathscr{A}^{-1})$ does not vanish identically on any single component of C (otherwise it would yield a section in $H^0(\Gamma_i, \omega_{\Gamma_i} \otimes \mathscr{A}^{-1}) \cong H^0(\mathbb{P}^1, -\alpha)$ with $\alpha \geq 1$). Therefore we can proceed exactly as in the irreducible case.

Now assume $C = \Gamma_1 + \Gamma_2$, $\deg_{\Gamma_2} \mathscr{A} \leq -2$ and $\deg_{\Gamma_1} \mathscr{A} \geq 6$. In this case we can apply (iii) of Prop. 25 taking $B = \Gamma_2$. Indeed, $h^0(\Gamma_1, \omega_{\Gamma_1} \otimes \mathscr{A}^{-1}) = h^0(\Gamma_1, \omega_{\Gamma_1}) = 0$ and we have the following maps

$$H^0(C, \omega_C) \cong H^0(\Gamma_2, \omega_C) \ ; \ \ H^0(C, \omega_C \otimes \mathscr{A}^{-1}) \hookrightarrow H^0(\Gamma_2, \omega_C \otimes \mathscr{A}^{-1}).$$

To complete the proof it remains to show that $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) \leq h^0(C, \omega_C) - 2 = p_a(C) - 2$. This follows by the following exact sequence

$$0 \to H^0(\Gamma_2, \omega_{\Gamma_2} \otimes \mathscr{A}^{-1}) \to H^0(C, \omega_C \otimes \mathscr{A}^{-1}) \to H^0(\Gamma_1, \omega_C \otimes \mathscr{A}^{-1}) \to 0.$$

In fact if $\deg_{\Gamma_1}(\omega_C \otimes \mathscr{A}^{-1}) \geq 0$ we have $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) = p_a(C) - 1 - \deg \mathscr{A}$, whereas $h^0(C, \omega_C \otimes \mathscr{A}^{-1}) = h^0(\Gamma_2, \omega_{\Gamma_2} \otimes \mathscr{A}^{-1}) = -\deg_{\Gamma_2} \mathscr{A} - 1 < p_a(C) - 2$ if $\deg_{\Gamma_1}(\omega_C \otimes \mathscr{A}^{-1}) < 0$ since $\deg_{\Gamma_2}(\omega_C + \mathscr{A}) \geq 1$ by the ampleness of $\omega_C \otimes \mathscr{A}$.

□

If one considers a curve C with many components another useful tool is the following long exact sequences for Koszul groups.

Proposition 27 *Let* $C = A + B$ *and let* $|\mathscr{F}|$ *be a complete base point free system on C such that*

- $H^0(C, \mathscr{F}) \twoheadrightarrow H^0(A, \mathscr{F})$,
- $H^0(C, \mathscr{F}^{\otimes k}) \twoheadrightarrow H^0(B, \mathscr{F}^{\otimes k})$ *for every* $k \geq 2$,
- $H^0(A, \mathscr{F}(-B)) = 0$.

Then we have a long exact sequence

$$\begin{array}{l} \cdots \to \mathscr{K}_{p+1,q-1}(C, \mathscr{F}) \to \mathscr{K}_{p+1,q-1}(B, W, \mathscr{F}) \to \mathscr{K}_{p,q}(A, W, \mathscr{O}_A(-B), \mathscr{F}) \\ \quad \to \quad \mathscr{K}_{p,q}(C, \mathscr{F}) \quad \to \mathscr{K}_{p,q}(B, W, \mathscr{F}) \qquad \to \cdots \end{array}$$

where $W \cong H^0(C, \mathscr{F})$.

Proof. With a slight abuse of notation we identify W with $H^0(C,\mathscr{F})$. Consider

$$B^1=\bigoplus_{q\geq 0}H^0(A,\mathscr{F}^{\otimes q}(-B)),\ \ B^2=\bigoplus_{q\geq 0}H^0(C,\mathscr{F}^{\otimes q}),\ \ B^3=W\oplus(\bigoplus_{q\neq 1}H^0(B,\mathscr{F}^{\otimes q}))$$

By our hypotheses the above vector spaces can be seen as $S(W)-$modules and moreover they fit into the following exact sequence

$$0\to\ B^1\to\ B^2\to\ B^3\to 0$$

where the maps preserve the grading. By the long exact sequence for Koszul Cohomology (cf. [12, Corollary 1.4.d, Thm. 3.b.1]) we can conclude.

□

Remark 28 *We point out that in this case, when considering B^1 as an $S(W)$-module we have to take in account the complex whose terms are the vector spaces $\bigwedge^p W\otimes H^0(\mathscr{F}^{\otimes q}(-B))$, i.e., we must consider the splitting $W=H^0(A,\mathscr{F}_{|A})\oplus U$, with $U\cong H^0(B,\mathscr{F}(-A))$ the subspace given by the sections of W vanishing on A.*

Setting $u=\dim U$ and $s=\max\{0,p-u\}$ then, arguing as in [12, Proof of Thm. (3.b.7)], *it is immediately seen that we have a decomposition*

$$\mathscr{K}_{p,q}(A,W,\mathscr{O}_A(-B),\mathscr{F})\ \cong\bigoplus_{s\leq p'\leq p}\left[\mathscr{K}_{p',q}(A,\mathscr{O}_A(-B),\mathscr{F}_{|A})\otimes\bigwedge^{p-p'}U\right]$$

Notice that if C is numerically connected, $\mathscr{F}\cong\omega_C$, B is numerically connected and A is the disjoint union of irreducible rational curves then the above hypotheses are satisfied.

We point out that if $\mathscr{F}$ is very ample but the restriction map $H^0(C,\mathscr{F})\to H^0(B,\mathscr{F})$ is not surjective then, following the notation of [1], we can talk of "Weak Property N_p" for the curve B embedded by the system $W=H^0(C,\mathscr{F})_{|B}$.

2.4 Divisors normally generated on algebraic curves

To conclude this preliminary section we recall a theorem proved in [11] on the normal generation of invertible sheaves of high degree.

Theorem 29 ([11, Thm. A]) *Let C be a curve contained in a smooth algebraic surface and let $\mathscr{H}\overset{\text{num}}{\sim}\mathscr{F}\otimes\mathscr{G}$, where $\mathscr{F},\mathscr{G}$ are invertible sheaves such that*

$$\begin{array}{ll}\deg\mathscr{F}_{|B}\geq\ p_a(B)+1 & \forall\ \textit{subcurve}\ \ B\subseteq C\\ \deg\mathscr{G}_{|B}\geq\ p_a(B) & \forall\ \textit{subcurve}\ \ B\subseteq C\end{array}$$

Then for every $n\geq 1$ the natural multiplication map $(H^0(C,\mathscr{H}))^{\otimes n}\to H^0(C,\mathscr{H}^{\otimes n})$ is surjective.

Moreover, applying the same arguments used in [11, Proof of Thm. A, p. 327] we have

Proposition 210 *Let C be a curve contained in a smooth algebraic surface and let $\mathscr{H}_1$, $\mathscr{H}_2$ be two invertible sheaves such that $\mathscr{H}_1\overset{\text{num}}{\sim}\mathscr{F}\otimes\mathscr{G}_1$, $\mathscr{H}_2\overset{\text{num}}{\sim}\mathscr{F}\otimes\mathscr{G}_2$ with*

$$\begin{array}{ll}\deg\mathscr{F}_{|B}\geq\ p_a(B)+1 & \forall\ \textit{subcurve}\ \ B\subseteq C\\ \deg\mathscr{G}_{1\,|B}\geq\ p_a(B) & \forall\ \textit{subcurve}\ \ B\subseteq C\\ \deg\mathscr{G}_{2\,|B}\geq\ p_a(B) & \forall\ \textit{subcurve}\ \ B\subseteq C\end{array}$$

Then $H^0(C,\mathscr{H}_1)\otimes H^0(C,\mathscr{H}_2)\twoheadrightarrow H^0(C,\mathscr{H}_1\otimes\mathscr{H}_2)$.

For even invertible sheaves of high degree theorem 29 yields as a corollary the following

Theorem 211 *Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a curve contained in a smooth algebraic surface and let $\mathscr{H}$ be an even invertible sheaf on C such that*

$$\deg_B \mathscr{H} \geq 2p_a(B)+2 \quad \forall \textit{ subcurve } B \subseteq C$$

Then for every $n \geq 1$ the natural multiplication map $(H^0(C,\mathscr{H}))^{\otimes n} \to H^0(C,\mathscr{H}^{\otimes n})$ is surjective.

Proof. First of all notice that $\mathscr{H}$ is very ample by [6, Thm. 1.1]. Moreover since $\mathscr{H}$ is even there exists an invertible sheaf $\mathscr{F}$ such that $\mathscr{F}^{\otimes 2} \overset{\text{num}}{\sim} \mathscr{H}$. By our numerical assumptions for every subcurve $B \subseteq C$ we have $\deg_B \mathscr{F} \geq p_a(B)+1$ and $\deg_B(\mathscr{H}\otimes\mathscr{F}^{-1}) \geq p_a(B)+1$, whence we can conclude by Thm. 29.

