Quasi-variational equilibrium models for network flow problems

G. Mastroeni[1] and M. Pappalardo[2]

Abstract. We consider a formulation of a network equilibrium problem given by a suitable quasi-variational inequality where the feasible flows are supposed to be dependent on the equilibrium solution of the model. The Karush-Kuhn-Tucker optimality conditions for this quasi-variational inequality allow us to consider dual variables, associated with the constraints of the feasible set, which may receive interesting interpretations in terms of the network, extending the classic ones existing in the literature.

Keywords: Network flows, quasi-variational inequalities, equilibrium problems, Karush-Kuhn-Tucker multipliers.

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1 Introduction

In the context of traffic networks we consider a model where the equilibrium flow is given by the solution of a quasi-variational inequality (QVI).
In general, optimal flows are defined as the minimizers of a suitable global cost functional, which sometimes is hard to be determined since it requires a deterministic knowledge of the behaviour of the users. On the contrary, the optimal flows of a variational model are defined as the solutions of a variational inequality which do not necessarily coincide with the minimum of the cost functional. Variational models give rise to more general kinds of equilibria which reflect the dynamic behaviour of the users and, for this reason, have been considered by several authors [2, 3, 4, 5, 8, 11, 12, 14, 15, 17].

The dependence of the costs of the arcs on the flows, is widely considered in the literature as well as models with elastic demands or elastic travel time in dynamic traffic assignment problems (see e.g. [6, 10, 3]); indeed, it is well known that very often the traffic demand and the travel time are influenced by the actual conditions of the network and therefore must be considered as a function of the equilibrium solution. Models with elastic data are formulated, in our context, by means of a QVI where the feasible set very generally depends

[1]Department of Computer Science, University of Pisa, Largo B.Pontecorvo 2, 56127 PISA, Italy.
Tel.: +39-50-2212708
Fax: +39-50-2212726
email: gmastroeni@di.unipi.it
[2]Department of Computer Science, University of Pisa, Largo B.Pontecorvo 2, 56127 PISA, Italy.
Tel.: +39-50-2212750
Fax: +39-50-2212726
email: massimo.pappalardo@unipi.it

on the equilibrium solution. Following this line, in this paper we consider a more general case than those existing in the literature, assuming that even the capacities of the flows on the arcs or on the paths of the network depend on the equilibrium flows. In fact, if a path is very crowded, it is reasonable that its capacity should be reduced according to the value of the equilibrium solution in order to guarantee an efficient circulation on the network. We analyse in details the capacity constraints and the role of the associated Lagrange multipliers which may lead to equivalent unconstrained reformulations of the considered models.

The paper is organized as follows. Section 2 is devoted to preliminary definitions and results concerning QVI and traffic networks. In Section 3, we consider the arc-flow quasi-variational model formulated by means of a QVI where the operator represents the cost associated with the arcs of the network. We show that the dual variables associated with the conservation flow constraints can be interpreted in terms of potentials associated with the nodes of the network, while the multipliers associated with the capacity constraints on the arcs can be considered as an additional cost to be added to the given one in order to achieve the equality with the difference of potentials at the nodes, which provides a generalization of the classic results existing in the literature.

In Section 4 we present the path-flow model, where the variables represent the flows on the paths of the network and show the relationships with the arc-flow model. In Section 5 we consider a reformulation of a model with capacity constraints by means of a model without capacities, performing a suitable perturbation of the operator depending on the Karush-Kuhn-Tucker (KKT) multipliers associated with the capacity constraints. The models presented in this paper are a generalization of those considered in [12] which can be recovered in case the feasible set does not depend on the solution of the problem. Section 6 is devoted to concluding remarks with a brief mention to computational methods for QVI.

2 Preliminaries

Given a network $G = (N, A)$, where $N := \{1, .., p\}$ is the set of nodes and $A := \{A_1, .., A_n\}$ is the set of the arcs, we will consider two models: in the first one called the arc-flow model the variables are the flows on the arcs, while in the second one, called the path-flow model, the variable are the flows on the paths. In the first case we will consider the following assumptions and notations:

- f_i is the flow on the arc $A_i := (r, s), r, s \in N,\ i = 1, .., n$.
- $f := (f_1, \dots, f_n)^T$ is the vector of the flows on all arcs.
- We assume that each arc A_i is associated with an upper bound $d_i(f)$ on its capacity which may depend on the flow f, and we set $d(f) := (d_1(f), \dots, d_n(f))$.

