On the connections between semidefinite optimization and vector optimization

Laura Carosi[*] Johannes Jahn[†] Laura Martein[*]

Abstract

This paper works out connections between semidefinite optimization and vector optimization. It is shown that well-known semidefinite optimization problems are scalarized versions of a general vector optimization problem. This scalarization leads to the minimization of the trace or the maximal eigenvalue.

Mathematics Subject Classification 2000: 90C22, 90C29

1 Introduction

Semidefinite optimization is currently a rapidly growing branch of mathematical programming (for instance, see [1], [15], [12] and [17]). Special optimization problems in statistics (e.g., see [16] and [3]), structural optimization (e.g., see [15] and [2]) and combinatorial optimization (e.g., see [1], [15], [4] and [5]) lead to semidefinite optimization problems. It is the aim of this paper to work out the connections between semidefinite optimization and vector optimization. For instance, it is shown in this paper that special semidefinite optimization problems are scalarized vector optimization problems. Therefore, vector optimization seems to be an important tool for semidefinite optimization. For the formulation of a semidefinite optimization problem in a standard form we consider the finite dimensional Hilbert space $\mathcal{H}$ of all symmetric real (n, n) matrices $(n \in N)$ with the scalar product

$$\langle X, Y \rangle := \text{ trace } (X \cdot Y) \quad \text{for all } X, Y \in \mathcal{H}.$$

[*]Department of Statistics and Applied Mathematics,University of Pisa, Via Cosimo Ridolfi, 10, 56124 Pisa, Italy, e-mail: lcarosi@ec.unipi.it, lmartein@ec.unipi.it

[†]Institute of Applied Mathematics, University of Erlangen-Nŭrnberg, Martensstr. 3, 91058 Erlangen, Germany, e-mail:~jahn@am.uni-erlangen.de

The natural ordering cone in this Hilbert space is defined as

$$C := \{X \in \mathcal{H} \quad | \quad X \text{ positive semidefinite}\}.$$

This pointed convex cone induces a partial ordering $\leq_c$ in $\mathcal{H}$. The ordering cone C has a rich mathematical structure. Recall that it is self dual, i.e. the dual cone C^* equals C; its interior is given by

$$\text{int}(C) := \{X \in \mathcal{H} \quad | \quad X \text{ positive definite}\};$$

the quasi-interior of C^*

$$C^\# := \{X \in \mathcal{H} \quad | \quad \langle X, Y \rangle > 0 \quad \text{for all } Y \in C \setminus \{O_\mathcal{H}\}\}$$

is nonempty (for instance, the identity matrix I belongs to $C^\#$ because

$$\begin{array}{rcl} \langle I, Y \rangle & = & \text{trace } (I \cdot Y) \\ & = & \text{trace } (Y) \\ & = & \sum \text{ eigenvalues } (Y) \\ & > 0 & \text{ for all } Y \in C \setminus \{0_\mathcal{H}\}; \end{array}$$

and, therefore, a base B for C (e.g., see [7, Def. 1.10,d)]) is given by

$$\begin{array}{rl} B & = \{X \in C \quad | \quad \langle I, X \rangle = 1\} \\ & = \{X \in C \quad | \quad \sum \text{ eigenvalues } (X) = 1\} \end{array}$$

(e.g., see [7, Lemma 3.3]). For these properties we refer to [6, 11].

Next, we assume that S is a subset of the Hilbert space $\mathcal{H}$ with the property $S \cap C \neq \emptyset$ and $f : S \cap C \longrightarrow \Re$ is a given objective function. Then a semidefinite optimization problem in standard form can be written as

$$\min_{X \in S \cap C} f(X). \tag{1.1}$$

The set S represents a part of the constraint set being possibly defined by inequality and/or equality constraints; the additional condition $X \in C$ means that the matrix X is positive semidefinite.

If the set S describes a set of covariance matrices (from statistics), then the problem of the determination of minimal matrices (see Definition 2.1) of $S \cap C$ is a special vector optimization problem (see [16]) being closely related to problem (1.1) (in fact, in this case the real-valued objective in (1.1) is replaced by the vector-valued objective X).

2 Minimization of the trace

In this section we consider the semidefinite optimization problem (1.1) with the special objective function $f : S \cap C \longrightarrow \Re$ given by

$$f(X) = \text{ trace } (X) \quad \text{ for all } X \in S \cap C.$$

Recall the standard definition of minimality known from vector optimization (e.g., see [7, Definition 4.1]).