□

3 Disconnecting components of numerically connected curves

Taking an an irreducible component $\Gamma \subset C$ one problem is that the restriction map

$$H^0(C,\omega_C) \to H^0(\Gamma,\omega_{C|\Gamma})$$

is not surjective if $h^0(C-\Gamma,\mathscr{O}_{C-\Gamma}) = h^1(C-\Gamma,\omega_{C-\Gamma}) \geq 2$.

Nevertheless, if there exists a curve Γ with this property, it plays a special role in the proof of our main result.

To be more explicit, let us firstly consider the natural notion of disconnecting subcurve.

Definition 31 *Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a numerically connected curve. A subcurve $B \subset C$ is said to be a* disconnecting subcurve *if $h^0(C-B,\mathscr{O}_{C-B}) \geq 2$.*

If B is a disconnecting curve then by the exact sequence

$$H^0(C-B,\omega_{C-B}) \to H^0(C,\omega_C) \to H^0(B,\omega_C) \to H^1(C-B,\omega_{C-B}) \to H^1(C,\omega_C)$$

we deduce that the restriction map $H^0(C,\omega_C) \to H^0(B,\omega_{C|B})$ can not be surjective. In this case following the arguments pointed out by Konno in [14] one can consider an “intermediate” curve G such that $B \subseteq G \subseteq C$ and $H^0(C,\omega_C) \twoheadrightarrow H^0(G,\omega_{C|G})$.

We restrict our attention to the case of an irreducible and reduced disconnecting subcurve Γ, since in this situation we can use the approach and the results given in [14] and we can apply (*iii*) of Prop. 25. Moreover when dealing with an irreducible and reduced disconnecting subcurve we are able to construct invertible sheaves satisfying the degree assumptions of Prop. 210.

We have the following useful Lemma.

Lemma 32 *Let $C = \sum_{i=1}^{s} n_i\Gamma_i$ be a m-connected curve ($m \geq 1$) and $\Gamma \subset C$ be an irreducible and reduced disconnecting subcurve. Let G be a minimal subcurve of C such that $H^0(C,\omega_C) \twoheadrightarrow H^0(G,\omega_{C|G})$ and $\Gamma \subseteq G \subseteq C$.*

Setting $E := C-G$, $G' := G-\Gamma$, then

(a) E is a maximal subcurve of $C-\Gamma$ such that $h^1(E,\omega_E) = h^0(E,\mathscr{O}_E) = 1$;
(b) Γ is of multiplicity 1 in G, $\omega_G \otimes (\omega_C)^{-1} \cong \mathscr{O}_G(-E)$ is nef on G';
(c) $\deg_\Gamma(E) = \deg_{G'}(-E) + e$ with $e \geq m$;

(d) $h^1(E+\Gamma,\omega_{E+\Gamma})=1$*, hence* $H^0(C,\omega_C)\twoheadrightarrow H^0(G',\omega_{C|G'})$*;*
(e) *G is m-connected and in particular* $h^1(G,\omega_G)=1$*;*

Proof. By hypotheses $H^0(C,\omega_C)\not\twoheadrightarrow H^0(\Gamma,\omega_{C|\Gamma})$ and G is a minimal subcurve such that $H^0(C,\omega_C)\twoheadrightarrow H^0(G,\omega_{C|G})$. Therefore $E=C-G$ is a maximal subcurve of $C-\Gamma$ such that $h^1(E,\omega_E)=h^1(C,\omega_C)=1$, proving (a).

Moreover by [14, Lemma 2.2.1] either $\omega_G\otimes(\omega_C)^{-1}$ is nef on G, or Γ is of multiplicity one in G and $\omega_G\otimes(\omega_C)^{-1}$ is nef on $G-\Gamma=G'$.

Now by adjunction $\omega_G\otimes(\omega_C)^{-1}\cong\mathscr{O}_G(-E)$, which has negative degree on G since C is numerically connected by assumption. Therefore we can exclude the first case and by [14, Lemma 2.2.1] we conclude that Γ is of multiplicity one in G, $\omega_G\otimes(\omega_C)^{-1}\cong\mathscr{O}_G(-E)$ is nef on $G':=G-\Gamma$, and $\deg_\Gamma(E)=\deg_{G'}(-E)+e$ with $e\geq m$, proving (b) and (c).

To prove (d) consider the two curves E and Γ. Now $h^1(E,\omega_E)=1$ by (a) and $h^1(\Gamma,\omega_{E+\Gamma})=0$ because $\deg_\Gamma(\omega_{E+\Gamma})\geq 2p_a(\Gamma)-1$, whence we conclude considering the exact sequence

$$H^1(E,\omega_E)\to H^1(E+\Gamma,\omega_{E+\Gamma})\to H^1(\Gamma,\omega_{E+\Gamma})=0$$

(e) follows since $\mathscr{O}_{G'}(-E)$ is nef on G'. In fact if $B\subset G$ without loss of generality we may assume $B\subset G'$, and then we obtain $B\cdot(G-B)=B\cdot(C-B)-E\cdot B\geq B\cdot(C-B)\geq m$ since $G=C-E$, that is G is m-connected. $h^1(G,\omega_G)=1$ follows by [6, Thm. 3.3].

□

With an abuse of notation we will call a subcurve $E\subset C$ as in Lemma 32 a *maximal connected subcurve* of $C-\Gamma$.

The above Lemma allows us to consider the splitting $C=G+E$ since by connectedness both the restriction maps $H^0(C,\omega_C)\to H^0(G,\omega_{C|G})$ and $H^0(C,\omega_C)\to H^0(E,\omega_{C|E})$ are surjective.

Concerning the subcurve G we have the following theorem.

Theorem 33 *Let* $C=\sum_{i=1}^s n_i\Gamma_i$ *be an even* 4*-connected curve and assume there exists an irreducible and reduced disconnecting subcurve* $\Gamma\subset C$*.*

Let G be a minimal subcurve of C such that $H^0(C,\omega_C)\twoheadrightarrow H^0(G,\omega_C)$ *and* $\Gamma\subseteq G\subseteq C$*.*

Then on G the multiplication map $H^0(G,\omega_C)\otimes H^0(G,\omega_G)\to H^0(G,\omega_C\otimes\omega_G)$ *is surjective.*

To simplify the notation, for every subcurve $B\subset C$ by $H^0(B,\omega_C)$ we will denote the space of sections of $\omega_{C|B}$.

If there exists a disconnecting component Γ and a decomposition $C=G'+\Gamma+E$ as in Lemma 32 such that $h^1(G',\omega_{G'})\geq 2$ then we need an auxiliary Lemma.

Lemma 34 *Let* $C=\sum_{i=1}^s n_i\Gamma_i$ *be an even* 4*-connected curve and assume there exists an irreducible and reduced disconnecting subcurve* $\Gamma\subset C$*.*

If there exists a decomposition $C=G'+\Gamma+E$ *as in Lemma 32 such that* $h^1(G',\omega_{G'})\geq 2$ *then there exist a decomposition* $C=E+\Gamma+G_1+G_2$ *s.t.*

(a) $G_2+\Gamma$ *is 4-connected*
(b) $h^1(G_1,\omega_{G_1})=1$
(c) $\mathscr{O}_{G_2}(-G_1)$ *is nef on* G_2

(d) $H^0(G,\omega_G) \twoheadrightarrow H^0(G_2+\Gamma,\omega_G)$.

Proof. Let $C=E+G$ and $G=\Gamma+G'$ be as in Lemma 32. By *(e)* of Lemma 32 G is 4-connected and by our hypothesis $h^1(G',\omega_{G'})\geq 2$, i.e., the irreducible curve Γ is a disconnecting component for G too. Therefore by Lemma 32 applied to G, there exists a maximal connected subcurve $G_1\subset G'$ and a decomposition $G=\Gamma+G_1+G_2$ such that *(a)*, *(b)*, *(c)*, *(d)* hold.

□

Proof of Thm. 33. The proof of theorem 33 will be treated considering separately the case $h^1(G',\omega_{G'})=1$ and $h^1(G',\omega_{G'})\geq 2$.

Case 1: *There exists a disconnecting component Γ and a decomposition $C=G'+\Gamma+E$ as in Lemma 32 such that $h^1(G',\omega_{G'})=1$.*

Let $G=G'+\Gamma$. On Γ both the invertible sheaves $\omega_\Gamma(E)$ and ω_G have degree $\geq 2p_a(\Gamma)+2$. In particular we have the following exact sequence

$$0\to H^0(\Gamma,\omega_\Gamma(E))\to H^0(G,\omega_C)\to H^0(G',\omega_C)\to 0$$

Twisting with $H^0(\omega_G)=H^0(G,\omega_G)$ we get the following commutative diagram:

$$\begin{array}{ccccc} H^0(\Gamma,\omega_\Gamma(E))\otimes H^0(\omega_G) & \hookrightarrow & H^0(G,\omega_C)\otimes H^0(\omega_G) & \twoheadrightarrow & H^0(G',\omega_C)\otimes H^0(\omega_G)\\ r_1\downarrow & & r_2\downarrow & & r_3\downarrow\\ H^0(\Gamma,\omega_\Gamma(E)\otimes\omega_G) & \hookrightarrow & H^0(G,\omega_C\otimes\omega_G) & \twoheadrightarrow & H^0(G',\omega_C\otimes\omega_G)\end{array}$$

Now, since by our hypothesis $h^1(G',\omega_{G'})=1$ then $H^0(G,\omega_G)\twoheadrightarrow H^0(\Gamma,\omega_G)$ and we have the surjection $H^0(\Gamma,\omega_\Gamma(E))\otimes H^0(\Gamma,\omega_G)\twoheadrightarrow H^0(\Gamma,\omega_\Gamma(E)\otimes\omega_G)$ by [19, Thm.6] .