- $c_i(f)$ is the cost of the arc A_i and $c(f) := (c_1(f), \dots, c_n(f))^T$; we assume that $c(f) \geq 0$.
- $q_j(f)$ is the balance at the node j, $j = 1, \dots, p$, which may depend on the flow f, and $q(f) := (q_1(f), \dots, q_p(f))^T$.
- $\Gamma = (\gamma_{ij}) \in \mathbb{R}^p \times \mathbb{R}^n$ is the node-arc incidence matrix whose elements are

$$\gamma_{ij} = \begin{cases} -1, & \text{if } i \text{ is the initial node of the arc } A_j, \\ +1, & \text{if } i \text{ is the final node of the arc } A_j, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

The following notations will be used in the path-flow model:

- m is the total number of the considered paths and $F := (F_1, \dots, F_m)^T$ is the vector of the relative flows.
- We will suppose that, the nodes of the couple origin-destination W_j are connected by the (oriented) paths, R_i, $i \in P_j \subseteq \{1, \dots, m\}$, $\forall j = 1, \dots, \ell$.
- $C_i(F)$ is the cost on the path R_i, $i = 1, .., m$, and $C(F) := (C_1(F), ..., C_m(F))$.
- $\rho_j = \rho_j(F)$ is the traffic demand for W_j, $j = 1, \dots, \ell$, which may depend on the flow F, and $\rho(F) := (\rho_1(F), \dots, \rho_\ell(F))^T$.
- $\Phi = (\phi_{ij}) \in \mathbb{R}^\ell \times \mathbb{R}^m$ is the couples-paths incidence matrix whose elements are

$$\phi_{ij} = \begin{cases} 1, & \text{if } W_i \text{ is connected by the path } R_j, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

- $D_i = D_i(F)$ is the capacity on the path R_i, $i = 1, ..m$, and $D(F) := (D_1(F), \dots, D_m(F))$.

We refer to Examples 1 and 2 in sections 3 and 4, respectively, for a concrete explanation of the notation related to the previous models.

As mentioned in the introduction, we will consider a quasi-variational model for a traffic equilibrium problem. We recall that, in a general setting, a quasi-variational inequality consists in finding $x^* \in K(x^*) := \{x \in \mathbb{R}^n : g(x, x^*) \leq 0,\ h^*(x, x^*) = 0\}$, such that

$$\langle \Omega(x^*), x - x^* \rangle \geq 0, \quad \forall x \in K(x^*) \qquad QVI(\Omega, K)$$

where $\Omega : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $g : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $h : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^p$. When $K(x^*) = K$ is a fixed set for any $x^* \in \mathbb{R}^n$, then $QVI(\Omega, K)$ reduces to the classic variational inequality (VI).

In our setting, given the traffic network G, the equilibrium flow is given by the solution of a QVI where Ω is the cost functional associated with the arcs or with the paths of the network, g and h are the capacity and the flow conservation constraints of the network.

In particular, for the arc-flow model given a flow $f^* \in \mathbb{R}^n$ and setting in $QVI(\Omega, K)$

$$\Omega(f) := c(f), \quad g(f, f^*) := (f - d(f^*), -f), \quad h(f, f^*) := \Gamma f - q(f^*),$$

then the feasible set is given by $K(f^*) := \{f \in \mathbb{R}^n \ : \ \Gamma f = q(f^*), \ 0 \leq f \leq d(f^*)\}$, and the equilibrium condition is formulated by $QVI(c, K)$:

$$\text{find } f^* \in K(f^*) \ \text{ s.t. } \ \langle c(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K(f^*).$$

As regards the path-flow model, given a flow $F^* \in \mathbb{R}^m$, replacing in $QVI(\Omega, K)$ f with F and setting

$$\Omega(F) := C(F), \quad g(F, F^*) := (F - D(F^*), -F), \quad h(F, F^*) := \Phi F - \rho(F^*),$$

then the feasible path flow set is defined by $K(F^*) := \{F \in \mathbb{R}^m \ : \ \Phi F = \rho(F^*), \ 0 \leq F \leq D(F^*)\}$, and the equilibrium condition of the quasi-variational path flow model is given by $QVI(C, K)$:

$$\text{find } F^* \in K(F^*) \ \text{ s.t. } \ \langle C(F^*), F - F^* \rangle \geq 0, \quad \forall F \in K(F^*).$$