Definition 2.1 A matrix $\bar{X}$ is called a *minimal* element of the set $S \cap C$ if

$$(\{\bar{X}\} - C) \cap S \cap C = \{\bar{X}\}$$

or, equivalently

$$X \leq_C \bar{X}, \quad X \in S \cap C \quad \Longrightarrow \quad X = \bar{X}.$$

Then we obtain the following result.

Proposition 2.1 *Every solution of the semidefinite optimization problem*

$$\min_{X \in S \cap C} \ \textit{trace } (X) \tag{2.1}$$

is a minimal element of the set $S \cap C$.

Proof. For every $X \in \mathcal{H}$ the trace of X can be written as

$$\text{trace } (X) = \text{ trace } (I \cdot X) = \langle I, X \rangle.$$

Since $I \in C^{\#}$, a standard scalarization result known from vector optimization (e.g., see [7, Thm. 5.18, b]) leads to the assertion. ■

The preceding proposition shows an essential property of solutions of the special semidefinite optimization problem (2.1): they are minimal matrices. Hence, Problem (2.1) is actually an auxiliary problem for the solution of the vector optimization problem:

$$\text{Determine a minimal element of the set } S \cap C. \tag{2.2}$$

Problem (2.1) is also used in statistics for determination of minimal covariance matrices (see [16]). Now we turn our attention to a linear problem. We consider the following semidefinite program in equality standard form

$$\begin{array}{rl} \min & \langle X, Q_0 \rangle \\ \langle X, Q_i \rangle & = c_i, \quad i = 1, \ldots, m \\ X & \geq_C 0_{\mathcal{H}} \end{array} \tag{2.3}$$

where Q_0, Q_i are symmetric matrices. Let us set $S = \{X \in \mathcal{H} \mid \langle X, Q_i \rangle = c_i,\ i = 1, \ldots, m\}$. The following theorem points out the relationship between Problem (2.3) and the problem of finding the minimal matrices of $S \cap C$ (see Definition 2.1).

Proposition 2.2 *(a) If $\bar{X}$ is the unique optimal solution to the Problem (2.3) then $\bar{X}$ is a minimal element of the set $S \cap C$.*
(b) Consider Problem (2.3) where Q_0 is positive definite. Then every solution of (2.3) is a minimal point of the set $S \cap C$.

Proof. (a) Suppose on the contrary that $\bar{X}$ is not minimal. Then there exists $Z \in S$ such that $Z = \bar{X} - A$, with positive semidefinite A and $A \neq 0_{\mathcal{H}}$. $Z \in S$ implies $\langle Z, Q_i \rangle = \langle \bar{X}, Q_i \rangle - \langle A, Q_i \rangle = c_i$ for every $i = 1, ..., m$, so that since $\bar{X}$ is feasible and $\langle \bar{X}, Q_i \rangle = c_i$ we get $\langle A, Q_i \rangle = 0$ for every $i = 1, ..., m$. It follows that $Z_1 = \bar{X} + A$ is feasible since it is the sum of two positive semidefinite matrices and $\langle Z_1, Q_i \rangle = \langle \bar{X}, Q_i \rangle + \langle A, Q_i \rangle = c_i$. The optimality of $\bar{X}$ implies $\langle X, Q_0 \rangle \geq \langle \bar{X}, Q_0 \rangle$ for every $X \in S$. As particular cases we get

$$\begin{array}{l} \langle Z, Q_0 \rangle = \langle \bar{X} - A, Q_0 \rangle = \langle \bar{X}, Q_0 \rangle - \langle A, Q_0 \rangle \geq \langle \bar{X}, Q_0 \rangle \\ \langle Z_1, Q_0 \rangle = \langle \bar{X} + A, Q_0 \rangle = \langle \bar{X}, Q_0 \rangle + \langle A, Q_0 \rangle \geq \langle \bar{X}, Q_0 \rangle \end{array}$$

Consequently $\langle A, Q_0 \rangle = 0$ and so $Z_1 = \bar{X} + A$ is optimal and this contradicts the uniqueness of the solution for Problem (2.3).
(b) With the same argument of (a), if there exists $Z \in S$ such that $Z = \bar{X} - A$, with positive semidefinite A and $A \neq 0_{\mathcal{H}}$, then it results $\langle A, Q_0 \rangle = 0$ and this cannot be true since $\langle A, Q_0 \rangle > 0$, being Q_0 positive definite. ■

3 Minimization of the maximal eigenvalue

Again, we consider the semidefinite optimization problem (1.1), but now the objective function $f : S \cap C \longrightarrow \Re$ is given by

$$f(X) = \text{ max. eigenvalue } (X) \quad \text{ for all } X \in S \cap C.$$

Problems of this type play an important role in semidefinite optimization (e.g., see [14], [15] and [9]). In order to work out the connection between this special semidefinite optimization problem and the vector optimization problem (2.2) we recall the known definition of weakly minimal elements (e.g., see [7, Def. 4.12]).