The theorem follows since also r_3 is surjective by Prop. 210. Indeed, $\omega_{G|G'}\cong\omega_{C|G'}(-E)$ with $\mathcal{O}_{G'}(-E)$ nef and $\omega_{C|G'}$ is an even invertible sheaf whose degree on every subcurve $B\subseteq G'$ satisfies $\deg_B(\omega_C)\geq 2p_a(B)+2$.

Case 2: *There exists a disconnecting component Γ and a decomposition $C=G'+\Gamma+E$ as in Lemma 32 such that $h^1(G',\omega_{G'})\geq 2$.*

Let $C=E+\Gamma+G_1+G_2$ and $G=\Gamma+G_1+G_2$, be a decomposition as in Lemma 34. We proceed as in Case 1, considering the curve $G_2+\Gamma$ instead of the irreducible Γ.

First of all let us prove that $H^1(G_2+\Gamma,\omega_{G_2+\Gamma}(E))=0$.

We have $(\omega_{G_2+\Gamma}(E))_{|G_2}\cong(\omega_C(-G_1))_{|G_2}$ and in particular for every subcurve $B\subseteq G_2$ $\deg_B(\omega_{G_2+\Gamma}(E))\geq 2p_a(B)+2$ since $\mathcal{O}_{G_2}(-G_1)$ is nef .

If $B\not\subseteq G_2$, we can write $B=B'+\Gamma$, with $B'\subseteq G_2$, obtaining

$$\deg_B(\omega_{G_2+\Gamma}(E))=\deg_B(\omega_{G_2+\Gamma})+E\cdot B\geq 2p_a(B)+2$$

since $E\cdot B=E\cdot(B'+\Gamma)\geq E\cdot(G'+\Gamma)\geq 4$ and $\deg_B(\omega_{G_2+\Gamma})\geq 2p_a(B)-2$ by connectedness. Therefore $H^1(G_2+\Gamma,\omega_{G_2+\Gamma}(E))=0$ by [5, Lemma 2.1] and we have the following exact sequence

$$0\to H^0(G_2+\Gamma,\omega_{G_2+\Gamma}(E))\to H^0(G,\omega_C)\to H^0(G_1,\omega_C)\to 0$$

Twisting with $H^0(\omega_G)=H^0(G,\omega_G)$ we can argue as in Case 1. Indeed, consider the commutative diagram:

$$\begin{array}{ccccc} H^0(G_2+\Gamma,\omega_{G_2+\Gamma}(E))\otimes H^0(\omega_G) & \hookrightarrow & H^0(G,\omega_C)\otimes H^0(\omega_G) & \twoheadrightarrow & H^0(G_1,\omega_C)\otimes H^0(\omega_G)\\ r_1\downarrow & & r_2\downarrow & & r_3\downarrow\\ H^0(G_2+\Gamma,\omega_{G_2+\Gamma}(E)\otimes\omega_G) & \hookrightarrow & H^0(G,\omega_C\otimes\omega_G) & \twoheadrightarrow & H^0(G_1,\omega_C\otimes\omega_G)\end{array}$$

The map r_3 is onto by Prop. 210 since $\omega_G \cong \omega_C(-E)$, $\mathscr{O}_{G_1}(-E)$ is nef and by Lemma 34 we have the surjection $H^0(\omega_G) \twoheadrightarrow H^0(G_1, \omega_G)$.

The Theorem follows if we show that r_1 is surjective too. Notice that we can write the multiplication map r_1 as follows:

$$H^0(G_2+\Gamma, \omega_{G_2+\Gamma}(E)) \otimes H^0(G_2+\Gamma, \omega_{G_2+\Gamma}(G_1)) \to H^0(G_2+\Gamma, \omega_{G_2+\Gamma}^{\otimes 2}(E+G_1))$$

that is, r_1 is symmetric in E and G_1.

Assume firstly that $(G_1-E)\cdot\Gamma \geq 0$. We proceed considering a general effective Cartier divisor Υ on $G_2+\Gamma$ such that

$$\begin{cases} (\mathscr{O}_{G_2}(\Upsilon))^{\otimes 2} \stackrel{\text{num}}{\sim} \omega_C(-2E)_{|G_2} \\ \deg(\mathscr{O}_\Gamma(\Upsilon)) = \frac{1}{2}\deg(\omega_{C|\Gamma}) - E\cdot\Gamma - \delta \end{cases}$$

with $\delta = \lceil \frac{-G_1\cdot G_2}{2} \rceil$. We remark that by connectedness of C and nefness of $\mathscr{O}_{G_2}(-2E)$ $\deg(\mathscr{O}_{\Gamma_i}(\Upsilon)) \geq 1$ for every $\Gamma_i \subseteq G_2$ and by our numerical conditions

$$\deg(\mathscr{O}_\Gamma(\Upsilon)) = p_a(\Gamma) - 1 + \frac{1}{2}(G_1-E)\cdot\Gamma + \frac{1}{2}G_2\cdot\Gamma - \delta \geq 1$$

since $(G_1-E)\cdot\Gamma \geq 0$ by our assumptions and by 4-connectedness of C we have $G_2\cdot\Gamma - 2\delta \geq (\Gamma+G_1)\cdot G_2 \geq (\Gamma+G_1+E)\cdot G_2 \geq 4$.

Now let $\mathscr{F} := \omega_G(-\Upsilon)$. $\mathscr{F}$ is a general invertible subsheaf of $\omega_{G|G_2+\Gamma}$ s.t.

$$\begin{cases} \deg_B \mathscr{F} = \frac{1}{2}\deg_B \omega_C \ \ \forall B \subseteq G_2 \\ \deg_\Gamma \mathscr{F} = \frac{1}{2}\deg_\Gamma \omega_C + \delta \end{cases}$$

By 4-connectedness on every subcurve $B \subseteq G_2+\Gamma$ $\deg_B \mathscr{F} \geq p_a(B)+1$. By Theorem 21 we conclude that $|\mathscr{F}|$ is base point free and $h^1(G_2+\Gamma, \mathscr{F}) = 0$.

Therefore we have the following exact sequence

$$0 \to H^0(G_2+\Gamma, \mathscr{F}) \to H^0(G_2+\Gamma, \omega_G) \to H^0(\mathscr{O}_\Upsilon) \to 0$$

and we have the surjection $H^0(\mathscr{O}_\Upsilon) \otimes H^0(G, \omega_{G_2+\Gamma}(E)) \twoheadrightarrow H^0(G, \mathscr{O}_\Upsilon \otimes \omega_{G_2+\Gamma}(E))$ since $\mathscr{O}_\Upsilon$ is a skyscraper sheaf and $|\omega_{G_2+\Gamma}(E)|$ is base point free by [6, Thm.3.3]. Whence the map r_1 is surjective if we prove that

$$H^0(G_2+\Gamma, \omega_{G_2+\Gamma}(E)) \otimes H^0(G_2+\Gamma, \mathscr{F}) \twoheadrightarrow H^0(G_2+\Gamma, \omega_{G_2+\Gamma}(E) \otimes \mathscr{F})$$

With this aim we are going to apply *(iii)* of Prop. 25. First notice that

$$H^0(G_2+\Gamma, \mathscr{F}) \hookrightarrow H^0(\Gamma, \mathscr{F})$$

Indeed, by adjunction and Serre duality the kernel of this map is isomorphic to $H^0(G_2, \mathscr{F} - \Gamma) \cong H^1(G_2, \omega_C(-E-G_1) \otimes \mathscr{F}^{-1})$, which vanishes by Thm. 21 since it is the first cohomology group of a general invertible sheaf whose degree on every component $B \subseteq G_2$ satisfies

$$\deg_B(\omega_C(-E-G_1) \otimes \mathscr{F}^{-1}) = \frac{\deg_B(\omega_C)}{2} + (-E-G_1)\cdot B \geq \frac{\deg_B(\omega_C)}{2} \geq p_a(B)$$

because C is 4-connected and $\mathscr{O}_{G_2}(-E-G_1)$ is nef .

Moreover we have also the embedding

$$H^0(G_2+\Gamma,\omega_{G_2+\Gamma}\otimes[\omega_{G_2+\Gamma}(E)]^{-1}\otimes\mathscr{F})\hookrightarrow H^0(\Gamma,\omega_{G_2+\Gamma}\otimes[\omega_{G_2+\Gamma}(E)]^{-1}\otimes\mathscr{F})$$

since $H^0(G_2,\mathscr{F}-E-\Gamma)\cong H^1(G_2,\omega_C(-G_1)\otimes\mathscr{F}^{-1})=0$ because

$$\deg_B(\omega_C(-G_1)\otimes\mathscr{F}^{-1})=\frac{\deg_B(\omega_C)}{2}+(-G_1\cdot B)\geq\frac{\deg_B(\omega_C)}{2}\geq p_a(B)$$

by 4-connectedness of C and nefness of $\mathscr{O}_{G_2}(-G_1)$. In order to conclude we are left to compute $h^0(G_2+\Gamma,\omega_{G_2+\Gamma}\otimes[\omega_{G_2+\Gamma}(E)]^{-1}\otimes\mathscr{F})=h^0(G_2+\Gamma,\mathscr{F}(-E))$.