Similarly to constrained extremum problems, we can associate with $QVI(\Omega, K)$ KKT-type optimality conditions in order to obtain primal-dual formulations of $QVI(\Omega, K)$ as stated by the following well-known result:

Theorem 2.1 *Let $g(\cdot, y)$ and $h(\cdot, y)$ be affine functions for every $y \in X := \{y \in \mathbb{R}^n : y \in K(y)\}$. Then $x^* \in \mathbb{R}^n$ is a solution of $QVI(\Omega, K)$ iff there exist $\lambda^* \in \mathbb{R}^p$, $\mu^* \in \mathbb{R}^m$ such that (x^*, μ^*, λ^*) is a solution of the system*

$$\begin{cases} \Omega(x) \ + \ \sum_{i=1}^m \mu_i \nabla_x g_i(x, x) \ + \sum_{j=1}^p \lambda_j \nabla_x h_j(x, x) = \ 0 \\ \langle \mu, g(x, x) \rangle \ = \ 0 \\ \mu \geq 0, \ g(x, x) \leq 0, \ h(x, x) = 0. \end{cases} \tag{3}$$

Proof. Note that $QVI(\Omega, K)$ is equivalent to the problem of finding $x^* \in K(x^*)$ which is an optimal solution of the problem

$$\min_{x \in K(x^*)} \langle \Omega(x^*), x \rangle. \tag{4}$$

For $x = x^*$, the system (3) represents the classic KKT conditions for (4) which, under the assumptions of the theorem, are necessary and sufficient for the optimality of the point x^*.

□

In the paper, we will emphasize the role of the multipliers λ^* and μ^* which may receive suitable interpretations when $QVI(\Omega, K)$ represents the equilibrium condition of a network flow problem.

Note that our analysis is a direct extension of the classic one where the equilibrium solution is obtained by solving a suitable constrained extremum problem. Indeed, when $\Omega = \nabla f$ and K does not depend on x^*, then (3) collapses to the KKT conditions for the problem

$$\min \ f(x) \quad s.t. \quad x \in K.$$

3 The arc-flow quasi-variational model

In this section we consider the arc-flow quasi-variational model where the cost function depends on the flow passing through the arcs. This model extends to the variational context the classic minimum cost flow problem whose applications are well-known in the field of traffic equilibrium problems.

As stated in the previous section, the equilibrium condition of the arc-flow quasi-variational model is formulated by $QVI(c, K)$: find $f^* \in K(f^*)$ s.t.

$$\langle c(f^*), f - f^* \rangle \geq 0, \quad \forall f \in K(f^*) := \{f \in \mathbb{R}^n \ : \ \Gamma f = q(f^*), \ 0 \leq f \leq d(f^*)\}. \tag{5}$$

By means of the KKT conditions given by Theorem 2.1, we can state the following characterization of $QVI(c, K)$.

Theorem 3.1 *The following statements are equivalent:*
(i) f^* *is a solution of* $QVI(c, K)$*;*

(ii) there exist $(\lambda^*, \mu^*) \in \mathbb{R}^{p+n}$ *such that* (f^*, λ^*, μ^*) *is a solution of the system*

$$\begin{cases} c(f) \ + \ \Gamma^\top \lambda \ + \ \mu \ \geq \ 0 \\ \langle c(f) + \Gamma^\top \lambda + \mu, f \rangle \ = \ 0 \\ \langle f - d(f), \mu \rangle = 0 \\ 0 \leq f \leq d(f) \ , \ \Gamma f = q(f), \ \mu \geq 0 \end{cases} \tag{6}$$

(3i) $f^* \in K(f^*)$ *and there exist* $\lambda^* \in \mathbb{R}^p$ *and* $\mu^* \in \mathbb{R}^n_+$ *such that,* $\forall (i,j) \in A$*:*

$$0 < f^*_{ij} < d_{ij}(f^*) \Longrightarrow c_{ij}(f^*) = \lambda^*_i - \lambda^*_j, \quad \mu^*_{ij} = 0, \tag{7}$$

$$f^*_{ij} = 0 \Longrightarrow c_{ij}(f^*) \geq \lambda^*_i - \lambda^*_j, \quad \mu^*_{ij} = 0, \tag{8}$$

$$f^*_{ij} = d_{ij}(f^*) \Longrightarrow c_{ij}(f^*) = \lambda^*_i - \lambda^*_j - \mu^*_{ij}. \tag{9}$$

Proof. (i) $\Leftrightarrow$ (ii). It is enough to note that system (6) is equivalent to the KKT conditions associated with (5), which are equivalent to $QVI(c, K)$, according to Theorem 2.1.