Definition 3.1 A matrix $\bar{X}$ is called a *weakly minimal* element of the set $S \cap C$ if

$$(\{\bar{X}\} - \text{ int } (C)) \cap S \cap C = \emptyset.$$

It is well-known that every minimal matrix of $S \cap C$ is also weakly minimal but the converse is not true, in general.

Proposition 3.1 *Every solution of the semidefinite optimization problem*

$$\min_{X \in S \cap C} \textit{max. eigenvalue } (X) \tag{3.1}$$

is a weakly minimal element of the set $S \cap C$. Every unique solution of (3.1) is a minimal element of the set $S \cap C$.

Proof. For every $X \in C$ the maximal eigenvalue of X equals the spectral norm $\| X \|$. Therefore, problem (3.1) can be written as

$$\min_{X \in S \cap C} \| X \|$$

being equivalent to the problem

$$\min_{X \in S \cap C} \| X + I \| \tag{3.2}$$

(actually the spectrum of X is shifted by 1). So, a solution $\bar{X}$ of (3.1) solves problem (3.2) as well. Applying Cor. 3.2 in [8] to this case $\bar{X}$ is a weakly minimal element of the set $S \cap C$, and it is also a minimal element, if it is uniquely determined. ■

Consequently, the semidefinite optimization problem (3.1) is an auxiliary problem for the vector optimization problem (2.2), and solutions of (3.1) can be interpreted as minimal or at least as weakly minimal elements of the set $S \cap C$.

Next, we investigate the question whether it is possible to characterize an arbitrary minimal or weakly minimal element of the set $S \cap C$ as a solution of a semidefinite optimization problem, or in other words: which class of semidefinite optimization problems can generate the set of minimal or weakly minimal elements of the set $S \cap C$?

Proposition 3.2 *Let $\alpha > 0$ be an arbitrarily chosen number.*

(a) If $\bar{X}$ is a weakly minimal element of the set $S \cap C$, then there is a positive definite matrix $A \in \mathcal{H}$ so that $\bar{X}$ is a solution of the semidefinite optimization problem

$$\min_{X \in S \cap C} \quad \textit{max. eigenvalue } (A^{-1}(X + \alpha\, I)). \tag{3.3}$$

(b) If $\bar{X}$ is a minimal element of the set $S \cap C$, then there is a positive definite matrix $A \in \mathcal{H}$ so that $\bar{X}$ is a unique solution of problem (3.3).

Proof. (a) If $\bar{X}$ is a weakly minimal element of the set $S \cap C$, by [8, Cor. 3.2] (where I is replaced by $\alpha\ I$) there is a positive definite matrix $A \in \mathcal{H}$ so that $\bar{X}$ solves

$$\min_{X \in S \cap C} \parallel X + \alpha\ I \parallel_A \tag{3.4}$$

where $\| \cdot \|_A$ denotes the wheighted spectral norm

$$\parallel Y \parallel_A := \sup_{x \neq 0_{\Re^n}} \left\{ \frac{x^T Y x}{x^T A x} \right\} \qquad \text{for all } Y \in \mathcal{H}.$$

If we define $\varphi : \Re^n \backslash \{0_{\Re^n}\} \longrightarrow \Re$ for an arbitrary $Y \in \mathcal{H}$ by

$$\varphi(x) := \frac{x^T Y x}{x^T A x} \qquad \text{for all } x \in \Re^n \backslash \{0_{\Re^n}\},$$

then

$$\begin{aligned} \nabla\varphi(x) &= \tfrac{1}{(x^T A x)^2}(2Yxx^T Ax - x^T Yx2Ax) \\ &= \tfrac{2}{x^T A x}(Yx - \tfrac{x^T Y x}{x^T A x} Ax) \\ &= \tfrac{2}{x^T A x}(Yx - \varphi(x)Ax) \text{ for all } x \in \Re^n \backslash \{0_{\Re^n}\}, \end{aligned}$$

and the equation $\nabla\varphi(x) = 0_{\Re^n}$ is equivalent to the eigenvalue equation

$$A^{-1} Y x = \varphi(x) x.$$

Consequently, we obtain

$$\begin{aligned} \parallel Y \parallel_A &= \sup_{x \neq 0_{\Re^n}} \varphi(x) \\ &= \text{max. eigenvalue } (A^{-1} Y). \end{aligned}$$

Since $\bar{X}$ solves (3.4), we then conclude that $\bar{X}$ also solves problem (3.3).