$\mathscr{F}(-E)$ is a general invertible sheaf s.t.

$$\begin{cases}\deg_B(\mathscr{F}(-E))=\frac{1}{2}\deg_B\omega_C-E\cdot B & \forall B\subseteq G_2\\ \deg_\Gamma(\mathscr{F}(-E))=\frac{1}{2}\deg_\Gamma(\omega_{G_2+\Gamma})+\frac{1}{2}(G_1-E)\cdot\Gamma+\delta\end{cases}$$

Therefore we obtain immediately that $\deg_B\mathscr{F}(-E)\geq p_a(B)$ on every $B\subseteq G_2$, whereas if $B=B'+\Gamma$ with $B'\subseteq G_2$

$$\deg_B(\mathscr{F}(-E))=\frac{1}{2}\deg_B(\omega_{G_2+\Gamma})+\frac{1}{2}(G_1-E)\cdot B'+\frac{1}{2}(G_1-E)\cdot\Gamma+\delta\geq p_a(B)$$

since by our assumptions $G_2+\Gamma$ is numerically connected, $(G_1-E)\cdot\Gamma\geq 0$, $E\cdot B'\leq 0$ and $\delta\geq\frac{1}{2}(-G_1\cdot G_2)\geq\frac{1}{2}(-G_1\cdot B')$.

In particular by Theorem 21 we get $H^1(G_2+\Gamma,\mathscr{F}(-E))=0$ and by Riemann-Roch theorem we have $h^0(G_2+\Gamma,\mathscr{F}(-E))=h^0(G_2+\Gamma,\mathscr{F})-E\cdot(\Gamma+G_2)$. Finally, since $\mathscr{O}_{G_1}(-E)$ is nef we have $E\cdot(\Gamma+G_2)\geq E\cdot(\Gamma+G_2+G_1)\geq 4$ because C is 4-connected, that is, $h^0(G_2+\Gamma,\mathscr{F}(-E))\leq h^0(G_2+\Gamma,\mathscr{F})-4$. Whence all the hypotheses of (*iii*) of Prop. 25 are satisfied and we can conclude.

If $(E-G_1)\cdot\Gamma<0$ we simply exchange the role of $\mathscr{O}_{G_2+\Gamma}(E)$ with the one of $\mathscr{O}_{G_2+\Gamma}(G_1)$ and we reply the proof "verbatim", since our numerical conditions are symmetric in E and G_1.

□

4 The canonical ring of an even 4-connected curve

In this section we are going to show Theorem 11. We recall that under our assumptions ω_C is very ample by [6, Thm. 3.6].

Proof of Theorem 11. For all $k\in\mathbb{N}$ we have to show the surjectivity of the maps

$$\rho_k:(H^0(C,\omega_C))^{\otimes k}\longrightarrow H^0(C,\omega_C^{\otimes k})$$

For $k=0,1$ it is obvious. For $k\geq 3$ it follows by an induction argument applying Prop. 25 to the sheaves $\omega_C^{\otimes(k-1)}$ and ω_C.

For $k=2$ the proof is based on the above results. If C is irreducible and reduced the result is (almost) classical. For the general case we separate the proof in three different parts, depending on the existence of suitable irreducible components.

Case A: *There exists a not disconnecting irreducible curve Γ of arithmetic genus $p_a(\Gamma)\geq 1$.*

In this case, writing $C=\Gamma+E$, we have the surjections $H^0(C,\omega_C)\twoheadrightarrow H^0(\Gamma,\omega_{C|\Gamma})$ and $H^0(C,\omega_C)\twoheadrightarrow H^0(E,\omega_{C|E})$, whence we can conclude by the following commutative diagram:

$$\begin{array}{ccccc} H^0(\Gamma,\omega_\Gamma)\otimes H^0(\omega_C) & \hookrightarrow & H^0(\omega_C)\otimes H^0(\omega_C) & \twoheadrightarrow & H^0(E,\omega_C)\otimes H^0(\omega_C) \\ r_1\downarrow & & \rho_2\downarrow & & r_3\downarrow \\ H^0(\Gamma,\omega_\Gamma\otimes\omega_C) & \hookrightarrow & H^0(C,\omega_C^{\otimes 2}) & \twoheadrightarrow & H^0(E,\omega_C^{\otimes 2}) \end{array}$$

(where $H^0(\omega_C)=H^0(C,\omega_C)$). Indeed, since C is 4-connected and ω_C is an even divisor we get the surjection of the map r_3 by Theorem 211, while Proposition 26 ensure the surjectivity of the map r_1, forcing ρ_2 to be surjective too (cf. also [19, Thm. 6]).

Case B: *There exists a disconnecting irreducible component Γ.*

Let us consider the decomposition $C=E+G$ introduced in Lemma 32. Then we have the exact sequence

$$0\to H^0(G,\omega_G)\to H^0(C,\omega_C)\to H^0(E,\omega_C)\to 0$$

and furthermore by Lemma 32 *(a)* also the map $H^0(C,\omega_C)\to H^0(G,\omega_C)$ is onto. Replacing Γ with G we can build a commutative diagram analogous to the one shown in case A. Keeping the notation r_1,r_3 for the analogous maps, by Theorem 211 and Theorem 33 the maps r_3 and r_1 are surjective, whence also ρ_2 is onto.

Case C: *Every irreducible component Γ_i of C has arithmetic genus $p_a(\Gamma_i)=0$ and it is not disconnecting.*

First of all notice that by connectedness for every irreducible Γ_h there exists at least one Γ_k such that $\Gamma_h\cdot\Gamma_k\geq 1$.

Moreover if every component Γ_i has arithmetic genus $p_a(\Gamma_i)=0$ then the condition $h^0(B,\mathscr{O}_B)=1$ for a curve $B=\sum a_i\Gamma_i\subset C$ implies $\Gamma_i\cdot(B-\Gamma_i)\geq 1$ for every $\Gamma_i\subset B$. Indeed, if it were $B=\Gamma_i+(B-\Gamma_i)$ with $\Gamma_i\cdot(B-\Gamma_i)\leq 0$, then we would get $h^0(\Gamma_i,\mathscr{O}_{\Gamma_i}(-(B-\Gamma_i)))\geq 1$ and $h^1(\Gamma_i,\mathscr{O}_{\Gamma_i}(-(B-\Gamma_i)))=0$ because $\Gamma_i\cong\mathbb{P}^1$. Whence we would obtain $h^1(B,\mathscr{O}_B)\geq 2$ by the following exact sequence

$$0\to H^0(\Gamma_i,\mathscr{O}_{\Gamma_i}(-(B-\Gamma_i)))\to H^0(B,\mathscr{O}_B)\to H^0(B-\Gamma_i,\mathscr{O}_{B-\Gamma_i})\to 0$$

We will consider separately the different situations that may happen.

C.1. *There exist two components Γ_h,Γ_k (possibly $h=k$ if* $\operatorname{mult}_C\Gamma_h\geq 2$*) such that $\Gamma_h\cdot\Gamma_k\geq 2$, and $\Gamma=\Gamma_h+\Gamma_k\subset C$ is not disconnecting.*

C.1.1 C has exactly two components.

If the two components are distinct then C is a binary curve (see [3] for the definition and main properties) and our result follows from [4, Prop. 3].

If $\Gamma_1=\Gamma_2=\Gamma$ we use a slight generalization of the classical argument used by Saint-Donat in [21].

For simplicity let $r=p_a(C)-1=\deg_\Gamma(\omega_C)=\Gamma^2-2$ and let us identify Γ and C with their images in $\mathbb{P}^r$. Notice that we have $H^0(C,\omega_C)\cong H^0(\Gamma,\omega_C)$.

Following the cited paper [21] we take $P_1,\cdots,P_{r-1}$ general points on C and we set $S=P_1\cup\cdots\cup P_{r-1}$. Notice that S can be seen as a subscheme of Γ. Indeed, for every point P, denoting by $\mathscr{M}$, $\tilde{\mathscr{M}}$ the maximal ideals of $\mathscr{O}_C$, respectively $\mathscr{O}_\Gamma$, at P, we have $\mathscr{O}_C/\mathscr{M}\cong\mathscr{O}_\Gamma/\tilde{\mathscr{M}}$ and then, since $\mathscr{I}_S\cong\prod\mathscr{M}_{P_i}$, we obtain the isomorphism $\mathscr{O}_C/\mathscr{I}_S\cong\mathscr{O}_\Gamma/\mathscr{I}_{S_{|\Gamma}}$.