(ii) $\Leftrightarrow$ (3i) It is straightforward.

□

Theorem 3.1 (3i) is a generalization of the classic Bellman conditions for minimum-cost flow problems and an interpretation of the multipliers in terms of potentials can be developed.

Observe that if there exists a positive flow between the nodes i and j then there is a positive difference of potentials between i and j. Indeed, by (3i) of Theorem 3.1, we can derive the following relations:

$$f_{ij}^* > 0 \Longrightarrow \lambda_i^* - \lambda_j^* \geq 0,$$

$$\lambda_i^* - \lambda_j^* < 0 \Longrightarrow f_{ij}^* = 0.$$

Moreover, the multipliers μ_{ij} in (9) can be interpreted as an additional cost that added to $c_{ij}(f^*)$ allows one to obtain the equality with the difference of potentials $\lambda_i^* - \lambda_j^*$.

Example 3.1 Consider the network in Fig.1:

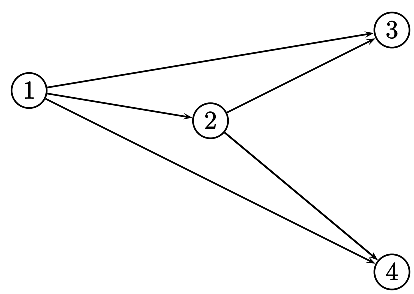

Figure 1: Traffic network in Example 3.1.

Set $q(f) := (-5, 0, 1 + f_{24}, 4 - f_{24})^\top$ and

$$d(f) := (7 - \frac{f_{12}}{5}, 6 - \frac{f_{13}}{4}, 4 - \frac{f_{14}}{4}, 5 - \frac{f_{23}}{3}, 4 + \frac{f_{23}}{6} - \frac{f_{24}}{4})^\top,$$

where $q(f)$ represents the balance at the nodes $\{1, .., 5\}$ and $d(f)$ the capacity on the arcs displayed in the lexicographic order. Note that the capacity decreases as the flow on the arc increases, which seems to be reasonable in the context of traffic problems. Let the cost function be $c(f) := Cf + b$ where C is the diagonal matrix with components on the diagonal given by the vector $c := (1, 4, 4, 2, 1)^\top$ and $b := (2, 23, 10, 8, 1)^\top$. Let us look for a solution f^* such that $0 \leq f^* < d(f^*)$ with $f_{13}^* = 0$, which implies that $f_{12}^* > 0$, $f_{23}^* > 0$. In such a case the system

(6) becomes:

$$\begin{cases} f^*_{12} + 2 \; - \lambda_1 + \lambda_2 = 0 \\ 23 - \lambda_1 + \lambda_3 \geq 0 \\ 4f^*_{14} + 10 \; - \lambda_1 + \lambda_4 \geq 0 \\ 2f^*_{23} + 8 \; - \lambda_2 + \lambda_3 = 0 \\ f^*_{24} + 1 \; - \lambda_2 + \lambda_4 \geq 0 \\ (4f^*_{14} + 10 \; - \lambda_1 + \lambda_4)f^*_{14} = 0 \\ (f^*_{24} + 1 \; - \lambda_2 + \lambda_4)f^*_{24} = 0 \\ f^*_{12} + f^*_{14} = 5 \\ f^*_{12} = f^*_{23} + f^*_{24} \\ f^*_{23} = 1 + f^*_{24} \\ f^*_{24} + f^*_{14} = 4 - f^*_{24} \\ 0 \leq f^* < d(f^*) \end{cases} \tag{10}$$

It is easy to show that $f^* = (5, 0, 0, 3, 2)$ with $\lambda = (7, 0, -14, -3)$ is a solution of the system (10) and therefore of (6). By Theorem 3.1, f^* is a solution of $QVI(c, K)$.

4 The path-flow quasi-variational model

In this section, we describe the path-flow quasi-variational model, where the variables are the flows on the paths of the network joining given couples origin/destination, and we relate it to the arc-flow model.