(b) If $\bar{X}$ is a minimal element of the set $S \cap C$, by [8, Cor. 3.2, (a)] there is a positive definite matrix $A \in \mathcal{H}$ so that

$$\parallel \bar{X} + \alpha\ I \parallel_A < \parallel X + \alpha\ I \parallel_A \qquad \text{for all } X \in S \cap C \qquad \text{with } X \neq \bar{X}.$$

Here $\| \cdot \|_A$ again denotes the wheighted spectral norm. So, $\bar{X}$ is a unique solution of problem (3.4). This leads to the assertion. ■

The converse of the implication in Proposition 3.2 immediately follows from [8, Cor. 3.2] and the characterization of the wheighted spectral norm given in the previous proof.

Proposition 3.3 *Let $\alpha > 0$ be an arbitrarily chosen number, and let $A \in \mathcal{H}$ be an arbitrarily chosen positive definite matrix.*

(a) Every solution of problem (3.3) is a weakly minimal element of the set $S \cap C$.

(b) Every unique solution of problem (3.3) is a minimal element of the set $S \cap C$.

Remark 3.1 (a) In the Propositions 3.2 and 3.3 the positive real number α can be chosen arbitrarily small. Therefore, for calculations on a computer it makes sense to set $\alpha = 0$ and to work with the simpler problem

$$\min_{x \in S \cap C} \text{ max. eigenvalue } (A^{-1}X). \tag{3.5}$$

In this case we see that (3.1) is a special problem of the afore-mentioned form (simply set $A = I$). Hence, from the point of view of vector optimization it seems to be better to work with problem (3.5) for an arbitrary positive definite matrix A instead of (3.1).

(b) The proof of Proposition 3.2, (a) shows that for an arbitrary $\alpha > 0$ the objective function of problem (3.4) can be written as

$$\begin{aligned} \| X + \alpha\ I \|_A &= \text{ max. eigenvalue } (A^{-1}(X + \alpha\ I)) \\ &= \| A^{-1}(X + \alpha\ I) \| \quad \text{ for all } X \in S \cap C. \end{aligned}$$

Here $\| \cdot \|$ again denotes the (unwheighted) spectral norm. Then we get

$$\begin{aligned} \| A^{-1}(X + \alpha\ I) \| &\leq \| A^{-1} \| \cdot \| X + \alpha\ I \| \\ &= \| A^{-1} \| \cdot \| X \| \text{ for all } X \in S \cap C. \end{aligned}$$

Consequently, we obtain an upper bound for the minimal value of problem (3.4) being independent of $\alpha > 0$, i.e.

$$\min_{X \in S \cap C} \| X + \alpha\ I \|_A \leq \| A^{-1} \| \cdot \min_{X \in S \cap C} \| X \| \qquad \text{for all } \alpha > 0.$$

4 Eigenvalue transformation

We now consider the semidefinite optimization problem (1.1) in the very special form

$$\left.\begin{array}{l} \min\ f(\lambda_i(X)) \\ \text{subject to the constraints} \\ (\lambda_1(X), \ldots, \lambda_n(X)) \in \tilde{S} \\ X \geq_C 0_{\mathcal{H}} \end{array}\right. \tag{4.1}$$

$(\lambda_1(X), \ldots, \lambda_n(X)$ denote the eigenvalues of X, and now $f : \Re^n_+ \longrightarrow \Re$ and $\emptyset \neq \tilde{S} \subset \Re^n$). In this case we consider a transformation from the Hilbert space $\mathcal{H}$ to the eigenvalue coordinate system being identical with the $\Re^n$ space.

For every $X \in \mathcal{H}$ satisfying the constraints of problem (4.1) we have for the eigenvalues

$$(\lambda_1(X), \ldots, \lambda_n(X)) \in \tilde{S}$$

and

$$\lambda_1(X), \ldots, \lambda_n(X) \geq 0.$$

Then the problem (4.1) is equivalent to the standard optimization problem in $\Re^n$

$$\begin{array}{l} \min f(\lambda_1, \ldots, \lambda_n) \\ \text{subject to the constraints} \\ (\lambda_1, \ldots, \lambda_n) \in \tilde{S} \\ \lambda_1, \ldots, \lambda_n \geq 0. \end{array}$$