Our claim is that for general $P_1,\cdots,P_{r-1}$:

(i) $h^0(C,\mathscr{I}_S\omega_C)=2$ and $h^1(C,\mathscr{I}_S\omega_C)=1$;
(ii) the evaluation map $H^0(C,\mathscr{I}_SK_C)\otimes\mathscr{O}_C\to\mathscr{I}_S\omega_C$ is surjective;
(iii) $H^0(C,\mathscr{I}_S\omega_C)\otimes H^0(C,\omega_C)\to H^0(C,\mathscr{I}_S\omega_C^{\otimes 2})$ is surjective.

(i) follows since $H^0(C,\omega_C)\cong H^0(\Gamma,\omega_C)$, $\mathscr{O}_{S_{|\Gamma}}\cong\mathscr{O}_S$ and $H^0(\Gamma,\omega_C)\twoheadrightarrow\mathscr{O}_{S_{|\Gamma}}$ because $\Gamma=C_{red}$ is a rational curve and $\deg_\Gamma(\mathscr{I}_S\omega_C)=1$. Whence by Riemann-Roch $h^0(C,\mathscr{I}_S\omega_C)=2$ and $h^1(C,\mathscr{I}_S\omega_C)=1$.

Concerning (ii) we have to prove that $H^0(C,\omega_C)\twoheadrightarrow\mathscr{O}_{S'}$ for every 0-dimensional scheme S' containing S with $\mathrm{length}(S')=\mathrm{length}(S)+1$.

If $S'=S\cup Q$, with Q a point distinct from $P_1,\cdots P_{r-1}$, then we obtain the surjection $H^0(C,\omega_C)\cong H^0(\Gamma,\omega_C)\twoheadrightarrow\mathscr{O}_{S'}$ since $S'\cong S'_{|\Gamma}$ too.

If $\mathrm{Supp}(S')=\mathrm{Supp}(S)$, i.e., there exists a point P_i such $S'=Z_i\bigcup_{j\neq i}P_j$ with Z_i a 0-dimensional scheme of length 2 supported at P_i, we consider

$$W_{i,k}=\big\{(P_1,\cdots P_{r-1})\in C^{(r-1)}\ :\ P_k\in\langle T_{P_i}(C),\bigcup_{j\neq i,k}P_j\rangle\big\}\subset C^{(r-1)}$$

(where $T_{P_i}(C)$ denotes the affine 2-dimensional tangent space to C at P_i and $C^{(r-1)}$ is the Cartesian product of $r-1$ copies of C).

Since the linear span $\langle T_{P_i}(C),\bigcup_{j\neq i,k}P_j\rangle$ has codimension at least one in $\mathbb{P}^r$, then its intersection with Γ contains at most 2 more points, that is $W_{i,k}$ is a closed subvariety of $C^{(r-1)}$. Therefore for $(P_1,\cdots,P_{r-1})\in C^{(r-1)}\setminus\big[\bigcup_k W_{i,k}\big]$ the 0-dimensional scheme $S'=Z_i\bigcup_{j\neq i}P_j$ imposes independent conditions on $H^0(C,\omega_C)$ since $Z_i\subset T_{P_i}(C)$. Taking

$$(P_1,\cdots,P_{r-1})\in C^{(r-1)}\setminus\big[\bigcup_{i,k}W_{i,k}\big]$$

we conclude that for every S' containing S with $\mathrm{length}(S')=\mathrm{length}(S)+1$ we obtain the surjection $H^0(C,\omega_C)\twoheadrightarrow\mathscr{O}_{S'}$.

To prove (iii) we improve a generalization of the classical base point free pencil trick. Consider the evaluation map $H^0(C,\mathscr{I}_S\omega_C)\otimes\omega_C\xrightarrow{ev}\mathscr{I}_S\omega_C^{\otimes 2}$ and its kernel $\mathscr{K}$. By the following exact sequence

$$0\to\mathscr{K}\to H^0(C,\mathscr{I}_S\omega_C)\otimes\omega_C\to\mathscr{I}_S\omega_C^{\otimes 2}\to 0$$

we obtain the required surjection if and only if $h^1(C,\mathscr{K})=2$ since $h^1(C,\mathscr{I}_S\omega_C^{\otimes 2})=0$ by [6, Thm. 1.1]. Firstly let us show that $\mathscr{K}\cong\mathscr{H}om(\mathscr{I}_S\omega_C,\omega_C)$. Indeed consider a basis $\{x_0,x_1\}$ for $H^0(C,\mathscr{I}_S\omega_C)$ and define the map

$$\begin{aligned}\iota:\ \mathscr{H}om(\mathscr{I}_S\omega_C,\omega_C)&\to H^0(C,\mathscr{I}_S\omega_C)\otimes\omega_C\\ \varphi&\mapsto x_0\otimes\varphi(x_1)-x_1\otimes\varphi(x_0).\end{aligned}$$

ι is injective since the sheaf $\mathscr{I}_S\omega_C$ is generated by its sections and $\mathrm{im}(\iota) \subseteq \mathscr{K}$ by our construction. Since over the generic points the two sheaves are isomorphic, we conclude that ι induces an isomorphism because the Euler characteristic of both sheaves coincide:

$$\chi(\mathscr{H}om(\mathscr{I}_S\omega_C, \omega_C)) = \deg S - (p_a(C) - 1) = \chi(\mathscr{K}).$$

Secondly, it is easy to see that $\mathscr{I}_S\omega_C$ is reflexive, which implies $\mathscr{H}om(\mathscr{K}, \omega_C) \cong \mathscr{I}_S\omega_C$ and then

$$h^1(C, \mathscr{K}) = \dim(\mathrm{Hom}(\mathscr{K}, \omega_C)) = h^0(C, \mathscr{H}om(\mathscr{K}, \omega_C)) = h^0(C, \mathscr{I}_S\omega_C) = 2.$$

Finally considering the exact sequence

$$\begin{array}{ccccc} H^0(C, \mathscr{I}_S\omega_C) \otimes H^0(C, \omega_C) & \hookrightarrow & H^0(C, \omega_C)^{\otimes 2} & \twoheadrightarrow & H^0(S, \mathscr{O}_S) \otimes H^0(C, \omega_C) \\ \downarrow m & & \downarrow r & & \downarrow \\ H^0(C, \mathscr{I}_S\omega_C^{\otimes 2}) & \hookrightarrow & H^0(C, \omega_C^{\otimes 2}) & \twoheadrightarrow & H^0(S, \mathscr{O}_S) \end{array}$$

we can conclude applying the same argument adopted by Saint-Donat in [21, Thm. 2.10, p. 164].

C.1.2 If $C - \Gamma \neq \emptyset$ then, setting $E = C - \Gamma$ by *(ii)* of Proposition 26 we can proceed exactly as in *Case A*.

C.2. *There exist two components Γ_h, Γ_k (possibly $h = k$ if* $\mathrm{mult}_C \Gamma_h \geq 2$*) such that $\Gamma := \Gamma_h + \Gamma_k \subset C$ is disconnecting and $\Gamma_h \cdot \Gamma_k \geq 0$.*

In this case take E a maximal subcurve of $C - \Gamma_k - \Gamma_h$ such that $h^0(E, \mathscr{O}_E) = 1$ and let $G = C - E$. Then we obtain a decomposition $C = E + \Gamma + G'$ with $\mathscr{O}_{G'}(-E)$ nef on G'.

Firstly let us point out some useful remarks about this decomposition.

We have $h^0(E + \Gamma_k + G', \mathscr{O}_{E+\Gamma_k+G'}) = 1$ since Γ_k is not disconnecting in C and it is immediately seen that also $h^0(\Gamma_k + G', \mathscr{O}_{\Gamma_h+G'}) = 1$ because $\mathscr{O}_{G'}(-E)$ is nef on G'. But $p_a(\Gamma_k) = 0$, whence by the remark given at the beginning of Case C $\Gamma_k \cdot G' \geq 1$. In particular $H^0(G, \omega_G) \twoheadrightarrow H^0(\Gamma_h, \omega_G)$. Similarly we obtain $\Gamma_h \cdot G' \geq 1$, $H^0(G, \omega_G) \twoheadrightarrow H^0(\Gamma_k, \omega_G)$, and considering $\mathscr{O}_\Gamma(E)$ we have $E \cdot \Gamma_h \geq 1$ and $E \cdot \Gamma_k \geq 1$. Furthermore, since $E \cdot \Gamma \geq 4$, may assume $E \cdot \Gamma_h \geq 2$.

We will consider firstly the subcase where $\Gamma_h \cdot \Gamma_k \geq 1$ and secondly the case where the product is null.