As stated in Section 2, the equilibrium condition of the quasi-variational path-flow model is given by $QVI(C, K)$: find $F^* \in K(F^*)$ s.t.

$$\langle C(F^*), F - F^* \rangle \geq 0, \quad \forall F \in K(F^*) := \{F \in \mathbb{R}^m \; : \; \Phi F = \rho(F^*), \; 0 \leq F \leq D(F^*)\}. \tag{11}$$

For the sake of simplicity, we use the same symbol K employed for the arc-flow model, to identify the feasible set.

Theorem 4.1 *The following statements are equivalent:*

(i) F^* is a solution of $QVI(C, K)$;

(ii) $F^* \in K(F^*)$ and, for every $R_s, R_r \in P_j$

$$F^*_s < D_s(F^*) \; , \; C_s(F^*) < C_r(F^*) \quad \Rightarrow \; F^*_r = 0, \quad \forall j = 1, \ldots, \ell;$$

(3i) There exists $\lambda^* \in \mathbb{R}^\ell$ and $\mu^* \in \mathbb{R}^m$ such that (F^*, λ^*, μ^*) is a solution of system

$$\begin{cases} C(F) \; - \; \Phi^\top \lambda \; + \; \mu \; \geq \; 0 \\ \langle C(F) - \Phi^\top \lambda + \mu, F \rangle \; = \; 0 \\ \langle F - D(F), \mu \rangle = 0 \\ 0 \leq F \leq D(F) \; , \; \Phi F = \rho(F), \; \mu \geq 0 \end{cases} \tag{12}$$

Proof. $(i) \Leftrightarrow (ii)$. It is analogous to Theorem 2.1 in [13].
$(i) \Leftrightarrow (3i)$. It is enough to note that (12) is equivalent to the KKT conditions for $QVI(C,K)$ so that by Theorem 2.1 we complete the proof. □

Remark 4.1 The equivalence between the first two statements in the previous theorem is a generalization to the quasi-variational case of an analogous result proved in [13].
In case $D_i = +\infty$, for every $i = 1, .., m$, statement (ii) is a generalization of the classic Wardrop principle, which states that a vector $F^* \in K(F^*)$ is an equilibrium flow if $\forall R_q, R_s \in P_j$ we have:

$$C_s(F^*) > C_q(F^*) \;\Rightarrow\; F_s^* = 0, \quad \forall j = 1, .., \ell.$$

We can give an interpretation of the KKT multipliers in the presence of capacity constraints.

Let $I_j(F^*) := \{i \in P_j : 0 < F_i^* < D_i(F^*)\}$, $I_j^0(F^*) := \{i \in P_j : F_i^* = 0\}$,

$$I_j^+(F^*) := \{i \in P_j : F_i^* = D_i(F^*)\}.$$

Then

$$\lambda_j^* = \begin{cases} \min\limits_{i \in I_j(F^*)} C_i(F^*) & \text{if } I_j(F^*) \neq \emptyset \\ \min\limits_{i \in I_j^0(F^*)} C_i(F^*) & \text{if } I_j(F^*) = \emptyset, \quad I_j^0(F^*) \neq \emptyset \\ \max\limits_{i \in I_j^+(F^*)} C_i(F^*) & \text{if } I_j(F^*) = \emptyset, \quad I_j^0(F^*) = \emptyset \end{cases}$$

for $j = 1, \ldots, \ell$, and

$$\mu_i^* = \begin{cases} \lambda_j^* - C_i(F^*), & \text{if } i \in P_j \textit{ and } F_i^* = D_i(F^*) \\ 0, & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, m$.

Example 4.1 Consider the network in Fig.2:
Set:

- $W = (W_1, W_2) = \{1/4, 1/5\}$ (couples origin-destination)
- $\rho(F^*) = (\rho_1(F^*), \rho_2(F^*))^T = (0.5 + F_2^* + F_3^*, 2F_1^* + F_2^* + 2.5)^T$ (traffic demand)
- $P_1 = \{R_1, R_2\} = \{(1,2,4), (1,3,4)\}$, $P_2 = \{R_3, R_4\} = \{(1,2,5), (1,3,5)\}$
- $D(F^*) = (5 - \dfrac{F_3^*}{2}, 4 - \dfrac{F_3^*}{2}, 6 - \dfrac{F_1^* + F_2^*}{2}, 10 - F_2^*)^T$.
- $C(F) = (C_1(F), C_2(F), C_3(F), C_4(F))^T$, where
  $C_1(F) = 3F_1 + F_3 + 1$, $C_2(F) = F_2 + F_4 + 2$, $C_3(F) = 3F_3 + F_1$, $C_4(F) = F_2 + F_4$.