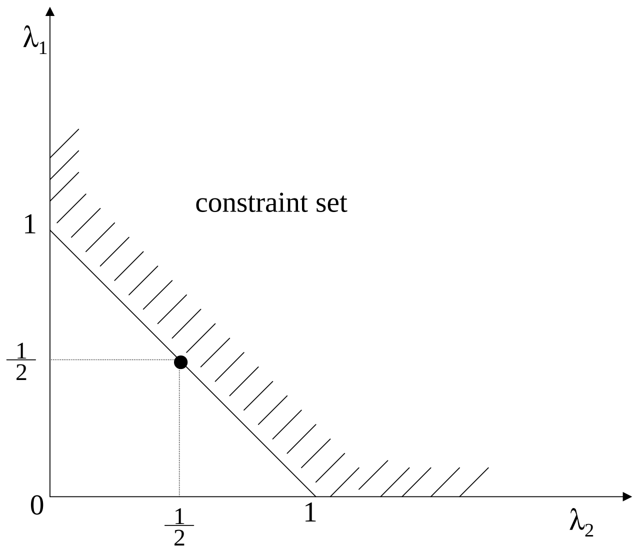
Example 4.1 Consider the special semidefinite optimization problem

$$\begin{array}{l} \min\ \text{max. eigenvalue } (X) \\ \text{subject to the constraints} \\ \text{trace } (X) \geq 1 \\ X \geq_C 0_{\mathcal{H}}. \end{array} \tag{4.2}$$

This problem is equivalent to

$$\begin{array}{l} \min \max\limits_{1 \leq i \leq n} \{\lambda_i\} \\ \text{subject to the constraints} \\ \lambda_1 + \cdots + \lambda_n \geq 1 \\ \lambda_1, \ldots, \lambda_n \geq 0. \end{array} \tag{4.3}$$

Obviously, this problem has the unique solution $\left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. The following picture illustrates the constraint set for $n = 2$.



So, every positive semidefinite symmetric matrix with the eigenvalues $\frac{1}{n}, \ldots, \frac{1}{n}$, e.g.

$$\bar{X} := \begin{pmatrix} \frac{1}{n} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{n} \end{pmatrix},$$

is a solution of problem (4.2). By Proposition 3.1 this matrix is a minimal element of the set $S \cap C$.

5 Conclusion

This paper shows that standard semidefinite optimization problems are special scalarized versions of a general vector optimization problem. This theoretical connection between semidefinite optimization and vector optimization shows concrete applications in a favourable light because one can also interpret solutions as minimal matrices among a constrained set of matrices.

References

[1] F. Alizadeh, Interior point methods in semidefinite programming with applications to combinatorial optimization, SIAM J. Optim. 5 (1995) 13 - 51.

[2] A. Ben-Tal and A. Nemirovski, Structural design, in: [17], 443 - 468.

[3] V. Fedorov and J. Lee, Design of experiments in statistics, in: [17], 511 - 532.

[4] M. X. Goemans, Semidefinite programming in combinatorial optimization, Math. Programming 79 (1997) 143 - 161.

[5] M. Goemans and F. Rendl, Combinatorial optimization, in: [17], 343 - 360.

[6] B. Iochum, Cônes autopolaires et algèbres de Jordan (Lecture Notes in Mathematics 1049, Springer, Berlin, 1984).

[7] J. Jahn, Mathematical vector optimization in partially ordered linear spaces (Peter Lang, Frankfurt, 1986).

[8] J. Jahn, Parametric approximation problems arising in vector optimization, J. Optim. Theory Appl. 54 (1987) 503 - 516.

[9] F. Jarre, Eigenvalue problems and nonconvex minimization, in: [17], 547 - 562.

[10] J. R. Magnus, H. Neudeker, Matrix Differential Calculus with Applications in Statistics and Econometrics, (J. Wiley & Sons, 1994)

[11] M. Petschke, Ein Maximumprinzip für konvexe vektorwertige Funktionen (manuscript, Technical University of Darmstadt, Germany, 1986).

[12] F. Potra, C. Roos and T. Terlaky, Special issue on interior point methods, Optimization Meth. & Soft. 11 & 12 (1999) 1 - 690.

[13] C.R. Rao, M.B.Rao, Matrix Algebra and its Applications to Statistics and Econometrics, (World Scientific, 1998)

[14] A. Shapiro and M. K. H. Fan, On eigenvalue optimization, SIAM J. Optim. 5 (1995) 552 - 569.

[15] L. Vandenberghe and S. Boyd, Semidefinite programming, SIAM Review 38 (1996), 49 - 95.

[16] W. Vogel, Vektoroptimierung in Produkträumen (Anton Hain, Meisenheim am Glan, 1977).

[17] H. Wolkowicz, R. Saigal and L. Vandenberghe (eds.), Handbook on semidefinite programming (Kluwer, 2000).