C.2.1. If $\Gamma_h \cdot \Gamma_k \geq 1$ and $\Gamma = \Gamma_h + \Gamma_k \subset C$ is disconnecting, arguing as in *Case B*, the theorem follows if we have the surjection of the multiplication map $r_1 : H^0(G, \omega_C) \otimes H^0(G, \omega_G) \to H^0(G, \omega_C \otimes \omega_G)$. Considering the diagram

$$\begin{array}{ccccc} H^0(\Gamma, \omega_\Gamma(E)) \otimes H^0(\omega_G) & \hookrightarrow & H^0(G, \omega_C) \otimes H^0(\omega_G) & \twoheadrightarrow & H^0(G', \omega_C) \otimes H^0(\omega_G) \\ s_1 \downarrow & & r_1 \downarrow & & t_1 \downarrow \\ H^0(\Gamma, \omega_\Gamma(E) \otimes \omega_G) & \hookrightarrow & H^0(G, \omega_C \otimes \omega_G) & \twoheadrightarrow & H^0(G', \omega_C \otimes \omega_G) \end{array}$$

it is sufficient to show that s_1 is onto since t_1 is surjective by Prop. 210. With this aim we take the splitting

$$0 \to H^0(\Gamma_h, \omega_{\Gamma_h}(E)) \to H^0(\Gamma, \omega_\Gamma(E)) \to H^0(\Gamma_k, \omega_{\Gamma_k}(\Gamma_h + E)) \to 0$$

Twisting with $H^0(G,\omega_G)=H^0(\omega_G)$ (notice that $H^0(G,\omega_G)\twoheadrightarrow H^0(\Gamma_h,\omega_G)$ and similarly for Γ_k by the above remark), we can conclude since we have the surjections

$$H^0(\Gamma_h,\omega_{\Gamma_h}(E))\otimes H^0(\Gamma_h,\omega_G)\twoheadrightarrow H^0(\Gamma_h,\omega_{\Gamma_h}(E)\otimes\omega_G)$$
$$H^0(\Gamma_k,\omega_{\Gamma_k}(\Gamma_h+E))\otimes H^0(\Gamma_k,\omega_G)\twoheadrightarrow H^0(\Gamma_k,\omega_{\Gamma_k}(\Gamma_h+E)\otimes\omega_G)$$

because $\Gamma_h\cong\Gamma_k\cong\mathbb{P}^1$, and all the sheaves have positive degree on both the curves (see [12, Corollary 3.a.6] for details).

C.2.2. Assume now $\Gamma_h\cdot\Gamma_k=0$ and $\Gamma=\Gamma_h+\Gamma_k\subset C$ to be disconnecting.

If $E\cdot\Gamma_h\geq 2$, $E\cdot\Gamma_k\geq 2$ and $G'\cdot\Gamma_h\geq 2$, $G'\cdot\Gamma_k\geq 2$ then we consider the exact sequence

$$0\to\omega_\Gamma(E)\to\omega_{C|(G'+\Gamma)}\to\omega_{C|G'}\to 0$$

and we operate as in *C.2.1.*

Otherwise, without loss of generality, we may assume $E\cdot\Gamma_h=1$ or $G'\cdot\Gamma_h=1$.

If $E\cdot\Gamma_h=1$ then by 4-connectedness of C $E\cdot\Gamma_k\geq 3$, $G'\cdot\Gamma_h\geq 3$, $G'\cdot\Gamma_k\geq 3$.

Let $G=G'+\Gamma_h+\Gamma_k$ and consider the splitting $C=E+G$: as in the previous case it is enough to prove the surjection of $r_1: H^0(G,\omega_C)\otimes H^0(G,\omega_G)\to H^0(G,\omega_C\otimes\omega_G)$.

With this aim we take the following exact sequence

$$0\to H^0(\Gamma_k,\omega_{\Gamma_k}(E))\to H^0(G,\omega_C)\to H^0(G'+\Gamma_h,\omega_C)$$

By our numerical conditions $|\omega_G|$ is base point free and by connectedness we have the surjection $H^0(\omega_G)\twoheadrightarrow H^0(\Gamma_k,\omega_G)$.

By [12, (2.a.17), (3.a.6)] (or simply since we have sheaves of positive degree on a rational curve) the multiplication map $H^0(\Gamma_k,\omega_G)\otimes H^0(\Gamma_k,\omega_{\Gamma_k}(E))\twoheadrightarrow H^0(\Gamma_k,\omega_G\otimes\omega_{\Gamma_k}(E))$ is onto.

On the contrary $\deg\mathscr{O}_{G'+\Gamma_h}(-E)\geq -1$. Therefore we can consider a subsheaf $\mathscr{F}\subset\omega_{C|G'+\Gamma_h}$ such that $(\mathscr{F}_{|G'+\Gamma_h})^{\otimes 2}\overset{\text{num}}{\sim}\omega_{C|G'+\Gamma_h}$. Then for every $B\subset G'+\Gamma_h$ $\mathscr{F}_{|B}$ has degree at least $p_a(B)+1$ whilst $\omega_G\otimes\mathscr{F}^{-1}$ is an invertible sheaf of degree at least $p_a(B)$. Whence by Prop. 210 $H^0(G'+\Gamma_h,\omega_G)\otimes H^0(G'+\Gamma_h,\omega_C)\twoheadrightarrow H^0(G'+\Gamma_h,\omega_G\otimes\omega_C)$ and then r_1 is onto.

If $G'\cdot\Gamma_h=1$ then by 4-connectedness of C $E\cdot\Gamma_h\geq 3$, $E\cdot\Gamma_k\geq 3$, $G'\cdot\Gamma_k\geq 3$.

In this case we write $\tilde{E}:=E+\Gamma_h$, $\tilde{G}:=G'+\Gamma_k$. We have a decomposition $C=\tilde{E}+\tilde{G}$ where $\tilde{E}$ is connected, $\tilde{G}$ is 3-connected. Moreover by adjunction we have the isomorphism $\omega_{\tilde{G}}=\omega_C(-\tilde{E})_{|\tilde{G}}$, where $\deg\mathscr{O}_{\tilde{G}}(-\tilde{E})\geq -1$.

Arguing as above the theorem follows if we prove the surjection of the multiplication map $\tilde{r}_1: H^0(\tilde{G},\omega_C)\otimes H^0(\tilde{G},\omega_{\tilde{G}})\to H^0(G,\omega_C\otimes\omega_{\tilde{G}})$.

With this aim let us show firstly that $h^0(G',\mathscr{O}_{G'})=1$. Indeed, since Γ_k is not disconnecting for C, whilst it is disconnecting for $C-\Gamma_h$ $h^0(G'+\Gamma_h,\mathscr{O}_{G'+\Gamma_h})=1$. Therefore since $\deg\mathscr{O}_{\Gamma_h}(-G')=-1$ and $\Gamma_h\cong\mathbb{P}^1$ by the exact sequence

$$H^0(\Gamma_h,\mathscr{O}_{\Gamma_h}(-G'))\to H^0(G'+\Gamma_h,\mathscr{O}_{G'+\Gamma_h})\to H^0(G',\mathscr{O}_{G'})\to H^1(\Gamma_h,\mathscr{O}_{\Gamma_h}(-G'))$$

we obtain $h^0(G',\mathscr{O}_{G'})=1$ because $H^0(\Gamma_h,\mathscr{O}_{\Gamma_h}(-G'))=H^1(\Gamma_h,\mathscr{O}_{\Gamma_h}(-G'))=0$. Going back to $\tilde{G}:=G'+\Gamma_k$ let us take the exact sequence

$$0\to H^0(\Gamma_k,\omega_{\Gamma_k}(E))\to H^0(\tilde{G},\omega_C)\to H^0(G',\omega_C)\to 0.$$

We have $H^0(\Gamma_k, \omega_{\Gamma_k}(E)) \otimes H^0(\Gamma_k, \omega_{\tilde{G}}) \twoheadrightarrow H^0(\Gamma_k, \omega_{\Gamma_k}(E) \otimes \omega_{\tilde{G}})$ since $\Gamma_k \cong \mathbb{P}^1$.

Finally, taking a subsheaf $\mathscr{F} \subset \omega_{C|\tilde{G}}$ such that $(\mathscr{F}_{\tilde{G}})^{\otimes 2} \overset{\text{num}}{\sim} \omega_{C|\tilde{G}}$, we can consider the two sheaves $\mathscr{G}_1 := \omega_C \otimes \mathscr{F}^{-1} \overset{\text{num}}{\sim} \mathscr{F}$, $\mathscr{G}_2 := \omega_{\tilde{G}} \otimes \mathscr{F}^{-1}$.

$\mathscr{F}, \mathscr{G}_1, \mathscr{G}_2$ satisfy the assumptions of Prop. 210, whence $\tilde{r}_1$ is surjective and we can conclude.

C.3. *There exists two distinct irreducible components* Γ_h, Γ_k *such that* $\Gamma_h \cdot \Gamma_k = 0$, *and* $\Gamma = \Gamma_h + \Gamma_k$ *is not disconnecting.*

In this case the situation is slightly different.

By §2.3 $\rho_2 : (H^0(C, \omega_C))^{\otimes 2} \twoheadrightarrow H^0(C, \omega_C^{\otimes 2})$ iff $\mathscr{K}_{0,1}(C, \omega_C, \omega_C) = 0$, and by [12, (2.a.17)] $\mathscr{K}_{0,1}(C, \omega_C, \omega_C) = \mathscr{K}_{0,2}(C, \omega_C)$.

Write $C = A + \Gamma_h + \Gamma_k$: since $p_a(\Gamma_i) = 0$, $p_a(\Gamma_h + \Gamma_k) = -1$ and all these curves are not disconnecting then we can consider the long exact sequence of Koszul groups (Prop. 27) for the decomposition $C = A + \Gamma_h + \Gamma_k$ (respectively for the decompositions $C - \Gamma_h = A + \Gamma_k$, $C - \Gamma_k = A + \Gamma_h$).