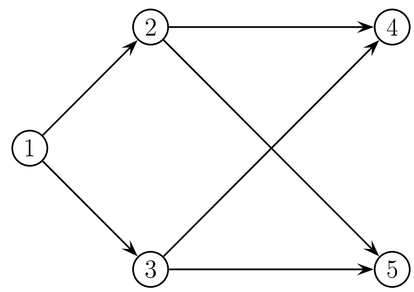

Figure 2: Traffic network in Example 4.1.

The cost function is $C(F) = BF + d$ where

$$B = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad d = (1, 2, 0, 0)^T$$

The feasible set is given by

$$K(F^*) := \{F \in \mathbb{R}^4 : \Phi F = \rho(F^*),\ F \geq 0\},$$

where $\Phi = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

A solution of system (12) is the following: $F^* = (2.5, 1.5, 2, 7)^T$, $(\lambda_1^*, \lambda_2^*) = (10.5, 8.5)^T, \mu^* = 0$.

By Theorem 4.1, F^* is a solution of $QVI(C, K)$.

The optimal cost is given by: $C(F^*) = (10.5, 10.5, 8.5, 8.5)^T$. Note that the multiplier λ_j^* coincides with the cost of any optimal path, having positive flow, which joins the couple W_j, $j = 1, 2$.

It is possible to obtain an arc-flow formulation of the path-flow model defining the feasible set by

$$\hat{K}(F^*) := \{f \in \mathbb{R}^n : f = \Delta F, \Phi F = \rho(F^*),\ F \geq 0\}.$$

where $\Delta = \{\delta_{is}\}$ is the Kronecker matrix, defined by

$$\delta_{is} = \begin{cases} 1, & \text{if } A_i \in R_s \\ 0, & \text{if } A_i \notin R_s \end{cases}, \quad i = 1, \dots n,\ s = 1, \dots m.$$

Next results provides a formulation of the path-flow model in terms of a QVI where the cost function depends on the flows on the arcs.

Theorem 4.2 *Assume that the cost* $C_s(F)$ *is given by*

$$C_s(F) = \sum_{i=1}^{n} \delta_{is} c_i(f), \;\; s = 1, .., m.$$

Then, F^* *is a solution of* $QVI(C, K)$ *if and only if it is a solution of the following problem:*

$$\text{find } f^* \in \hat{K}(F^*) \quad s.t. \quad \langle c(f^*), f - f^* \rangle \geq 0, \quad \forall f \in \hat{K}(F^*), \tag{13}$$

with $f^* := \Delta F^*$, $F^* \in K(F^*)$.

Proof. By the Kronecker matrix the flows on the arcs can be expressed in terms of the flows on the paths $f_i = \sum_{s=1}^{m} \delta_{is} F_s$. Therefore $f = \Delta F$ and $C(F) = \Delta^T c(f)$. Recalling that $f^* := \Delta F^*$ with $F^* \in K(F^*)$, the previous relations lead to the following equalities:

$$\langle C(F^*), F - F^* \rangle = c^T(f^*)\Delta(F - F^*) = \langle c(f^*), f - f^* \rangle,$$

which proves our assertion. □

The equivalent formulation of the path-flow model given by (13) can be generalized by adding, for example, further capacity constraints on the arcs or flow conservation constraints at the nodes in the feasible set $\hat{K}(F^*)$, even though the traffic demand is related to the couples origin-destination.

5 Equivalence between a model with capacities and one without capacities

In this section we analyse the possibility of reformulating a model with capacity constraints on the arcs or on the paths, with a model without capacities, but with a different operator.
Let us consider, at first, the arc-flow model with capacity constraints. We shall denote by $K_u(f^*)$ the uncapacitated feasible set (obtained by dropping the constraint $f \leq d(f)$), i.e.,

$$K_u(f^*) := \{f \in \mathbb{R}^n : \Gamma f = q(f^*), \; f \geq 0\},$$

where u stands for "unbounded". We preliminarily show that f^* is a solution of $QVI(c, K_u)$ if and only if it is a solution of the following quasi-variational inequality: find $f^* \in K_u(f^*)$ such that

$$\langle c(f^*) - \Psi'_f(\alpha; f^*, f^*), f - f^* \rangle \geq 0, \quad \forall f \in K_u(f^*), \qquad QVI_\alpha(c, K_u)$$

where $\Psi : \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ is a suitable function depending on the parameter $\alpha \in \mathbb{R}^k$ and Ψ'_f denotes the gradient of Ψ, w.r.t. its third component.
Let $X := \{f \in \mathbb{R}^n : f \in K(f)\}$ and consider the following preliminary result.