We set $W = H^0(\omega_C)$. By Thm. 211 for $i \in \{h, k\}$ and every $q \geq 1$ $\mathscr{K}_{0,q}(A + \Gamma_i, \omega_C) = 0$; consequently $\mathscr{K}_{0,q}(A, W, \omega_C) = 0\ \forall q \geq 1$.

Therefore it is sufficient to prove that the following sequence is exact

$$\mathscr{K}_{1,1}(C, \omega_C) \xrightarrow{\ \iota\ } \mathscr{K}_{1,1}(A, W, \omega_C) \xrightarrow{\ \pi\ } \mathscr{K}_{0,2}(\Gamma_h + \Gamma_k, W, \mathscr{O}_{\Gamma_h + \Gamma_k}(-A), \omega_C).$$

First of all, since $\Gamma_h \cap \Gamma_k = \{\emptyset\}$ we have $\mathscr{O}_{\Gamma_k}(-A - \Gamma_h) \cong \mathscr{O}_{\Gamma_k}(-A)$ and moreover we get the splitting of the exact sequence of invertible sheaves

$$0 \to \mathscr{O}_{\Gamma_k}(-A) \to \mathscr{O}_{\Gamma_h + \Gamma_k}(-A) \to \mathscr{O}_{\Gamma_h}(-A) \to 0.$$

In particular for every p, q $\mathscr{K}_{p,q}(\Gamma_h + \Gamma_k, W, \mathscr{O}_{\Gamma_h + \Gamma_k}(-A), \omega_C)$ is isomorphic to

$$\mathscr{K}_{p,q}(\Gamma_h, W, \mathscr{O}_{\Gamma_h}(-A), \omega_C) \bigoplus \mathscr{K}_{p,q}(\Gamma_k, W, \mathscr{O}_{\Gamma_k}(-A), \omega_C)$$

Now we consider Γ_k.

By Remark 28 $\mathscr{K}_{0,2}(\Gamma_k, W, \mathscr{O}_{\Gamma_k}(-A), \omega_C) \cong \mathscr{K}_{0,2}(\Gamma_k, \mathscr{O}_{\Gamma_k}(-A), \omega_{C|\Gamma_k})$.

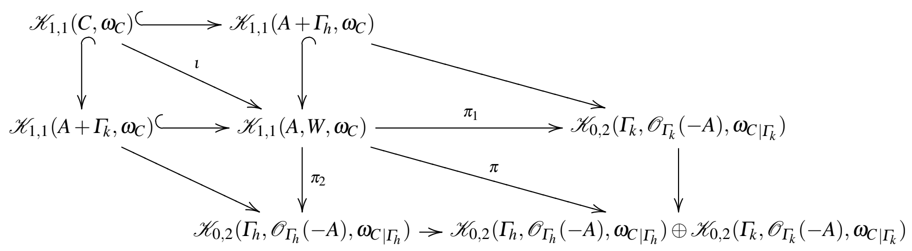
By [12, (2.a.17)] and Remark 28 we have

$$\mathscr{K}_{1,1}(\Gamma_k, W, \mathscr{O}_{\Gamma_k}(-A - \Gamma_h), \omega_C) \cong \mathscr{K}_{1,0}(\Gamma_k, W, \omega_{\Gamma_k}, \omega_C) \cong$$
$$\cong \left[\mathscr{K}_{0,0}(\Gamma_k, \omega_{\Gamma_k}, \omega_{C|\Gamma_k}) \otimes H^0(A + \Gamma_h, \omega_{A+\Gamma_h})\right] \oplus \mathscr{K}_{1,0}(\Gamma_k, \omega_{\Gamma_k}, \omega_{C|\Gamma_k}) = 0$$

since by [12, (3.a.6)] both the summands are zero because $\Gamma_k \cong \mathbb{P}^1$. By the same arguments we get $\mathscr{K}_{0,2}(\Gamma_h, W, \mathscr{O}_{\Gamma_h}(-A), \omega_C) \cong \mathscr{K}_{0,2}(\Gamma_h, \mathscr{O}_{\Gamma_h}(-A), \omega_{C|\Gamma_h})$ and $\mathscr{K}_{1,1}(\Gamma_h, W, \mathscr{O}_{\Gamma_h}(-A - \Gamma_k), \omega_C) = 0$.

Whence we obtain $\mathscr{K}_{1,1}(\Gamma_h + \Gamma_k, W, \mathscr{O}_{\Gamma_h + \Gamma_k}(-A), \omega_C)) = 0$, that is, ι is injective.

To prove the surjectivity of π, we consider the following commutative diagram:



Now π_1 is surjective since $\mathscr{K}_{0,2}(A+\Gamma_k,\omega_C)=0$. Analogously π_2 is surjective.

Moreover we can write $\pi=(\pi_1,\pi_2)$ by the above mentioned splitting and we have $\mathscr{K}_{1,1}(C,\omega_C)=\mathscr{K}_{1,1}(A+\Gamma_h,\omega_C)\cap\mathscr{K}_{1,1}(A+\Gamma_k,\omega_C)$ (that is, the space of quadrics vanishing along C is given considering the intersection of the quadrics vanishing along $A+\Gamma_h$, resp. along $A+\Gamma_h$). Therefore π is surjective, which implies $\mathscr{K}_{0,2}(C,\omega_C)=0$.

C.4. *For every irreducible subcurve Γ_i appearing with multiplicity bigger than 2 one has $\Gamma_i^2\leq 1$ and for every pair of distinct curves Γ_i,Γ_j one has $\Gamma_i\cdot\Gamma_j=1$; moreover the curve $(\Gamma_i+\Gamma_j)\subset C$ is always not disconnecting.*

C.4.1. Assume that C contains three components $\Gamma_1,\Gamma_2,\Gamma_3$ (possibly equal) such that $\Gamma_1\cdot\Gamma_2=\Gamma_2\cdot\Gamma_3=\Gamma_1\cdot\Gamma_3=1$ and $\Gamma:=\Gamma_1+\Gamma_2+\Gamma_3$ is not disconnecting. (Notice that if C has only one irreducible component, then we are exactly in this case since necessarily $\Gamma_1^2=1$ and $\mathrm{mult}_C(\Gamma_1)\geq 5$ by 4-connectedness of C).

In this case Γ is 2-connected with arithmetic genus =1, $E=C-G\neq\emptyset$ and then we can proceed as in *Case A*, since $\omega_\Gamma\cong\mathscr{O}_\Gamma$.

C.4.2. Assume that C contains three components $\Gamma_1,\Gamma_2,\Gamma_3$ (possibly equal) such that $\Gamma_1\cdot\Gamma_2=\Gamma_2\cdot\Gamma_3=\Gamma_1\cdot\Gamma_3=1$ and the curve $\Gamma=\Gamma_1+\Gamma_2+\Gamma_3$ is disconnecting.

In this case we can write $C-\Gamma_1-\Gamma_2=E+\Gamma_3+G'$ with E,G' as in Lemma 32, that is, we have a decomposition $C=E+\Gamma+G'$ with $\mathscr{O}_{G'}(-E)$ nef . Moreover $E\cdot\Gamma_3\geq 1$, $G'\cdot\Gamma_3\geq 1$ since $\Gamma_3\cong\mathbb{P}^1$ and $\Gamma_1+\Gamma_2$ is not disconnecting, and similar inequalities hold for Γ_1 and Γ_2.

Let $G=G'+\Gamma$. Since E is connected it is enough to prove that $r_1: H^0(G,\omega_C)\otimes H^0(G,\omega_G)\to H^0(G,\omega_C\otimes\omega_G)$ is onto.

Notice that for every $i\in\{1,2,3\}$ we have $\deg_{\Gamma_i}\omega_G\geq 0$ and $H^0(G,\omega_G)\twoheadrightarrow H^0(\Gamma_i,\omega_G)$.

Without loss of generality we may assume $E\cdot\Gamma_1\geq 2$ since $E\cdot(\Gamma_1+\Gamma_2+\Gamma_3)\geq 4$. We work as in *Case C.2.1*, i.e., we consider the splitting $G=\Gamma+G'$ and we take the sheaf $\omega_\Gamma(E)$. Then we have the exact sequence

$$0\to H^0(\Gamma_1,\omega_{\Gamma_1}(E))\to H^0(\Gamma,\omega_\Gamma(E))\to H^0(\Gamma_2+\Gamma_3,\omega_{\Gamma_2+\Gamma_3}(\Gamma_1+E))\to 0$$

and by the same degree arguments adopted in *Case C.2.1* it is immediately seen that we have the surjective maps

$$H^0(\Gamma_1,\omega_{\Gamma_1}(E))\otimes H^0(\Gamma_1,\omega_G)\twoheadrightarrow H^0(\Gamma_1,\omega_{\Gamma_1}(E)\otimes\omega_G)$$

$$H^0(\Gamma_2+\Gamma_3,\omega_{\Gamma_2+\Gamma_3}(\Gamma_1+E))\otimes H^0(\Gamma_2+\Gamma_3,\omega_G)\twoheadrightarrow H^0(\Gamma_2+\Gamma_3,\omega_{\Gamma_2+\Gamma_3}(\Gamma_1+E)\otimes\omega_G)$$

that is, we get the surjection $H^0(\Gamma,\omega_\Gamma(E))\otimes H^0(\omega_G)\twoheadrightarrow H^0(\Gamma,\omega_\Gamma(E)\otimes\omega_G)$. Finally, as in the previous cases $H^0(G,\omega_C)\otimes H^0(\omega_G)\twoheadrightarrow H^0(G,\omega_C\otimes\omega_G)$ since $\mathscr{O}_{G'}(-E)$ is nef, and then we can conclude that r_1 is onto.

C.4.3. Finally, we are left with the case where C has exactly two irreducible components, Γ_1,Γ_2 of nonpositive selfintersection: $C=n_1\Gamma_1+n_2\Gamma_2$, $\Gamma_1\cdot\Gamma_2=1$ and $\Gamma_i^2\leq 0$ for $i=1,2$.

We may assume $\Gamma_1^2\geq\Gamma_2^2$. Since C is 4-connected with an easy computation we obtain $\Gamma_1^2=0$ and $\mathrm{mult}_C(\Gamma_i)\geq 4$. Moreover n_2 is even since C is an even curve.

Notice that for every subcurve $B=\alpha_1\Gamma_1+\alpha_2\Gamma_2\subset C$

$$B\cdot(C-B)=\alpha_2(n_1+(n_2-1)\Gamma_2^2)-\alpha_2(\alpha_2-1)\Gamma_2^2+\alpha_1(n_2-2\alpha_2)\geq\alpha_2(n_1+(n_2-1)\Gamma_2^2)$$

(since we may assume $2\alpha_2\leq n_2$ by the symmetry of the intersection product), which implies $B\cdot(C-B)\geq 4\alpha_2$ because $\Gamma_2\cdot(C-\Gamma_2)=n_1+(n_2-1)\Gamma_2^2\geq 4$.

If $\Gamma_2^2 = 0$, we take $E = 2\Gamma_1 + 2\Gamma_2$. Then $p_a(E) = 1$ and applying a refinement of the above formula it is easy to see that $C - E$ is numerically connected. In this case we can conclude as in *Case A*.

If $\Gamma_2^2 < 0$, let $a_1 = \lceil \frac{n_1}{2} \rceil$, $a_2 = \frac{n_2}{2}$ and let $G := a_1\Gamma_1 + a_2\Gamma_2$, $E := C - G = (n_1 - a_1)\Gamma_1 + a_2\Gamma_2$.

Now E is numerically connected and G is 2-connected. Indeed, let us consider a subcurve $B = \alpha_1\Gamma_1 + \alpha_2\Gamma_2 \subset G$. Since $2G \cdot B \geq G \cdot B$ we have $B \cdot (G - B) \geq \frac{1}{4} 2B \cdot (C - 2B) \geq 2$ since $2B \cdot (C - 2B) \geq 8\alpha_2$ by the above formula, whereas if $B \subset E$ then $B \cdot (E - B) \geq \frac{1}{4} 2B \cdot (C - \Gamma_1 - 2B) \geq 1$.

Therefore it is enough to prove the surjection of

$$r_1 : H^0(G, \omega_C) \otimes H^0(G, \omega_G) \to H^0(G, \omega_G \otimes \omega_C).$$

We have the following exact sequence

$$0 \to H^0(a_1\Gamma_1 + \Gamma_2, \omega_{a_1\Gamma_1+\Gamma_2}(E)) \to H^0(G, \omega_C) \to H^0((a_2 - 1)\Gamma_2, \omega_C) \to 0$$

and moreover $H^0(G, \omega_G) \twoheadrightarrow H^0((a_2 - 1)\Gamma_2, \omega_G)$ since $a_1\Gamma_1 + \Gamma_2$ is numerically connected.

By [6, Thm. 3.3] $|\omega_G|$ is a base point free system on G since G is 2-connected. Let $W := \mathrm{im}\{H^0(G, \omega_G) \to H^0(a_1\Gamma_1 + \Gamma_2, \omega_G)\}$. Then W is a base point free system and moreover we have $H^1(a_1\Gamma_1 + \Gamma_2, \omega_{a_1\Gamma_1}(E) \otimes \omega_G{}^{-1}) \cong H^1(a_1\Gamma_1 + \Gamma_2, E - (a_2 - 1)\Gamma_2) = 0$ because $\Gamma_i \cong \mathbb{P}^1$, $\Gamma_1^2 = 0$, $\Gamma_1 \cdot \Gamma_2 = 1$ and $(E - (a_2 - 1)\Gamma_2) \cdot \Gamma_1 = 1$, $(E - (a_2 - 1)\Gamma_2) \cdot \Gamma_2 \geq 1$ since E is 1-connected. Therefore by Prop. 25 we have the surjection

$$H^0(a_1\Gamma_1 + \Gamma_2, \omega_{a_1\Gamma_1+\Gamma_2}(E)) \otimes W \twoheadrightarrow H^0(a_1\Gamma_1 + \Gamma_2, \omega_{a_1\Gamma_1+\Gamma_2}(E) \otimes \omega_G)$$

Finally $H^0((a_2 - 1)\Gamma_2, \omega_G) \otimes H^0((a_2 - 1)\Gamma_2, \omega_C) \twoheadrightarrow H^0((a_2 - 1)\Gamma_2, \omega_G \otimes \omega_C)$ follows from (*i*) of Prop. 25 taking $\mathscr{H} = \omega_C, \mathscr{F} = \omega_G$ if $\mathscr{O}_{\Gamma_2}(E)$ is nef , or $\mathscr{F} = \omega_C, \mathscr{H} = \omega_G$ if $\mathscr{O}_{\Gamma_2}(-E)$ is nef .

Q.E.D. for Theorem 11

5 On the canonical ring of regular surfaces

In this section we prove Theorem 12. The arguments we adopt are very classical and based on the ideas developed in [8]. Essentially we simply restrict to a curve in the canonical system $|K_S|$. The only novelty is that now we do not make any requests on such a curve (i.e. we allow the curve $C \in |K_S|$ to be singular and with many components) since we can apply Thm. 11.

Proof of Theorem 12.

By assumption there exists a 3-connected not honestly hyperelliptic curve $C = \sum_{i=1}^s n_i\Gamma_i \in |K_S|$. Let $s \in H^0(S, K_S)$ be the corresponding section, so that C is defined by $(s) = 0$.

By adjunction we have $(K_S^{\otimes 2})_{|C} = (K_S + C)_{|C} \cong \omega_C$, that is, C is an even curve; in particular it is 4-connected. Thus we can apply Theorem 11, obtaining the surjection

$$(H^0(C, K_S^{\otimes 2}))^{\otimes k} \twoheadrightarrow H^0(C, K_S^{\otimes 2k}) \quad \forall k \in \mathbb{N}.$$

Now let us consider the usual maps given by multiplication of sections

$$A_{l,m} : H^0(S, K_S^{\otimes l}) \otimes H^0(S, K_S^{\otimes m}) \to H^0(S, K_S^{\otimes(l+m)})$$
$$a_{l,m} : H^0(C, K_S^{\otimes l}) \otimes H^0(C, K_S^{\otimes m}) \to H^0(C, K_S^{\otimes(l+m)})$$

and consider the following commutative diagram

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H^0(S, K_S^{\otimes(k-1)}) & \xrightarrow{\cong} & H^0(S, K_S^{\otimes(k-1)}) \\
\downarrow{\scriptstyle C_k} & & \downarrow{\scriptstyle c_k} \\
\bigoplus\limits_{\substack{l+m=k \\ 0<l\le m}} \left[H^0(S, K_S^{\otimes l}) \otimes H^0(S, K_S^{\otimes m})\right] & \xrightarrow{\overline{\rho}_k} & H^0(S, K_S^{\otimes k}) \\
\downarrow{\scriptstyle R_k} & & \downarrow{\scriptstyle r_k} \\
\bigoplus\limits_{\substack{l+m=k \\ 0<l\le m}} \left[H^0(C, K_S^{\otimes l}) \otimes H^0(C, K_S^{\otimes m})\right] & \xrightarrow{\rho_k} & H^0(C, K_S^{\otimes k}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Here the left hand column is a complex, while the right hand column is exact. Moreover

- C_k is given by tensor product with s while c_k is given by product with s
- $R_k = \bigoplus\limits_{\substack{l+m=k \\ 0<l\le m}} r_l \otimes r_m$ (where r_l, r_m are the usual restrictions)
- $\overline{\rho}_k = \bigoplus\limits_{\substack{l+m=k \\ 0<l\le m}} A_{l,m}$ and $\rho_k = \bigoplus\limits_{\substack{l+m=k \\ 0<l\le m}} a_{l,m}$

Note that $\mathrm{coker}(\rho_k) \cong \mathrm{coker}(\overline{\rho}_k)$ for every $k \in \mathbb{N}$.

Now, for S of general type, if $p_g \geq 1$ and $q = 0$ by [8, Thm. 3.4] $\overline{\rho}_k$ is surjective for every $k \geq 5$ except the case $p_g = 2$, $K^2 = 1$, which is not our case since otherwise C would be a curve of genus 2 contradicting our assumptions. For $k = 4$ the map $a_{2,2}$ is surjective by Thm. 11. Whence ρ_4 and $\overline{\rho}_4$ are surjective, and this proves the theorem.

Q.E.D. for Theorem 12

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