Theorem 5.1 *Assume that the function Ψ fulfils the following assumptions, for every $\alpha \in \mathbb{R}^k$:*

1. $\Psi(\alpha; f^*, f^*) = 0, \quad \forall f^* \in X;$
2. $\Psi(\alpha; f^*, \cdot)$ *is a differentiable concave function on* $K_u(f^*)$, $\quad \forall f^* \in X$.

Then f^* is a solution of $QVI_\alpha(c, K_u)$ if and only if it is a solution of the following problem: find $f^* \in K_u(f^*)$ such that

$$\langle c(f^*), f - f^* \rangle - \Psi(\alpha; f^*, f) \geq 0, \quad \forall f \in K_u(f^*), \tag{14}$$

Proof. Let f^* be a solution of (14). Taking into account assumption *1*, this is equivalent to the fact that f^* is an optimal solution of the problem

$$\min_{f \in K_u(f^*)} [\langle c(f^*), f - f^* \rangle - \Psi(\alpha; f^*, f)]. \tag{15}$$

By assumption *2*, we have that (15) is a convex problem, so that $QVI_\alpha(c, K_u)$ is a necessary and sufficient optimality condition for (15), which completes the proof. □

Let $\psi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ be defined by

$$\Psi(\alpha; f^*, f) := \sum_{i=1}^{n} \alpha_i (f_i^* - f_i), \quad \alpha_i \geq 0, \ i = 1, \ldots, n. \tag{16}$$

In this case $QVI_\alpha(c, K_u)$ becomes: find $f^* \in K_u(f^*)$ such that

$$\langle c(f^*) + \alpha, f - f^* \rangle \geq 0, \quad \forall f \in K_u(f^*).$$

Theorem 5.2 *Let ψ be defined by* (16)*. Then, f^* is a solution of $QVI(c, K)$ if and only if there exists $\alpha \in \mathbb{R}^n_+$ such that*

$$\begin{cases} f^* \textit{is a solution of } QVI_\alpha(c, K_u) \\ \sum_{i=1}^n \alpha_i (f_i^* - d_i(f^*)) = 0, \\ f^* \leq d(f^*) \end{cases}$$

Proof. First of all, we observe that Ψ fulfils the assumptions *1,2* of Theorem 5.1. Assume that f^* is a solution of $QVI_\alpha(c, K_f)$ for a suitable $\alpha \in \mathbb{R}^n_+$ with $\sum_{i=1}^n \alpha_i (f_i^* - d_i(f^*)) = 0$, $f^* \leq d(f^*)$.

By Theorem 5.1 we have that

$$\langle c(f^*), f - f^* \rangle \ \geq \ \Psi(\alpha; f^*, f), \quad \forall f \in K(f^*),$$

taking into account that $K(f^*) \subseteq K_u(f^*)$. Since, in our hypotheses,

$$\Psi(\alpha; f^*, f) = \sum_{i=1}^{n} \alpha_i (f_i^* - f_i) = \sum_{i=1}^{n} \alpha_i (d_i(f^*) - f_i) \geq 0, \ \forall f \in K(f^*),$$

and $f^* \le d(f^*)$, then f^* is a solution of $QVI(c, K)$.

Conversely, let f^* be a solution of $QVI(c, K)$. By Theorem 2.1, it is known that f^* is a solution of $QVI(c, K)$ if and only if there exists $(\mu^*, \lambda^*, s^*) \in \mathbb{R}^{n\times p\times n}$ such that $(f^*, \mu^*, \lambda^*, s^*)$ is a solution of the system

$$\begin{cases} c(f) \ + \ \Gamma^\top\lambda \ + \ \mu \ - s \ = \ 0 \\ \langle \mu, f - d(f)\rangle \ = \ \langle s, f\rangle = 0 \\ f \le d(f) \ , \ \Gamma f = q(f), \ f \ge 0 \\ \mu \ge 0, \ s \ge 0 \end{cases} \tag{17}$$

Consider now $QVI_\alpha(c, K_u)$. If we put $\alpha := \mu^*$, then $QVI_{\mu^*}(c, K_u)$ becomes

$$\langle c(f^*) + \mu^*, f - f^*\rangle \ge 0, \quad \forall f \in K_u(f^*).$$

Still by Theorem 2.1, we have that f^* is a solution of $QVI_{\mu^*}(c, K_u)$ if and only if there exists $(\lambda^0, s^0) \in \mathbb{R}^{p\times n}$ such that (f^*, λ^0, s^0) is a solution of the system

$$\begin{cases} c(f) \ + \ \mu^* + \ \Gamma^\top\lambda \ \ - s \ = \ 0 \\ \langle s, f\rangle = 0 \\ \Gamma f = q(f), \ f \ge 0, \ s \ge 0 \end{cases} \tag{18}$$

Therefore, if $(f^*, \mu^*, \lambda^*, s^*)$ is a solution of (17) then (f^*, λ^*, s^*) is a solution of (18) which yields that f^* solves $QVI_{\mu^*}(c, K_u)$. □

By the proof of Theorem 5.2, it follows that the parameter α, which ensures the equivalence between the capacitated and the uncapacitated QVI, can be chosen as the multiplier μ^*, associated with the constraint $f \le d(f)$.

With similar arguments used for the arc-flow model, it is possible to make a reduction of a path-flow model with capacity constraints to a model without capacities.

Under analogous assumptions to those considered for the arc-flow model, F^* is a solution of the quasi-variational inequality $QVI(C, K)$ if and only if it is a solution of the following $QVI_\alpha(C, K_U)$: find $F^* \in K_U(F^*) := \{F \in \mathbb{R}^m : \Phi F = \rho(F^*) : F \ge 0\}$ such that

$$\langle C(F^*) - \Psi'_F(\alpha; F^*, F^*), F - F^*\rangle \ge 0, \quad \forall F \in K_U(F^*),$$

for a suitable $\Psi : \mathbb{R}^k \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$ depending on the parameter $\alpha \in \mathbb{R}^k$ and where Ψ'_F denotes the gradient of Ψ, w.r.t. its third component. Let

$$\Psi(\alpha; F^*, F) := \sum_{i=1}^m \alpha_i(F_i^* - F_i), \quad \text{with } \alpha_i \ge 0, \ i = 1, \ldots, m.$$

In this case $QVI_\alpha(C, K_U)$ becomes: find $F^* \in K_U(F^*)$ such that

$$\langle C(F^*) + \alpha, F - F^*\rangle \ge 0, \quad \forall F \in K_U(F^*).$$

Theorem 5.3 *Let Ψ be defined as above. Then F^* is a solution of $QVI(C,K)$ if and only if there exists $\alpha \in \mathbb{R}^m_+$ such that*

$$\begin{cases} F^* \text{ is a solution of } QVI_\alpha(C, K_U), \\ \sum_{i=1}^m \alpha_i(F_i^* - D_i(F^*)) = 0, \\ F^* \leq D(F^*) \end{cases}$$

Note that the parameter α coincides with the multiplier μ^*, related to the constraint $F \leq D(F^*)$, in the system (12) associated with $QVI(C,K)$.

6 Concluding remarks

We have analysed a quasi-variational formulation of the minimum cost flow problem by exploiting its KKT optimality conditions given by system (3) which has allowed us to emphasize the role of the KKT multipliers in the context of traffic network problems. The importance of KKT optimality condition also arises in the solution methods for variational inequalities, since a great number of algorithms for VI is based on the solution of system (3) by means of classic methods of numerical analysis. Analogously in the quasi-variational case, it is possible to solve directly the KKT-system (3) (see e.g., [7]).

A further widely used class of algorithms is based on the reformulation of VI by means of a suitable constrained extremum problem, whose objective function is the so called gap function, i.e., a function which is non negative on the domain and that takes the value zero in correspondence of a solution of VI. A general theory concerning gap functions for QVI has been developed (see, e.g. [9, 16]) and therefore could be used here for solving our traffic problem. Line search algorithms for gap functions have shown to be particularly efficient in the context of VI: it might be an interesting research line to deepen the analysis of such methods for QVI.

A particular mention in the field of solution methods for QVI must be deserved to the recent paper [1] which provides a characterization of some classes of QVI whose solution set is a union of solution sets of suitable VI. It is possible to show that the QVI, introduced in Sections 3 and 4, belong to one of these classes. Finally, we recall projection methods which are a standard method for solving VI and have been extended under suitable assumptions to QVI, allowing to solve a QVI by means of a sequence of VI [3].

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